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BURGHELEA-FRIEDLANDER-KAPPELER'S GLUING FORMULA FOR THE ZETA-DETERMINANT AND ITS APPLICATIONS TO THE ADIABATIC DECOMPOSITIONS OF THE ZETA-DETERMINANT AND THE ANALYTIC TORSION

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Abstract. The gluing formula of the zeta-determinant of a Laplacian given by Burghelea, Friedlander and Kappeler contains an unknown constant. In this paper we compute this constant to complete the formula under an assumption that the product structure is given near the boundary. As applications of this result, we prove the adiabatic decomposition theorems of the zeta-determinant of a Laplacian with respect to the Dirichlet and Neumann boundary conditions and of the analytic torsion with respect to the absolute and relative boundary conditions.

1. Introduction

In [3], Burghelea, Friedlander and Kappeler established a gluing formula for the zeta determinant of an elliptic operator on a compact manifold. This formula contains an unknown constant which can be expressed in terms of the zero coefficients of some asymptotic expansions. In this paper we compute this constant in the case when the product structure is given near the boundary, and then we apply this result to prove the adiabatic decomposition theorems for the zeta determinant and the analytic torsion. Some results of this paper are known from the work of Klimek and Wojciechowski in [6], but our method is completely different from theirs.

Let M be a compact oriented m-dimensional manifold with boundary Z (Z may be empty), and Y a hypersurface of M such that $M - Y$ has two components and $Y \cap Z = \emptyset$. We denote by M_1, M_2 the closure of each component, *i.e.* $M = M_1 \cup_Y M_2$. Choose a collar neighborhood N of Y, which is diffeomorphic to $[-1, 1] \times Y$, $N \cap Z = \emptyset$, and choose a metric g on M that is a product metric on N. Suppose that $E \to M$ is a complex vector bundle such that $E|_N$ has the product structure, which means that $E|_N = p^*E|_Y$, where $p : [-1, 1] \times Y \to Y$ is the canonical projection. Let Δ_M be a Laplacian acting on smooth sections of E, and let Δ_{M_1}, Δ_{M_2} be the restrictions of Δ_M to M_1 and M_2. By a Laplacian we mean a positive semi-definite 2nd order differential operator whose principal symbol is $\sigma_L(\Delta_M)(x, \xi) = ||\xi||^2$. We

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assume that Δ_M is $-\partial_u^2 + \Delta_Y$ on N, where ∂_u is the unit normal vector field to Y on N, outward to M_1, and Δ_Y is a Laplacian on Y.

We denote by D, B the Dirichlet boundary conditions on Z, Y and by C the Neumann boundary condition on Y, defined as follows:

$$\begin{aligned} D : C^\infty(M_i) \to C^\infty(Z \cap M_i) \quad &\text{by} \quad D(\phi) = \phi|_Z, \\ B : C^\infty(M_i) \to C^\infty(Y) \quad &\text{by} \quad B(\phi) = \phi|_Y, \\ C : C^\infty(M_i) \to C^\infty(Y) \quad &\text{by} \quad C(\phi) = (\partial_u \phi)|_Y. \end{aligned}$$

Then the Laplacian $\Delta_{M,D}$ ($\Delta_{M_i,B,D}$, $\Delta_{M_i,C,D}$) with the Dirichlet condition on Z (the Dirichlet condition on Y and Z, the Neumann condition on Y and the Dirichlet condition on Z) is defined by the same operator Δ_M (Δ_{M_i}) with domains as follows:

$$\begin{aligned} Dom(\Delta_{M,D}) &= \{\phi \in C^\infty(M) \mid D(\phi) = 0\}, \\ Dom(\Delta_{M_i,B,D}) &= \{\phi \in C^\infty(M_i) \mid B(\phi) = 0, D(\phi) = 0\}, \\ Dom(\Delta_{M_i,C,D}) &= \{\phi \in C^\infty(M_i) \mid C(\phi) = 0, D(\phi) = 0\}. \end{aligned}$$

For computational reasons, we consider $\Delta^m_{M,D_m} + t^m$, $\Delta^m_{M_i,B_m,D_m} + t^m$ and $\Delta^m_{M_i,C_m,D_m} + t^m$ ($t \in \mathbb{R}^+$) rather than $\Delta_{M,D}$, $\Delta_{M_i,B,D}$ and $\Delta_{M_i,C,D}$, where D_m, B_m and C_m are the Dirichlet and the Neumann boundary conditions corresponding to Δ^m_M, $\Delta^m_{M_i}$ (or $\Delta^m_M + t^m$, $\Delta^m_{M_i} + t^m$) defined as follows:

$$\begin{aligned} D_m &= (D, D\Delta_M, \cdots, D\Delta_M^{m-1}), \\ B_m &= (B, B\Delta_{M_i}, \cdots, B\Delta_{M_i}^{m-1}), \\ C_m &= (C, C\Delta_{M_i}, \cdots, C\Delta_{M_i}^{m-1}). \end{aligned}$$

Note that

$$\Delta^m_{M,D} + t^m = \begin{cases} \prod_{k=-[\frac{m}{2}]}^{[\frac{m-1}{2}]} (\Delta_{M,D} + e^{i\frac{2k\pi}{m}} t) & \text{if } m \text{ is odd}, \\ \prod_{k=-[\frac{m}{2}]}^{[\frac{m-1}{2}]} (\Delta_{M,D} + e^{i\frac{(2k+1)\pi}{m}} t) & \text{if } m \text{ is even}. \end{cases}$$

For $-[\frac{m}{2}] \le k \le [\frac{m-1}{2}]$, let $\alpha_k = e^{i\frac{2k\pi}{m}}$ if m is odd, and $\alpha_k = e^{i\frac{(2k+1)\pi}{m}}$ if m is even.

Now we describe the so-called Dirichlet-to-Neumann operator $R(\alpha_k t) : C^\infty(Y) \to C^\infty(Y)$ associated to $\Delta_{M,D} + \alpha_k t$ on Y. Let $P_i(\alpha_k t) : C^\infty(Y) \to C^\infty(M_i)$ be the Poisson operator on Y associated to $\Delta_{M,D} + \alpha_k t$, which is characterized by the following equations (for details see [3], [4], [8]):

$$BP_i(\alpha_k t) = Id_Y, \qquad DP_i(\alpha_k t) = 0, \qquad (\Delta_M + \alpha_k t)P_i(\alpha_k t) = 0.$$

Then $R(\alpha_k t)$ is defined by the composition of the following maps:

$$\begin{aligned} C^\infty(Y) \xrightarrow{\delta_{ia}} C^\infty(Y) \oplus C^\infty(Y) &\xrightarrow{(P_1(\alpha_k t), P_2(\alpha_k t))} C^\infty(M_1) \oplus C^\infty(M_2) \\ &\xrightarrow{(C_1, C_2)} C^\infty(Y) \oplus C^\infty(Y) \xrightarrow{\delta_{if}} C^\infty(Y), \end{aligned}$$

where $\delta_{ia}(g) = (g, g)$, $C_1(\phi_1) = (\partial_u \phi_1)|_Y$, $C_2(\phi_2) = (\partial_u \phi_2)|_Y$ and $\delta_{if}(g, h) = g - h$. It is known that $R(\alpha_k t)$ is a ΨDO of order 1 (*cf.* Theorem 2.1) and by choosing π as an Agmon angle, $\log DetR(\alpha_k t)$ is well defined. The following theorem is due to Burghelea, Friedlander and Kappeler ([8], see also [3] and [4]).

Theorem 1.1.

$$\log Det(\Delta^m_{M,D_m} + t^m) - \log Det(\Delta^m_{M_1,B_m,D_m} + t^m) - \log Det(\Delta^m_{M_2,B_m,D_m} + t^m)$$
$$= -\sum_{k=-[\frac{m}{2}]}^{[\frac{m-1}{2}]} c_k + \sum_{k=-[\frac{m}{2}]}^{[\frac{m-1}{2}]} \log DetR(\alpha_k t),$$

where c_k is the zero coefficient in the asymptotic expansion of $\log DetR(\alpha_k t)$ as $t \to \infty$.

Remark. In [3] and [8], Theorem 1.1 was proved only in the case $Z = \emptyset$. However, the proof can be extended without any modification to the case that Z is non-empty.

The purpose of this paper is to compute the zero coefficients in Theorem 1.1 under the assumption of the product structures on N and $E|_N$, and then to apply this result to prove the adiabatic decomposition theorems for the zeta-determinant of a Laplacian and the analytic torsion. We first have the following theorem.

Theorem 1.2. *We assume the product structures of M and E on N and $\Delta_M = -\partial_u^2 + \Delta_Y$ on N. Then $\sum_k c_k = m \log 2 \cdot (\zeta_{\Delta_Y}(0) + dimKer\Delta_Y)$.*

Setting $t = 0$, we get the following corollary.

Corollary 1.3. *We further assume that $\Delta_{M,D}$ is invertible. Then*

$$\log Det\Delta_{M,D} - \log Det\Delta_{M_1,B,D} - \log Det\Delta_{M_2,B,D}$$
$$- \log 2 \cdot (\zeta_{\Delta_Y}(0) + dimKer\Delta_Y) + \log DetR.$$

Remarks. (1) If $dimY$ is odd, it is well-known that $\zeta_{\Delta_Y}(0) + dimKer\Delta_Y = 0$. In this case, the assertion in Corollary 1.3 can be written as follows:

$$\log Det\Delta_{M,D} - \log Det\Delta_{M_1,B,D} - \log Det\Delta_{M_2,B,D} = \log DetR,$$

which was observed in [7].

(2) Theorem 1.1, Theorem 1.2 and Corollary 1.3 also hold when we impose the absolute (or the relative) boundary condition on Z (see Theorem 5.2).

