

## TWISTED SUMS WITH $C(K)$ SPACES

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ABSTRACT. If  $X$  is a separable Banach space, we consider the existence of non-trivial twisted sums  $0 \rightarrow C(K) \rightarrow Y \rightarrow X \rightarrow 0$ , where  $K = [0, 1]$  or  $\omega^\omega$ . For the case  $K = [0, 1]$  we show that there exists a twisted sum whose quotient map is strictly singular if and only if  $X$  contains no copy of  $\ell_1$ . If  $K = \omega^\omega$  we prove an analogue of a theorem of Johnson and Zippin (for  $K = [0, 1]$ ) by showing that all such twisted sums are trivial if  $X$  is the dual of a space with summable Szlenk index (e.g.,  $X$  could be Tsirelson's space); a converse is established under the assumption that  $X$  has an unconditional finite-dimensional decomposition. We also give conditions for the existence of a twisted sum with  $C(\omega^\omega)$  with strictly singular quotient map.

### 1. INTRODUCTION AND PRELIMINARY REMARKS

Let  $X$  and  $Y$  be real Banach spaces. Then we say  $\text{Ext}(X, Y) = \{0\}$  if every short exact sequence  $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$  splits; informally this means that if  $Z$  is a Banach space containing  $Y$  and so that  $Z/Y \sim X$ , then there is a bounded projection of  $Z$  onto  $Y$ . A space  $Z$  with a subspace isomorphic to  $Y$  so that  $Z/Y$  is isomorphic to  $X$  is often called a twisted sum of  $Y$  and  $X$  (order is important!). Thus  $\text{Ext}(X, Y) = \{0\}$  if and only if every twisted sum of  $Y$  and  $X$  is trivial (i.e. reduces to  $Y \oplus X$ ).

Fundamental tools for us are the pushout and pullback constructions. These are well-known to algebraists and topologists, but less so to analysts. So we will describe them briefly in the Banach space setting. If  $T : E \rightarrow X$  and  $S : E \rightarrow Y$  are two operators defined on the same Banach space, then their pushout  $Z$  is defined as the quotient of  $X \oplus_1 Y$  by the closure of  $\{(Te, -Se) : e \in E\}$ , together with the natural mappings  $X \rightarrow Z$  and  $Y \rightarrow Z$  (i.e., the restrictions of the quotient mapping). In case one of the mappings, say  $S$ , is the inclusion mapping from a short exact sequence, then completing the diagram gives a second short exact sequence with the same quotient space  $F$ :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{S} & Y & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & F & \longrightarrow & 0. \end{array}$$

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Conversely, if we are given any commutative diagram as above, then  $Z$  must be isomorphic to the pushout of  $S$  and  $T$ ; this observation will be used several times in the sequel. Note also that the operator  $Y \rightarrow Z$  is an isomorphic embedding (respectively a quotient mapping) if and only if  $T$  is. Furthermore, the lower sequence splits if and only if  $T$  can be extended to  $Y$ . These well-known exercises follow from standard diagram-chasing arguments.

Dually, if  $S : X \rightarrow E$  and  $T : Y \rightarrow E$  are two operators into the same Banach space, then their pullback  $Z$  is defined as the subspace of all  $(x, y) \in X \oplus_\infty Y$  for which  $Sx = Ty$ , together with the natural mappings  $Z \rightarrow Y$  and  $Z \rightarrow X$ . In case one of the original mappings, say  $S$ , is the quotient mapping from a short exact sequence, then completing the diagram gives a second short exact sequence with the same subspace  $F$ :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow T & & \\ 0 & \longrightarrow & F & \longrightarrow & X & \xrightarrow{S} & E & \longrightarrow & 0. \end{array}$$

Conversely, if we are given any commutative diagram as above, then  $Z$  must be isomorphic to the pullback of  $S$  and  $T$ . Note again that the operator  $Z \rightarrow X$  is an isomorphic embedding (respectively a quotient mapping) if and only if  $T$  is. For further information, see [16, Chap. 1] and the references therein.

Let  $X$  be any separable Banach space and let  $Q_X : \ell_1 \rightarrow X$  be any quotient map. We will keep the notation  $\tilde{X}$  for the kernel of  $Q_X$  (which is unique up to automorphism provided it is infinite dimensional, see [35], [36, p. 108] or [15, p. 382]). The following theorem is well known:

**Theorem 1.1.** *Suppose  $X$  and  $Y$  are separable Banach spaces. Then the following are equivalent:*

- (1)  $\text{Ext}(X, Y) = \{0\}$ .
- (2) If  $T : \tilde{X} \rightarrow Y$  is a bounded operator, then there is a bounded extension  $\tilde{T} : \ell_1 \rightarrow Y$ .
- (3) If  $Z$  is a separable Banach space containing a subspace  $E$  so that  $Z/E \sim X$  and  $T : E \rightarrow Y$  is a bounded operator, then there is an extension  $\tilde{T} : Z \rightarrow Y$ .

*Proof.* It is trivial that (3) implies (1). For (1) implies (3) we use the pushout construction:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow S & & \parallel & & \\ 0 & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Now (1) implies the existence of a projection  $P : W \rightarrow Y$ , and then  $PS$  extends  $T$ .

That (2) is equivalent to (3) is clear from the proof of Corollary 1.1 of [26]. Alternatively, [30, Prop. 3.1] proves directly the equivalence of (1) and (2).  $\square$

*Remark.* Of course all separability assumptions can be removed if we simply replace  $\ell_1$  by  $\ell_1(I)$  for a suitable index set.

There is an immediate corollary, which essentially says that  $\text{Ext}(X, Y) = \{0\}$  is a three-space property of  $X$ :

**Corollary 1.2.** *Suppose  $Y$  is a Banach space and  $X$  is a Banach space with a subspace  $E$  so that  $\text{Ext}(E, Y) = \{0\}$ , and  $\text{Ext}(X/E, Y) = \{0\}$ . Then  $\text{Ext}(X, Y) = \{0\}$ .*

*Proof.* Let  $\widetilde{X}$  and  $Q_X$  be defined as above. Given  $T : \widetilde{X} \rightarrow Y$ , we need to find an extension to all of  $\ell_1$ . We will apply Theorem 1.1.

If  $Q : X \rightarrow X/E$  is the obvious mapping, we may choose  $\widetilde{X/E}$  to be the kernel of  $Q \circ Q_X$ . Then  $y \mapsto Q_X y$  is a quotient mapping from  $\widetilde{X/E}$  onto  $E$  with kernel  $\widetilde{X}$ . The implication (1)  $\Rightarrow$  (3) then gives us an extension  $\widetilde{T} : \widetilde{X/E} \rightarrow Y$  of  $T$ , which by the implication (1)  $\Rightarrow$  (2) admits a further extension  $\widetilde{\widetilde{T}} : \ell_1 \rightarrow Y$ .  $\square$

In this paper, we consider the case when the subspace of our twisted sum is  $C(K)$  for some compact metric space  $K$ . If  $K$  is uncountable, then the theorem of Milutin [40, Theorem 8.5] implies we may consider  $K = [0, 1]$ . The following result is due to Johnson and Zippin [26], in view of Theorem 1.1:

**Theorem 1.3.** *If  $X$  is isomorphic to the dual of a subspace of  $c_0$  (so that  $\widetilde{X}$  can be assumed weak\*-closed), then  $\text{Ext}(X, C(K)) = \{0\}$  for every compact  $K$ .*

In [28] the following converses were found. Throughout this paper, we will use (FDD) to indicate a finite-dimensional Schauder decomposition and (UFDD) to indicate an unconditional finite-dimensional Schauder decomposition. Recall also that  $X$  is said to have the *strong Schur property* if there is a constant  $c > 0$  so that for any normalized sequence  $(x_n)$  with  $\|x_m - x_n\| \geq \delta > 0$  for any  $m \neq n$ , there exists a subsequence  $(x_n)_{n \in \mathcal{M}}$  such that

$$\left\| \sum_{k \in \mathcal{M}} \alpha_k x_k \right\| \geq c\delta \sum_{k \in \mathcal{M}} |\alpha_k|$$

for any finitely supported sequence  $(\alpha_k)_{k \in \mathcal{M}}$ .

**Theorem 1.4.** *If  $X$  is separable and  $\text{Ext}(X, C[0, 1]) = \{0\}$ , then  $X$  has the strong Schur property. If  $X$  also has a (UFDD), then  $X$  is isomorphic to the dual of a subspace of  $c_0$ .*

Let us remark at this point that Bourgain and Pisier [9] (cf. [16, §1.8]) showed that for any separable Banach space  $X$  that is not an  $\mathcal{L}_\infty$ -space there is a space  $Y$  that is an  $\mathcal{L}_\infty$ -space so that  $Y$  contains  $X$  as an uncomplemented subspace and  $Y/X$  has the Schur property and the Radon-Nikodým property.

Recall that an operator is called strictly singular if its restriction to an infinite-dimensional subspace of its domain is never an isomorphic embedding. In Section 2 we consider the problem of characterizing those separable spaces  $X$  for which there is a short exact sequence  $0 \rightarrow C[0, 1] \rightarrow Z \rightarrow X \rightarrow 0$  so that the quotient map is strictly singular. We show in Theorem 2.3 that this is equivalent to the requirement that  $X$  contains no copy of  $\ell_1$ .

In Section 3 we consider quantitative results for the case  $K = \omega^N$ . In this case  $C(K)$  is isomorphic to  $c_0$ , so that  $\text{Ext}(X, C(K)) = \{0\}$  for every separable  $X$  by Sobczyk's theorem [43], but it is still worthwhile to consider projection constants. We need the following elementary result; we recall that  $Z$  is said to be separably injective if it is complemented in every separable superspace. As usual,  $I_X$  indicates the identity on a given Banach space  $X$ .

