TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 1, Pages 219–243
S 0002-9947(03)03116-7
Article electronically published on August 25, 2003

VASSILIEV INVARIANTS FOR BRAIDS ON SURFACES

JUAN GONZÁLEZ-MENESES AND LUIS PARIS

Abstract. We show that Vassiliev invariants separate braids on a closed oriented surface, and we exhibit a universal Vassiliev invariant for these braids in terms of chord diagrams labeled by elements of the fundamental group of the surface.

1. Definitions and statements

1.1. **Introduction.** Vassiliev knot invariants were introduced by V. A. Vassiliev ([17], [18]; see also [6], [2]), and they have been generalized to several other knot-like objects, such as links, braids, tangles, string links, knotted graphs, etc. The purpose of this paper is to consider Vassiliev invariants of braids on surfaces, and to extend some well-known results on Vassiliev invariants of Artin braids to the case of braids on surfaces.

Our study of Vassiliev invariants is inspired by Papadima's work [13] on Vassiliev invariants for Artin braids with values in $\mathbb{Z}$. However, the presence of the fundamental group of the surface changes the analysis substantially. Anyway, the Vassiliev theory for braids on surfaces, set forth in this paper, appears to be a natural generalization of the corresponding theory for Artin braids.

1.2. **Braids and singular braids on surfaces.** Throughout this paper M will denote a closed, orientable surface of genus $g \geq 1$, and $\mathcal{P} = \{P_1, \dots, P_n\}$ a set of n distinct points in M. Define a *n-braid based at $\mathcal{P}$* to be a collection $b = (b_1, \dots, b_n)$ of disjoint smooth paths in $M \times [0,1]$, called *strings* of b, such that the i-th string b_i runs monotonically in $t \in [0,1]$ from the point $(P_i, 0)$ to some point $(P_j, 1)$, $P_j \in \mathcal{P}$.

An *isotopy* in this context is a deformation through braids (which fixes the ends). Multiplication of braids is defined by concatenation, generalizing the construction of the fundamental group. The isotopy classes of braids with this multiplication form the group $B_n(M, \mathcal{P})$, called the *braid group with n strings on M based at $\mathcal{P}$*. Note that the group $B_n(M, \mathcal{P})$ does not depend, up to isomorphism, on the set $\mathcal{P}$ of points, but only on the cardinality $n = |\mathcal{P}|$. So we may write $B_n(M)$ in place of $B_n(M, \mathcal{P})$.

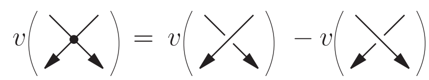
In the same way as Artin braid groups have been extended to singular braid monoids ([6],[1]), one can extend the braid group $B_n(M)$ to $SB_n(M)$, the *monoid of singular braids with n strings on M*. The strings of a singular braid are now

Received by the editors November 7, 2000 and, in revised form, May 20, 2002.
2000 *Mathematics Subject Classification.* Primary 20F36; Secondary 57M27, 57N05.
Key words and phrases. Braid, surface, Vassiliev invariant, finite type invariant.
The first author was supported in part by DGESIC-PB97-0723, by BFM2001-3207 and by the European network TMR Sing. Eq. Diff. et Feuill.

allowed to intersect transversely, but only in finitely many double points, called *singular points*.

As with braids, isotopy is a deformation through singular braids (which fixes the ends), and multiplication is by concatenation. Note that the isotopy classes of singular braids form a monoid and not a group: the singular braids with one or more singular points being non-invertible.

1.3. **Vassiliev invariants and Vassiliev filtration.** An *invariant* of braids on M with values in an abelian group A, is a set-mapping $v: B_n(M) \to A$. Just as for knots and Artin braids, one can extend v to singular braids by using the recursive rule



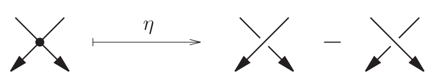
The picture on the left-hand side represents a small neighborhood of a singular point in a singular braid. Those on the right-hand side represent the braids which are obtained from the previous one by *resolution* of that singular point. That is, we modify the first braid inside the neighborhood of the singular point, in a *positive* and a *negative* way, to obtain two singular braids having one less singular point.

Let d be an integer. A *Vassiliev invariant of type* d is an invariant v such that $v(b) = 0$ for every singular braid b with more than d singular points.

