Abstract. It is proved that the Kähler angle of the pseudo-holomorphic sphere of constant curvature in complex Grassmannians is constant. At the same time we also prove several pinching theorems for the curvature and the Kähler angle of the pseudo-holomorphic spheres in complex Grassmannians with non-degenerate associated harmonic sequence.

1. Introduction

In this paper we study conformal minimal two-spheres in complex Grassmann manifolds by using the harmonic sequence. Given a harmonic map $\varphi$ of surfaces $M$ into the complex Grassmannian $G_{k,n}$, by using the $\partial'$-transform Chern and Wolfson ([3], [10]) obtained the following harmonic sequence associated to $\varphi$:

$$\varphi = \varphi_0 \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_j \xrightarrow{\partial'} \cdots,$$

where $\varphi_{j+1} = \partial' \varphi_j$, $j = 0, 1, \cdots$, and $\varphi_j : M \to G_{k_j,n}$ are harmonic maps, $k_j = \text{rank}(\varphi_j)$. If $\varphi_j$ is anti-holomorphic, then $k_j+1 = 0$. When $\varphi$ is holomorphic we call $\varphi_j$ a pseudo-holomorphic curve generated by $\varphi$. Such curves with the induced metrics from the associated complex Grassmann manifolds form a class of minimal immersions. When $k_j = k_j+1$ we say that $\varphi_j$ is non-degenerate. When $k_j = k_j+1$ for all $j$ we say that the harmonic sequence associated to the map $\varphi$ is non-degenerate.

When specialized to $G_{1,n} = \mathbb{CP}^{n-1}$, any pseudo-holomorphic curve is obtained from a holomorphic curve projected into $\mathbb{CP}^{n-1}$. Calabi ([2]) showed that any simply connected holomorphic curve in $\mathbb{CP}^{n-1}$ is completely determined, up to holomorphic isometries of $\mathbb{CP}^{n-1}$, by its induced metric. Calabi also showed that a simply connected holomorphic curve of constant curvature in $\mathbb{CP}^{n-1}$ is the Veronese curve, up to unitary equivalence. For a pseudo-holomorphic curve in $\mathbb{CP}^{n-1}$, Bolton, Jensen, Rigoli and Woodward ([1]) showed that, up to a holomorphic isometry of $\mathbb{CP}^{n-1}$, the harmonic sequence determined by any linearly full conformal minimal immersion of constant curvature in $\mathbb{CP}^{n-1}$ is the Veronese sequence, in which each map is a minimal immersion with constant curvature and constant Kähler angle.
It is well known that the rigidity fails for pseudo-holomorphic curves or holomorphic curves generalized to $G_{k;n}$ ([5], [14]). For example, Chi and Zheng ([5]) classified the holomorphic curves of the Riemann sphere into $G_{2;4}$ with the induced constant curvature 2 into two classes, up to unitary equivalence, in which none of the curves are congruent. Let $\varphi : S^2 \to G_{k;n}$ be a pseudo-holomorphic curve in a complex Grassmannian $G_{k;n}$. Problem: Is the Kähler angle $\theta(\varphi)$ of $\varphi$ constant when its Gauss curvature $K(\varphi)$ is constant? What are the relationships between the Kähler angle and the Gauss curvature of $\varphi$ and its ramification index? In this paper we will investigate these questions.

In the second and third sections of this paper we obtain some fundamental formulas for pseudo-holomorphic curves in complex Grassmann manifolds. In the fourth section, by using these formulas we prove that the curvatures of pseudo-holomorphic curves are equal to $4/N$ ($N$ is a positive integer) if these curvatures are constant (this result was proved by Chi and Zheng in [5])(Theorem 4.1), and prove that Kähler angles of pseudo-holomorphic curves of constant curvature are constant (Theorem 4.2). In this section, we also give a harmonic sequence, in which each map is a minimal immersion with constant curvature and constant Kähler angle.

In the final section, we give some pinching theorems for pseudo-holomorphic curves with the associated non-degenerate harmonic sequence for curvatures and Kähler angles (Theorems 5.2, 5.6 and 5.7). At the same time we also show that the Kähler angle of a pseudo-holomorphic curve is independent of its ramification index under the assumption of Theorem 5.2.

2. Minimal Immersions and Harmonic Sequences

Let $U(n)$ be the unitary group. Let $M$ be a simply connected domain in the unit sphere $S^2$ and let $(z, \overline{z})$ be a complex coordinate on $M$. We take the metric $ds_M^2 = dzd\overline{z}$ on $M$. Denote

$$\partial = \frac{\partial}{\partial z}, \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}}, \quad A_z = \frac{1}{2} s^{-1} \partial s, \quad A_{\overline{z}} = \frac{1}{2} s^{-1} \overline{\partial} s.$$  

Let $s : M \to U(n)$ be a smooth map; then $s$ is a harmonic map if and only if it satisfies the following equation ([9]):

$$\overline{\partial} A_z = [A_z, A_{\overline{z}}].$$  

If $s : S^2 \to U(n)$ is a harmonic map, then $s$ is a conformal map; so $s$ is a minimal immersion. Let $\omega = g^{-1}dg$ be a Maurer-Cartan form on $U(n)$, and let $ds^2_{U(n)} = \frac{1}{s} \text{tr} \omega\omega^*$ be the metric on $U(n)$. Then the metric induced by $s$ on $S^2$ is given by