The main idea of proving Theorem 1.2 is to show that under the assumption of the product structure, $R(\alpha_k t)$ can be expressed as $2\sqrt{\Delta_Y + \alpha_k t}$ + a smoothing operator (Theorem 2.1). We are going to show this fact in the next section by using an observation, due to I.M. Gelfand (probably unpublished), that the Dirichlet-to-Neumann operator satisfies a Ricatti type equation (*cf.* (2.2)).

Now we apply Corollary 1.3 to discuss the adiabatic decomposition of the zeta-determinant of a Laplacian into the zeta-determinants of Laplacians with the Dirichlet and Neumann boundary conditions. Recall that N is a collar neighborhood of Y, which is diffeomorphic to $[-1,1] \times Y$. We denote by M_r the compact manifold with boundary obtained by attaching $N_{r+1} = [-r-1, r+1] \times Y$ to $M - (-\frac{1}{2}, \frac{1}{2}) \times Y$ by identifying $[-1, -\frac{1}{2}] \times Y$ with $[-r-1, -r-\frac{1}{2}] \times Y$ and $[\frac{1}{2}, 1] \times Y$ with $[r+\frac{1}{2}, r+1] \times Y$. We also denote by $M_{1,r}$, $M_{2,r}$ the manifolds with boundary which are obtained by attaching $[-r, 0] \times Y$, $[0, r] \times Y$ to M_1, M_2 by identifying Y with $\{-r\} \times Y$ and Y with $\{r\} \times Y$, respectively. Then the bundle $E \to M$ and the Laplacian Δ_M on M can be extended naturally to the bundle $E_r \to M_r$ and the Laplacian Δ_{M_r} on M_r.

To describe the next result, we need to define the operators $Q_i : C^\infty(Y) \to C^\infty(Y)$ $(i = 1, 2)$ by slightly modifying the Dirichlet-to-Neumann operator. For

$f \in C^{\infty}(Y)$, choose $\phi_i \in C^{\infty}(M_i)$ satisfying $\Delta_{M_i}\phi_i = 0$, $\phi_i|_Z = 0$ and $\phi_i|_Y = f$. We define

$$Q_1(f) = (\partial_u \phi_1)|_Y, \qquad Q_2(f) = (-\partial_u \phi_2)|_Y.$$

Then each Q_i is an elliptic ΨDO of order 1 (*cf.* Theorem 2.1), and the Dirichlet-to-Neumann operator R is $R = Q_1 + Q_2$. The following is the second result of this paper.

Theorem 1.4. *We assume that both $Q_1 + \sqrt{\Delta_Y}$ and $Q_2 + \sqrt{\Delta_Y}$ are invertible operators and $k = dimKer\Delta_Y$. We further assume that $\Delta_{M_r,D}$ is invertible for r large enough. Then*

$$\lim_{r\to\infty} \left\{\log Det(\Delta_{M_r,D}) - \log Det(\Delta_{M_{1,r},B,D}) - \log Det(\Delta_{M_{2,r},B,D}) + k\log r\right\} = \frac{1}{2}\log Det\Delta_Y.$$

Remarks. (1) If Δ_Y has non-trivial kernel, we define $Det\Delta_Y$ from the zeta function $\zeta_{\Delta_Y}(s)$ consisting of only non-zero eigenvalues.

(2) If Δ_M is a connection Laplacian for a connection compatible with the inner product, each Q_i is a non-negative operator (Lemma 4.3).

(3) Suppose that $\Delta_M = A^2$ for a Dirac operator A which has the form $G(\partial_u + B)$ near Y with G a bundle automorphism satisfying

$$G^* = -G, \qquad G^2 = -Id, \qquad B^* = B, \qquad GB = -BG. \tag{1.1}$$

Here G and B do not depend on the normal coordinate u. Then the invertiblity of both $Q_1 + \sqrt{B^2}$ and $Q_2 + \sqrt{B^2}$ is equivalent to the non-existence of the extended L^2-solutions of $A_{M_{1,\infty}}$, $A_{M_{2,\infty}}$ on $M_{1,\infty}$ and $M_{2,\infty}$ (Corollary 4.5).

(4) Suppose that Δ_M is a connection Laplacian or a Dirac Laplacian for a connection compatible with the inner product, and $\Delta_{M,D}$ is invertible. Then the invertiblity of both $Q_1 + \sqrt{\Delta_Y}$ and $Q_2 + \sqrt{\Delta_Y}$ implies the invertiblity of $\Delta_{M_r,D}$ for r large enough (Lemma 4.6).

Let $\tilde{M}_{1,r}$ be the double of $M_{1,r}$. Then it is a well-known fact that

$$\log Det\Delta_{\tilde{M}_{1,r},D,D} = \log Det\Delta_{M_{1,r},C,D} + \log Det\Delta_{M_{1,r},B,D}.$$

Combining this fact with Corollary 1.3 and Theorem 1.4, we have the following result.

Corollary 1.5. *We assume the hypotheses in Theorem 1.4. Then:*

(1) $\lim_{r\to\infty} \left\{\log Det(\Delta_{M_{1,r},C,D}) - \log Det(\Delta_{M_{1,r},B,D}) + k\log r\right\} = \frac{1}{2}\log Det(\Delta_Y).$

(2) $\lim_{r\to\infty} \left\{\log Det(\Delta_{M_r,D}) - \log Det(\Delta_{M_{1,r},C,D}) - \log Det(\Delta_{M_{2,r},B,D})\right\} = 0.$

Finally we discuss the adiabatic decomposition of the analytic torsion into the analytic torsions with the absolute and relative boundary conditions.

Here we assume that M is a closed manifold with a hypersurface Y and M has a product structure near Y. We define M_r, $M_{1,r}$ and $M_{2,r}$ as above so that $M_r = M_{1,r} \cup_{\{0\}\times Y} M_{2,r}$. Suppose that ρ_{M_r} ($\rho_{M_{1,r}}$, $\rho_{M_{2,r}}$, ρ_Y) is an orthogonal representation of $\pi_1(M_r)$ ($\pi_1(M_{1,r})$, $\pi_1(M_{2,r})$, $\pi_1(Y)$) to $SO(n)$, respectively. Then we can define the analytic torsions $\tau(M_r, \rho_{M_r})$, $\tau_{abs}(M_{i,r}, \rho_{M_{i,r}})$, $\tau_{rel}(M_{i,r}, \rho_{M_{i,r}})$ ($i = 1, 2$), $\tau(Y, \rho_Y)$ in the standard way (for the definitions, see Section 5). Our goal is to recover the Klimek-Wojciechowski result about the analytic torsion in [6] as follows.

First, let us consider $M_{1,r}$ (a manifold with boundary Y) only. For a given representation $\rho_{M_{1,r}} : \pi_1(M_{1,r}) \to SO(n)$ and the natural homomorphism $\iota_Y : \pi_1(Y) \to \pi_1(M_{1,r})$, define $\rho_Y : \pi_1(Y) \to SO(n)$ by $\rho_Y = \rho_{M_{1,r}} \circ \iota_Y$. We denote by Δ_Y^q ($\Delta_{M_{1,r}}^q$) the Hodge Laplacian acting on q-forms on Y (on $M_{1,r}$) and valued in E_{ρ_Y} ($E_{\rho_{M_{1,r}}}$), where $E_{\rho_Y} = \tilde{Y} \times_{\rho_Y} \mathbb{R}^n$ with $\tilde{Y}$ the universal covering space of Y ($E_{\rho_{M_{1,r}}}$ is defined in the same way). We define Q_1^q the same way as in Theorem 1.4 with the bundle $E = \bigwedge^q T^*M_{1,r} \otimes E_{\rho_{M_{1,r}}}$. If necessary, by tensoring $\mathbb{C}$ on E, we regard E as a complex vector bundle. Then we have the following theorem.

Theorem 1.6. *Suppose that for each q,* $Q_1^q + \begin{pmatrix} \sqrt{\Delta_Y^q} & 0 \\ 0 & \sqrt{\Delta_Y^{q-1}} \end{pmatrix}$ *is an invertible operator on* $\{-r\} \times Y$ *and* $H^q(M_{1,r}; \rho_{M_{1,r}})$, $H^q(M_{1,r}, Y; \rho_{M_{1,r}})$ *are trivial groups. Then*

$$\lim_{r\to\infty} \left\{ \log \tau_{abs}(M_{1,r}, \rho_{M_{1,r}}) - \log \tau_{rel}(M_{1,r}, \rho_{M_{1,r}}) \right\} = \log \tau(Y; \rho_Y).$$

Remark. If $Q_1^q + \begin{pmatrix} \sqrt{\Delta_Y^q} & 0 \\ 0 & \sqrt{\Delta_Y^{q-1}} \end{pmatrix}$ is invertible, by Corollary 4.5 there are no extended L^2-solutions of $d_q + d_q^*$ on $M_{1,\infty}$, which implies that $Ker\Delta_Y^{q-1} = Ker\Delta_Y^q = 0$ (*cf.* [1], [2], [5]).

Next, we consider the closed manifold M_r and manifolds with boundary $M_{i,r}$ ($i = 1, 2$). For a given representation $\rho_{M_r} : \pi_1(M_r) \to SO(n)$ and the natural homomorphisms $\iota_{M_{i,r}} : \pi_1(M_{i,r}) \to \pi_1(M_r)$, $\iota_Y : \pi_1(Y) \to \pi_1(M_{i,r})$, define $\rho_{M_{i,r}} : \pi_1(M_{i,r}) \to SO(n)$, $\rho_Y : \pi_1(Y) \to SO(n)$ by $\rho_{M_{i,r}} = \rho_{M_r} \circ \iota_{M_{i,r}}$, $\rho_Y = \rho_{M_{i,r}} \circ \iota_Y$. We also define Δ_Y^q, Q_1^q and Q_2^q as in Theorem 1.6.

