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SESHADRI CONSTANTS ON JACOBIAN OF CURVES

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Abstract. We compute the Seshadri constants on the Jacobian of hyperelliptic curves, as well as of curves with genus three and four. For higher genus curves we conclude that if the Seshadri constants of their Jacobian are less than 2, then the curves must be hyperelliptic.

1. Introduction and Statement of Results

Let X be a smooth complex projective variety. Let L be an ample line bundle. Let $p \in X$ be a point. Define the Seshadri constant of L at p to be the real number

$$\epsilon(L,p) := \inf\left\{\frac{C\cdot L}{\operatorname{mult}_p C} \mid p \in C \subset X\right\}.$$

Here the infimum is taken over all reduced curves C passing through p, and $\operatorname{mult}_p C$ is the multiplicity of C at p. Another equivalent definition is

$$\epsilon(L,p) = \sup\{\epsilon \mid f^*L - \epsilon E \text{ is nef}\},$$

where $f : Bl_pX \to X$ is the blow-up of X at p and E is the exceptional divisor.

The Seshadri constant indicates how far the ample divisor is from the boundary of the ample cone near point p, and thus measures positivity, or ampleness locally. The study of Seshadri constants has drawn increasing interest during recent years. For properties of Seshadri constants see [1] and [6].

In the case of abelian varieties, it is known that a general element in the moduli space of principally polarized abelian varieties of dimension g has Seshadri constant very close to its maximum upper bound ([4]). On the other hand, there are some special abelian varieties, namely Jacobian, which have relatively small Seshadri constants. We will discuss some cases in this paper.

Let C be a smooth projective algebraic curve over $\mathbf{C}$ with genus $g = g(C) \geq 2$. Denote Θ to be the theta divisor of $J(C)$, its Jacobian (recall $J(C) = \operatorname{Pic}^0(C)$). Since abelian varieties are homogeneous spaces, we can define $\epsilon = \epsilon(\Theta, 0) = \epsilon(\Theta, p)$ for any $p \in J(C)$.

It is known that $1 < \epsilon \leq \sqrt{g}$, and if C is hyperelliptic, then $\epsilon \leq \frac{2g}{g+1}$ ([4], [5]). In particular, if $g = 2$, then C is hyperelliptic and it is known that $\epsilon = \frac{4}{3}$ ([6]). The problem becomes very interesting even when $g = 3$. The point here is to see if the Seshadri constants can be their maximum, i.e., $\sqrt{g}$, thus most of the time irrational, or always less than their maximum – and thus more likely rational. While all the

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existing examples suggest the latter, we investigate this problem in detail, mainly by looking at the cases when $\epsilon \le 2$.

Our main result is the following theorem:

Theorem 1.1. *Assume the Picard number of $J(C)$ is one. Then:*

(1) *If C is hyperelliptic, then $\epsilon = \frac{2g}{g+1}$.*
(2) *If $g = 3$ and C is not hyperelliptic, then $\epsilon = \frac{12}{7}$.*
(3) *If $g = 4$ and C is not hyperelliptic, then $\epsilon = 2$.*
(4) *If $g \ge 5$ and C is not hyperelliptic, then $\epsilon \ge 2$.*

Part (4) of the theorem can be restated as:

Corollary 1.2. *If $g \ge 5$ and $\epsilon < 2$, then C is hyperelliptic.*

Remark 1.3. (1) For the ease of calculation on the Neron-Severi group of the symmetric product C_2, we need that it is generated by a fiber and the diagonal, i.e., its Picard number is 2. That is true if C is of general moduli. We need this condition throughout this paper. But this restriction, however, seems to be not essential.

(2) We can also locate all the special curves that give relatively small ratios in cases (1) to (3).

2. Proof of Theorem: Hyperelliptic Case

The following observation, while straightforward, points out where we want to find special curves that give the exact value of the Seshadri constants.