$$ds^2 = -\text{tr} A_z A_{\overline{z}} dzd\overline{z}. $$  

Let $G_{k,n}$ be the complex Grassmann manifold consisting of all complex $k$-dimensional subspaces in $C^n$. Here we consider $G_{k,n}$ as the set of Hermitian orthogonal projections onto a $k$-dimensional subspace in $C^n$, i.e., $G_{k,n} = \{ \varphi \text{ is the Hermitian orthogonal projection onto a } k\text{-dimensional subspace in } C^n \}$. Then $\varphi : S^2 \to G_{k,n}$ is a Hermitian orthogonal projection onto a $k$-dimensional subbundle $\eta \subset S^2 \times C^n$, and $s = \varphi - \varphi^+$ is a map from $S^2$ into $U(n)$. It is well known that $\varphi$ is harmonic if and only if $s$ is harmonic. If $\varphi^+ \overline{\partial} \varphi = 0$ or $\varphi^+ \partial \varphi = 0$, we call $\varphi$ a holomorphic curve or an anti-holomorphic curve in $G_{k,n}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Using \( \varphi \), the harmonic sequences (see [3], [10]) are given by
\[
(3) \quad \varphi = \varphi_0 \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_{\alpha} \xrightarrow{\partial'} \cdots,
\]
and
\[
(4) \quad \varphi = \varphi_0 \xrightarrow{\partial''} \varphi_{-1} \xrightarrow{\partial''} \cdots \xrightarrow{\partial''} \varphi_{-\alpha} \xrightarrow{\partial''} \cdots,
\]
where \( \varphi_{\alpha} : S^2 \times C^n \to \text{Im}(\varphi_{\alpha-1}' \partial \varphi_{\alpha-1}) \) and \( \varphi_{-\alpha} : S^2 \times C^n \to \text{Im}(\varphi_{-\alpha+1}' \overline{\partial} \varphi_{-\alpha+1}) \) are Hermitian orthogonal projections, \( \alpha = 1, 2, \cdots \).

**Proposition 2.1 (\text{[7]}).** For (3) and (4), we have
\[
\varphi_\alpha \partial \varphi_\alpha = -\varphi_{\alpha-1}' \partial \varphi_{\alpha-1}, \quad \varphi_\alpha \overline{\partial} \varphi_\alpha = -\varphi_{\alpha-1} \overline{\partial} \varphi_{\alpha-1},
\]
where \( \alpha = \pm 1, \pm 2, \cdots \).

If \( \varphi_0 \) is a holomorphic curve in (3) or an anti-holomorphic curve in (4), then elements in (3) or (4) are finite and are mutually orthogonal. If there exists a holomorphic curve \( \varphi_0 \) in \( G_{k,n} \) such that \( \varphi \) is an element in the harmonic sequence (3), i.e., \( \varphi = \varphi_\alpha : S^2 \to G_{k,n} \) belongs to the harmonic sequence
\[
(5) \quad 0 \xrightarrow{\partial'} \varphi_0 \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi = \varphi_\alpha \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_{\alpha} \xrightarrow{\partial'} 0,
\]
then we call \( \varphi \) a pseudo-holomorphic curve in complex Grassmann manifolds, and \( \alpha_0 \) is called the length of the harmonic sequence (5).

Now we assume that \( \varphi = \varphi_{\alpha} : S^2 \to G_{k,n} \) is a pseudo-holomorphic curve. Then we may choose the local unitary frame \( e_1, e_2, \cdots, e_n \) on \( S^2 \times C^n \) such that \( e_{k_{\alpha+1}}, \cdots, e_{k_{\alpha}} \) span \( \text{Im}(\varphi_{\alpha-1}' \partial \varphi_{\alpha-1}) \), where \( k_\alpha = \text{rank}(\varphi_{\alpha-1}' \partial \varphi_{\alpha-1}), \alpha = 1, 2, \cdots, k_0 = \text{rank}(\varphi_0) \).

Let \( W_\alpha = (e_{k_{\alpha+1}}, e_{k_{\alpha}}, \cdots, e_{k_{\alpha}}) \) be an \((n \times k_{\alpha})\)-matrix. Then we have
\[
(6) \quad \varphi_\alpha = W_\alpha W_\alpha^*,
\]
\[
(7) \quad W_\alpha^* W_\alpha = I_{k_{\alpha} \times k_{\alpha}}, \quad W_\alpha^* W_{\alpha+1} = 0, \quad W_{\alpha}^* W_{\alpha-1} = 0.
\]
By (7) and a straightforward computation we obtain
\[
(8) \quad \begin{cases} 
\partial W_\alpha = W_{\alpha+1} \Omega_\alpha + W_\alpha \Psi_\alpha, \\
\overline{\partial} W_\alpha = -W_{\alpha-1} \Omega_{\alpha-1}^* + W_\alpha \Psi_\alpha^*,
\end{cases}
\]
where \( \Omega_\alpha \) is a \((k_{\alpha+1} \times k_{\alpha})\)-matrix and \( \Psi_\alpha \) is a \((k_{\alpha} \times k_{\alpha})\)-matrix, \( \alpha = 0, 1, 2, \cdots \).

It is well known that \( \Omega_\alpha = 0 \) or \( \Omega_{\alpha-1} = 0 \) in (8) if and only if \( \varphi_\alpha \) is anti-holomorphic or holomorphic. It is very evident that integrability conditions for (8) are
\[
(9) \quad \overline{\partial} \Omega_\alpha = \Psi_{\alpha+1}^* \Omega_\alpha - \Omega_\alpha \Psi_\alpha^*,
\]
\[
(10) \quad \overline{\partial} \Psi_\alpha + \partial \Psi_\alpha^* = \Omega_\alpha^* \Omega_\alpha + \Psi_\alpha^* \Psi_\alpha - \Omega_{\alpha-1} \Omega_\alpha^* - \Psi_\alpha \Psi_\alpha^*.
\]
By (8), \( A_{\alpha}^{(\alpha)} \) and \( A_{\alpha}^{(\alpha)} \) for \( \varphi_\alpha \) are given by
\[
(11) \quad A_{\alpha}^{(\alpha)} = -W_\alpha \Omega_{\alpha-1} W_{\alpha-1}^* - W_{\alpha+1} \Omega_\alpha W_\alpha^*,
\]
\[
(12) \quad A_{\alpha}^{(\alpha)} = W_\alpha \Omega_\alpha W_{\alpha+1}^* + W_{\alpha-1} \Omega_\alpha^* W_\alpha^*.
\]
It can easily be checked that (9) is equivalent to (1). An immediate consequence of (8) is
Proposition 2.2. Let \( \varphi = \varphi_\alpha : S^2 \to G_{k_\alpha, n} \) be a pseudo-holomorphic curve, with \( \Omega_\alpha \) and \( \Psi_\alpha \) determined by equations (8). Then \( \Omega_\alpha \) and \( \Psi_\alpha \) satisfy equations (9) and (10).