Theorem 1.7. *Suppose that, for each q,*

$$Q_1^q + \begin{pmatrix} \sqrt{\Delta_Y^q} & 0 \\ 0 & \sqrt{\Delta_Y^{q-1}} \end{pmatrix} \quad \textit{and} \quad Q_2^q + \begin{pmatrix} \sqrt{\Delta_Y^q} & 0 \\ 0 & \sqrt{\Delta_Y^{q-1}} \end{pmatrix}$$

are invertible operators on $\{-r\} \times Y$, $\{r\} \times Y$, *and* $H^q(M_r; \rho_{M_r})$, $H^q(M_{1,r}; \rho_{M_{1,r}})$, $H^q(M_{2,r}, Y; \rho_{M_{2,r}})$ *are trivial groups. Then:*

(1) $\lim_{r\to\infty} \left(\log Det\Delta_{M_r}^q - \log Det\Delta_{M_{1,r},abs}^q - \log Det\Delta_{M_{2,r},rel}^q \right) = 0.$

(2) $\lim_{r\to\infty} \left(\log \tau(M_r; \rho_{M_r}) - \log \tau_{abs}(M_{1,r}; \rho_{M_{1,r}}) - \log \tau_{rel}(M_{2,r}; \rho_{M_{2,r}}) \right) = 0.$

Remark. Recently J. Park and K. Wojciechowski proved the following result in [10]. Suppose that M is an odd-dimensional compact manifold with $M = M_1 \cup_Y M_2$ and D is a Dirac operator acting on smooth sections of a Clifford module bundle E with $D = G(\partial_u + B)$ near Y. Denote by $P_>$, $P_<$ the Atiyah-Patodi-Singer boundary conditions projecting the positive and negative eigenspaces of B, respectively. Assume that

$$KerB = \{0\}, \qquad Ker_{L^2}D_{1,\infty} = Ker_{L^2}D_{2,\infty} = \{0\},$$

where $Ker_{L^2}D_{i,\infty}$ is the set of all extended L^2-solutions of $D_{i,\infty}$ on $M_{i,\infty}$. Then

$$\lim_{r\to\infty} \left\{ \log DetD_r^2 - \log DetD_{M_{1,r},P_>}^2 - \log DetD_{M_{2,r},P_<}^2 \right\} = -\log 2 \cdot \zeta_{B^2}(0).$$

This result is the main motivation of this paper. In [9] we are going to recover this result by using the techniques in this paper.

2. Asymptotic symbol of $R(\alpha_k t)$

In this section, we are going to describe the asymptotic symbol of $R(\alpha_k t)$. The following method was observed by I.M. Gelfand.

We start by defining $Q_i(\alpha_k t) : C^\infty(Y) \to C^\infty(Y)$ $(i = 1, 2)$ as follows. For $f \in C^\infty(Y)$, choose $\phi_i \in C^\infty(M_i)$ such that

$$(\Delta_{M_i} + \alpha_k t)\phi_i = 0, \quad \phi_i|_Y = f, \quad \phi_i|_Z = 0.$$

Then we define

$$Q_1(\alpha_k t)(f) = (\partial_u \phi_1)|_Y \quad \text{and} \quad Q_2(\alpha_k t)(f) = (-\partial_u \phi_1)|_Y.$$

From this definition, we get

$$R(\alpha_k t) = Q_1(\alpha_k t) + Q_2(\alpha_k t),$$

and it's enough to consider $Q_1(\alpha_k t)$ only. From now on we denote $Q_1(\alpha_k t)$ simply by $Q(\alpha_k t)$.

For $f \in C^\infty(Y)$, let φ be a solution of $\Delta_{M_1} + \alpha_k t$ with $\varphi|_Y = f$ and $\phi|_Z = 0$. Then

$$\frac{d}{du}\varphi(u, y) = Q_u(\alpha_k t)\varphi(u, y),$$

where $Q_u(\alpha_k t)$ is defined similarly to $Q(\alpha_k t) = Q_0(\alpha_k t)$ at the level $\{u\} \times Y$:

$$\frac{d^2}{du^2}\varphi(u, y) = \left(\frac{d}{du}Q_u(\alpha_k t)\right)\varphi(u, y) + Q_u(\alpha_k t)^2\varphi(u, y).$$

For $0 \le u < 1$,

$$(\Delta_Y + \alpha_k t)\varphi(u, y) = \left(\frac{d}{du}Q_u(\alpha_k t)\right)\varphi(u, y) + Q_u(\alpha_k t)^2\varphi(u, y).$$

Consequently, for $0 \le u < 1$,

$$\frac{d}{du}Q_u(\alpha_k t) = -Q_u(\alpha_k t)^2 + (\Delta_Y + \alpha_k t). \tag{2.2}$$

Now let us consider the asymptotic symbol of $Q_u(\alpha_k t)$ as follows:

$$\sigma(Q_u(\alpha_k t)) \sim q_1(u, y, \xi) + q_0(u, y, \xi) + \cdots + q_{1-j}(u, y, \xi) + \cdots,$$

where $q_{1-j}(u, y, \xi)$ is the homogeneous part of $\sigma(Q_u(\alpha_k t))$ of order $1-j$ with respect to ξ. Then

$$\sigma\left(\frac{d}{du}Q_u(\alpha_k t)\right) \sim \frac{d}{du}q_1(u, y, \xi) + \frac{d}{du}q_0(u, y, \xi) + \cdots + \frac{d}{du}q_{1-j}(u, y, \xi) + \cdots. \tag{2.3}$$

Note that

$$\begin{aligned}\sigma\left(Q_u(\alpha_k t)^2\right) &\sim \sum_{k=0}^{\infty} \sum_{\substack{|\omega|+i+j=k \\ i,j\ge 0}} \frac{1}{\omega!} d_\xi^\omega q_{1-i}(u, y, \xi) \cdot D_y^\omega q_{1-j}(u, y, \xi) \\ &= q_1^2(u, y, \xi) + (d_\xi q_1 D_y q_1 + q_0 q_1 + q_1 q_0) + \cdots.\end{aligned} \tag{2.4}$$

Suppose that

$$\sigma(\Delta_Y + \alpha_k t) = (p_2(y, \xi) + \alpha_k t Id) + p_1(y, \xi) + p_0(y, \xi).$$

Since $\frac{d}{du}Q_u(\alpha_k t)$ is a ΨDO of order 1, $q_1^2(u,y,\xi) = p_2(y,\xi) + \alpha_k t Id$. Applying the argument of Lemma 3.3 in [8] to the double of a manifold with boundary, one can show that

$$q_1(u,y,\xi) = \sqrt{p_2(y,\xi) + \alpha_k t Id}. \tag{2.5}$$

Hence q_1 does not depend on u, and $\frac{d}{du}q_1(u,y,\xi) = 0$. Again, from (2.2), (2.3) and (2.4), since q_1 is a scalar matrix, $(d_\xi q_1 D_y q_1 + 2q_1 q_0) = p_1(y,\xi)$ and

$$q_0(u,y,\xi) = (2q_1(y,\xi))^{-1}\left(p_1(y,\xi) - d_\xi q_1(y,\xi)\cdot D_y q_1(y,\xi)\right).$$

Hence $q_0(u,y,\xi)$ does not depend on u, and $\frac{d}{du}q_0(u,y,\xi) = 0$. In general,

$$q_{-1} = (2q_1)^{-1}\left\{-\sum_{\substack{|\omega|+i+j=2\\ 0\le i,j\le 1}} \frac{1}{\omega!} d_\xi^\omega q_{1-i}(y,\xi)\cdot D_y^\omega q_{1-j}(y,\xi) + p_0(y,\xi)\right\}$$

and for $k \ge 3$,

$$q_{1-k} = (2q_1)^{-1}\left\{-\sum_{\substack{|\omega|+i+j=k\\ 0\le i,j\le k-1}} \frac{1}{\omega!} d_\xi^\omega q_{1-i}(y,\xi)\cdot D_y^\omega q_{1-j}(y,\xi)\right\}.$$

Hence, each q_{1-k} does not depend on u, and this implies that $\frac{d}{du}Q_u(\alpha_k t)$ is a smoothing operator. Setting $u=0$ in (2.2), we see that

$$Q(\alpha_k t)^2 = (\Delta_Y + \alpha_k t) + \text{ a smoothing operator}, \tag{2.6}$$

and we get the following theorem.

Theorem 2.1. *Under the assumption of the product structure near N, we have the following:*
(1) $Q(\alpha_k t) = \sqrt{\Delta_Y + \alpha_k t} +$ a smoothing operator.
(2) $R(\alpha_k t) = 2\sqrt{\Delta_Y + \alpha_k t} +$ a smoothing operator.

Proof. It's enough to show the first statement. From (2.5) we have

$$Q(\alpha_k t) = \sqrt{\Delta_Y + \alpha_k t} + A,$$

where A is an operator of order 0. Squaring both sides and using (2.6), we have

$$\begin{aligned} Q(\alpha_k t)^2 &= (\Delta_Y + \alpha_k t) + \sqrt{\Delta_Y + \alpha_k t}A + A\sqrt{\Delta_Y + \alpha_k t} + A^2 \\ &= (\Delta_Y + \alpha_k t) + \text{ a smoothing operator.} \end{aligned}$$

Hence $\sqrt{\Delta_Y + \alpha_k t}A + A\sqrt{\Delta_Y + \alpha_k t} + A^2$ is a smoothing operator, which implies that A is a smoothing operator. □

3. Computation of the zero coefficient of $\log Det R(\alpha_k t)$ as $t \to \infty$

It is shown in [3] that $\log Det R(\alpha_k t)$ has an asymptotic expansion as $t\to\infty$ and each coefficient can be computed by the asymptotic symbol of $R(\alpha_k t)$. Hence, from Theorem 2.1, $\log Det R(\alpha_k t)$ and $\log Det(2\sqrt{\Delta_Y + \alpha_k t})$ have the same asymptotic expansions as $t\to\infty$. In this section, we are going to compute the asymptotic expansion of $\log Det(2\sqrt{\Delta_Y + \alpha_k t})$ by using the method in [12].