There is an equivalent definition of a Vassiliev invariant, in terms of the so-called *Vassiliev filtration*. First, consider the group ring $\mathbb{Z}[B_n(M)]$. We can define a map

$$\eta: SB_n(M) \longrightarrow \mathbb{Z}[B_n(M)]$$

which "resolves" all the singular points of a given braid, with the corresponding signs. That is,



This map is a well-defined multiplicative morphism. Remark that a singular braid with d singular points is mapped to an alternate sum of 2^d non-singular braids, each one having coefficient $+1$ or -1 depending on the sign of its corresponding resolutions.

Let $S_dB_n(M)$ denote the set of isotopy classes of singular braids with d singular points. We denote by V_d the $\mathbb{Z}$-module generated by $\eta(S_dB_n(M))$. One can easily verify that V_d is a (two-sided) ideal of $\mathbb{Z}[B_n(M)]$ and that we have the inclusions $V_{d+1} \subset V_d$ and $V_{d_1}V_{d_2} = V_{d_1+d_2}$, for all $d_1, d_2, d \in \mathbb{N}$. We have then obtained a filtration

$$\mathbb{Z}[B_n(M)] = V_0 \supset V_1 \supset V_2 \supset \cdots,$$

which is called the *Vassiliev filtration* of $\mathbb{Z}[B_n(M)]$.

The definition of a Vassiliev invariant in terms of the Vassiliev filtration is as follows. One can extend any invariant $v: B_n(M) \to A$ by linearity to a morphism of $\mathbb{Z}$-modules $v: \mathbb{Z}[B_n(M)] \to A$. Note that the previous extension of v to singular braids can also be expressed by $v(b) = v(\eta(b))$, for $b \in SB_n(M)$. Then, v is a Vassiliev invariant of type d if and only if it vanishes on V_{d+1}. Therefore, the set of Vassiliev invariants of type d with values in A is equal to

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[B_n(M)]/V_{d+1}\,,\, A).$$

1.4. **Statements.** We have two goals in this paper. The first one is to show that Vassiliev invariants separate braids on surfaces, that is, to prove the following.

Theorem 1.1. *Given two non-equivalent braids b and c on M, there exist an integer $N \geq 1$ and a Vassiliev invariant v_N of type N such that $v_N(b) \neq v_N(c)$. Moreover, v_N can be chosen to take values in $\mathbb{Z}$.*

This result is known to hold for Artin braids ([3], [10], [13]), but it is still a conjecture for knots. Actually, Theorem 1.1 is a corollary of the following theorem.

Theorem 1.2. *Let $\{V_d\}_{d=1}^{\infty}$ be the Vassiliev filtration of $\mathbb{Z}[B_n(M)]$. Then*

(1) $\bigcap_{d=0}^{\infty} V_d = \{0\}$*, and*
(2) V_d/V_{d+1} *is a free $\mathbb{Z}$-module for all $d \geq 0$.*

Indeed, if Theorem 1.2 holds, then, given two non-equivalent braids $b, c \in B_n(M)$, there exists an integer N such that $b - c \notin V_{N+1}$. Then we can take v_N to be the canonical projection from $\mathbb{Z}[B_n(M)]$ to $\mathbb{Z}[B_n(M)]/V_{N+1}$. In addition, if V_d/V_{d+1} is a free $\mathbb{Z}$-module for all d, then

$$\mathbb{Z}[B_n(M)]/V_{N+1} \simeq (\mathbb{Z}[B_n(M)]/V_1) \oplus (V_1/V_2) \oplus \cdots \oplus (V_N/V_{N+1})$$

is also a free $\mathbb{Z}$-module, so we can obviously compose the above projection with a map from $\mathbb{Z}[B_n(M)]/V_{N+1}$ to $\mathbb{Z}$, in such a way that the image of $b - c$ is non-zero. Therefore, our first goal will be achieved by proving Theorem 1.2.

Our second goal is to define a universal Vassiliev invariant for $B_n(M)$ which generalizes the notion of chord diagrams for Artin braids. Recall that a *chord diagram* is a diagram made of n vertical lines and a finite number of horizontal segments, called chords, connecting the lines. An *M-labeled chord diagram* is a chord diagram such that each chord is labeled by an element of $\pi_1(M)$ (see Figure 1). Note that the set of M-labeled chord diagrams is equipped with a multiplication defined by concatenation. The free $\mathbb{Z}$-module generated by the chord diagrams is a $\mathbb{Z}$-algebra which can be identified with $\mathbb{Z}[t_{i,j,\gamma}]$, the free non-commutative $\mathbb{Z}$-algebra freely generated by the $t_{i,j,\gamma}$, where $i, j \in \{1, \ldots, n\}$, $i \neq j$, $\gamma \in \pi_1(M)$, and where $t_{i,j,\gamma} = t_{j,i,\gamma^{-1}}$ (see Figure 1).