**Proposition 1.5.** *Let  $X$  be any separable Banach space, let  $Z$  be a separably injective Banach space and let  $k$  be a constant. Then the following are equivalent:*

(1) If  $Y$  is a separable Banach space and  $E$  is a closed subspace with  $Y/E$  isometric to  $X$ , then for any bounded linear operator  $T : E \rightarrow Z$  and any  $\varepsilon > 0$ , there is an extension  $\tilde{T} : Y \rightarrow Z$  with  $\|\tilde{T}\| < k\|T\| + \varepsilon$ .

(2) If  $0 \rightarrow Z \xrightarrow{j} Y \xrightarrow{q} X \rightarrow 0$  is an (isometric) exact sequence and any  $\varepsilon > 0$ , then there is a linear operator  $P : Y \rightarrow Z$  with  $Pj = I_Z$  and  $\|P\| \leq k + \varepsilon$ .

*Proof.* It is clear from the definition that if the short exact sequence is given, then we may find such a  $P$  with  $\|P\| \leq k + \varepsilon$ . Conversely, suppose  $Y$  is a separable Banach space and  $E$  is a closed subspace with  $Y/E$  isometric to  $X$ . If  $T : E \rightarrow Z$  is an operator with  $\|T\| \leq 1$ , we form the pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{j} & Y & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow s & & \parallel & & \\ 0 & \longrightarrow & Z & \xrightarrow{j'} & PO & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Then, if  $P : PO \rightarrow Z$  satisfies  $Pj' = I_Z$ , we see that  $PS = \tilde{T}$  extends  $T$  and  $\|PS\| \leq \|P\|$ .  $\square$

Our results build on earlier work of Amir and Baker, who showed that the separable projection constant of  $C(\omega^N)$  is  $2N + 1$ , [2], [3] and [4]. In particular, we show that, given any  $\varepsilon > 0$ , there is a space  $Z$  containing  $C(\omega^N)$  isometrically so that  $X/C(\omega^N)$  is isometric to  $c_0$  and the norm of any projection is at least  $2N + 1 - \varepsilon$ . However, our main motivation in Section 3 is to provide the necessary groundwork to study the case  $K = \omega^\omega$ , which is done in Section 4. Here we show results parallel to Theorems 1.3 and 1.4 above. We show that if  $X$  is the dual of a space with summable Szlenk index [31], [23, §2], then  $\text{Ext}(X, C(\omega^\omega)) = \{0\}$ , and this condition is necessary if  $X$  has a (UFDD). An example of such an  $X$  is Tsirelson's space [31].

We also consider the possibility of  $\text{Ext}(X, C(\omega^\omega))$  being large in the sense that there is a twisted sum  $0 \rightarrow C(\omega^\omega) \rightarrow Z \rightarrow X \rightarrow 0$  for which the quotient map is strictly singular. We show that a sufficient condition for the construction of such a short exact sequence is that  $X$  has a shrinking (UFDD) and contains no subspace that is the dual of a space with summable Szlenk index. This leads to new counterexamples for several old problems.

We refer to [16] and [29] for a discussion of twisted sums in general. Let us note that in Section 3 it is important to consider twisted sums in the *isometric* category rather than the isomorphic category; hence the standard pushout and pullback constructions were defined above isometrically. Of course any isomorphic twisted sum can be equivalently renormed to an isometric twisted sum.

## 2. A UNIVERSAL TWISTED SUM

**Theorem 2.1.** *Suppose  $X$  is a separable Banach space. Then there is a universal short exact sequence  $0 \rightarrow C[0, 1] \rightarrow Y \rightarrow X \rightarrow 0$  such that every short exact sequence  $0 \rightarrow C[0, 1] \rightarrow Z \rightarrow X \rightarrow 0$  can be identified with a pushout, i.e., there exist linear operators  $S : C[0, 1] \rightarrow C[0, 1]$  and  $S_1 : Y \rightarrow Z$  so that the following*

diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C[0, 1] & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow S & & \downarrow S_1 & & \parallel & & \\ 0 & \longrightarrow & C[0, 1] & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

*Proof.* Let  $Q_X : \ell_1 \rightarrow X$  be a quotient mapping and let  $\tilde{X}$  be the kernel of this map. Consider the collection  $\{L_j : j \in J\}$  of all linear operators  $L_j : \tilde{X} \rightarrow C[0, 1]$  with  $\|L_j\| \leq 1$ . Then let  $L : \tilde{X} \rightarrow \ell_\infty(J : C[0, 1])$  be defined by  $L\xi = (L_j\xi)_{j \in J}$ . Since  $L$  has separable range, we can find a subspace of  $\ell_\infty(J : C[0, 1])$  isomorphic to  $C[0, 1]$  and containing the range of  $L$ . In this way we induce a bounded linear operator  $A : \tilde{X} \rightarrow C[0, 1]$  such that every bounded operator  $B : \tilde{X} \rightarrow C[0, 1]$  factors through  $A$ , i.e.,  $B = VA$ , where  $V : C[0, 1] \rightarrow C[0, 1]$  is bounded.

Next we use the pushout construction to construct our twisted sum:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{X} & \longrightarrow & \ell_1 & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow A & & \downarrow A_1 & & \parallel & & \\ 0 & \longrightarrow & C[0, 1] & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0; \end{array}$$

it remains to verify its universality.

So let  $0 \rightarrow C[0, 1] \rightarrow Z \rightarrow X \rightarrow 0$  be any twisted sum of  $C[0, 1]$  and  $X$ . Then, using the projective property of  $\ell_1$ , we can construct a quotient mapping  $T_1 : \ell_1 \rightarrow Z$ . Since it is unique up to automorphism, we may choose  $\tilde{X} = T^{-1}(C[0, 1])$ . If  $T$  is the restriction of  $T_1$  to  $\tilde{X}$ , then the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{X} & \longrightarrow & \ell_1 & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow T_1 & & \parallel & & \\ 0 & \longrightarrow & C[0, 1] & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

This means simply that  $Z$  is obtained by the pushout of  $0 \rightarrow \tilde{X} \rightarrow \ell_1 \rightarrow X \rightarrow 0$  using  $T$ . Now we can write  $T = SA$  for some  $S : C[0, 1] \rightarrow C[0, 1]$ , and it follows that  $Z$  is obtained from  $Y$  by the pushout construction using  $S$ .  $\square$

We need the well-known result that there is a non-trivial twisted sum of  $C[0, 1]$  and  $c_0$ . The first published reference we know is [22, Theorem 6]. In [1] a stronger statement about the non-existence of Lipschitz liftings is proved; a non-separable version is to be found in [18]. The example, also studied in [27], can be described as follows. Let  $Q = (q_n)$  be any dense sequence in  $[0, 1]$ . We could for example order the rationals in  $(0, 1)$  into a sequence  $(q_n)$ , but we prefer not to be specific. Denote by  $D$  the set of all functions from  $[0, 1]$  into  $\mathbb{R}$  that are continuous at every  $t \notin Q$  and left continuous with right limits at every  $t \in Q$ . Routine arguments show that all such functions are bounded and that the sup-norm makes  $D$  into a Banach space. Clearly  $C = C[0, 1]$  is a closed subspace and  $D/C$  is isometric to  $c_0$ . More precisely, let us denote by  $J : D \rightarrow c_0$  the “jump function”  $Jf = \frac{1}{2}(f(q_n+) - f(q_n))$ . Then  $J$  maps  $D$  onto  $c_0$ , and  $d(f, C) = \|Jf\|$  for all  $f$  in  $D$ . We denote by  $e_n$  the usual basis in  $c_0$ . It is well known [6, p. 33], [27, p. 20] that  $D$  is isometric to the space of continuous functions on the Cantor set, but we do not need this representation.

**Lemma 2.2.** *Let  $(f_n)$  be any sequence of functions in  $D$  for which  $J(f_n) = e_n$  for all  $n$ . Then the sequence  $(f_n)$  is not weakly Cauchy.*

*Proof.* The assumption  $J(f_n) = e_n$  means that  $f_n(q_n+) - f_n(q_n) = 2$  for all  $n$ . Let us assume  $(f_n)$  is weakly Cauchy and hence bounded. We first note that if  $I$  is any nonempty open interval in  $(0, 1)$ ,  $\alpha \in \mathbb{R}$  and  $m \in \mathbb{N}$ , then there exist  $n > m$  and a nonempty open interval  $J$  with  $\bar{J} \subset I$  such that for some  $\beta$  with  $|\beta - \alpha| \geq 1$  we have  $|f_n(t) - \beta| \leq \frac{1}{4}$  for  $t \in J$ . Indeed, we just pick  $n > m$  so that  $q_m \in I$ , and then let  $\beta$  be either  $f_n(q_n)$  or  $f_n(q_n+)$ . The interval  $J$  can then be chosen using the left- or right-hand limit condition.

Now we can use this inductively to create a subsequence  $(f_{n_k})$  of  $(f_n)$ , a sequence of nonempty intervals  $(I_k)$  with  $\bar{I}_{k+1} \subset I_k$ , and a sequence of reals  $(\alpha_k)$  with  $|\alpha_{k+1} - \alpha_k| \geq 1$  so that  $|f_{n_k}(t) - \alpha_k| \leq \frac{1}{4}$  for  $t \in I_k$ . If we pick  $t_0 \in \bigcap_{k=1}^\infty I_k$  (which is nonempty by compactness), it is clear that  $|f_{n_k}(t_0) - f_{n_{k+1}}(t_0)| \geq \frac{1}{2}$  for all  $k$ , and this gives us a contradiction.  $\square$

**Theorem 2.3.** *Suppose  $X$  is a separable Banach space. Then there is a twisted sum*

$$0 \longrightarrow C[0, 1] \longrightarrow Y \xrightarrow{Q} X \longrightarrow 0$$

with  $Q$  strictly singular if and only if  $X$  contains no copy of  $\ell_1$ .