Note that

$$\log Det(2\sqrt{\Delta_Y + \alpha_k t}) = \log 2 \cdot \zeta_{(\Delta_Y + \alpha_k t)}(0) + \frac{1}{2}\log Det(\Delta_Y + \alpha_k t), \tag{3.1}$$

and we are going to consider $\log Det(\Delta_Y + \alpha_k t)$. Since $Re(\alpha_k)$ is possibly negative, we avoid this difficulty as follows. Put $\alpha_k = e^{i\theta_k}$ with $\theta_k = \frac{2k\pi}{m}$ for m odd and $\frac{(2k+1)\pi}{m}$ for m even. Choose an angle ϕ_k with $0 \le |\phi_k| < \frac{\pi}{2}$ so that $Re(e^{i(\theta_k - \phi_k)}) > 0$. (In fact, if $0 \le |\theta_k| < \frac{\pi}{2}$, we choose $\phi_k = 0$.) Then

$$\begin{aligned} \log Det(\Delta_Y + \alpha_k t) &= \log Det\{e^{i\phi_k}(e^{-i\phi_k}\Delta_Y + e^{i(\theta_k-\phi_k)}t)\} \\ &= -\frac{d}{ds}|_{s=0}\left\{e^{-i\phi_k s}\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i(\theta_k-\phi_k)}t)}(s)\right\} \\ &= i\phi_k \zeta_{(e^{-i\phi_k}\Delta_Y + e^{i(\theta_k-\phi_k)}t)}(0) + \log Det(e^{-i\phi_k}\Delta_Y + e^{i(\theta_k-\phi_k)}t). \end{aligned} \tag{3.2}$$

Put $\tilde{\theta}_k = \theta_k - \phi_k$. Then

$$\begin{aligned} \zeta_{(e^{-i\phi_k}\Delta_Y + e^{i(\theta_k-\phi_k)}t)}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty r^{s-1} Tr e^{-r(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)} dr \\ &= \frac{1}{\Gamma(s)}\int_0^\infty r^{s-1} e^{-rte^{i\tilde{\theta}_k}} Tr e^{-re^{-i\phi_k}\Delta_Y} dr. \end{aligned}$$

The following lemma is a well-known fact.

Lemma 3.1. *As $r \to 0$, we have the following asymptotic expansion:*

$$Tr e^{-re^{-i\phi_k}\Delta_Y} \sim b_1 r^{-\frac{m-1}{2}} + b_2 r^{-\frac{m-2}{2}} + \cdots + b_m + b_{m-1} r^{\frac{1}{2}} + \cdots$$

with $b_m = \zeta_{\Delta_Y}(0) + dim Ker \Delta_Y$.

Now we are going to compute the asymptotic expansion of $\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(s)$ as $t \to \infty$:

$$\begin{aligned} \zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty r^{s-1} e^{-rte^{i\tilde{\theta}_k}} Tr e^{-re^{-i\phi_k}\Delta_Y} dr \\ &= \frac{1}{\Gamma(s)}\int_0^\infty (\frac{u}{t})^{s-1} e^{-ue^{i\tilde{\theta}_k}} Tr e^{-\frac{u}{t}e^{-i\phi_k}\Delta_Y} \frac{1}{t} du \\ &= t^{-s}\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1} e^{-ue^{i\tilde{\theta}_k}} Tr e^{-\frac{u}{t}e^{-i\phi_k}\Delta_Y} du. \end{aligned}$$

As $t \to \infty$,

$$\begin{aligned} \zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(s) &\sim t^{-s}\sum_{j=1}^\infty \frac{1}{\Gamma(s)} b_j \int_0^\infty u^{s-1}(\frac{u}{t})^{\frac{j-m}{2}} e^{-ue^{i\tilde{\theta}_k}} du \\ &= \sum_{j=1}^\infty b_j t^{-s+\frac{m-j}{2}} \frac{1}{\Gamma(s)}\int_0^\infty u^{s+\frac{j-m}{2}-1} e^{-ue^{i\tilde{\theta}_k}} du \\ &= \sum_{j=1}^\infty b_j t^{-s+\frac{m-j}{2}} \frac{1}{\Gamma(s)} (e^{-i\tilde{\theta}_k})^{s+\frac{j-m}{2}} \int_0^\infty (ue^{i\tilde{\theta}_k})^{s+\frac{j-m}{2}-1} e^{-ue^{i\tilde{\theta}_k}} (e^{i\tilde{\theta}_k}) du. \end{aligned}$$

Consider the contour integral $\int_C z^{s+\frac{j-m}{2}-1} e^{-z} dz$ for $Res > \frac{m-j}{2}$, where

$$\begin{aligned} C = \{re^{i\tilde{\theta}_k} \mid \epsilon \le r \le R\} &\cup \{\epsilon e^{i\theta} \mid 0 \le \theta \le \tilde{\theta}_k\} \\ &\cup \{r \mid \epsilon \le r \le R\} \cup \{Re^{i\theta} \mid 0 \le \theta \le \tilde{\theta}_k\} \end{aligned}$$

and oriented counterclockwise. Then one can check that

$$\int_0^\infty (ue^{i\tilde{\theta}_k})^{s+\frac{j-m}{2}-1} e^{-ue^{i\tilde{\theta}_k}} (e^{i\tilde{\theta}_k}) du = \int_0^\infty r^{s+\frac{j-m}{2}-1} e^{-r} dr = \Gamma(s + \frac{j-m}{2}).$$

We therefore obtain the following asymptotic expansion for $t \to \infty$:

$$\begin{aligned}\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(s) &\sim \sum_{j=1}^{\infty} b_j (e^{-i\tilde{\theta}_k})^{s+\frac{j-m}{2}} \frac{\Gamma(s+\frac{j-m}{2})}{\Gamma(s)} t^{-s+\frac{m-j}{2}} \\ &= s \sum_{\substack{j=1 \\ j\neq m}}^{\infty} b_j (e^{-i\tilde{\theta}_k})^{s+\frac{j-m}{2}} \frac{\Gamma(s+\frac{j-m}{2})}{\Gamma(s+1)} t^{-s+\frac{m-j}{2}} + b_m e^{-i\tilde{\theta}_k s} t^{-s}.\end{aligned}$$

This gives the asymptotic expansion of $\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(s)$ as $t \to \infty$. In view of Theorem 1.1 we are mainly interested in the zero coefficients in the asymptotic expansions of $\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0)$ and $\zeta'_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0)$ as $t \to \infty$.

First, setting $s = 0$, the zero coefficient $\pi_0(\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0))$ in the asymptotic expansion of $\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0)$ is the following:

$$\pi_0(\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0)) = b_m = \zeta_{\Delta_Y}(0) + dimKer\Delta_Y. \tag{3.3}$$

Taking the derivative at $s = 0$, the zero coefficient of $\zeta'_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0)$ can be obtained only in the term $b_m e^{-i\tilde{\theta}_k s} t^{-s}$. Hence, by (3.2) and (3.3), the zero coefficient $\pi_0(\Delta_Y + \alpha_k t)$ in the asymptotic expansion of $\log Det(\Delta_Y + \alpha_k t)$ as $t \to \infty$ is

$$\begin{aligned}\pi_0(\Delta_Y + \alpha_k t) &= i\phi_k(\zeta_{\Delta_Y}(0) + dimKer\Delta_Y) + i(\theta_k - \phi_k)(\zeta_{\Delta_Y}(0) + dimKer\Delta_Y) \\ &= i\theta_k(\zeta_{\Delta_Y}(0) + dimKer\Delta_Y).\end{aligned} \tag{3.4}$$

We summarize the above computations as follows.

Proposition 3.2. *The zero coefficients in the asymptotic expansions of*

$$\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0) \quad \textit{and} \quad \log Det(\Delta_Y + \alpha_k t)$$

as $t \to \infty$ are the following:

(1) $\pi_0(\zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0)) = \zeta_{\Delta_Y}(0) + dimKer\Delta_Y$.

(2) $\pi_0(\Delta_Y + \alpha_k t) = i\theta_k(\zeta_{\Delta_Y}(0) + dimKer\Delta_Y)$, *where* $\alpha_k = e^{i\theta_k}$.

Now we are ready to compute $c = \sum_k c_k$ in Theorem 1.1. Since $\zeta_{(\Delta_Y + \alpha_k t)}(0) = \zeta_{(e^{-i\phi_k}\Delta_Y + e^{i\tilde{\theta}_k}t)}(0)$, from (3.1) and Proposition 3.2 we get

$$c_k = \log 2 \cdot (\zeta_{\Delta_Y}(0) + dimKer\Delta_Y) + \frac{1}{2} i\theta_k(\zeta_{\Delta_Y}(0) + dimKer\Delta_Y),$$

and hence

$$\sum_k c_k = m \log 2 \cdot (\zeta_{\Delta_Y}(0) + dimKer\Delta_Y).$$

This completes the proof of Theorem 1.2.

4. The adiabatic decomposition of the zeta-determinant of a Laplacian

In this section we are going to prove Theorem 1.4. Recall that

$$M_{1,r} = M_1 \cup_Y [-r, 0] \times Y, \quad M_{2,r} = M_2 \cup_Y [0, r] \times Y,$$

where we identify Y with $\{-r\} \times Y$ and Y with $\{r\} \times Y$. Then

$$M_r = M_{1,r} \cup_{\{0\}\times Y} M_{2,r}. \tag{4.1}$$

Throughout this section we denote $\{r\}\times Y$ by Y_r, the Dirichlet (Neumann) condition on Y_r by B_r (C_r) and the Dirichlet condition on Z by D. We assume that $\Delta_{M,D}$

is invertible. Then, under certain conditions, $\Delta_{M_r,D}$ is also invertible for r large enough (Lemma 4.6).