Let \( \varphi^{(\alpha)} = \varphi_0 \oplus \cdots \oplus \varphi_\alpha \) for (5) and \( k^{(\alpha)} = k_0 + \cdots + k_\alpha \). Then by Proposition 2.1 we have
\[
\partial \varphi^{(\alpha)} = \varphi^{(\alpha)} \partial \varphi_\alpha.
\]
Hence \( \varphi^{(\alpha)} : S^2 \to G_{k^{(\alpha)}, n} \) is a holomorphic map, and the harmonic map sequence (5) becomes
\[
0 \to \varphi^{(\alpha)} \to \varphi^{(\alpha)} \to \cdots \to \varphi^{(\alpha)} \to 0.
\]
If \( k_\alpha = k_{\alpha+1} \), i.e., \( \text{rank}(\varphi_\alpha) = \text{rank}(\varphi^{(\alpha+1)}) \), then \( \varphi^{(\alpha)} \) is called non-degenerate. If \( \varphi^{(\alpha)} \) is non-degenerate for \( \alpha = 0, 1, \cdots, \alpha_0 - 1 \) in (5), i.e., \( k_0 = k_1 = \cdots = k_{\alpha_0} \), then the harmonic sequence (5) is called the non-degenerate harmonic sequence associated to the harmonic map \( \varphi = \varphi^{(\alpha)} \). Now we assume that \( \varphi^{(\alpha)} \) is non-degenerate; then 
\[
\det(\Omega_\alpha) \text{ is a well-defined invariant on } S^2 \text{ and has only isolated zeros. Let }
\]
\[
l_\alpha = \text{tr}(\Omega_\alpha \Omega_\alpha^*),
\]
and we have

Proposition 2.3. If \( \varphi = \varphi_\alpha : S^2 \to G_{k_\alpha, n} \) is a non-degenerate pseudo-holomorphic curve, then
\[
2 \partial \overline{\partial} \log |\det(\Omega_\alpha)| = l_{\alpha-1} - 2l_\alpha + l_{\alpha+1}.
\]

Proof. By (9) and the rule of differentiating a determinant, we get
\[
\overline{\partial} \log |\det(\Omega_\alpha)| = \text{tr}(\Omega^{-1}_\alpha \partial \Omega_\alpha) = \text{tr} \Psi^{(\alpha+1)} - \text{tr} \Psi^{(\alpha)},
\]
\[
\partial \log |\det(\Omega_\alpha)| = \text{tr}((\Omega_\alpha^*)^{-1} \partial \Omega_\alpha) = \text{tr} \Psi^{(\alpha+1)} - \text{tr} \Psi^{(\alpha)}.
\]
It is not difficult to obtain (16) by (10).

Remark. If \( \varphi^{(\alpha)} \) is non-degenerate for all \( \alpha \) in (5), then
\[
2 \partial \overline{\partial} \log |\det(\Omega_\alpha)| = l_{\alpha-1} - 2l_\alpha + l_{\alpha+1}
\]
for \( \alpha = 0, 1, \cdots, \alpha_0 - 1 \), where \( l_1 = l_{\alpha_0} = 0 \). When \( k_\alpha = 1 \) for all \( \alpha \), then \( l_\alpha = |\det(\Omega_\alpha)|^2 \), and (17) is just the unintegrated Plücker formulae for \( l_\alpha \) derived by Bolton, Jensen, Rigoli and Woodward in [1].

3. Kähler Angles and Gauss Curvatures

If \( \varphi : M \to G_{k_\alpha, n} \) is a conformal immersion of a Riemann surface \( M \), we define the Kähler angle of \( \varphi \) to be the function \( \theta : M \to [0, \pi] \) given in terms of a complex coordinate \( z \) on \( M \) by
\[
\tan \frac{\theta(p)}{2} = \frac{|d\varphi(\partial/\partial \overline{z})|}{|d\varphi(\partial/\partial z)|}, \quad p \in M.
\]
It is clear that \( \theta \) is globally defined and is smooth at \( p \) unless \( \theta(p) = 0 \) or \( \pi \). Let \( z = x + \sqrt{-1}y \), and let \( J \) denote the complex structure on \( G_{k_\alpha, n} \); then \( \theta \) is the angle between \( Jd\varphi(\partial/\partial x) \) and \( d\varphi(\partial/\partial y) \). The importance of the Kähler angle in
the theory of minimal immersions of surfaces into Kähler manifolds was pointed out by Chern and Wolfson [4]. Indeed, \( \varphi \) is holomorphic if and only if \( \theta(p) = 0 \) for all \( p \in M \), while \( \varphi \) is anti-holomorphic if and only if \( \theta(p) = \pi \) for all \( p \in M \).

Now suppose that \( \varphi : S^2 \to G_{k,n} \) is a conformal minimal immersion in the harmonic sequence (5). Then each \( \varphi_\alpha : S^2 \to G_{k_\alpha,n} \) is a conformal minimal immersion. So there exists a finite set \( X_\alpha \) (see [1]) such that the Kähler angle

\[
\theta_\alpha : S^2 \setminus X_\alpha \to [0, \pi]
\]

is well defined, and is smooth on \( S^2 \setminus X_\alpha \).