Lemma 2.1. *If C' is an irreducible curve in $J(C)$ such that $\frac{C' \cdot \Theta}{\mathrm{mult}_0 C'} \le 2$, then for any divisor D with $D \equiv k\Theta$ and $\mathrm{mult}_0 D \ge 2k$, we have $C' \subset D$.*

If C is hyperelliptic, then the case of $k = 1$ in Lemma 2.1 reads $D \equiv \Theta$ and $\mathrm{mult}_0 D \ge 2$, which we denote as (*). For $d \ge 2$, let C_d be the d-fold symmetric product of C (the set of effective divisors of degree d on C).

Proposition 2.2. *Let $u : C_d \to J(C)$ be the Abel-Jacobi map. Then*

$$\bigcap_{(*)} D = u(C_2).$$

Proof. Let L be a hyperelliptic line bundle on C. Let p_0 be a ramification point of the g^1_2; so $L = \mathcal{O}_C(2p_0)$. We fix a translation of the Abel-Jacobi map $u : C_d \to J(C)$ by sending $Y \in C_d$ to $Y - \deg(Y) \cdot p_0 \in J(C)$, and for simplicity we ignore the p_0 part for representation of points in $J(C)$ in our proof. Recall that $\phi : C_{g-3} \to C_{g-1}, Y \to Y + L$ maps C_{g-3} birationally and surjectively to $\mathrm{Sing}(\Theta)$ in this case ([3]).

For any $Y \in C_{g-3}$, define $D_Y = \Theta - Y$. It translates $Y + 2p_0 \in \mathrm{Sing}(\Theta)$ to $0 \in D_Y$. Thus $D_Y \equiv \Theta$ and $\mathrm{mult}_0 D \ge 2$. It suffices to show that $\bigcap D_Y = u(C_2)$.

It is obvious that $u(C_2) \subset \bigcap D_Y$, since for any point $(p, q) \in C_2$ we can rewrite it as $(p + q + Y) - Y \in D_Y$ for any $Y \in C_{g-3}$.

On the other side, any points in $\bigcap D_Y$ can be represented as $D - Y$ for some $D \in C_{g-1}$ and $Y \in C_{g-3}$. Also, since it is in the intersection, for any $F \in C_{g-3}$, there exists $E \in C_{g-1}$ such that $D - Y = E - F$, i.e., $D - Y + F$ is (equivalent to) an effective divisor for any F. We claim $D - Y$ itself must be effective, and since it has degree 2, it is in $u(C_2)$.

Pick a representation of $D - Y$ such that Y contains no ramification point of g^1_2. First assume that D contains no hyperelliptic pair. If $D - Y$ is not effective, pick $p \in Y$ but $p \notin D$, and let $L = \mathcal{O}_C(p + p')$. Choose $F = Y - p + p'$. Then $D - Y + F = D - p + p'$. But the linear system $|D - p + p'|$ is empty, since otherwise it must contain a multiple of hyperelliptic pairs and base points, which will lead to $p \in D$.

If D has some hyperelliptic pairs, then cancel as many points in $D - Y$ as possible until either $D - Y$ is effective or D runs out of hyperelliptic pairs and reduce to a similar situation in the first case. □

If C is hyperelliptic, and $\mathrm{rk}(NS(C_2)) = 2$, then $NS(C_2)$ is generated by F and Δ, the image of a fiber and the diagonal from the natural map $C \times C \to C_2$. There is a rational curve, call it $\mathbf{P}^1$, that consists of hyperelliptic pairs $\{(p, q) \in C_2 \mid \mathcal{O}_C(p+q) = L\}$. Also denote $u^*(\Theta)$ still as Θ. We list the numerical properties of $NS(C_2)$ below.

Lemma 2.3. *With notation as above, we have:*

(1) $\Theta = (g + 1)F - \frac{1}{2}\Delta$ *and* $\mathbf{P}^1 = 2F - \frac{1}{2}\Delta$.
(2) $F^2 = 1, F \cdot \Delta = 2, \Delta^2 = 4 - 4g$.