From the decomposition (4.1) and Corollary 1.3, we have

$$\log Det\Delta_{M_r,D} = \log Det\Delta_{M_{1,r},B_0,D} + \log Det\Delta_{M_{2,r},B_0,D} - \log 2\cdot(\zeta_{\Delta_Y}(0) + dim Ker\Delta_Y) + \log Det R_{M_r}. \tag{4.2}$$

From the decomposition $M_r = (M_1 \cup M_2) \cup N_r$ with $N_r = [-r,r]\times Y$, we have

$$\begin{aligned}\log Det\Delta_{M_r,D} &= \log Det\Delta_{(M_1\cup M_2),B_{-r},B_r,D} + \log Det\Delta_{N_r,B_{-r},B_r}\\ &\quad - \log 2\cdot(\zeta_{\Delta_{Y\cup Y}}(0) + dim Ker\Delta_{Y\cup Y}) + \log Det R_{-r,r}\\ &= \log Det\Delta_{M_1,B,D} + \log Det\Delta_{M_2,B,D} + \log Det\Delta_{N_r,B_{-r},B_r}\\ &\quad - 2\log 2\cdot(\zeta_{\Delta_Y}(0) + dim Ker\Delta_Y) + \log Det R_{-r,r},\end{aligned} \tag{4.3}$$

where $R_{-r,r} : C^\infty(Y_{-r}) \oplus C^\infty(Y_r) \to C^\infty(Y_{-r}) \oplus C^\infty(Y_r)$ is the Dirichlet-to-Neumann operator corresponding to the decomposition $(M_1 \cup M_2) \cup N_r$.

Put $N_{-r,0} = [-r,0]\times Y$ and $N_{0,r} = [0,r]\times Y$. Since $M_{1,r} = M_1 \cup N_{-r,0}$ and $M_{2,r} = M_2 \cup N_{0,r}$, we have

$$\log Det\Delta_{M_{1,r},B_0,D} = \log Det\Delta_{M_1,B,D} + \log Det\Delta_{N_{-r,0},B_{-r},B_0} - \log 2\cdot(\zeta_{\Delta_Y}(0) + dim Ker\Delta_Y) + \log Det R_{M_{1,r}}, \tag{4.4}$$

$$\log Det\Delta_{M_{2,r},B_0,D} = \log Det\Delta_{M_2,B,D} + \log Det\Delta_{N_{0,r},B_0,B_r} - \log 2\cdot(\zeta_{\Delta_Y}(0) + dim Ker\Delta_Y) + \log Det R_{M_{2,r}}. \tag{4.5}$$

Here $\Delta_{N_{-r,0},B_{-r},B_0} = -\partial_u^2 + \Delta_Y$ with the domain $\{\phi \in C^\infty(N_{-r,0}) \mid \phi|_{Y_{-r}} = \phi|_{Y_0} = 0\}$ and $R_{M_{1,r}}$ is the Dirichlet-to-Neumann operator corresponding to the decomposition $M_{1,r} = M_1 \cup ([-r,0]\times Y)$. $\Delta_{N_{0,r},B_0,B_r}$ and $R_{M_{2,r}}$ are defined similarly.

Then from (4.2)–(4.5), we have

$$\begin{aligned}&- \log 2\cdot(\zeta_{\Delta_Y}(0) + dim Ker\Delta_Y) + \log Det R_{M_r}\\ &= \log Det\Delta_{N_r,B_{-r},B_r} - \log Det\Delta_{N_{-r,0},B_{-r},B_0} - \log Det\Delta_{N_{0,r},B_0,B_r}\\ &\quad + \log Det R_{-r,r} - \log Det R_{M_{1,r}} - \log Det R_{M_{2,r}}.\end{aligned} \tag{4.6}$$

From the decomposition of N_r as

$$N_r = ([-r,0]\times Y) \cup ([0,r]\times Y),$$

we have

$$\log Det\Delta_{N_r,B_{-r},B_r} - \log Det\Delta_{N_{-r,0},B_{-r},B_0} - \log Det\Delta_{N_{0,r},B_0,B_r} = -\log 2\cdot(\zeta_{\Delta_Y}(0) + dim Ker\Delta_Y) + \log Det R_{N_r}, \tag{4.7}$$

where $R_{N_r} : C^\infty(Y_0) \to C^\infty(Y_0)$ is defined as follows. For $f \in C^\infty(Y_0)$, choose $\phi(u,y)$ so that $(-\partial_u^2 + \Delta_Y)\phi = 0$ on $N_r - Y_0$, $\phi|_{Y_0} = f$, $\phi|_{Y_{-r}} = \phi|_{Y_r} = 0$. Then, $R_{N_r}(f) = \left(\partial_u(\phi|_{N_{-r,0}}) - \partial_u(\phi|_{N_{0,r}})\right)|_{Y_0}$. Hence, we obtain from (4.6) and (4.7)

$$\log Det R_{M_r} = \log Det R_{N_r} + \log Det R_{-r,r} - \log Det R_{M_{1,r}} - \log Det R_{M_{2,r}}. \tag{4.8}$$

Now we are going to find the spectrum of $R_{N_r} : C^\infty(Y_0) \to C^\infty(Y_0)$. For $f_k \in C^\infty(Y_0)$ with $\Delta_Y f_k = \lambda_k f_k$, we have

$$\phi(u,y) = \begin{cases} \left(e^{\sqrt{\lambda_k}u} + \dfrac{e^{-\sqrt{\lambda_k}r}}{e^{\sqrt{\lambda_k}r} - e^{-\sqrt{\lambda_k}r}}(e^{\sqrt{\lambda_k}u} - e^{-\sqrt{\lambda_k}u})\right) f_k(y) \\ \hfill \text{for } (u,y) \in N_{-r,0}, \\ \left(e^{-\sqrt{\lambda_k}u} - \frac{e^{-\sqrt{\lambda_k}r}}{e^{\sqrt{\lambda_k}r} - e^{-\sqrt{\lambda_k}r}}(e^{\sqrt{\lambda_k}u} - e^{-\sqrt{\lambda_k}u})\right) f_k(y) \\ \hfill \text{for } (u,y) \in N_{0,r}. \end{cases}$$

Hence,

$$R_{N_r}(f_k) = \left(2\sqrt{\lambda_k} + \frac{4\sqrt{\lambda_k}e^{-\sqrt{\lambda_k}r}}{e^{\sqrt{\lambda_k}r} - e^{-\sqrt{\lambda_k}r}}\right) f_k,$$

where we interpret $\frac{4\sqrt{\lambda_k}e^{-\sqrt{\lambda_k}r}}{e^{\sqrt{\lambda_k}r} - e^{-\sqrt{\lambda_k}r}}$ as $\frac{2}{r}$ when $\lambda_k = 0$. The spectrum of R_{N_r} is

$$\left\{2\sqrt{\lambda_k} + \frac{4\sqrt{\lambda_k}e^{-\sqrt{\lambda_k}r}}{e^{\sqrt{\lambda_k}r} - e^{-\sqrt{\lambda_k}r}} \mid \lambda_k \in Spec(\Delta_Y)\right\}.$$

Let $P_{Ker\Delta_Y} : C^\infty(Y) \to C^\infty(Y)$ be the orthogonal projection onto $Ker\Delta_Y$. Then

$$\begin{aligned} &\zeta_{R_{N_r}}(s) - \zeta_{(2\sqrt{\Delta_Y} + \frac{2}{r}P_{Ker\Delta_Y})}(s) \\ &\quad = \sum_{\lambda_k \neq 0} \left\{\left(2\sqrt{\lambda_k} + \frac{4\sqrt{\lambda_k}e^{-\sqrt{\lambda_k}r}}{e^{\sqrt{\lambda_k}r} - e^{-\sqrt{\lambda_k}r}}\right)^{-s} - \left(2\sqrt{\lambda_k}\right)^{-s}\right\}. \end{aligned}$$

The following lemma can be checked easily.

Lemma 4.1. *Let A be an invertible elliptic operator of order > 0, and K_r a one-parameter family of trace class operators such that $\lim_{r\to\infty} Tr(K_r) = 0$. Then*

$$\lim_{r\to\infty} \log Det(A + K_r) = \log Det A.$$

Proof. Note that

$$\begin{aligned} \log Det(A + K_r) - \log Det A &= \int_0^1 \frac{d}{dt} \log Det(A + tK_r) dt \\ &= \int_0^1 Tr\left((A + tK_r)^{-1} K_r\right) dt. \end{aligned}$$

If we denote by λ_0 the smallest eigenvalue of $|A|$, for r large enough we have

$$|\log Det(A + K_r) - \log Det A| \leq \frac{1}{2\lambda_0} Tr(K_r)$$

and hence the result follows. □

Applying Lemma 4.1 with $A = 2\sqrt{\Delta_Y}$ and $K_r = g_r(\Delta_Y)$ with

$$g_r(x) = \frac{4\sqrt{x}e^{-\sqrt{x}r}}{e^{\sqrt{x}r} - e^{-\sqrt{x}r}}$$

on the orthogonal complement of $Ker\Delta_Y$, we get the following equation:

$$\lim_{r\to\infty}\left\{\log Det R_{N_r} - \log Det(2\sqrt{\Delta_Y} + \frac{2}{r}P_{Ker\Delta_Y})\right\} = 0.$$

Since

(4.9)

$$\log Det(2\sqrt{\Delta_Y} + \frac{2}{r}P_{Ker\Delta_Y}) = \log 2 \cdot (\zeta_{\Delta_Y}(0) + dimKer\Delta_Y) + \frac{1}{2}\log Det\Delta_Y - (dimKer\Delta_Y)\log r,$$

we get the following corollary.