*Proof.* If  $\ell_1$  embeds into  $X$ , then, by the well-known lifting property of  $\ell_1$  [36, p. 107],  $Q$  cannot be strictly singular.

Conversely, suppose  $\ell_1$  does not embed into  $X$ . We will argue that the universal twisted sum  $Y$  given by Theorem 2.1 has a strictly singular quotient map  $Q : Y \rightarrow X$ . First we show that whenever  $E$  is an infinite-dimensional closed subspace of  $X$ , then there is a twisted sum  $0 \rightarrow C[0, 1] \rightarrow Z \rightarrow E \rightarrow 0$  so that the pullback by the inclusion  $E \rightarrow X$  does not split. Since  $X$  does not contain  $\ell_1$ , any such subspace  $E$  contains a weakly null basic sequence  $(x_n)_{n=1}^\infty$  [36, p. 5, Remark] spanning a subspace  $E_0$ . By considering the basis expansion we thus obtain a map  $T_0 : E_0 \rightarrow c_0$  so that  $T_0(x_n) = e_n$ , the  $n^{\text{th}}$ -basis vector in  $c_0$ . Since  $c_0$  is separably injective, we can extend  $T_0$  to a bounded operator  $T : X \rightarrow c_0$ .

We now use the twisted sum of  $C[0, 1]$  and  $c_0$  constructed above and form the pullback using  $T$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & C[0, 1] & \longrightarrow & D & \xrightarrow{J} & c_0 & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow T & & \\ 0 & \longrightarrow & C[0, 1] & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

We now need only show that the further pullback via the inclusion  $E \rightarrow X$  does not split. Thus we consider

$$\begin{array}{ccccccc} 0 & \longrightarrow & C[0, 1] & \longrightarrow & D & \xrightarrow{J} & c_0 & \longrightarrow & 0 \\ & & \parallel & & \uparrow v & & \uparrow T|_E & & \\ 0 & \longrightarrow & C[0, 1] & \longrightarrow & Z_0 & \longrightarrow & E & \longrightarrow & 0. \end{array}$$

Now if  $L : E \rightarrow Z_0$  is a lifting, then  $VLx_n$  is weakly null. However,  $JVLx_n = e_n$ , and so we contradict Lemma 2.2.

Finally, Theorem 2.1 implies that the sequence  $0 \rightarrow C[0, 1] \rightarrow Z_0 \rightarrow E \rightarrow 0$  can be obtained from the sequence  $0 \rightarrow C[0, 1] \rightarrow Y \rightarrow X \rightarrow 0$  by first taking the pushout via  $S : C[0, 1] \rightarrow C[0, 1]$  and then taking the pullback via  $E \rightarrow X$ . This procedure is equivalent to first taking the pullback via  $E \rightarrow X$ , and then taking the pushout via  $S : C[0, 1] \rightarrow C[0, 1]$ . Since the final sequence does not split, neither

does the intermediate sequence  $0 \rightarrow C[0, 1] \rightarrow Q^{-1}(E) \rightarrow E \rightarrow 0$ . Since  $E$  was arbitrary, we conclude that  $Q$  is strictly singular.  $\square$

A simplification of this argument shows that if  $X$  is separable but fails the Schur property, then  $\text{Ext}(X, C[0, 1]) \neq \{0\}$ . Of course Theorem 1.4 is stronger.

This essentially formal construction gives an interesting corollary:

**Corollary 2.4.** *There is a twisted sum  $Y$  of  $C[0, 1]$  and  $c_0$  that is necessarily an  $\mathcal{L}_\infty$ -space but is not isomorphic to a quotient of  $C(K)$  for any compact  $K$ .*

*Proof.* Taking  $X = c_0$  in Theorem 2.3 gives us an example with  $Q : Y \rightarrow c_0$  strictly singular. Since  $c_0$  is not reflexive,  $Q$  cannot be weakly compact. By a well-known result of Pełczyński [39, Theorem 1],  $Y$  cannot be isomorphic to a quotient of any  $C(K)$  space.  $\square$

Note here that  $Y^*$  is isomorphic to an  $L_1(\mu)$ -space, but  $Y$  cannot be renormed so that  $Y^*$  is isometric to an  $L_1(\mu)$  by a result of Johnson and Zippin [25]. This easily gives a counterexample to the old problems 3c and 3e of Lindenstrauss and Rosenthal [35], although other much more sophisticated counterexamples have been known for some time [5], [8]. For a stronger example, see the end of §4.

### 3. TWISTED SUMS WITH $C(\omega^N)$

If  $N \in \mathbb{N}$ , then the space  $C(\omega^N)$  is isomorphic to  $c_0$ , and so for any separable Banach space  $X$ , we have  $\text{Ext}(X, C(\omega^N)) = \{0\}$ . In this case it is natural to introduce the extension constant  $\pi_N(X)$ , which we define to be the least constant so that if  $Y$  is a separable Banach space and  $E$  is a closed subspace with  $Y/E$  isometric to  $X$ , then for any bounded linear operator  $T : E \rightarrow C(\omega^N)$  and  $\varepsilon > 0$ , there is an extension  $\tilde{T} : Y \rightarrow C(\omega^N)$  with  $\|\tilde{T}\| < \pi_N(X)\|T\| + \varepsilon$ . In view of Proposition 1.5,  $\pi_N(X)$  is also the least constant such that if

$$0 \rightarrow C(\omega^N) \xrightarrow{j} Y \xrightarrow{q} X \rightarrow 0$$

is an (isometric) exact sequence and  $\varepsilon > 0$ , then there is a linear operator  $P : Y \rightarrow C(\omega^N)$  with  $Pj = I_{C(\omega^N)}$  and  $\|P\| \leq \pi_N(X) + \varepsilon$ .

The following theorem is due to Amir [2], [3] and Baker [4]:

**Theorem 3.1.** *For any separable Banach space  $X$  we have  $\pi_N(X) \leq 2N + 1$ , and there is a separable Banach space  $X$  such that  $\pi_N(X) = 2N + 1$ .*

In fact, it follows from the arguments in [3] that we may take  $X = C(\omega^{N-1})$ . The main purpose of this section is to show that  $X$  may be chosen independently of  $N$ , more precisely that  $\pi_N(c_0) = 2N + 1$ . This will be needed in the next section, where it will also be useful to introduce an alternative constant  $\rho_N(X)$ , defined as the least constant such that if  $T : X \rightarrow \ell_\infty(\omega^N)$  is a bounded operator satisfying  $d(Tx, C(\omega^N)) \leq \|x\|$  for  $x \in X$ , and  $\varepsilon > 0$ , there is a linear operator  $L : X \rightarrow C(\omega^N)$  with  $\|T - L\| \leq \rho_N(X) + \varepsilon$ .

**Lemma 3.2.** *For any separable Banach space  $X$  we have  $\rho_N(X) \leq \pi_N(X) \leq \rho_N(X) + 1$ .*

*Proof.* First suppose  $Y$  is a Banach space containing  $C(\omega^N)$  and such that  $Y/C(\omega^N)$  is isometric to  $X$ . Then there is a bounded projection  $P_0 : Y \rightarrow C(\omega^N)$ . (We may suppose  $\|P_0\| \leq 2N + 1$ , but this is not necessary.) We can also find a linear operator

$S : Y \rightarrow \ell_\infty(\omega^N)$  with  $\|S\| = 1$  extending the identity on  $C(\omega^N)$ . Now  $P_0 - S = Tq$  for some  $T : X \rightarrow \ell_\infty(\omega^N)$ , where  $q : Y \rightarrow X$  is the quotient map. It is easy to check that  $T$  satisfies  $d(Tx, C(\omega^N)) \leq \|x\|$ . Hence, for  $\varepsilon > 0$ , we can find a linear operator  $L : X \rightarrow C(\omega^N)$  with  $\|T - L\| \leq \rho_N(X) + \varepsilon$ . Now  $P = P_0 - Lq$  is a projection onto  $C(\omega^N)$ . If  $y \in Y$ , then  $Py = P_0y - Tqy + (T - L)qy = Sy + (T - L)qy$ , so that  $\|P\| \leq 1 + \rho_N(X) + \varepsilon$ . Hence  $\pi_N(X) \leq 1 + \rho_N(X)$ .