The Abel-Jacobi map $u : C_2 \to J(C)$ contracts $\mathbf{P}^1$ and is isomorphic outside of $\mathbf{P}^1$. Now let C'' be an irreducible curve in C_2 not contracted by u and let $C' = u(C'')$. Then

$$\frac{C'' \cdot \Theta}{C'' \cdot \mathbf{P}^1} = \frac{C' \cdot \Theta}{\mathrm{mult}_0 C'}.$$

So our theorem in the hyperelliptic case follows from the following proposition.

Proposition 2.4. *Among all irreducible curves in* C_2 *not contracted by* u, Δ *is the only curve with minimum ratio* $\frac{\Delta \cdot \Theta}{\Delta \cdot \mathbf{P}^1} = \frac{2g}{g+1}$.

Proof. Since $\Delta \cdot \Theta = 4g$ and $\Delta \cdot \mathbf{P}^1 = 2g + 2$, we have $\frac{\Delta \cdot \Theta}{\Delta \cdot \mathbf{P}^1} = \frac{4g}{2g+2} = \frac{2g}{g+1}$.

Let $C_0 = aF + b\Delta \subset C_2$ be an irreducible curve not contracted by u. Then $C_0 \cdot \mathbf{P}^1 = a + (2g + 2)b \geq 0$ and $C_0 \cdot \Theta = (a + 4b)g > 0$. Then

$$\frac{\Delta \cdot \Theta}{\Delta \cdot \mathbf{P}^1} = \frac{(a + 4b)g}{a + (2g + 2)b} > \frac{2g}{g + 1} \iff a > 0.$$

But if $a \leq 0$, then we must have $b > 0$ since $C_0 \cdot \Theta = (a + 4b)g > 0$. Now we have $C_0 \cdot \Delta = 2a + b(4 - 4g) < 0$. Since both C_0 and Δ are irreducible, $C_0 = \Delta$. □

Remark 2.5. A little more detailed calculation shows that Δ is actually the only curve whose corresponding ratio is less than two.

3. Proof of the Theorem: Non-hyperelliptic Case

If C is non-hyperelliptic, then choose the case $k = 2$ in Lemma 2.1 which reads $D \equiv 2\Theta$ and $\mathrm{mult}_0 D \geq 4$. Denote this linear system as $|2\Theta|_{00}$. So we look at the base locus of $|2\Theta|_{00}$. Here we need the following result of Welters.

Proposition 3.1 (Welters [7]). $Bs(|2\Theta|_{00}) = \lambda(C \times C)$. *Here* $\lambda : C \times C \to J(C)$, $\lambda(p, q) = p - q$, *is the difference map.*

Remark 3.2. Welters' theorem is true for all curves with $g = 3$ or $g \geq 5$. For $g = 4$ the base locus has two more isolated points, which will not affect our proof since we are looking at curves inside the base locus.

In this case we look at the Neron-Severi group in $C \times C$. It is generated by fibers F_1, F_2 and the diagonal Δ. We list their numerical properties below.

Lemma 3.3. *With notation as above, we have:*

(1) $\lambda^*\Theta = (g-1)(F_1+F_2)+\Delta$.
(2) $F_i^2 = 0$, $F_1 \cdot F_2 = F_i \cdot \Delta = 1$, $\Delta^2 = 2-2g$, $i = 1, 2$.

Since C is non-hyperelliptic, the difference map λ contracts the diagonal Δ to $0 \in J(C)$ and is isomorphic outside Δ. This enables us, as similarly in the hyperelliptic case, to shift the computation from the ratio $\frac{C\cdot\Theta}{\mathrm{mult}_0 C}$ in $J(C)$ to the ratio of the intersection numbers in the Neron-Severi group, which are well understood. Specifically, let C'' be an irreducible curve in $C \times C$ not contracted by λ and let $C' = u(C'')$. Then

$$\frac{C''\cdot\lambda^*\Theta}{C''\cdot\Delta} = \frac{C'\cdot\Theta}{\mathrm{mult}_0 C'}.$$

So our theorem in the non-hyperelliptic case follows from the following proposition.