Corollary 4.2.

$$\lim_{r\to\infty} (\log DetR_{N_r} + (dimKer\Delta_Y)\log r) = \log 2 \cdot (\zeta_{\Delta_Y}(0) + dimKer\Delta_Y) + \frac{1}{2}\log Det\Delta_Y.$$

Now we discuss the operators $R_{M_{1,r}}$, $R_{M_{2,r}}$ and $R_{-r,r}$. First, we can describe $R_{M_{1,r}} : C^\infty(Y_{-r}) \to C^\infty(Y_{-r})$ as follows. For $f_k \in C^\infty(Y_{-r})$ with $\Delta_Y f_k = \lambda_k f_k$, we choose the section $\phi \in C^0(M_{1,r})$ satisfying $\Delta_{M_{1,r}}\phi = 0$ on $M_{1,r} - Y_{-r}$, $\phi|_{Y_{-r}} = f_k$ and $\phi|_Z = \phi|_{Y_0} = 0$. Then one can check that

$$R_{M_{1,r}}(f_k) = Q_1(f_k) - (\partial_u(\phi|_{N_{-r,0}}))|_{Y_{-r}} = Q_1(f_k) + \left(\sqrt{\lambda_k} + \frac{2\sqrt{\lambda_k}e^{-\sqrt{\lambda_k}r}}{e^{\sqrt{\lambda_k}r} - e^{-\sqrt{\lambda_k}r}}\right) f_k.$$

In the same way,

$$R_{M_{2,r}}(f_k) = Q_2(f_k) + (\partial_u(\phi|_{N_{0,r}}))|_{Y_r} = Q_2(f_k) + \left(\sqrt{\lambda_k} + \frac{2\sqrt{\lambda_k}e^{-\sqrt{\lambda_k}r}}{e^{\sqrt{\lambda_k}r} - e^{-\sqrt{\lambda_k}r}}\right) f_k.$$

Similarly, $R_{-r,r} : C^\infty(Y_{-r})\oplus C^\infty(Y_r) \to C^\infty(Y_{-r})\oplus C^\infty(Y_r)$ is described as follows:

$$R_{-r,r}(f_k, 0) = \left(Q_1(f_k) + \left(\sqrt{\lambda_k} + \frac{2\sqrt{\lambda_k}e^{-2\sqrt{\lambda_k}r}}{e^{2\sqrt{\lambda_k}r} - e^{-2\sqrt{\lambda_k}r}}\right) f_k,\ \ -\frac{2\sqrt{\lambda_k}}{e^{2\sqrt{\lambda_k}r} - e^{-2\sqrt{\lambda_k}r}} f_k\right),$$

$$R_{-r,r}(0, f_k) = \left(-\frac{2\sqrt{\lambda_k}}{e^{2\sqrt{\lambda_k}r} - e^{-2\sqrt{\lambda_k}r}} f_k,\ \ Q_2(f_k) + \left(\sqrt{\lambda_k} + \frac{2\sqrt{\lambda_k}e^{-2\sqrt{\lambda_k}r}}{e^{2\sqrt{\lambda_k}r} - e^{-2\sqrt{\lambda_k}r}}\right) f_k\right).$$

We therefore have

$$R_{-r,r} = \begin{pmatrix} Q_1 + \sqrt{\Delta_Y} & 0 \\ 0 & Q_2 + \sqrt{\Delta_Y} \end{pmatrix} + h_r(\Delta_Y)\begin{pmatrix} e^{-2r\sqrt{\Delta_Y}} & -1 \\ -1 & e^{-2r\sqrt{\Delta_Y}} \end{pmatrix},$$

where $h_r(x) = \frac{2\sqrt{x}}{e^{2r\sqrt{x}} - e^{-2r\sqrt{x}}}$ and $h_r(\Delta_Y)$ acts on $Ker\Delta_Y$ as multiplication by $\frac{1}{2r}$.

We are going to discuss the operators Q_i and $Q_i + \sqrt{\Delta_Y}$. The following lemma can be checked by using integration by parts (*cf.* Proposition 4.3 in [2]).

Lemma 4.3. *Suppose that ∇ is a connection which is compatible to the inner product on M.* i.e. *for any sections s_1, $s_2 \in C^\infty(E)$ and a tangent vector w, $w(s_1, s_2) = (\nabla_w s_1, s_2) + (s_1, \nabla_w s_2)$. If $\Delta_M = \nabla^*\nabla$, then each Q_i is a non-negative, self-adjoint operator.*

Next, let us consider a Dirac Laplacian for a Dirac operator A which has the form $G(\partial_u + B)$ near the boundary Y, where G is a bundle automorphism satisfying the conditions (1.1), and both G and B do not depend on the normal coordinate u. We refer to [5] for the following lemma (*cf.* Lemma 3.1 in [5]).

Lemma 4.4. *Let ϕ and ψ be smooth sections on M_j ($j = 1, 2$). Then*

$$\langle A_{M_j}\phi, \psi\rangle_{M_j} - \langle \phi, A_{M_j}\psi\rangle_{M_j} = \epsilon_j \langle \phi|_Y, G(\psi|_Y)\rangle_Y,$$

where $\epsilon_j = 1$ for $j = 2$ and $\epsilon_j = -1$ for $j = 1$.

Suppose that for $f \in C^\infty(Y)$, ϕ_j is the solution of $A^2_{M_j}$ with $\phi_j|_Y = f$, $\phi_j|_Z = 0$. Then by Lemma 4.4

$$\langle (Q_1 + |B|)f, f\rangle_Y = \langle A_{M_1}\phi_1, A_{M_1}\phi_1\rangle_{M_1} + \langle (|B| - B)f, f\rangle_Y, \tag{4.10}$$

$$\langle (Q_2 + |B|)f, f\rangle_Y = \langle A_{M_2}\phi_2, A_{M_2}\phi_2\rangle_{M_2} + \langle (|B| + B)f, f\rangle_Y. \tag{4.11}$$

As a consequence, $f \in Ker(Q_1 + |B|)$ if and only if $A_{M_1}\phi_1 = 0$ and $f \in ImP_{\geq}$; and hence on the cylinder part we can express ϕ_1 as

$$\phi_1 = \sum_{j=1}^{k} a_j g_j + \sum_{\lambda_j > 0} b_j e^{-\lambda_j u} h_j,$$

where $Bg_j = 0$, $Bh_j = \lambda_j h_j$. This implies that ϕ_1 is the restriction of an extended L^2-solution of $A_{M_{1,\infty}}$ on $M_{1,\infty} := M_1 \cup_Y Y \times [0, \infty)$. We can make a similar assertion for ϕ_2 and have the following corollary (*cf.* Theorem 2.2 in [5], see also [1], [2]).

Corollary 4.5. *The invertibility of $Q_1 + \sqrt{B^2}$ and $Q_2 + \sqrt{B^2}$ is equivalent to the non-existence of the extended L^2-solutions of $A_{M_{1,\infty}}$ and $A_{M_{2,\infty}}$ on $M_{1,\infty}$ and $M_{2,\infty}$. In particular, this condition implies that $KerB = 0$.*

Lemma 4.6. *Suppose that Δ_M is either a connection Laplacian or a Dirac Laplacian for a connection compatible to the inner product as above, and $\Delta_{M,D}$ is invertible. If both $Q_1 + \sqrt{\Delta_Y}$ and $Q_2 + \sqrt{\Delta_Y}$ are invertible, then $R_{-r,r}$ and $\Delta_{M_{r,D}}$ are invertible for r large enough.*

Proof. We are going to show first that $R_{-r,r}$ is injective. Then this implies that $\Delta_{M_{r,D}}$ is injective. Since $\Delta_{M_{r,D}}$ is self-adjoint, $\Delta_{M_{r,D}}$ is invertible, and this implies again that $R_{-r,r}$ is also invertible ([3], [8]).

Putting $A_r = h_r(\Delta_Y)$ with $h_r(x) = \frac{2\sqrt{x}}{e^{2r\sqrt{x}} - e^{-2r\sqrt{x}}}$, we get

$$\left\langle R_{-r,r}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{L^2(Y)}$$

$$= \langle (Q_1 + \sqrt{\Delta_Y})f, f\rangle + \langle (Q_2 + \sqrt{\Delta_Y})g, g\rangle + \langle A_r e^{-2r\sqrt{\Delta_Y}} f, f\rangle + \langle A_r e^{-2r\sqrt{\Delta_Y}} g, g\rangle - \langle A_r g, f\rangle - \langle A_r f, g\rangle.$$

Note that each $Q_i + \sqrt{\Delta_Y}$ is a non-negative operator by Lemma 4.3 and (4.10), (4.11). Let λ_0 be the minimum of the eigenvalues of $Q_1 + \sqrt{\Delta_Y}$ and $Q_2 + \sqrt{\Delta_Y}$. Since $\lim_{r\to\infty} ||A_r||_{L^2} = 0$, one can choose r_0 so that $||A_r||_{L^2} < \lambda_0$ for $r \geq r_0$. Then $R_{-r,r}$ is injective for $r \geq r_0$ and this completes the proof. □

In case both $Q_1 + \sqrt{\Delta_Y}$ and $Q_2 + \sqrt{\Delta_Y}$ are invertible, we can apply Lemma 4.1 directly.

Corollary 4.7. *Assume that both* $Q_1 + \sqrt{\Delta_Y}$ *and* $Q_2 + \sqrt{\Delta_Y}$ *are invertible. Then:*
(1) $\lim_{r\to\infty} \log DetR_{M_{1,r}} = \log Det(Q_1 + \sqrt{\Delta_Y})$.
(2) $\lim_{r\to\infty} \log DetR_{M_{2,r}} = \log Det(Q_2 + \sqrt{\Delta_Y})$.
(3) $\lim_{r\to\infty} \log DetR_{-r,r} = \log Det(Q_1 + \sqrt{\Delta_Y}) + \log Det(Q_2 + \sqrt{\Delta_Y})$.