Conversely, suppose  $T : X \rightarrow \ell_\infty(\omega^N)$  is a bounded operator with

$$d(Tx, C(\omega^N)) \leq \|x\|$$

for  $x \in X$ . Let  $Z$  be the space  $X \oplus C(\omega^N)$  normed by

$$\|(x, h)\| = \max(\|x\|, \|h - Tx\|).$$

Then the map  $(x, h) \rightarrow x$  defines a quotient mapping of  $Y$  onto  $X$  (since  $d(Tx, C(\omega^N)) \leq \|x\|$ ) with kernel  $E = \{0\} \oplus C(\omega^N)$ . Hence, if  $\varepsilon > 0$ , there is a projection  $P : Y \rightarrow E$  with  $\|P\| \leq \pi_N(X) + \varepsilon$ . Then  $P$  takes the form  $P(x, h) = (0, h - Lx)$ , where  $L : X \rightarrow C(\omega^N)$  is bounded. Now if  $x \in X$ , we have  $P(x, Tx) = (0, Tx - Lx)$ , so that  $\|Tx - Lx\| \leq \|P\|\|x\|$ . Hence  $\rho_N(X) \leq \pi_N(X)$ .  $\square$

**Lemma 3.3.** *Suppose  $K$  is a compact Hausdorff space and  $h \in \ell_\infty(K)$ . Then*

$$d(h, C(K)) = \frac{1}{2} \sup_{s \in K} (\limsup_{t \rightarrow s} h(t) - \liminf_{t \rightarrow s} h(t)).$$

*Proof.* Define  $f(s) = \liminf_{t \rightarrow s} h(t)$  and  $g(s) = \limsup_{t \rightarrow s} h(t)$  for  $s \in K$ . It is routine to check that  $f$  is upper semicontinuous and that  $g$  is lower semicontinuous. If  $R = \frac{1}{2} \sup_{s \in K} (g(s) - f(s))$ , then a classical interpolation theorem gives us a continuous function  $\tilde{h}$  satisfying  $g - R \leq \tilde{h} \leq f + R$ . Clearly  $f \leq h \leq g$ , and so  $-R \leq \tilde{h} - h \leq R$ , as required.  $\square$

We now need a representation of  $\omega^N$ . To this end we consider the power set of  $\mathbb{N}$ , i.e.,  $2^{\mathbb{N}}$ , which is homeomorphic to the Cantor set in the standard product topology. Let  $\mathcal{F}_N$  be the subset of all sets  $a$  with cardinality  $|a| \leq N$ . Then  $\mathcal{F}_N$  is homeomorphic to  $\omega^N$ . Indeed,  $\{\sum_{n \in a} 2^{-n} : a \in \mathcal{F}_N\}$  is order isomorphic and homeomorphic to  $\omega^N$ .

Any nonempty finite subset  $a$  of  $\mathbb{N}$  will be written in increasing order, i.e.,  $a = \{n_1, \dots, n_k\}$ , where  $n_1 < n_2 < \dots < n_k$ . We write  $\max a = n_k$ . We write  $a < b$  if either  $a$  is empty and  $b$  is not, or if  $a = \{n_1, \dots, n_k\}$  and  $b = \{m_1, \dots, m_l\}$ , where  $l > k$  and  $m_j = n_j$  for  $j \leq k$ . For each nonempty finite  $a = \{n_1, \dots, n_k\} \in 2^{\mathbb{N}}$  we define  $a^- = \{n_1, \dots, n_{k-1}\} = a \setminus \{n_k\}$ . We define  $a^+$  as the collection of all  $a \vee m = \{n_1, \dots, n_k, m\}$ , where  $m > n_k$ ;  $\emptyset^+$  is simply  $\mathbb{N}$ . Although we do not need it in this section, we define here a subset  $\mathcal{A}$  of  $\mathcal{F}_N$  to be *full* if the following three conditions hold:

- (1)  $\emptyset \in \mathcal{F}_N$ .
- (2) If  $\emptyset \neq a \in \mathcal{A}$ , then  $a^- \in \mathcal{A}$ .
- (3) If  $a \in \mathcal{A}$  and  $|a| < N$ , then  $\mathcal{A} \cap a^+$  is infinite.

It is then easy to see that any full subset of  $\mathcal{F}_N$  is also homeomorphic to  $\omega^N$ .

Next let  $\mathcal{A}$  be a full subset of  $\mathcal{F}_N$  and let  $X$  be a fixed separable Banach space. We consider a bounded map  $a \mapsto x_a^*$  of  $\mathcal{A}$  into  $X^*$ .

**Lemma 3.4.** *If  $T : X \rightarrow \ell_\infty(\mathcal{A})$  is defined by  $Tx(a) = x_a^*(x)$ , then we have*

$$d(Tx, C(\mathcal{A})) \leq \|x\| \quad \forall x \in X$$

*if and only if  $\limsup_{b,c \rightarrow a} \|x_b^* - x_c^*\| \leq 2$  for each  $a \in A$  with  $|a| < N$ .*

*Proof.* This follows easily from Lemma 3.3, since we require  $\limsup_{b \rightarrow a} x_b^*(a) - \liminf_{b \rightarrow a} x_b^*(x) \leq 2\|x\|$  for all  $x \in X$ . We omit the details. Note that if  $|a| = N$ , then any sequence converging to  $a$  will be eventually constant.  $\square$

We conclude this section with a minor variation of Amir’s part of the Amir-Baker Theorem:

**Theorem 3.5.** *For each  $N$  we have  $\pi_N(c_0) = 2N + 1$ .*

*Proof.* Let us choose  $\varepsilon > 0$  and  $r \in \mathbb{N}$ , and let  $m = 2^r$ . Then let  $G$  be the dyadic group  $\{-1, 1\}^r$ , with its usual normalized measure, and let  $u_1, \dots, u_m$  denote the characters of this group. Let  $\bar{u} = \frac{1}{m}(u_1 + \dots + u_m)$ , so that  $\bar{u}$  is actually the function that is one at the identity and zero elsewhere. Let  $v_k = u_k - \bar{u} \in L_\infty(G)$  and  $v_k^* = u_k$ , regarded as an element of  $L_1(G) = L_\infty(G)^*$ . Then  $\|v_k\| = \|v_k^*\| = 1$  for all  $k$ , and if  $j \neq k$ , then  $\|v_j^* - v_k^*\| = 1$ .

Now consider  $X = c_0(\mathcal{F}_{N-1}; L_\infty(G))$  so that  $X$  is isometric to  $c_0$ . We define a linear operator  $T : X \rightarrow \ell_\infty(\mathcal{F}_N)$ . Consider any element  $x = (w_a)_{a \in \mathcal{F}_{N-1}} \in X$ , where  $w_a \in L_\infty(G)$ . We define  $Tx(\emptyset) = 0$ , and then

$$Tx(a) = Tx(a-) + 2v_j^*(w_{a-}),$$

where  $j \equiv \max a \pmod{m}$ . Now let  $Z$  be the set of all  $(x, h) \in X \oplus_\infty \ell_\infty(\mathcal{F}_N)$  such that  $h - Tx \in C(\mathcal{F}_N)$ , and put  $E = \{(0, h) : h \in C(\mathcal{F}_N)\}$ ; it is easy to see that the quotient space  $Z/E$  is isometric to  $X$  (since  $d(Tx, C(\mathcal{F}_N)) \leq \|x\|$  by Lemma 3.4). Let  $P$  be a bounded projection of  $Z$  onto  $E$ , and write  $P(x, Tx) = (0, Sx)$ , where  $S : X \rightarrow C(\mathcal{F}_N)$ .

For notational purposes, if  $a \in \mathcal{F}_{N-1}$  and  $j \leq m$ , we define  $H(a, j)$  to be the set of  $b \geq a \vee n$ , where  $n > \max a$  and  $n \equiv j \pmod{m}$ , and  $x_{j,a} = v_j \chi_{\{a\}} \in X$ . For any  $a \in \mathcal{F}_N$  we put  $K(a) = \{b : b \geq a\}$ .

We now claim that if  $a \in \mathcal{F}_{N-1}$ , then there exists  $j = j(a)$  so that  $x = x_{j,a}$  satisfies  $Sx(a) \leq 0$ . Indeed,  $\sum_{j=1}^m x_{j,a} = 0$ , and so  $\sum_{j=1}^m Sx_{j,a}(a) = 0$ . Considering the topology on  $\mathcal{F}_N$ , it follows that there exists  $k = k(a) > \max a$  so that if  $b \geq a \vee l$ , where  $l \geq k(a)$ , then  $Sx(b) \leq \varepsilon$ .

Let us take  $n_1 = j(\emptyset) + mk(\emptyset)$  and then define inductively  $n_2, \dots, n_N$  so that  $n_s \geq k(\{n_1, \dots, n_{s-1}\})$  and  $n_s \equiv j(\{n_1, \dots, n_{s-1}\}) \pmod{m}$  for  $1 < s \leq N$ . Let  $a = \{n_1, \dots, n_N\}$ . Then we let

$$x = \sum_{\emptyset \leq b < a} x_{j(b),b}.$$

It is easy to see that

$$Sx(a) \leq N\varepsilon.$$

It is routine to check that if  $c \geq b \vee n$ , with  $n \equiv j \pmod{m}$ , then

$$T(v_{j(b)} \chi_{\{b\}})(c) = 2v_j^*(v_{j(b)}),$$

and  $T(v_{j(b)} \chi_{\{b\}})(c) = 0$  for all other  $c \in \mathcal{F}_N$ . Since  $v_j^*(v_k) = \delta_{jk} - \frac{1}{m}$ , where  $\delta_{jk}$  is the Kronecker delta, this implies that

$$T(x_{j(b),b}) = 2\chi_{H(b,j(b))} - \frac{2}{m}\chi_{K(b) \setminus \{b\}}.$$

Summing, we obtain

$$Tx = 2 \sum_{\emptyset \leq b < a} \left( \chi_{H(b,j(b))} - \frac{1}{m} \chi_{K(b) \setminus \{b\}} \right).$$

Let  $h = \chi_{K(\emptyset)} + 2 \sum_{\emptyset < b \leq a} \chi_{K(b)}$ . By construction  $H(b, j(b)) \subseteq K(b) \subseteq H(b-, j(b-))$  for each  $b \leq a$ . A short calculation then yields

$$\|Tx - h\| \leq 1 + \frac{2N}{m}.$$

Since  $\|v_{j(b)}\| = 1$ , we also have  $\|(x, Tx - h)\| \leq 1 + \frac{2N}{m}$ , and thus  $\|Sx - h\| \leq \|P\|(1 + \frac{2N}{m})$ . But  $h(a) = 2N + 1$ . Thus

$$2N + 1 - N\varepsilon \leq (h - Sx)(a) \leq \|P\|(1 + \frac{2N}{m}).$$

Since we can choose  $m$  arbitrarily large and  $\varepsilon$  arbitrarily small, this implies that  $\pi_N(c_0) \geq 2N + 1$ . □