Proposition 3.4. *With notation as above:*

(1) *If $g = 3$, the minimum ratio $\frac{C''\cdot\lambda^*\Theta}{C''\cdot\Delta}$ is $\frac{12}{7}$ for curves in $C \times C$, and is achieved by one curve.*
(2) *If $g = 4$, the minimum ratio $\frac{C''\cdot\lambda^*\Theta}{C''\cdot\Delta}$ is 2 for curves in $C\times C$, and is achieved by more than one curve.*
(3) *If $g \geq 5$, then $\frac{C''\cdot\lambda^*\Theta}{C''\cdot\Delta} \geq 2$ for all curves in $C \times C$ not contracted by λ.*

Proof. (1) $g = 3$: In this case, the canonical system embeds C as a plane quartic. Let $\mathcal{O}_C(1)$ be its hyperplane section. Consider the curve

$$C_0 = \{(p,q) \mid \mathcal{O}_C(p+q+2r) = \mathcal{O}_C(1) \text{ for some } r \in C\} \subset C_2.$$

Write $C_0 = aF + b\Delta$. Since $C_0 \cdot \Delta = 56$ (twice the number of bitangents) and $C_0 \cdot F = 10$ (degree of the ramification divisor of the dual curve's g_3^1), we can solve for a and b and get $C_0 = 16F - 3\Delta$. C_0 is irreducible since it is isomorphic to C via $p+q \to r$. Pulling it back to $C \times C$ we get a curve $C_0^{''} = 16(F_1+F_2)+6\Delta$. Now

$$\frac{C_0^{''}\cdot\lambda^*\Theta}{C_0^{''}\cdot\Delta} = \frac{[16(F_1+F_2)-6\Delta]\cdot[2(F_1+F_2)+\Delta]}{[16(F_1+F_2)-6\Delta]\cdot\Delta} = \frac{96}{56} = \frac{12}{7}.$$

To claim that $\frac{12}{7}$ is the minimum ratio, let $C^{''} = aF_1 + bF_2 + C\Delta$ be any irreducible curve in $C \times C$ not contracted by λ. If $C^{''} \neq C_0^{''}$, then $C^{''} \cdot C_0^{''} = 10(a+b)+56c \geq 0$. So if $c \geq 0$, then

$$\frac{C^{''}\cdot\lambda^*\Theta}{C''\cdot\Delta} = \frac{3(a+b)}{a+b-4c} \geq 3 > \frac{12}{7}.$$

If $c < 0$, then

$$\frac{C^{''}\cdot\lambda^*\Theta}{C''\cdot\Delta} = \frac{3(a+b)}{a+b-4c} \geq \frac{3(a+b)}{a+b+\frac{5}{7}(a+b)} = \frac{7}{4} > \frac{12}{7}.$$

This shows that the only curve that achieves the minimum ratio $\frac{12}{7}$ is $C_0^{''}$.

(2) $g = 4$: In this case C has two g_3^1's. Let L be one g_3^1. Consider the curve $C_0 = \{(p,q) \mid |L-p-q| > 0\} \subset C_2$. Since $C_0 \cdot F = 2$ and $C_0 \cdot \Delta = 12$ (degree

of ramification divisor of L), we find that $C_0 = 3F - \frac{1}{2}\Delta$. Lift to $C \times C$ to get $C_0^{''} = 3(F_1 + F_2) - \Delta$. A similar calculation as above shows that

$$\frac{C_0^{''} \cdot \lambda^* \Theta}{C_0^{''} \cdot \Delta} = \frac{24}{12} = 2,$$

and it is the minimum ratio that can be achieved on $C \times C$.

Note that in this case there is another curve (from the other g_3^1) that gives the minimum ratio. The reason is that in this case $C_0^2 = 0$, while in the case of $g = 3$ we have $C_0^2 < 0$ (thus unique).