Let \( t_\alpha = \left( \tan \frac{\theta_\alpha}{2} \right)^2 \). Then, in terms of a local complex coordinate \( z \),

\[
t_\alpha = \frac{|d\varphi_\alpha(\partial/\partial \overline{z})|^2}{|d\varphi_\alpha(\partial/\partial z)|^2} = \frac{l_{\alpha - 1}}{l_\alpha}.
\]

Let \( ds_\alpha^2 \) and \( ds_{(\alpha)}^2 \) be the metrics on \( S^2 \setminus X_\alpha \) induced by \( \varphi_\alpha \) and \( \varphi^{(\alpha)} \) respectively. Then by (11), (12) and (13) we have

\[
ds_\alpha^2 = (l_{\alpha - 1} + l_\alpha)dz d\overline{z}, \quad ds_{(\alpha)}^2 = l_\alpha dz d\overline{z}.
\]

The Laplacians \( \Delta_\alpha \) and \( \Delta_{(\alpha)} \) for \( ds_\alpha^2 \) and \( ds_{(\alpha)}^2 \) are given by

\[
\Delta_\alpha = \frac{4}{l_{\alpha - 1} + l_\alpha} \partial \overline{\partial}, \quad \Delta_{(\alpha)} = \frac{4}{l_\alpha} \partial \overline{\partial},
\]

and the curvatures \( K_\alpha, K_{(\alpha)} \) of \( \varphi_\alpha \) and \( \varphi^{(\alpha)} \) by

\[
K_\alpha = -\frac{2}{l_{\alpha - 1} + l_\alpha} \partial \overline{\partial} \log(l_{\alpha - 1} + l_\alpha), \quad K_{(\alpha)} = -\frac{2}{l_\alpha} \partial \overline{\partial} \log l_\alpha,
\]

the area forms \( dv_\alpha \) and \( dv_{(\alpha)} \) by

\[
dv_\alpha = (l_{\alpha - 1} + l_\alpha) \frac{d\overline{z} \wedge dz}{2\sqrt{-1}}, \quad dv_{(\alpha)} = l_\alpha \frac{d\overline{z} \wedge dz}{2\sqrt{-1}}.
\]

Choose holomorphic sections \( f_1, \ldots, f_{k_{(\alpha)}} \) in \( \Gamma(S^2 \times \mathbb{C}^{(\alpha)}) \) so that they span \( \text{Im}(\varphi^{(\alpha)}) \) and

\[
f_1 \wedge \cdots \wedge f_{k_{(\alpha)}} : S^2 \to \mathbb{C}^{(n)}
\]

is a nowhere zero holomorphic curve.

Let \( F^{(\alpha)} = f_1 \wedge \cdots \wedge f_{k_{(\alpha)}} \). Now consider the Plücker embedding (see [12], [13])

\[
[F^{(\alpha)}] : S^2 \to \mathbb{CP}^{(k_{(\alpha)})^{-1}},
\]

which is a holomorphic isometry, and

\[
[F^{(\alpha)}]^* ds^2_{\mathbb{CP}^{(k_{(\alpha)})^{-1}}} = l_\alpha dz d\overline{z}.
\]

By [1], we have

\[
\partial \overline{\partial} \log |F^{(\alpha)}|^2 = l_\alpha,
\]

and the degree \( \delta_\alpha \) of \( F^{(\alpha)} \) is given by

\[
\delta_\alpha = \frac{1}{2\pi \sqrt{-1}} \int_{S^2} \partial \overline{\partial} \log |F^{(\alpha)}|^2 d\overline{z} \wedge dz = \frac{1}{2\pi \sqrt{-1}} \int_{S^2} l_\alpha d\overline{z} \wedge dz,
\]
which is equal to the degree of the polynomial function $F^{(\alpha)}$ in $z$. We call $\delta_\alpha$ the degree of the holomorphic curve $\varphi^{(\alpha)}$. Thus from (17) and (27) we get

**Proposition 3.1.** If $\varphi = \varphi_\alpha : S^2 \to G_{k_\alpha,n}$ is a non-degenerate pseudo-holomorphic curve, then

$$-\tau_\alpha = \delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1},$$

where $\tau_\alpha = -\frac{1}{\pi \sqrt{-1}} \int_{S^2} \partial \bar{\partial} \log |\det \Omega_\alpha| dz \wedge \bar{dz}$ is the number of singular points of $\Omega_\alpha$, i.e., the number of zeros of $\det \Omega_\alpha$.

**Remark.** If $\varphi_\alpha$ is non-degenerate for $\alpha = 0, 1, \cdots, \alpha_0 - 1$, then $-\tau_\alpha = \delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1}$ for all $\alpha$, and $\delta_{\alpha-1} = \delta_{\alpha_0} = 0$; in particular, when $k_0 = \cdots = k_{\alpha_0} = 1$, (28) is the global Plucker formula (see [6]).

Let $ds^2 = |\det \Omega_\alpha|^2 dz \wedge \bar{dz} = \psi_\alpha \bar{\psi}_\alpha$, where $\psi_\alpha$ is a type $(1, 0)$ analytic 1-form. Then $ds^2 = \psi_\alpha \oplus \bar{\psi}_\alpha$ is a singular Hermitian metric. Let $D_S = \sum_{p \in S^2} \text{ord}_p(\psi_\alpha)_p$ be the singular divisor of $(S^2, ds^2)$, i.e., the zero divisor of $\psi_\alpha$. By the Gauss-Bonnet-Chern theorem we have

$$\tau_\alpha = \tau_\alpha + 2,$$

where $\tau_\alpha = \deg D_S$.

We say that $\tau_\alpha$ is the ramification index of $\varphi_\alpha$. Evidently, $\tau_\alpha$ is a non-negative integer. If $\tau_\alpha = 0$, $\varphi_\alpha$ is called unramified by Bolton et al. ([1]).