#### 4. TWISTED SUMS WITH $C(\omega^\omega)$

Our motivation for studying the constants  $\pi_N(X)$  comes from the following theorem:

**Theorem 4.1.** *Suppose  $X$  is a separable Banach space. Then  $\text{Ext}(X, C(\omega^\omega)) = \{0\}$  if and only if  $\sup_N \pi_N(X) < \infty$ .*

*Proof.* To simplify notation we will work with  $C_0(\omega^\omega) = \{f \in C(\omega^\omega) : f(\omega^\omega) = 0\}$ , which is clearly isomorphic to  $C(\omega^\omega)$ . Since  $C(\omega^N)$  is isomorphic to a one-complemented subspace of  $C_0(\omega^\omega)$  for each  $N$ , necessity is obvious. Conversely, suppose  $Y$  is a separable Banach space and  $E$  is a closed subspace of  $Y$  so that  $Y/E$  is isometric to  $X$ . Suppose  $T : E \rightarrow C_0(\omega^\omega)$  is bounded with  $\|T\| \leq 1$ . Let  $M = \sup_N \pi_N(X) + 1$ . For  $n \in \mathbb{N}$  let  $R_n$  be the restriction map  $R_n : C_0(\omega^\omega) \rightarrow C(K_n)$ , where  $K_1 = [1, \omega]$  and  $K_n = [\omega^{n-1} + 1, \omega^n]$  for  $n \geq 2$ .

Let  $F_k$  be an increasing sequence of finite-dimensional subspaces of  $Y$  such that  $\bigcup F_k$  is dense in  $Y$ . Let  $G_k$  be finite-dimensional subspaces of  $E$  so that if  $x \in F_k$ , then  $d(x, G_k) \leq 2d(x, E)$ . Let  $q : Y \rightarrow Y/E$  be the quotient map and let  $q(F_k) = H_k$ .

For each  $k$  let  $n(k)$  be the least integer such that if  $e \in (F_k + G_k) \cap E$ , then  $\|R_n T e\| \leq 2^{-k} \|e\|$ . Then, since  $T$  maps  $E$  into  $C_0(\omega^\omega)$ , we see that  $n(k)$  is well defined.

For fixed  $k$ , letting  $n = n(k)$ , we can, since  $C(K_n)$  is an  $\mathcal{L}_{\infty,1}$ -space, find an operator  $S_n : F_k + G_k \rightarrow C(K_n)$  so that  $\|S_n\| \leq 2^{1-k}$  and  $S_n e = R_n T e$  for  $e \in E \cap (F_k + G_k)$ . Also we can find an operator  $V_n : Y \rightarrow C(K_n)$  such that  $\|V_n\| \leq M$  and  $V_n e = R_n T e$  for  $e \in E$ .

Now if  $y \in F_k + G_k$ , then there exists  $g \in G_k$  so that  $\|y - g\| \leq 2d(y, E)$ . Then

$$\|V_n y - S_n y\| = \|V_n(y - g) - S_n(y - g)\| \leq 2(M + 2)d(y, E).$$

It follows that there is an operator  $U_n : H_n \rightarrow C(K_n)$  with  $\|U_n\| \leq 2M + 4$  and  $U_n q = V_n - S_n$ . Since  $U_n(H_n)$  is finite dimensional, this may be extended to an operator  $\tilde{U}_n : X \rightarrow C(K_n)$  with  $\|\tilde{U}_n\| \leq 2M + 5$ . Next set  $\tilde{T}_n = V_n - \tilde{U}_n q$ . Then  $\|\tilde{T}_n\| \leq 3M + 6$ ,  $\tilde{T}_n$  extends  $R_n T$ , and  $\tilde{T}_n y = S_n y$  for  $y \in F_k + G_k$ , so that  $\|R_n T y\| \leq 2^{1-k} \|y\|$  for  $y \in F_k + G_k$ .

We finally extend the operator  $T$  by setting

$$\tilde{T}y(\alpha) = R_nTy(\alpha) \quad \text{if } \alpha \in K_n.$$

This provides an extension with  $\|\tilde{T}\| \leq 3M + 6$ . □

Next we recall some ideas from [23]. Suppose  $\mathcal{A}$  is a full subset of  $\mathcal{F}_N$ . We say that a map  $a \mapsto u_a^* : \mathcal{A} \rightarrow X^*$  is a weak\*-null tree map if  $u_\emptyset^* = 0$  and  $\lim_{b \in a^+} u_b^* = 0$  (weak\*) whenever  $|a| < N$ . If  $E$  is a closed subspace of  $X^*$ , we will define  $\alpha_N(E)$  to be the infimum of all  $\lambda$  such that whenever  $a \mapsto u_a^*$  is a weak\*-null tree map with  $u_a^* \in E$  and  $\|u_a^*\| \leq 1$  for all  $a$ , then there is a  $b \in \mathcal{A}$  with  $|b| = N$  and

$$\left\| \sum_{a \leq b} u_a^* \right\| \leq \lambda.$$

We shall say that a weak\*-null tree map is strongly weak\*-null if

$$\lim_{\max a \rightarrow \infty} u_a^* = 0$$

weak\*. The next lemma allows us to replace weak\*-null by strongly weak\*-null in the above definition of  $\alpha_N(E)$ .

**Lemma 4.2.** *If  $a \mapsto u_a^*$  is a bounded weak\*-null tree map on a full subset  $\mathcal{A}$  of  $\mathcal{F}_N$ , then there is a full subset  $\mathcal{B}$  of  $\mathcal{A}$  so that  $a \mapsto u_a^*$  is strongly weak\*-null on  $\mathcal{A}$ .*

*Proof.* Let  $(V_n)$  be a base of weak\*-neighborhoods of 0 such that  $V_{n+1} + V_{n+1} \subset V_n$  for all  $n$ . Let  $\mathcal{B} = \{b \in \mathcal{A} : u_a^* \in V_{\max a} \text{ for each } a \text{ with } \emptyset < a \leq b\}$ . It is easily verified that  $\mathcal{B}$  works. □

Now suppose  $X$  is a separable Banach space with a finite-dimensional Schauder decomposition  $(F_n)$ . We denote by  $S(m, n)$ , where  $0 \leq m \leq n \leq \infty$  and  $m < \infty$ , the operator

$$S(m, n) \left( \sum_{k=1}^{\infty} f_k \right) = \sum_{k=m+1}^n f_k$$

if  $f_k \in F_k$ . Note that  $S(n, n) = 0$  for all  $n$ . We say that  $(F_n)$  is *bi-monotone* if  $\|S(m, n)\| \leq 1$  for all  $m, n$ .

We shall let  $E(m, n)$  be the range of  $S(m, n)^*$  in  $X^*$ ; we refer to such subspaces as block subspaces. We let  $E$  be the closure of  $\bigcup_{m < n < \infty} E(m, n)$ .

**Theorem 4.3.** *Suppose  $X$  is a separable Banach space with a bi-monotone FDD  $(F_n)$ . Then:*

- (1)  $\rho_{2N}(X) \leq 4\alpha_N(E)$ .
- (2) *If  $(F_n)$  is 1-unconditional and shrinking (so that  $E = X^*$ ), then  $\alpha_N(X^*) \leq 2\rho_N(X)$ .*

*Proof.* (1) Suppose  $\lambda > 0$ . We define a notion of  $\lambda$ -acceptable subsets of  $B_E$  of cardinality at most  $N$ . A subset  $\{x_1^*, \dots, x_N^*\}$  of cardinality  $N$  is  $\lambda$ -acceptable if  $\|x_1^* + \dots + x_N^*\| \leq \lambda$ . We define acceptable sets of cardinality  $0 \leq k < N$  by reverse induction. For each  $0 \leq k < N$ , a subset  $\{x_1^*, \dots, x_k^*\}$  is  $\lambda$ -acceptable if there is a weak\*-neighborhood  $V$  of zero so that if  $x_{k+1}^* \in B_E \cap V$ , then  $\{x_1^*, \dots, x_{k+1}^*\}$  is  $\lambda$ -acceptable. It is easily seen that if  $\lambda > \alpha_N = \alpha_N(E)$ , then the empty set is  $\lambda$ -acceptable. More precisely it is easy to show that if this fails, then one can construct a weak\*-null tree map on  $\mathcal{F}_N$  denoted by  $a \mapsto u_a^*$  with  $u_a^* \in B_E$  so that

for every  $a$  with  $|a| = N$  we have  $\|\sum_{b \leq a} u_b^*\| > \lambda$ . This contradicts the definition of  $\alpha_N$ .

Next we shall say that a collection of  $k \leq N$  block subspaces  $\{G_1, \dots, G_k\}$  is  $\lambda$ -good if for some  $\mu < \lambda$  and every  $x_j^* \in B_{G_j}$  the set  $\{x_1^*, \dots, x_k^*\}$  is  $\mu$ -acceptable.

*Claim.* Suppose  $\lambda > \alpha_N$ . There is a function  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  so that if  $\{G_1, \dots, G_k\}$  is a  $\lambda$ -good family of block subspaces of  $E(0, n)$  with  $k < N$ , then for any block subspace  $G_{k+1}$  of  $E(\psi(n), \infty)$  the collection  $\{G_1, \dots, G_{k+1}\}$  is  $\lambda$ -good.