Combining Corollary 4.2 with Corollary 4.7 and (4.2), (4.8), we complete the proof of Theorem 1.4.

5. The adiabatic decomposition of the analytic torsion

In this section, we are going to prove Theorem 1.6 and Theorem 1.7. Recall that M is a closed manifold of dimension m with the product structure near a hypersurface Y. We define M_r, $M_{1,r}$ and $M_{2,r}$ as in Section 4 and suppose that ρ_{M_r} ($\rho_{M_{1,r}}$, $\rho_{M_{2,r}}$, ρ_Y) is an orthogonal representation of $\pi_1(M_r)$ ($\pi_1(M_{1,r})$, $\pi_1(M_{2,r})$, $\pi_1(Y)$) to $SO(n)$, respectively. Then we can construct a flat bundle $E_{\rho_{M_r}} = \tilde{M}_r \times_{\rho_{M_r}} \mathbb{R}^n$, where $\tilde{M}_r$ is the universal cover of M_r. The flat bundles $E_{\rho_{M_{1,r}}}$, $E_{\rho_{M_{2,r}}}$ and E_{ρ_Y} are defined in the same way.

For each q, denote by $\Delta^q_{M_r} := (d_q + d_q^*)^2$ the Hodge Laplacian acting on q-forms valued in $E_{\rho_{M_r}}$. Then the analytic torsion $\tau(M_r, \rho_{M_r})$ is defined by

$$\log \tau(M_r, \rho_{M_r}) = \frac{1}{2}\sum_{q=0}^{m}(-1)^q \cdot q \cdot \log Det\Delta^q_{M_r}.$$

To define the analytic torsion on $M_{i,r}$, we choose the absolute or the relative boundary condition on Y_0. Near Y_0, a differential q-form ω can be expressed by

$$\omega = \omega_1 + du \wedge \omega_2, \tag{5.1}$$

where ω_1 and ω_2 do not contain du.

Definition 5.1. Suppose that a q-form ω in $M_{1,r}$ is expressed as in (5.1).
(1) ω satisfies the *absolute boundary condition* if $(\partial_u \omega_1)|_{Y_0} = 0$ and $\omega_2|_{Y_0} = 0$.
(2) ω satisfies the *relative boundary condition* if $\omega_1|_{Y_0} = 0$ and $(\partial_u \omega_2)|_{Y_0} = 0$.

We denote by $\Omega^q_{abs}(M_{i,r})$, $\Omega^q_{rel}(M_{i,r})$ the sets of all q-forms valued in $E_{M_{i,r}}$ satisfying the absolute and the relative boundary conditions, respectively. We also denote by $\Delta^q_{M_{i,r},abs}$, $\Delta^q_{M_{i,r},rel}$ the Laplacian acting on q-forms valued in $E_{M_{i,r}}$ with

$$Dom(\Delta^q_{M_{i,r},abs}) = \Omega^q_{abs}(M_{i,r}), \qquad Dom(\Delta^q_{M_{i,r},rel}) = \Omega^q_{rel}(M_{i,r}).$$

Then the analytic torsions $\tau_{abs}(M_{i,r}, \rho_{M_{i,r}})$ and $\tau_{rel}(M_{i,r}, \rho_{M_{i,r}})$ are defined by

$$\log \tau_{abs}(M_{i,r}, \rho_{M_{i,r}}) = \frac{1}{2}\sum_{q=0}^{m}(-1)^q \cdot q \cdot \log Det\Delta^q_{M_{i,r},abs},$$

$$\log \tau_{rel}(M_{i,r}, \rho_{M_{i,r}}) = \frac{1}{2}\sum_{q=0}^{m}(-1)^q \cdot q \cdot \log Det\Delta^q_{M_{i,r},rel}.$$

It is a well-known fact (*cf.* [11]) that

$$Ker\Delta^q_{M_{i,r},abs} \cong H^q(M_{i,r}; \rho_{M_{i,r}}), \qquad Ker\Delta^q_{M_{i,r},rel} \cong H^q(M_{i,r}, Y; \rho_{M_{i,r}}).$$

We consider $M_{1,r}$ (a manifold with boundary Y) first. Recall that $M_{1,r} = M_1 \cup_{Y_{-r}} N_{-r,0}$ with $N_{-r,0} = [-r, 0] \times Y$, and $Y_{-r} = \{-r\} \times Y$, $Y_0 = \{0\} \times Y$. We denote by B, D the Dirichlet boundary conditions on Y_{-r}, Y_0, respectively.

For a given representation $\rho_{M_{1,r}} : \pi_1(M_{1,r}) \to SO(n)$, define $\rho_Y : \pi_1(Y) \to SO(n)$ by $\rho_Y = \rho_{M_{1,r}} \circ \iota_Y$, where $\iota_Y : \pi_1(Y) \to \pi_1(M_{1,r})$ is the natural homomorphism. Then the restriction of the bundle $E_{\rho_{M_{1,r}}}$ to Y is isomorphic to E_{ρ_Y}, (*cf.* [11]).

The set $\Omega^q(N_{-r,0}, E_{\rho_{M_{1,r}}}|_{N_{-r,0}})$ of q-forms valued in $E_{\rho_{M_{1,r}}}|_{N_{-r,0}}$ can be decomposed as follows:

$$\Omega^q(N_{-r,0}, E_{\rho_{M_{1,r}}}|_{N_{-r,0}}) = C^\infty([-r,0], E_{\rho_{M_{1,r}}}|_{N_{-r,0}}) \otimes \Omega^q(Y, E_{\rho_Y}) \oplus du \wedge C^\infty([-r,0], E_{\rho_{M_{1,r}}}|_{N_{-r,0}}) \otimes \Omega^{q-1}(Y, E_{\rho_Y}). \tag{5.2}$$

From this decomposition, the Laplacian $\Delta^q_{M_{1,r}}$, when restricted to $N_{-r,0}$, can be expressed as

$$\Delta^q_{M_{1,r}} = -\partial_u^2 + \begin{pmatrix} \Delta^q_Y & 0 \\ 0 & \Delta^{q-1}_Y \end{pmatrix}, \tag{5.3}$$

where Δ^q_Y is the Laplacian acting on q-forms on Y, valued in E_{ρ_Y}. Here and throughout this section we use the convention that $\Delta^q_Y = 0$ for $q < 0$ or $q \geq m$.

To describe the gluing formula of the type of Theorem 1.1 (or Corollary 1.3) in this context, we need to define modified Dirichlet-to-Neumann operators Q^q_1, $Q^q_{N_{-r,0},abs}$ and $Q^q_{N_{-r,0},rel}$ as follows. For simplicity, set $E = (\bigwedge^q T^*M_{1,r}) \otimes E_{\rho_{M_{1,r}}}$. For a given $f \in C^\infty(E|_{Y_{-r}})$, choose smooth sections $\phi \in C^\infty(E|_{M_1})$, $\psi_{abs} \in C^\infty(E|_{N_{-r,0}})$ and $\psi_{rel} \in C^\infty(E|_{N_{-r,0}})$ such that

$$\Delta^q_{M_1}\phi = 0, \quad \Delta^q_{N_{-r,0}}\psi_{abs} = \Delta^q_{N_{-r,0}}\psi_{rel} = 0, \quad \phi|_{Y_{-r}} = \psi_{abs}|_{Y_{-r}} = \psi_{rel}|_{Y_{-r}} = f,$$

and ψ_{abs} (ψ_{rel}) satisfies the absolute (relative) boundary condition on Y_0, respectively. Then we define

$$Q^q_1(f) = (\partial_u\phi)|_{Y_{-r}},$$
$$Q^q_{N_{-r,0},abs}(f) = (-\partial_u\psi_{abs})|_{Y_{-r}}, \quad Q^q_{N_{-r,0},rel}(f) = (-\partial_u\psi_{rel})|_{Y_{-r}},$$

and

$$R^q_{B,abs} = Q^q_1 + Q^q_{N_{-r,0},abs}, \quad R^q_{B,rel} = Q^q_1 + Q^q_{N_{-r,0},rel}.$$

Then the following theorem can be proved in the same way as Theorem 1.1 (*cf.* the Remark after Corollary 1.3).

Theorem 5.2. *We denote $k_q = dimKer\Delta^q_Y$. Then:*

(1) $$\log Det\Delta^q_{M_{1,r},D} - \log Det\Delta^q_{M_1,B} - \log Det\Delta^q_{N_{-r,0},B,D} = -\log 2(\zeta_{\Delta^{q-1}_Y}(0) + \zeta_{\Delta^q_Y}(0) + k_{q-1} + k_q) + \log DetR^q_{B,D}.$$

(2) $$\log Det\Delta^q_{M_{1,r},abs} - \log Det\Delta^q_{M_1,B} - \log Det\Delta^q_{N_{-r,0},B,abs} = -\log 2(\zeta_{\Delta^{q-1}_Y}(0) + \zeta_{\Delta^q_Y}(0) + k_{q-1} + k_q) + \log DetR^q_{B,abs}.$$

(3) $$\log Det\Delta^q_{M_{1,r},rel} - \log Det\Delta^q_{M_1,B} - \log Det\Delta^q_{N_{-r,0},B,rel} = -\log 2(\zeta_{\Delta^{q-1}_Y}(0) + \zeta_{\Delta^q_Y}(0) + k_{q-1} + k_q) + \log DetR^q_{B,rel}.$$