Let (5) be the non-degenerate harmonic sequence; if $\tau_\alpha = 0$ for $\alpha = 0, 1, \cdots, \alpha_0 - 1$, the harmonic sequence (5) is called totally unramified. Let $\varphi = \varphi_\alpha : S^2 \to G_{k_\alpha,n}$ be the pseudo-holomorphic conformal immersion with the non-degenerate associated harmonic sequence (5); we say that $\varphi$ is a totally unramified pseudo-holomorphic conformal immersion if $\varphi_0, \cdots, \varphi_{\alpha_0}$ is totally unramified.

If $\varphi_\alpha : S^2 \to G_{k_\alpha,n}$ is a conformal minimal immersion with constant Kähler angle, then we have

$$t_\alpha = \frac{\delta_{\alpha-1}}{\delta_\alpha},$$

and from (19) and (22) it follows that

$$K_\alpha = -\frac{2}{t_{\alpha-1} + t_\alpha} \partial \bar{\partial} \log l_{\alpha-1} = \frac{2}{t_{\alpha-1} + t_\alpha} \partial \bar{\partial} \log l_\alpha.$$

4. Conformal Minimal Immersions with Constant Curvatures

It is well known that any complex submanifold of a (simply-connected, complete) space of constant holomorphic curvature is completely determined, up to holomorphic isometries of the ambient space, by its induced metric (see [2], [8]). The Veronese sequence is the harmonic sequence

$$(31) \quad 0 \rightarrow \varphi_0 \rightarrow \varphi_1 \rightarrow \cdots \rightarrow \varphi_n \rightarrow 0,$$

where $n = \deg(\varphi_0)$, and each $\varphi_\alpha = [g_{\alpha,0}, \cdots, g_{\alpha,n}] : S^2 \to \mathbb{C}P^n$ is given by

$$g_{\alpha,j} = \frac{\alpha!}{(1 + z \bar{z})^n} \sqrt{\binom{n}{j} z^{j-n}} \sum_{k} (-1)^k \binom{j}{\alpha-k} \binom{n-j}{k} \bar{z}^k, \quad \alpha, j = 0, 1, \ldots, n.
Each map \( \varphi_\alpha \) in the Veronese sequence (31) is a conformal minimal immersion with constant curvature
\[
K(\varphi_\alpha) = \frac{4}{n + 2\alpha(n - \alpha)}
\]
and constant Kähler angle \( \theta_\alpha \) given by
\[
(\tan \frac{\theta_\alpha}{2})^2 = \frac{\alpha(n - \alpha + 1)}{(\alpha + 1)(n - \alpha)}.
\]

Bolton, Jensen, Rigoli and Woodward ([1]) showed that, up to a holomorphic isometry of \( \mathbb{CP}^n \), the harmonic sequence determined by \( \varphi : S^2 \to \mathbb{CP}^n \), which is a linearly full conformal minimal immersion of constant curvature, is the Veronese sequence. It is very complicated for pseudo-holomorphic curves in complex Grassmann manifolds; for example, rigidity fails, but we still believe that there are some good geometric properties. In this section we discuss pseudo-holomorphic curves of constant curvature in complex Grassmann manifolds, and Kähler angles.

Let \( \varphi_\alpha : S^2 \to G_{k,n} \) be a pseudo-holomorphic curve with constant curvature. Then we know that
\[
[F^{(\alpha-1)}] : S^2 \to \mathbb{CP}^{(k_{(\alpha-1)})}^{-1}, \quad [F^{(\alpha)}] : S^2 \to \mathbb{CP}^{(k_{(\alpha)})}^{-1}
\]
are two holomorphic curves with degrees \( \delta_{\alpha-1} \) and \( \delta_\alpha \) respectively. Consider the tensor product of \([F^{(\alpha-1)}]\) and \([F^{(\alpha)}]\),
\[
T^{(\alpha)} = F^{(\alpha-1)} \otimes F^{(\alpha)}.
\]
Then
\[
[T^{(\alpha)}] : S^2 \to \mathbb{CP}^{(k_{(\alpha-1)})} \otimes \mathbb{CP}^{(k_{(\alpha)})}^{-1}
\]
is a well-defined holomorphic curve, and from (25) the metric induced by \([T^{(\alpha)}]\) is given by
\[
[T^{(\alpha)}]^* ds^2_{\mathbb{CP}^{(k_{(\alpha-1)})} \otimes \mathbb{CP}^{(k_{(\alpha)})}^{-1}} = [F^{(\alpha-1)}]^* ds^2_{\mathbb{CP}^{(k_{(\alpha-1)})}^{-1}} + [F^{(\alpha)}]^* ds^2_{\mathbb{CP}^{(k_{(\alpha)})}^{-1}},
\]
i.e.,
\[
[T^{(\alpha)}]^* ds^2_{\mathbb{CP}^{(k_{(\alpha-1)})} \otimes \mathbb{CP}^{(k_{(\alpha)})}^{-1}} = (l_{\alpha-1} + l_\alpha)dzd\Sigma.
\]
Hence the curvature \( K_\alpha \) of \( \varphi_\alpha \) is equal to the curvature of \([T^{(\alpha)}]\). From [1], an immediate consequence is

**Theorem 4.1.** If \( \varphi : S^2 \to G_{k,n} \) is a pseudo-holomorphic curve with constant curvature \( K(\varphi) \), then \( K(\varphi) = 4/N \), where \( N \) is a positive integer.

This theorem was proved by Chi and Zheng ([5]) by the method of the moving frame. In the following we will prove

**Theorem 4.2.** If \( \varphi_\alpha : S^2 \to G_{k,n} \) is a pseudo-holomorphic curve with constant curvature \( K_\alpha \), then the Kähler angle \( \theta_\alpha \) of \( \varphi_\alpha \) is constant.