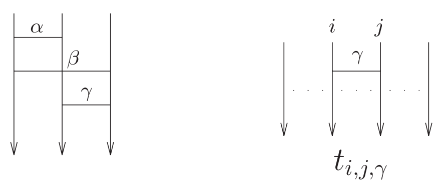


FIGURE 1. An M-labeled chord diagram and the generator $t_{i,j,\gamma}$.

We denote by $\mathcal{A}_n$ the quotient $\mathbb{Z}$-algebra obtained from $\mathbb{Z}[t_{i,j,\gamma}]$ by imposing the relations

- $[t_{i,j,\gamma}\ ,\ t_{k,l,\delta}] = 0,$ for all distinct $i, j, k, l \in \{1, \ldots, n\}$ and all $\gamma, \delta \in \pi_1(M)$,
- $[t_{i,j,\gamma}\ ,\ t_{j,k,\delta} + t_{i,k,(\gamma\delta)}] = 0,$ for all distinct $i, j, k \in \{1, \ldots, n\}$ and all $\gamma, \delta \in \pi_1(M)$,

and we denote by $\widehat{\mathcal{A}}_n$ its natural completion.

Note that the symmetric group Σ_n acts on $\pi_1(M)^n$ by permuting coordinates, so we can consider the induced semi-direct product $H_n = \pi_1(M)^n \rtimes \Sigma_n$. In addition, it is straightforward to show that H_n acts on $\widehat{\mathcal{A}}_n$, defining the semi-direct product $\widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n]$. The action is defined by the following relations:

- $\sigma\ t_{i,j,\gamma}\ \sigma^{-1} = t_{\sigma(i),\sigma(j),\gamma}$, for all $\sigma \in \Sigma_n$,
- $\mu(k)\ t_{i,j,\gamma}\ \mu(k)^{-1} = t_{i,j,\gamma}$, for all $\mu \in \pi_1(M)$ and all $k \neq i, j$,
- $\mu(i)\ t_{i,j,\gamma}\ \mu(i)^{-1} = t_{i,j,(\mu\gamma)}$, for all $\mu \in \pi_1(M)$,

where $\mu(i) = (1, \dots, 1, \overset{(i)}{\mu}, 1, \dots, 1) \in \pi_1(M)^n$. Note that one also has the following relation:

$$\mu(j)\ t_{i,j,\gamma}\ \mu(j)^{-1} = \mu(j)\ t_{j,i,\gamma^{-1}}\ \mu(j)^{-1} = t_{j,i,(\mu\gamma^{-1})} = t_{i,j,(\gamma\mu^{-1})}.$$

The $\mathbb{Z}$-algebra $\widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n]$ carries the filtration induced by that of $\widehat{\mathcal{A}}_n$, so its associated graded algebra is $\mathcal{A}_n \rtimes \mathbb{Z}[H_n]$. We also have $\mathrm{gr}_V \mathbb{Z}[B_n(M)] = \bigoplus_{d=0}^{\infty}(V_d/V_{d+1})$. Our second main result will be

Theorem 1.3. *There exists a homomorphism of $\mathbb{Z}$-modules $u : \mathbb{Z}[B_n(M)] \to \widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n]$ such that the corresponding graded map*

$$\mathrm{gr}u : \mathrm{gr}_V \mathbb{Z}[B_n(M)] \longrightarrow \mathcal{A}_n \rtimes \mathbb{Z}[H_n]$$

is an isomorphism of graded $\mathbb{Z}$-algebras.

We end this section by showing why u is called a *universal Vassiliev invariant* for $B_n(M)$.

Corollary 1.4. *Every Vassiliev invariant of $B_n(M)$ factors through u in a unique way.*

Proof. By Theorem 1.2, we know that $\mathbb{Z}[B_n(M)]/V_{N+1}$ is a free $\mathbb{Z}$-module for all $N \geq 0$; hence, $\mathbb{Z}[B_n(M)] \simeq (\mathbb{Z}[B_n(M)]/V_{N+1}) \oplus V_{N+1}$. Recall that

$$\mathrm{gr}_V \mathbb{Z}[B_n(M)] = \bigoplus_{d=0}^{\infty} (V_d/V_{d+1}) \simeq (\mathbb{Z}[B_n(M)]/V_{N+1}) \oplus \left(\bigoplus_{d>N} (V_d/V_{d+1}) \right).$$