Let us prove the claim. Since the family of block subspaces of  $E(0, n)$  is finite, it is clear there exists  $\mu < \lambda$  so that every  $\lambda$ -good collection  $\{G_1, \dots, G_k\}$  of block subspaces is actually  $\mu$ -good. Then pick  $\varepsilon > 0$  so that  $\mu + N\varepsilon < \lambda$ . Choose in each block subspace  $G$  an  $\varepsilon$ -net for the unit ball  $B_G$ . In this way we produce a finite collection  $\mathcal{G}$  of  $\mu$ -acceptable sets  $\{x_1^*, \dots, x_k^*\}$  so that whenever  $\{G_1, \dots, G_k\}$  is any  $\lambda$ -good collection of block subspaces of  $E(0, n)$  and whenever  $g_j^* \in B_{G_j}$ , then there is a  $\{x_1^*, \dots, x_k^*\} \in \mathcal{G}$  with  $\|g_j^* - x_j^*\| \leq \varepsilon$  for  $1 \leq j \leq k$ . Now it is clear from the definition of acceptability that we can find  $\psi(n)$  so that if  $x^* \in B_E \cap E(\psi(n), \infty)$  and  $\{x_1^*, \dots, x_k^*\} \in \mathcal{G}$  with  $k < N$ , then  $\{x_1^*, \dots, x_k^*, x^*\}$  is  $\mu$ -acceptable. Now it is easy to see by a perturbation argument that if  $\{G_1, \dots, G_k\}$  is  $\lambda$ -good with  $k < N$  and each  $G_j$  is contained in  $E(0, n)$ , then for any block subspace  $G$  of  $E(n, \infty)$  the collection  $\{G_1, \dots, G_k, G\}$  is  $(\mu + N\varepsilon)$ -good and hence also  $\lambda$ -good. This proves the claim.

We now fix  $\lambda > \alpha_N$  and suppose  $\theta > 1$ . Now suppose  $Tx = (x_a^*(x))_{a \in \mathcal{F}_{2N}}$  is a linear operator  $T : X \rightarrow \ell_\infty(\mathcal{F}_{2N})$  with  $d(Tx, C(\mathcal{F}_{2N})) \leq \|x\|$  for all  $x \in X$ . We use Lemma 3.4. For each  $a \in A$  with  $a > \emptyset$  we define  $\nu = \nu(a)$  to be the greatest natural number so that if  $b \in \mathcal{F}_{2N}$  and  $b \geq a$ , then  $\|S(0, \nu)x_b^* - S(0, \nu)x_{a-}^*\| \leq 2\theta$ . It follows from Lemma 3.4 that  $\lim_{b \in a+} \nu(b) = \infty$  for all  $a$  with  $|a| < N$ .

Next we inductively construct a map  $\varphi : \mathcal{F}_{2N} \rightarrow \mathbb{N}$ . Let  $\varphi(\emptyset) = \psi(\emptyset)$ . Then we define  $\varphi(a)$  by induction on  $|a|$ . If  $\nu(a) < \psi(\varphi(a-))$ , we let  $\varphi(a) = \varphi(a-)$ . If  $\nu(a) \geq \psi(\varphi(a-))$ , we let  $\varphi(a) = \nu(a)$ .

Now we define  $z_a^*$  for  $a \in \mathcal{F}_{2N}$  by putting  $z_\emptyset^* = x_\emptyset^*$ , and then if  $|a| > 0$  we define

$$z_a^* = \sum_{\emptyset < b \leq a} S(\varphi(b-), \varphi(b))^* x_{b-}^* + S(\varphi(a), \infty)^* x_a^*.$$

We claim that  $a \mapsto z_a^*$  is weak\*-continuous. In fact, if  $b > a$ , let  $c$  be the unique element in  $a+$  with  $a < c \leq b$ . Then

$$z_b^* - z_a^* = \sum_{c < d \leq b} S(\varphi(d-), \varphi(d))^* x_{d-}^* - S(\varphi(c), \infty)^* x_a^*.$$

Now  $\lim_{c \in a+} \mu(c) = \infty$ , and so  $\lim_{c \in a+} \varphi(c) = \infty$  and  $\varphi(d) \geq \varphi(c)$  whenever  $c \leq d \leq b$ . Hence as  $b \rightarrow a$  we have  $z_b^* - z_a^* \rightarrow 0$  weak\*.

Suppose now  $a = \{n_1, \dots, n_k\} \in \mathcal{F}_{2N}$ . Let  $m_0 = \varphi(\emptyset)$ , and then put  $m_j = \varphi\{n_1, \dots, n_j\}$  for  $1 \leq j \leq k$ . Consider the subspaces

$$\{E(m_0, m_1), E(m_1, m_2), \dots, E(m_{k-1}, m_k)\}.$$

If we delete those subspaces where  $m_j = m_{j-1}$  (i.e., where the subspace reduces to  $\{0\}$ ), then it is clear by induction that the remaining subspaces can be split into two  $\lambda$ -good collections by taking them alternately. Hence, if  $u_j^* \in E(m_{j-1}, m_j)$  with  $\|u_j^*\| \leq 1$  for  $1 \leq j \leq k$ , then  $\|\sum_{j=1}^k u_j^*\| \leq 2\lambda$ .

Next we estimate  $\|x_a^* - z_a^*\|$ . We have

$$x_a^* - z_a^* = \sum_{\emptyset < b \leq a} S(\varphi(b-), \varphi(b))^*(x_a^* - x_{b-}^*).$$

If  $\varphi(b) > \varphi(b-)$ , then  $\varphi(b) = \mu(b)$ , and so  $\|S(\varphi(b-), \varphi(b))^*(x_a^* - x_{b-}^*)\| \leq 2\theta$ . By the above remarks we have

$$\|x_a^* - z_a^*\| \leq 4\lambda\theta.$$

Our conclusion is that there is a bounded operator  $Lx = (z_a^*(x))_{a \in \mathcal{F}_{2N}}$  into  $C(\mathcal{F}_{2N})$  with  $\|L - T\| \leq 2\lambda\theta$ . Thus  $\rho_{2N}(X) \leq 2\alpha_N(E)$ . This concludes the proof of (1).

(2) Let us suppose  $a \mapsto u_a^*$  is a strongly weak\*-null tree map on  $\mathcal{F}_N$  with  $\|u_a^*\| \leq 1$  for  $a \in \mathcal{F}_N$ . Let  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  be any surjective map so that for each  $k \in \mathbb{N}$  the set  $\gamma^{-1}\{k\}$  is infinite. Let  $\mathcal{A}$  be the subset of  $\mathcal{F}_N$  consisting of the empty set and all  $\{n_1, \dots, n_k\}$  such that  $\gamma(n_j) \geq n_{j-1}$  for  $2 \leq j \leq k$ . It is clear that  $\mathcal{A}$  is full. Let  $\sigma\{n_1, \dots, n_k\} = \{\gamma(n_1), \dots, \gamma(n_k)\}$  for  $\{n_1, \dots, n_k\} \in \mathcal{A}$ . We then define  $a \mapsto x_a^*$  for  $a \in \mathcal{A}$  by

$$x_a^* = \sum_{\emptyset < b \leq a} u_{\sigma(b)}^*.$$

Note that if  $d > a$  with  $d \in \mathcal{A}$ , then

$$x_d^* - x_a^* = u_{\sigma(c)}^* + \sum_{c < b \leq d} u_{\sigma(b)}^*,$$

where  $a < c = c(d) \leq d$  and  $|c| = |a| + 1$ . Then it follows from the strong weak\*-nullity of  $a \mapsto u_a^*$  that

$$\lim_{d \rightarrow a} \sum_{c < b \leq d} u_{\sigma(b)}^* = 0$$

weak\*, since  $\max(\sigma(b)) \geq \max c$ . Hence we have

$$\limsup_{d \rightarrow a} \|x_d^* - x_a^*\| \leq 1.$$

By Lemma 3.4 and the definition of  $\rho_N(X)$ , for any  $\lambda > \rho_N(X)$  we can find a weak\*-continuous map  $a \mapsto z_a^*$  on  $\mathcal{A}$  such that  $\|x_a^* - z_a^*\| \leq \lambda$  for all  $a$ .

Now fix  $\varepsilon > 0$ . We determine an increasing sequence  $n_1, \dots, n_N$  so that  $\{n_1, \dots, n_N\} \in \mathcal{A}$  and an increasing sequence  $m_1, \dots, m_{2N} \in \mathbb{N}$  by induction. Suppose  $a = \{n_1, \dots, n_{k-1}\}$  has been chosen in  $\mathcal{A}$  (where if  $k = 1$ , we take  $a = \emptyset$ ) and that  $m_1, \dots, m_{2k-2}$  have been chosen. Then pick  $m_{2k-1} > m_{2k-2}$  (if  $k \geq 2$ ) so that  $\|S(m_{2k-1}, \infty)^*(x_a^* - z_a^*)\| < \varepsilon/(6N)$ . This is possible since the (FDD) is shrinking. Now pick  $c \in \sigma(a)^+$  with  $\|S(0, m_{2k-1})^*u_c^*\| < \varepsilon/(6N)$ ; this is possible since  $\lim_{c \in \sigma(a)^+} u_c^* = 0$  weak\*. Pick  $m_{2k} > m_{2k-1}$  so that  $\|S(m_{2k}, \infty)^*u_c^*\| < \varepsilon/(6N)$ . Now there are infinitely many  $b \in a^+$  with  $\sigma(b) = c$ ; amongst these we may choose  $b$  so that  $\|S(0, m_{2k})^*(z_b^* - z_a^*)\| < \varepsilon/(6N)$ , since  $\lim_{b \rightarrow a} z_b^* = z_a^*$  weak\*. We then let  $b = \{n_1, \dots, n_k\}$ . This completes the inductive construction.