**Proof.** From (22) we have
\[
K_\alpha(l_{\alpha-1} + l_\alpha) = -2\partial\overline{\partial}\log(l_{\alpha-1} + l_\alpha).
\]
When \( K_\alpha \) is constant, from (22), (23), (27) and the Gauss-Bonnet theorem it follows that
\[
K_\alpha = \frac{4}{\delta_{\alpha-1} + \delta_\alpha}.
\]
Hence from (22) we obtain
\[
-\frac{2}{l_{\alpha-1} + l_\alpha} \partial \bar{\partial} \log(l_{\alpha-1} + l_\alpha) = \frac{4}{\delta_{\alpha-1} + \delta_\alpha}.
\]
Choose a complex coordinate \( z \) on \( S^2 \setminus \{pt\} \) so that
\[
l_{\alpha-1} + l_\alpha = \frac{\delta_{\alpha-1} + \delta_\alpha}{(1 + z \bar{z})^2}.
\]
From (38), (39) and (26) we obtain
\[
\frac{\partial}{\partial \log} \left( \frac{|F^{(\alpha-1)}|^2|F^{(\alpha)}|^2}{(1 + z \bar{z})^{\delta_{\alpha-1} + \delta_\alpha}} \right) = 0.
\]
Since we can choose holomorphic sections \( f_1, \ldots, f_{k_\alpha} \) in \( \Gamma(S^2 \times C^n) \) such that the maps \( F^{(\alpha-1)} \) and \( F^{(\alpha)} \) are polynomial functions on \( C \) of degrees \( \delta_{\alpha-1} \) and \( \delta_\alpha \) respectively, it follows that
\[
\frac{|F^{(\alpha-1)}|^2|F^{(\alpha)}|^2}{(1 + z \bar{z})^{\delta_{\alpha-1} + \delta_\alpha}}
\]
is globally defined on \( C \) and has a non-zero constant limit \( c \), as \( z \to \infty \). So from (40) we get
\[
\frac{|F^{(\alpha-1)}|^2|F^{(\alpha)}|^2}{(1 + z \bar{z})^{\delta_{\alpha-1} + \delta_\alpha}} = c.
\]
Then we have
\[
|F^{(\alpha-1)}|^2 = c_{\alpha-1}(1 + z \bar{z})^{\delta_{\alpha-1}}, \quad |F^{(\alpha)}|^2 = c_\alpha(1 + z \bar{z})^{\delta_\alpha},
\]
where \( c_{\alpha-1} \) and \( c_\alpha \) are constants.

Hence, \( l_{\alpha-1} = \frac{\delta_{\alpha-1}}{(1 + z \bar{z})^2} \) and \( l_\alpha = \frac{\delta_\alpha}{(1 + z \bar{z})^2} \), namely, \( \varphi_\alpha \) is of constant curvature and constant Kähler angle.

From (19) and (22) we know that if \( \varphi_\alpha : S^2 \to G_{k_\alpha,n} \) is a pseudo-holomorphic curve with constant Kähler angle \( \theta_\alpha \), then \( K_\alpha = \frac{1}{1 + l_\alpha} K^{(\alpha)} \).

Remark. We do not need to assume that \( \varphi_\alpha : S^2 \to G_{k_\alpha,n} \) is non-degenerate in Theorem 4.2.

To conclude this section, we give an example. This example is a harmonic sequence, in which the Gauss curvature and the Kähler angle of each element are constant.

Let \( f_0(z) = (1, 0, \sqrt{2}z, 0, z^2) \) and \( g_0(z) = (0, 1, 0, z, 0) \); then
\[
\varphi_0 = \frac{1}{(1 + z \bar{z})^2} \begin{pmatrix}
1 & 0 & \sqrt{2}z & 0 & z^2 \\
0 & 1 + z \bar{z} & 0 & z(1 + z \bar{z}) & 0 \\
\sqrt{2}z & 0 & 2z \bar{z} & 0 & \sqrt{2}z \bar{z} \\
0 & z(1 + z \bar{z}) & 0 & z \bar{z}(1 + z \bar{z}) & 0 \\
z^2 & 0 & \sqrt{2}z \bar{z}^2 & 0 & z^2 \bar{z}^2
\end{pmatrix} : S^2 \to G_{2,5}
\]
determined by \( f_0(z) \) and \( g_0(z) \) is a holomorphic map.
An immediate computation shows that

\[ f_1(z, \overline{z}) = \varphi_0^+ (\partial f_0(z)) = \left( -\frac{2\pi}{1 + z\overline{z}}, \frac{1}{1 + z\overline{z}}, \frac{2\pi(1 - z\overline{z})}{1 + z\overline{z}}, 0 \right), \]

\[ g_1(z, \overline{z}) = \varphi_0^+ (\partial g_0(z)) = \left( 0, -\frac{\pi}{1 + z\overline{z}}, 0, \frac{1}{1 + z\overline{z}} \right), \]

and \( \varphi_1(z, \overline{z}) \) determined by \( f_1 \) and \( g_1 \) is given by

\[
\varphi_1 = \frac{1}{(1 + z\overline{z})^2} \begin{pmatrix}
2\pi & 0 & \overline{z}(z\overline{z} - 1) & 0 & -2z^2 \\
0 & \overline{z}(1 + z\overline{z}) & 0 & -z(1 + z\overline{z}) & 0 \\
\overline{z}(1 + z\overline{z}) & 0 & 2\overline{z}(z\overline{z} - 1) & 0 & \sqrt{2}z(z\overline{z} - 1) \\
-2z^2 & 0 & \overline{z}(z\overline{z} - 1) & 0 & \overline{z} \\
\end{pmatrix},
\]

which is obviously a pseudo-holomorphic curve into \( G_{2,5} \). Similarly, we have

\[ f_2 = \varphi_1^+ (\partial f_1) = \left( \frac{2\pi}{(1 + z\overline{z})^2}, 0, -\frac{2\sqrt{2}\pi}{(1 + z\overline{z})^2}, 0, \frac{2}{(1 + z\overline{z})^2} \right), \]

\[ g_2 = \varphi_1^+ (\partial g_1) = (0, 0, 0, 0, 0), \]

and \( \varphi_2 \), determined by \( f_2 \) and \( g_2 \), is given by

\[
\varphi_2 = \frac{1}{(1 + z\overline{z})^2} \begin{pmatrix}
z^2\overline{z} & 0 & -\sqrt{2}z^2 \overline{z} & 0 & z^2 \\
0 & 0 & 0 & 0 & 0 \\
\sqrt{2}z^2 & 0 & 2z\overline{z} & 0 & \overline{z} \\
0 & 0 & 0 & 0 & 0 \\
\overline{z}^2 & 0 & -\sqrt{2}\overline{z} & 0 & 1 \\
\end{pmatrix}.
\]