Now, since $\mathrm{gr}u$ is an isomorphism, we conclude that, for all $N \geq 0$, $\mathcal{A}_n^{(\leq N)} \rtimes \mathbb{Z}[H_n]$ is also a free $\mathbb{Z}$-module. Therefore,

$$\widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n] \simeq (\mathcal{A}_n^{(\leq N)} \rtimes \mathbb{Z}[H_n]) \oplus (\widehat{\mathcal{A}}_n^{(>N)} \rtimes \mathbb{Z}[H_n])$$

and

$$\mathcal{A}_n \rtimes \mathbb{Z}[H_n] \simeq (\mathcal{A}_n^{(\leq N)} \rtimes \mathbb{Z}[H_n]) \oplus (\mathcal{A}_n^{(>N)} \rtimes \mathbb{Z}[H_n]).$$

Every Vassiliev invariant $v \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[B_n(M)]/V_{N+1}\ , A)$ can then be seen as a linear map from $\mathrm{gr}_V \mathbb{Z}[B_n(M)]$ to A which vanishes on $\bigoplus_{d>N} (V_d/V_{d+1})$. Via $\mathrm{gr}u$, this means that v is a linear map from $\mathcal{A}_n \rtimes \mathbb{Z}[H_n]$ to A, which vanishes on $\mathcal{A}_n^{(>N)} \rtimes \mathbb{Z}[H_n]$. Therefore, if v is a Vassiliev invariant of type N, it can be lifted in a unique way to a linear map $\widehat{v} : \widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n] \to A$, which satisfies $v = \widehat{v} \circ u$. □

2. Vassiliev invariants separate braids

Our strategy for proving Theorem 1.2 is the following. In a first subsection, we introduce some ideal J of $\mathbb{Z}[B_n(M)]$ given by its generators, and we prove that $V_d = J^d$ for all $d \geq 0$. In a second subsection, we consider an exact sequence $1 \to K_n \to B_n(M) \to H_n \to 1$, and we prove that J^d is equal in some sense to $I(K_n)^d \otimes \mathbb{Z}[H_n]$, where $I(K_n)$ denotes the augmentation ideal of K_n. In a third subsection, we prove that K_n can be expressed as an iterated semi-direct product of free groups (of infinite rank). Finally, in the fourth subsection, we use the results of the previous ones to prove Theorem 1.2.

2.1. **The Vassiliev filtration coincides with the J-adic filtration.** The aim of this subsection is to introduce an ideal J of $\mathbb{Z}[B_n(M)]$ defined by its generators, and to show that the Vassiliev filtration coincides with the J-adic filtration (i.e., $V_d = J^d$ for all $d \in \mathbb{N}$).

We begin by explaining our "visualization" of (singular) braids, and by exhibiting generators for $SB_n(M)$.

We represent the surface M as a polygon of $4g$ sides which are identified as shown in Figure 2.

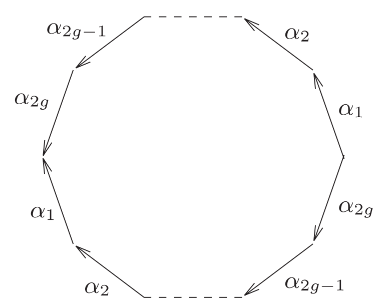


Figure 2. A representation of the surface M.

We draw braids over M in this polygon, as if we looked at the cylinder $M \times [0,1]$ from above; that is, we project the braid onto $M \times \{0\}$. As for the planar representations of knots, we see over- and under-crossings, and we can always move our braid via a suitable isotopy to avoid triple crossing points in the projection. See Figure 3 for an example.

Now, for every $i \in \{1,\dots,n\}$ and every $r \in \{1,\dots,2g\}$, we define the braid $a_{i,r}$ as follows. All the strings of $a_{i,r}$ are trivial except the i-th one, which goes through the r-th wall as shown in Figure 4. It goes upwards if r is odd and downwards if r is even.

We also define, for all $j = 1,\dots,n-1$, the braid σ_j as follows. All the strings of σ_j are trivial except the j-th one and the $(j+1)$-th one. The j-th string goes from $(P_j,0)$ to $(P_{j+1},1)$ and the $(j+1)$-th string goes from $(P_{j+1},0)$ to $(P_j,1)$, according to Figure 4. Note that $\sigma_1,\dots,\sigma_{n-1}$ are the classical generators of the braid group B_n of the disc.