Let  $a_k = \{n_1, \dots, n_k\}$  for  $0 \leq k \leq N$ . Then

$$\begin{aligned} \left\| \sum_{k=1}^N u_{\sigma(a_k)}^* \right\| &\leq \frac{\varepsilon}{3} + \left\| \sum_{k=1}^N S(m_{2k-1}, m_{2k})^* u_{\sigma(a_k)}^* \right\| \\ &\leq \frac{\varepsilon}{3} + \left\| \sum_{k=1}^N \left( S(m_{2k-1}, m_{2k})^* u_{\sigma(a_k)}^* + S(m_{2k-2}, m_{2k-1})^* (z_{\sigma(a_k)}^* - z_{\sigma(a_{k-1})}^*) \right) \right\| \\ &\leq \varepsilon + \left\| \sum_{k=1}^N (u_{\sigma(a_k)}^* + z_{\sigma(a_k)}^* - z_{\sigma(a_{k-1})}^*) \right\| \\ &\leq \varepsilon + \|x_{a_N}^* - z_{a_N}^* + z_{\emptyset}^* - x_{\emptyset}^*\| \\ &\leq \varepsilon + 2\lambda. \end{aligned}$$

Hence by the definition of  $\alpha_N(X^*)$  we have  $\alpha_N(X^*) \leq 2\lambda + \varepsilon$ . The theorem follows.  $\square$

We are now in a position to prove our main result:

**Theorem 4.4.** (1) Suppose  $X$  is a separable Banach space with summable Szlenk index. Then  $\text{Ext}(X^*, C(\omega^\omega)) = \{0\}$ .

(2) If  $Y$  is a separable Banach space with  $\text{Ext}(Y, C(\omega^\omega)) = \{0\}$  and  $Y$  has a (UFDD), then  $Y$  is the dual of a space  $X$  with summable Szlenk index.

*Remark.* For the definition and general properties of the Szlenk index, see for example [23, §2]. The original space constructed by Tsirelson [44] is a reflexive space with summable Szlenk index [31]. Its dual is the space usually referred to nowadays as Tsirelson's space [14].

*Proof.* If  $X$  has a shrinking (FDD), then (1) follows directly from Theorem 4.3. We can assume via renorming that the (FDD) is bi-monotone. We consider the dual (FDD) of  $X^*$ . In this case the subspace  $E$  of  $X^{**}$  is identified with  $X$  and the condition  $\sup_n \alpha_n(E) < \infty$  is equivalent (using [23, Theorem 4.10]) to the fact that  $X$  has summable Szlenk index, and this implies that  $\sup_N \pi_N(X^*)$  is finite.

For the general case we use a theorem of Johnson and Rosenthal [24], [36, Theorem 1.g.2 p.48], that  $X$  has a subspace  $Y$  so that  $X/Y$  and  $Y$  both have shrinking (FDD)s. It is easy to check that having summable Szlenk index is a property that passes to quotients, and it follows from renorming results in [23] (Theorem 4.10 (ii)) that it passes also to subspaces. Thus  $Y$  and  $X/Y$  must both have summable Szlenk index. Hence we have  $\text{Ext}(Y^\perp, C(\omega^\omega)) = \{0\}$  and  $\text{Ext}(X^*/Y^\perp, C(\omega^\omega)) = \{0\}$ , and so by Corollary 1.2 we have  $\text{Ext}(X, C(\omega^\omega)) = \{0\}$ . This concludes the proof of (1).

For (2) we may assume the (UFDD) is 1-unconditional. We observe that Theorem 4.3 implies  $\text{Ext}(c_0, C(\omega^\omega)) \neq \{0\}$ . (Direct constructions are also available.) Hence if  $\text{Ext}(Y, C(\omega^\omega)) = \{0\}$  and  $Y$  is separable, then  $Y$  contains no (necessarily complemented) copy of  $c_0$ . In particular, the (UFDD) of  $Y$  must be boundedly complete, and so  $Y = X^*$ , where  $X = E$  as defined in Theorem 4.3. Then we have by Theorem 4.1,  $\sup_N \pi_N(Y) < \infty$ , and hence by Lemma 3.2,  $\sup_N \rho_N(Y) < \infty$ . Applying Theorem 4.3 (2), we obtain  $\sup_n \alpha_n(X) < \infty$ . It follows again from Theorem 4.10 of [23] that  $X$  has summable Szlenk index.  $\square$

If  $X$  is any separable Banach space, we define a tree map  $a \mapsto v_a^* : \mathcal{F}_N \rightarrow X^*$  to be of *dense type* if the following conditions are satisfied:

- (1)  $v_\emptyset^* = 0$ .
- (2)  $\|v_a^*\| \leq 1$  for all  $a \in \mathcal{F}_N$ .
- (3) For each  $a$  with  $|a| < N$  there is a weak\*-neighborhood  $V$  of 0 so that the weak\*-closure of  $\{v_b^* : b \in a+\}$  contains  $V$ .
- (4) If  $b_n \rightarrow a$  and  $|b_n| \geq |a| + 2$  for all  $n$ , then  $v_{b_n}^* \rightarrow 0$  weak\*.

Next let  $y_a^* = \sum_{b \leq a} v_b^*$ . We can define  $Tx = (y_a^*(x))_{a \in \mathcal{F}_N}$ , so that  $T : X \rightarrow \ell_\infty(\mathcal{F}_N)$ .

**Lemma 4.5.** *Suppose  $X$  has a (UFDD). Suppose  $L : X \rightarrow C(\omega^\omega)$ , and  $T : X \rightarrow \ell_\infty(\mathcal{F}_N)$  is an operator induced by a tree map of dense type. Then  $\rho_N(X) \leq 2\|L - T\|$ .*

*Proof.* This essentially follows from the argument in Theorem 4.3. Let  $a \mapsto u_a^*$  be any strongly weak\*-null tree map with  $\|u_a^*\| \leq 1$  for all  $a$ . Let  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  be any surjective map so that for each  $k \in \mathbb{N}$  the set  $\gamma^{-1}\{k\}$  is infinite. Let  $\mathcal{A}$  be the subset of  $\mathcal{F}_N$  consisting of the empty set and all  $\{n_1, \dots, n_k\}$  such that  $\gamma(n_j) \geq n_{j-1}$  for  $2 \leq j \leq k$ . It is clear that  $\mathcal{A}$  is full. Let  $\sigma\{n_1, \dots, n_k\} = \{\gamma(n_1), \dots, \gamma(n_k)\}$  for  $\{n_1, \dots, n_k\} \in \mathcal{A}$ .

We now build a map  $\psi : \mathcal{A} \rightarrow \mathcal{F}_N$ . Define  $\psi(\emptyset) = \emptyset$ . If  $\psi(a)$  has been defined and  $|a| < N$ , we define  $\psi(b)$  for each  $b \in a+$  so that  $\psi(b) \in \psi(a)+$ ,  $\psi$  is one-one and  $\lim_{b \in a+} u_{\sigma(b)}^* - v_{\psi(b)}^* = 0$  weak\*.

Let  $x_a^* = \sum_{b \leq a} u_{\sigma(b)}^*$ . Then we claim that  $x_a^* - y_{\psi(a)}^*$  is weak\*-continuous. Indeed, if  $b \geq a$ ,

$$x_b^* - x_a^* - y_{\psi(b)}^* + y_{\psi(a)}^* = \sum_{a < c \leq b} u_{\sigma(c)}^* - v_{\psi(c)}^*.$$

Now if  $b_n \rightarrow a$  and we let  $d_n$  be chosen so that  $b_n \leq d_n \leq a$  and  $|d_n| = |a| + 1$ , we have

$$\sum_{d_n < c < b} (u_{\sigma(c)}^* - v_{\psi(c)}^*) \rightarrow 0 \quad \text{weak}^*$$

by the assumptions on both tree maps. On the other hand,

$$u_{\sigma(d_n)}^* - v_{\psi(d_n)}^* \rightarrow 0 \quad \text{weak}^*$$

by construction.

Now if  $Lx = (z_a^*(x))_{a \in \mathcal{F}_N}$ , then  $\|z_a^* - y_a^*\| \leq \|L - T\|$ . Now  $a \mapsto z_{\psi(a)}^* + x_a^* - y_{\psi(a)}^*$  is weak\*-continuous, and we can repeat the argument of Theorem 4.3 to deduce the conclusion.  $\square$

It is clear that we can always construct a tree map of dense type. Simply let  $(V_n)$  be a base of weak\*-neighborhoods of  $\{0\}$  in  $X^*$  with  $V_{n+1} + V_{n+1} \subset V_n$ . Then for  $a$  with  $|a| < N$ , simply choose  $\{u_{a \vee m}^*\}$  for  $m > \max a$  to be any sequence that is weak\*-dense in  $V_{\max a} \cap B_{X^*}$ . It is also clear that if  $Y$  is a subspace of  $X$  and  $j : Y \rightarrow X$  is the inclusion, then  $a \mapsto j^*u_a^*$  is a tree map of dense type in  $Y^*$ . This leads us to the following:

**Proposition 4.6.** *Let  $X$  be a separable Banach space with a shrinking 1-unconditional (UFDD). Then there is a bounded operator  $T : X \rightarrow \ell_\infty(\omega^N)$  so that*

$$d(Tx, C(\omega^N)) \leq \|x\|$$

for all  $x \in X$  and so that if  $E$  is a subspace of  $X$  with a (UFDD), then  $\rho_N(E) \leq 2\|L - T\|$  for any bounded operator  $L : E \rightarrow C(\omega^N)$ .

It is obvious from Theorem 4.4 that the existence of a twisted sum  $0 \rightarrow C(\omega^\omega) \rightarrow Y \rightarrow X \rightarrow 0$  with the quotient map strictly singular implies that  $X$  contains no subspace that is isomorphic to the dual of a space with summable Szlenk index. We now establish a partial converse.