We next describe the operators $\Delta^q_{N_{-r,0},B,abs}$ and $\Delta^q_{N_{-r,0},B,rel}$. From the decomposition (5.2), we have

$$\Delta^q_{N_{-r,0},B,abs} = \begin{pmatrix} (-\partial_u^2 + \Delta^q_Y)_{N_{-r,0},B,C} & 0 \\ 0 & (-\partial_u^2 + \Delta^{q-1}_Y)_{N_{-r,0},B,D} \end{pmatrix},$$
$$\Delta^q_{N_{-r,0},B,rel} = \begin{pmatrix} (-\partial_u^2 + \Delta^q_Y)_{N_{-r,0},B,D} & 0 \\ 0 & (-\partial_u^2 + \Delta^{q-1}_Y)_{N_{-r,0},B,C} \end{pmatrix},$$

where C means the Neumann boundary condition on Y_0 and B (D) means the Dirichlet boundary condition on Y_{-r} (Y_0). Hence, we have

$$\begin{aligned} (5.4)\quad & \log Det\Delta^q_{N_{-r,0},B,abs} - \log Det\Delta^q_{N_{-r,0},B,D} \\ &= \log Det(-\partial_u^2 + \Delta^q_Y)_{N_{-r,0},B,C} - \log Det(-\partial_u^2 + \Delta^q_Y)_{N_{-r,0},B,D}, \end{aligned}$$

$$\begin{aligned} (5.5)\quad & \log Det\Delta^q_{N_{-r,0},B,rel} - \log Det\Delta^q_{N_{-r,0},B,D} \\ &= \log Det(-\partial_u^2 + \Delta^{q-1}_Y)_{N_{-r,0},B,C} - \log Det(-\partial_u^2 + \Delta^{q-1}_Y)_{N_{-r,0},B,D}. \end{aligned}$$

Now we assume that $Q^q_1 + \begin{pmatrix} \sqrt{\Delta^q_Y} & 0 \\ 0 & \sqrt{\Delta^{q-1}_Y} \end{pmatrix}$ is invertible. Since the Hodge Laplacian $\Delta^q_{M_r}$ is a Dirac Laplacian satisfying (1.1), by Corollary 4.5 we have $k_{q-1} = k_q = 0$ (*cf.* the Remark below Theorem 1.6). By Corollary 1.5 and (5.4), (5.5) we have

$$(5.6)\qquad \lim_{r\to\infty} \left\{ \log Det\Delta^q_{N_{-r,0},B,abs} - \log Det\Delta^q_{N_{-r,0},B,D} \right\} = \frac{1}{2} \log Det\Delta^q_Y,$$

$$(5.7)\qquad \lim_{r\to\infty} \left\{ \log Det\Delta^q_{N_{-r,0},B,rel} - \log Det\Delta^q_{N_{-r,0},B,D} \right\} = \frac{1}{2} \log Det\Delta^{q-1}_Y.$$

From Theorem 5.2 we have

$$\begin{aligned} & \log Det\Delta^q_{M_{1,r},abs} - \log Det\Delta^q_{M_{1,r},D} \\ &= \left(\log Det\Delta^q_{N_{-r,0},B,abs} - \log Det\Delta^q_{N_{-r,0},B,D} \right) \\ (5.8)\quad &\quad + \log DetR^q_{B,abs} - \log DetR^q_{B,D}, \end{aligned}$$

and

$$\begin{aligned} & \log Det\Delta^q_{M_{1,r},rel} - \log Det\Delta^q_{M_{1,r},D} \\ &= \left(\log Det\Delta^q_{N_{-r,0},B,rel} - \log Det\Delta^q_{N_{-r,0},B,D} \right) \\ (5.9)\quad &\quad + \log DetR^q_{B,rel} - \log DetR^q_{B,D}. \end{aligned}$$

Lemma 5.3. *Suppose that* $Q^q_1 + \begin{pmatrix} \sqrt{\Delta^q_Y} & 0 \\ 0 & \sqrt{\Delta^{q-1}_Y} \end{pmatrix}$ *is invertible. Then*

$$\begin{aligned} \lim_{r\to\infty} \log DetR^q_{B,abs} &= \lim_{r\to\infty} \log DetR^q_{B,rel} = \lim_{r\to\infty} \log DetR^q_{B,D} \\ &= \log Det \left(Q^q_1 + \begin{pmatrix} \sqrt{\Delta^q_Y} & 0 \\ 0 & \sqrt{\Delta^{q-1}_Y} \end{pmatrix} \right). \end{aligned}$$

Proof. The last equality is exactly the assertion (1) in Corollary 4.7. We are going to show that

$$\lim_{r\to\infty} \log DetR^q_{B,abs} = \log Det \left(Q^q_1 + \begin{pmatrix} \sqrt{\Delta^q_Y} & 0 \\ 0 & \sqrt{\Delta^{q-1}_Y} \end{pmatrix} \right).$$

The case of $\log Det R^q_{B,rel}$ can be proved in the same way.

By a direct computation one can check the following. For $f \in \Omega^q(Y, E_{\rho_Y})$ with $\Delta^q_Y f = \lambda f$,

$$R^q_{B,abs}(f) = Q^q_1(f) + \left(\sqrt{\lambda} - \frac{2\sqrt{\lambda}e^{-\sqrt{\lambda}r}}{e^{\sqrt{\lambda}r} + e^{-\sqrt{\lambda}r}}\right) f.$$

For $g \in \Omega^{q-1}(Y, E_{\rho_Y})$ with $\Delta^{q-1}_Y g = \mu g$,

$$R^q_{B,abs}(du \wedge g) = Q^q_1(du \wedge g) + \left(\sqrt{\mu} + \frac{2\sqrt{\mu}e^{-\sqrt{\mu}r}}{e^{\sqrt{\mu}r} - e^{-\sqrt{\mu}r}}\right) du \wedge g.$$

Then the result follows from Lemma 4.1.

From (5.6)–(5.9) and Lemma 5.3, we have the following corollary.

Corollary 5.4. *Suppose that* $Q^q_1 + \begin{pmatrix} \sqrt{\Delta^q_Y} & 0 \\ 0 & \sqrt{\Delta^{q-1}_Y} \end{pmatrix}$ *is invertible for each* q. *Then the following equalities hold:*

(1)
$$\lim_{r\to\infty} \left\{\log Det \Delta^q_{M_{1,r},abs} - \log Det \Delta^q_{M_{1,r},D}\right\} = \begin{cases} \frac{1}{2}\log Det \Delta^q_Y & (0 \le q \le m-1), \\ 0 & (q = m). \end{cases}$$

(2)
$$\lim_{r\to\infty} \left\{\log Det \Delta^q_{M_{1,r},rel} - \log Det \Delta^q_{M_{1,r},D}\right\} = \begin{cases} \frac{1}{2}\log Det \Delta^{q-1}_Y & (1 \le q \le m), \\ 0 & (q = 0). \end{cases}$$

Now we are ready to prove Theorem 1.6. We have

$$\begin{aligned}
&\lim_{r\to\infty} \left\{\log \tau_{abs}(M_{1,r}, \rho_{M_{1,r}}) - \log \tau_{rel}(M_{1,r}, \rho_{M_{1,r}})\right\} \\
&= \lim_{r\to\infty} \frac{1}{2}\sum_{q=0}^{m} (-1)^q \cdot q \cdot \left(\log Det \Delta_{M_{1,r},abs} - \log Det \Delta_{M_{1,r},D}\right) \\
&\quad - \lim_{r\to\infty} \frac{1}{2}\sum_{q=0}^{m} (-1)^q \cdot q \cdot \left(\log Det \Delta_{M_{1,r},rel} - \log Det \Delta_{M_{1,r},D}\right) \\
&= \frac{1}{4}\sum_{q=0}^{m-1} (-1)^q \cdot q \cdot \log Det(\Delta^q_Y) - \frac{1}{4}\sum_{q=1}^{m} (-1)^q \cdot q \cdot \log Det(\Delta^{q-1}_Y) \\
&= \frac{1}{2}\sum_{q=0}^{m-1} (-1)^q \cdot q \cdot \log Det(\Delta^q_Y) + \frac{1}{4}\sum_{q=0}^{m-1} (-1)^q \cdot \log Det(\Delta^q_Y) \\
&= \tau(Y, \rho_Y).
\end{aligned}$$

This completes the proof of Theorem 1.6. □

Next, we take care of the closed manifold $M_r = M_{1,r} \cup_{Y_0} M_{2,r}$. From Theorem 1.4, we have

$$\lim_{r\to\infty} \left\{\log Det \Delta^q_{M_r} - \log Det \Delta^q_{M_{1,r},D} - \log Det \Delta^q_{M_{2,r},D}\right\} = \frac{1}{2}\left(\log Det \Delta^q_Y + \log Det \Delta^{q-1}_Y\right).$$

On the other hand,

$$\begin{aligned}&\lim_{r\to\infty}\left\{\log Det\Delta^q_{M_r}-\log Det\Delta^q_{M_{1,r},D}-\log Det\Delta^q_{M_{2,r},D}\right\}\\&=\lim_{r\to\infty}\left\{\left(\log Det\Delta^q_{M_r}-\log Det\Delta^q_{M_{1,r},abs}-\log Det\Delta^q_{M_{2,r},rel}\right)\right.\\&\qquad+\left(\log Det\Delta^q_{M_{1,r},abs}-\log Det\Delta^q_{M_{1,r},D}\right)\\&\qquad\left.+\left(\log Det\Delta^q_{M_{2,r},rel}-\log Det\Delta^q_{M_{2,r},D}\right)\right\}.\end{aligned}$$

From Corollary 5.4 we have

$$\lim_{r\to\infty}\left\{\log Det\Delta^q_{M_r}-\log Det\Delta^q_{M_{1,r},abs}-\log Det\Delta^q_{M_{2,r},rel}\right\}=0,$$

and therefore we obtain

$$\lim_{r\to\infty}\left\{\tau(M_r,\rho_{M_r})-\tau_{abs}(M_{1,r},\rho_{M_{1,r}})-\tau_{rel}(M_{2,r},\rho_{M_{2,r}})\right\}=0,$$

which completes the proof of Theorem 1.7.

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