(3) $g \geq 5$: Assume C has a g_d^1 ($d \geq 3$), call it L. As in (2), consider the curve $C_0 = \{(p, q) \mid |L - p - q| > 0\} \subset C_2$. Then $C_0 \cdot F = d - 1$ and $C_0 \cdot \Delta = 2d + 2g - 2$ (degree of ramification divisor of L). Thus $C_0 = dF - \frac{1}{2}\Delta$. Lifting to $C \times C$ we get $C_0^{''} = d(F_1 + F_2) - \Delta$. Now first we have

$$\frac{C_0^{''} \cdot \lambda^* \Theta}{C_0^{''} \cdot \Delta} = \frac{dg}{d + g - 1} > 2.$$

Secondly, for any irreducible $C^{''} = aF_1 + bF_2 + c\Delta \subset C \times C$ not contracted by λ, either $C^{''} \cdot C_0^{''} < 0$, or

$$\frac{C^{''} \cdot \lambda^* \Theta}{C^{''} \cdot \Delta} \geq \frac{d + g - 1}{d} \geq 2 \text{ if } d \leq g - 1.$$

Since the Brill-Noether number for g_d^1 is non-negative if $d \geq \frac{g+2}{2}$, both ratios above are at least 2. If the minimum ratio $\frac{C^{''} \cdot \lambda^* \Theta}{C^{''} \cdot \Delta} < 2$, then all the curves C_0 must be reducible and contain an irreducible component C_1 whose lift to $C \times C$ gives a small ratio. It is easy to see that $C_1^2 < 0$, thus unique in C_2. This is certainly impossible. (For example, if there are two different g_d^1 for some d, then to have a common component for corresponding $C_0 \subset C_2$, one coordinate has to be a base point of g_d^1. Thus it is a linear combination of fibers, and since the component is irreducible, it is a fiber. But for a fiber, the corresponding ratio is $g > 2$.)

Note that the minimum ratio exists and can be achieved if $\frac{dg}{d+g-1} \leq \frac{d+g-1}{d}$, which is equivalent to $\frac{dg}{d+g-1} \leq \sqrt{g}$, or $d \leq \sqrt{g} + 1$. □

4. Other Related Problems of Seshadri Constants

For non-hyperelliptic cases when $g \geq 5$, to find the Seshadri constants, the first step is to look at the curves in C_2. It is related to the problem whether the cone of effective curves of C_2 is closed. If it is, the curve from the boundary will give a better upper bound of $\epsilon(\Theta)$ which is less than $\sqrt{g}$. In all special cases that we have discussed (hyperelliptic, small genus, curves with g_d^1 for small d), the cone is closed. For the general case there is some indication that the curve from one boundary (the other one being the diagonal), if closed, will give the ratio $\frac{gq}{p}$ where (p, q) is the primitive solution of Pell's equation $x^2 - gy^2 = 1$, if g is not a square. The following example gives some indication that it could be true.

Example 4.1. If C is a plane quintic (i.e., genus is 6), consider the curve $C_0 = \{(p, q) \mid |\mathcal{O}_C(1) - p - q - 2r| > 0\} \subset C_2$. Then C_0 is irreducible and $C_0 = 50F - 7\Delta$. Since any curve $C^{'} \subset C_2$ satisfies $C_0 \cdot C^{'} \geq 0$, a calculation shows that on $C \times C$, $\frac{C^{''} \cdot \lambda^* \Theta}{C^{''} \cdot \Delta} \geq \frac{12}{5}$ holds for all irreducible curves (except Δ). Note that the bound $\frac{12}{5}$ is what the conjecture gives. Also note that one expects small values for plane curves that are special in the moduli of curves, so general curves of genus 6 must also satisfy that bound.

For $g \geq 5$, if $\epsilon < 2$, then it follows that C is hyperelliptic. In all cases, hyperelliptic curves give us the smallest Seshadri constants. From the known result ([2]) of $Bs(|2\Theta|_{00})$ in dimension 4, it is easy to see that:

If A is an indecomposable principally polarized abelian variety of dimension 4 and $\epsilon(\Theta) < 2$, then A is the Jacobian of a hyperelliptic curve C of genus 4.

It is very reasonable to ask the same question for any genus and seems like it could be true.

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