**Theorem 4.7.** *Suppose  $X$  has a shrinking (UFDD) and contains no subspace that is isomorphic to the dual of a space with summable Szlenk index. Then there is a short exact sequence*

$$0 \rightarrow C(\omega^\omega) \rightarrow V \xrightarrow{q} X \rightarrow 0$$

with  $q$  strictly singular.

*Proof.* We may assume  $X$  has a 1-unconditional (UFDD). For each  $N$  we construct  $T_N : X \rightarrow \ell_\infty(\omega^N)$  as given in Proposition 4.6. Let  $Z_N$  be the space  $X \oplus C(\omega^N)$  normed by  $\|(x, h)\| = \|x\| + \|h - Tx\|$ ; then there is a quotient map  $q_N : Z_N \rightarrow X$  defined by  $q_N(x, h) = x$ . We now construct an operator  $S_N : \tilde{X} \rightarrow C(\omega^N)$  in the usual way. Precisely, we fix a quotient map  $Q : \ell_1 \rightarrow X$  and define  $\hat{S}_N : \ell_1 \rightarrow Z_N$  so that  $\|\hat{S}_N\| \leq 2$  and  $q_N \hat{S}_N = Q$ . Now let  $S_N$  be the restriction of  $\hat{S}_N$  to  $\tilde{X}$ .

Let  $(F_n)$  be an increasing sequence of finite-dimensional subspaces so that  $\bigcup F_n$  is dense in  $\tilde{X}$ . Then, since  $C(\omega^N)$  is an  $\mathcal{L}_\infty$ -space, we can find a finite-rank projection  $P_N$  on  $C(\omega^N)$  whose range includes  $S_N(F_N)$  and with  $\|P_N\| \leq 2$ . Now let  $R_N = S_N - P_N S_N$ . Thus  $\|R_N\| \leq 6$ , and  $\lim_{N \rightarrow \infty} \|R_N \xi\| = 0$  for  $\xi \in \tilde{X}$ .

We now define a map  $R : \tilde{X} \rightarrow W = c_0(C(\omega^N)_{N=1}^\infty)$  by  $R\xi = (R_N \xi)_{N=1}^\infty$ . Note that the latter space is isomorphic to  $C(\omega^\omega)$ . We can now construct a pushout

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{X} & \longrightarrow & \ell_1 & \xrightarrow{Q} & X \longrightarrow 0 \\ & & \downarrow R & & \downarrow Q_V & & \parallel \\ 0 & \longrightarrow & W & \longrightarrow & V & \xrightarrow{q_X} & X \longrightarrow 0. \end{array}$$

We claim that  $q_X$  is strictly singular. If not, we can find a subspace  $E$  of  $X$  with a 1-unconditional shrinking (UFDD) so that there is a bounded operator  $\Lambda : E \rightarrow V$  so that  $q_X \Lambda = I_E$ . Then on  $Q^{-1}E$  we have  $q_X(Q_V - \Lambda Q) = 0$ , so that  $Q_V - \Lambda Q : Q^{-1}(E) \rightarrow W$  is an extension of  $R$  to  $Q^{-1}(E)$ . It follows that there exists a uniformly bounded sequence of operators  $\tilde{R}_N : Q^{-1}(E) \rightarrow C(\omega^N)$  which extend  $R_N$ . Put  $M = \sup \|\tilde{R}_N\| < \infty$ .

Note that  $P_N S_N$  has an extension to  $Q^{-1}(E)$  with  $\|P_N S_N\| \leq 5$ , since it is a finite-rank operator taking values in  $C(\omega^N)$ . Hence  $S_N$  has an extension  $\tilde{S}_N : Q^{-1}(E) \rightarrow C(\omega^N)$  with  $\|\tilde{S}_N\| \leq M + 5$ . Then  $\hat{S}_N - \tilde{S}_N$  factors through an operator  $e \mapsto (e, L_N e)$  from  $E$  into  $Z_N$  with norm at most  $M + 7$ . This implies that  $\|L_N - T\| \leq M + 7$ , and so  $\rho_N(E) \leq 2M + 14$ . Theorem 4.3 and [23, Theorem 4.10] then show that  $E$  must have summable Szlenk index.  $\square$

It now follows that there is a twisted sum of  $C(\omega^\omega)$  and  $c_0$  so that the quotient map is strictly singular. This space is not a quotient of a  $C(K)$ -space, and yet its dual must be isomorphic to  $\ell_1$ . This shows that the main result of [25] does not admit an isomorphic version. The space  $Y$  constructed in [8] also serves as a counterexample.

5. FINAL REMARKS

In [21] (cf. [29]) it is shown that  $\text{Ext}(\ell_2, \ell_2) \neq \{0\}$ . It follows without difficulty that  $\text{Ext}(\ell_p, \ell_q) \neq \{0\}$  when  $1 < p, q < \infty$ , since each space contains uniformly complemented copies of  $\ell_2^3$ . The following result is implicitly proved in [10], but it is heavily disguised; so we give a simple and direct diagram-chasing argument. For a nonlinear argument, see [12].

**Theorem 5.1.**  $\text{Ext}(c_0, \ell_1) \neq \{0\}$ .

*Proof.* In fact we will argue that  $\text{Ext}(C[0, 1], L_1) \neq \{0\}$ . It then follows from local arguments that  $\text{Ext}(X, Y) \neq \{0\}$  whenever  $X$  is an  $\mathcal{L}_\infty$ -space and  $Y = L_1(\mu)$  for some measure  $\mu$  (see, e.g., [12, Theorem 2]). Alternatively, one may carry out the ensuing argument locally.

We begin by considering some non-trivial twisted sum of  $\ell_2$  and  $\ell_2$ . By using the pushout and pullback constructions we build the following diagram:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \ell_2 & \xrightarrow{j_1} & Z & \xrightarrow{q_1} & \ell_2 & \longrightarrow & 0 \\
 & & \downarrow j_4 & & \downarrow j_5 & & \parallel & & \\
 0 & \longrightarrow & L_1 & \xrightarrow{j_2} & V & \xrightarrow{q_2} & \ell_2 & \longrightarrow & 0 \\
 & & \parallel & & \uparrow q_5 & & \uparrow q_4 & & \\
 0 & \longrightarrow & L_1 & \xrightarrow{j_3} & W & \xrightarrow{q_3} & C[0, 1] & \longrightarrow & 0.
 \end{array}$$

Here linear embeddings are denoted by  $j$  and quotient maps by  $q$ . First we recall that  $Z$  is of cotype  $p$  and type  $q$  whenever  $q < 2 < p$  [21, §3]. From the construction of the pushout,  $V$  is of cotype  $p$  for every  $p > 2$ .

We claim that the third row of this diagram cannot split. Suppose it does split. Then we can find an operator  $T : C[0, 1] \rightarrow W$  so that  $q_3T = I_{C[0,1]}$ . Then  $q_5T : C[0, 1] \rightarrow V$  must factor through some  $L_r$ -space, where  $r > 2$  since  $V$  has finite cotype. (This result can be traced to Maurey [37]; cf. also [42] or [20, Theorem 11.14(b)].) Since  $L_r$  has type 2 and  $L_1$  has cotype 2, every map from a subspace of  $L_r$  to  $L_1$  factors through a Hilbert space (this is Maurey’s generalization of Kwapien’s theorem [32] and [33]) and hence extends to a bounded operator from  $L_r$  into  $L_1$  by Maurey’s Extension theorem [38] (cf. [20, Theorem 12.22]). Applying all this to  $(q_5T)^{-1}(j_2L_1)$ , we can find an operator  $R : C[0, 1] \rightarrow j_2(L_1)$  so that  $Rf = q_5Tf$  if  $q_2q_5Tf = 0$ . But  $q_2q_5 = q_4q_3$ . Then  $q_5T - R = T_1q_4$  for some bounded operator  $T_1 : \ell_2 \rightarrow V$ . Thus the second row splits.

The conclusion of the argument was given in the proof of [30, Theorem 4.1]. If the second row splits, then  $V$  has cotype 2. Hence  $Z$  also has cotype 2, and also has type  $p > 1$ . But then  $Z^*$  is type 2 [41], and the Maurey Extension theorem guarantees that the dual exact sequence  $0 \rightarrow \ell_2 \rightarrow Z^* \rightarrow \ell_2 \rightarrow 0$  splits. By reflexivity the first row splits, contrary to our choice of  $Z$ .  $\square$

Finally, let us mention a non-separable problem related to the results of this paper. If  $X$  is a separable Banach space, then  $\text{Ext}(X, c_0) = \{0\}$  by Sobczyk’s theorem: we do not know, however, if there is a non-metrizable compact Hausdorff space  $K$  such that  $\text{Ext}(C(K), c_0) = \{0\}$ . It is known that if  $\Gamma$  is uncountable, then  $\text{Ext}(c_0(\Gamma), c_0) \neq \{0\}$ ; this is essentially contained in one proof of the fact that  $c_0$  is uncomplemented in  $\ell_\infty$ ; see also [1], [19, p. 260] and [13, §3]. It was noted in [17, Theorem 3.4] that if  $X$  is any non-separable WCG-space, then  $\text{Ext}(X, c_0) \neq \{0\}$ , and this settles the case when  $K$  is an Eberlein compact; similar arguments can

be used for Corson compact spaces. At the other extreme, if  $K$  is extremally disconnected, then  $C(K)$  contains a complemented  $\ell_\infty$  and  $\text{Ext}(\ell_\infty, c_0) \neq \{0\}$  was shown in [12]. Finally, the case of uncountable ordinal spaces can be reduced to  $K = [0, \omega_1]$ , and in this case Parovičenko's theorem [7] shows that  $\text{Ext}(C(K), c_0) \neq \{0\}$ .

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