\( \varphi_2 \) is an anti-holomorphic curve, which is isomorphic to the Veronese curve, in \( \mathbb{CP}^2 \).

Hence we obtain a harmonic sequence from \( \varphi_0 \):

\[ 0 \xrightarrow{\partial'} \varphi = \varphi_0 \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \varphi_2 \xrightarrow{\partial'} 0. \]

By a straightforward computation we obtain

\[ l_0 = \frac{3}{(1 + z\overline{z})^2}, \quad l_1 = \frac{2}{(1 + z\overline{z})^2}, \quad l_2 = 0. \]

It is very easy to see that \( K(\varphi_0) = 4/3 \), \( K(\varphi_1) = 4/5 \), \( K(\varphi_2) = 2 \) and \( t_1 = 3/2 \).

It is well known that the rigidity of holomorphic curves in Grassmannians fails; so this example is a special harmonic sequence.

5. Pinching Theorem for Curvature and Kähler Angle

In this section we will discuss curvature pinching and Kähler angle pinching of non-degenerate pseudo-holomorphic spheres in complex Grassmann manifolds.

Let \( \varphi = \varphi_0 : S^2 \to G_{k_n,n} \) be a pseudo-holomorphic curve with the non-degenerate associated harmonic sequence (5), and let \( \alpha_0 \) be the length of its associated harmonic sequence. Then from (28) we have

\[ \delta_\alpha = -\delta_{\alpha-2} + 2\delta_{\alpha-1} - \tau_{\alpha-1} - 2 \]

for \( \alpha = 1, \cdots, \alpha_0 \), and

\[ \tau_\alpha = (\delta_\alpha - \delta_{\alpha+1}) - (\delta_{\alpha-1} - \delta_\alpha) - 2 \]

for \( \alpha = 0, 1, \cdots, \alpha_0 - 1 \).
It is an immediate consequence of (41) and (42) that

$$\delta_{\alpha} = (\alpha + 1)(\delta_0 - \alpha) - \sum_{\beta=0}^{\alpha-1}(\alpha - \beta)\tau_{\beta}$$

for \(\alpha = 1, \cdots, \alpha_0\), and

$$\tau_0 + \cdots + \tau_\alpha = (\delta_\alpha - \delta_{\alpha+1}) + \delta_0 - 2(\alpha + 1)$$

for \(\alpha = 0, 1, \cdots, \alpha_0 - 1\), where \(\delta_0\) is the degree of the holomorphic map \(\varphi_0\) in (5).

From (43) and (44) we have also

$$\sum_{\alpha=0}^{\alpha_0-1}(\alpha - \alpha)\tau_{\alpha} = (\alpha_0 + 1)(\delta_0 - \alpha_0)$$

and

$$\delta_{\alpha} = (\alpha + 1)(\alpha_0 - \alpha) + \frac{\alpha_0 - \alpha}{\alpha_0 + 1} \sum_{\beta=0}^{\alpha-1}(\beta + 1)\tau_{\beta} + \frac{\alpha + 1}{\alpha_0 + 1} \sum_{\beta=\alpha}^{\alpha_0-1}(\alpha_0 - \beta)\tau_{\beta}.$$

Denoting \(\tau = \min \{\tau_0, \cdots, \tau_{\alpha_0-1}\} \geq 0\), we immediately obtain

$$\delta_0 \geq \alpha_0(1 + \frac{1}{2}\tau), \quad \delta_\alpha \geq (\alpha + 1)(\alpha_0 - \alpha)(1 + \frac{1}{2}\tau),$$

and “\(=\)” holds if and only if \(\tau_0 = \cdots = \tau_{\alpha_0-1}\), where \(\alpha = 0, 1, \cdots, \alpha_0 - 1\).

Obviously, \(\varphi\) is a totally unramified non-degenerate pseudo-holomorphic minimal immersion, i.e., the harmonic sequence \(\varphi_0, \cdots, \varphi_{\alpha_0} : S^2 \to G_{k,n}\) is non-degenerate and totally unramified if and only if the degree \(\delta_0\) of \(\varphi_0\) is \(\alpha_0\). For a totally unramified non-degenerate harmonic sequence \(\varphi_0, \cdots, \varphi_{\alpha_0} : S^2 \to G_{k,n}\) we have

$$\delta_{\alpha} = (\alpha + 1)(\alpha_0 - \alpha).$$

At first, by using the Gauss-Bonnet theorem we have

**Lemma 5.1.** Suppose that the curvature \(K_\alpha\) of \(\varphi_\alpha\) satisfies either \(K_\alpha \geq \frac{4}{\delta_{\alpha-1} + \delta_\alpha}\) or \(K_\alpha \leq \frac{1}{\delta_{\alpha-1} + \delta_\alpha}\). Then \(K_\alpha = \frac{4}{\delta_{\alpha-1} + \delta_\alpha}\).

**Remark.** In Lemma 5.1 we do not need to assume that \(\varphi_\alpha\) is non-degenerate.

**Theorem 5.2.** Let \(\varphi : S^2 \to G_{k,n}\) be a pseudo-holomorphic curve with non-degenerate associated harmonic sequence, and suppose that \(\varphi\) is the \(\alpha\)-th element \(\varphi_\alpha\) of its non-degenerate associated harmonic sequence.

(i) If \(K(\varphi) \geq \frac{4}{(\alpha_0 + 2\alpha(\alpha_0 - \alpha))(1 + \frac{1}{2}\tau)}\), then

$$K(\varphi) = \frac{4}{(\alpha_0 + 2\alpha(\alpha_0 - \alpha))(1 + \frac{1}{2}\tau)},$$

and \(\tau_\beta = \tau\) for all \(\beta\).

(ii) If \(K(\varphi) \leq \frac{4}{(\alpha_0 + 2\alpha(\alpha_0 - \alpha))(1 + \frac{1}{2}\tau)}\) and if \(\tau_\beta = \tau\) for all \(\beta\), then

$$K(\varphi) = \frac{4}{(\alpha_0 + 2\alpha(\alpha_0 - \alpha))(1 + \frac{1}{2}\tau)}.$$
Proof. From (47) we see that

\[ \delta_{a-1} + \delta_a \geq (\alpha_0 + 2\alpha(\alpha_0 - \alpha))(1 + \frac{1}{2}\tau), \]

with equality if and only if \( \tau_\beta = \tau \) for all \( \beta \). The result is now immediate from Lemma 5.1. \( \square \)

Remark. We have \( t_\alpha = \frac{\alpha(\alpha_0 - \alpha + 1)}{(\alpha + 1)(\alpha_0 - \alpha)} \) under the assumption of Theorem 5.2. This shows that the Kähler angle \( \theta_\alpha \) is independent of \( \tau \).

The following is an immediate consequence of Theorem 5.2.

Corollary 5.3. Let \( \varphi : S^2 \to G_{k,n} \) be a holomorphic curve with non-degenerate associated harmonic sequence. Suppose \( K(\varphi) \leq 4 \frac{1}{\alpha_0(1 + \frac{1}{2}\tau)} \) and \( \tau_\beta = \tau \) for all \( \beta \).

Then \( K(\varphi) = 4 \frac{1}{\alpha_0(1 + \frac{1}{2}\tau)} \), and \( \tau_\beta = \tau \) for all \( \beta \).

Similarly, the following theorem is also an immediate consequence of Theorem 5.2.

Corollary 5.4. Let \( \varphi : S^2 \to G_{k,n} \) be a holomorphic curve with non-degenerate associated harmonic sequence, and suppose \( K(\varphi) \geq 4 \frac{1}{\alpha_0(1 + \frac{1}{2}\tau)} \). Then \( K(\varphi) = 4 \frac{1}{\alpha_0(1 + \frac{1}{2}\tau)} \), and \( \tau_\beta = \tau \) for all \( \beta \).

We now prove a pinching theorem for the Kähler angle. Let \( \varphi : S^2 \to G_{k,n} \) be a pseudo-holomorphic sphere and let \( \varphi_0, \cdots, \varphi_\alpha \) be the associated harmonic sequence. We assume that \( \varphi = \varphi_\alpha \).

Lemma 5.5. If the Kähler angle \( t_\alpha \) of \( \varphi_\alpha \) satisfies either \( t_\alpha \geq \frac{\delta_{a-1}}{\delta_a} \) or \( t_\alpha \leq \frac{\delta_{a-1}}{\delta_a} \), then \( t_\alpha = \frac{\delta_{a-1}}{\delta_a} \).

Lemma 5.6. Let \( \varphi_\alpha \) be a pseudo-holomorphic curve with non-degenerate associated harmonic sequence. If \( \tau_\beta = \tau \) for all \( \beta \), and \( t_\alpha \) satisfies either \( t_\alpha \geq \frac{\alpha(\alpha_0 - \alpha + 1)}{(\alpha + 1)(\alpha_0 - \alpha)} \) or \( t_\alpha \leq \frac{\alpha(\alpha_0 - \alpha + 1)}{(\alpha + 1)(\alpha_0 - \alpha)} \), then \( t_\alpha = \frac{\alpha(\alpha_0 - \alpha + 1)}{(\alpha + 1)(\alpha_0 - \alpha)} \).

The proof of the above theorem is immediate from Lemma 5.5 and (47).

Using (46), we can also prove the following.

Theorem 5.7. Let \( \varphi : S^2 \to G_{k,n} \) be a pseudo-holomorphic curve with non-degenerate associated harmonic sequence, and suppose that \( \varphi \) is the \( \alpha \)-th element \( \varphi_\alpha \) of its non-degenerate associated harmonic sequence. If \( t_\alpha \leq \frac{1}{2} \) (resp. \( t_\alpha \geq 2 \)), then \( t_\alpha = 0 \) (resp. \( t_\alpha = \infty \)), i.e., \( \varphi \) is a holomorphic (resp. anti-holomorphic) curve.

Proof. When \( \alpha \neq 0 \) and \( \alpha_0 \), by (46) an immediate computation shows that

\[ \frac{1}{2} < \frac{\delta_{a-1}}{\delta_a} < 2. \]

Hence, by Lemma 5.5, if \( t_\alpha \leq \frac{1}{2} \) (resp. \( t_\alpha \geq 2 \)), then \( \alpha = 0 \) (resp. \( \alpha = \alpha_0 \)), i.e., \( \varphi \) is a holomorphic (resp. anti-holomorphic) curve. \( \square \)
We believe that $\tau \neq 0$ for the non-degenerate harmonic sequence associated to the holomorphic curve of constant curvature, except for the Veronese sequence.

References


Department of Mathematics, Graduate School, Chinese Academy of Sciences, Beijing 100039, China
E-mail address: xxj@gsacs.ac.cn

Department of Mathematics, Graduate School, Chinese Academy of Sciences, Beijing 100039, China
E-mail address: pengck@gsacs.ac.cn