It is easy to show that $\{a_{i,r};\ \ i=1,\dots,n,\ r=1,\dots,2g\} \cup \{\sigma_1,\dots,\sigma_{n-1}\}$ is a generator set for $B_n(M)$. Actually, there is no need to include $a_{i,r}$ if $i \geq 2$, but it is better for our purposes. One can find in [9] a presentation for $B_n(M)$ which involves these generators.

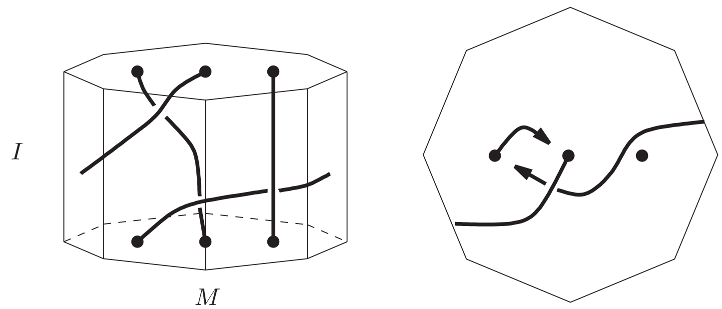


FIGURE 3. A braid with 3 strings on a surface of genus 2: two different viewpoints.

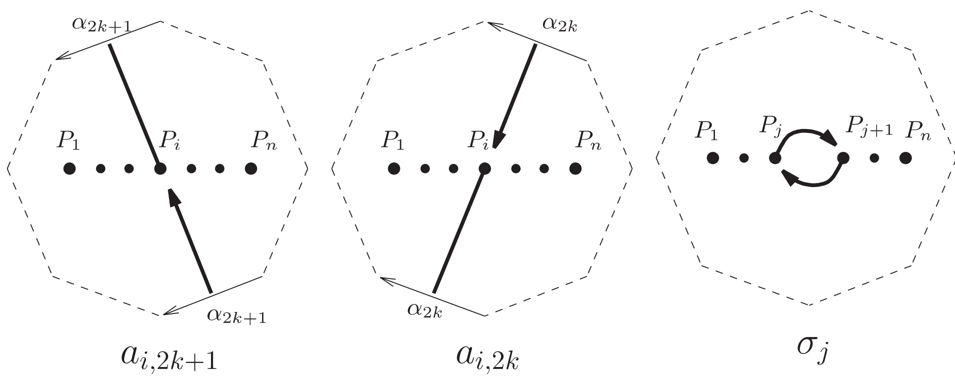


FIGURE 4. Generators for $B_n(M)$.

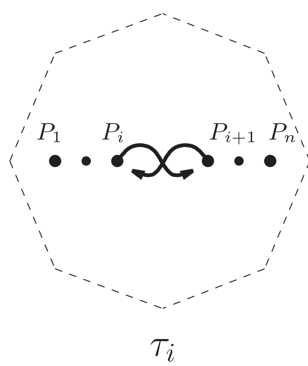


FIGURE 5. The singular braid τ_i.

For every $i = 1, \dots, n-1$, we define the singular braid $\tau_i \in S_1B_n(M)$ as in Figure 5. This singular braid has a unique singular point, which is the intersection of the i-th string and the $(i+1)$-th string. The i-th string goes from $(P_i, 0)$ to $(P_{i+1}, 1)$, and the $(i+1)$-th string goes from $(P_{i+1}, 0)$ to $(P_i, 1)$. The other strings are trivial.

By a suitable isotopy, any singular braid $b \in S_k B_n(M)$ can be written in the form

$$b = c_1 \tau_{j_1} c_2 \tau_{j_2} \cdots c_k \tau_{j_k} c_{k+1},$$

where $c_i \in B_n(M)$. So the following set generates $SB_n(M)$ (as a monoid):

$$\{a_{i,r}^{\pm 1};\ \ i = 1, \dots, n,\ r = 1, \dots, 2g\} \cup \{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\} \cup \{\tau_1, \dots, \tau_{n-1}\}.$$

Now, the morphism $\eta :\ SB_n(M) \to \mathbb{Z}[B_n(M)]$ sends $\sigma_i^{\pm 1}$ to $\sigma_i^{\pm 1}$, $a_{i,r}^{\pm 1}$ to $a_{i,r}^{\pm 1}$ and τ_i to $\sigma_i - \sigma_i^{-1}$. Recall that V_d denotes the $\mathbb{Z}$-submodule of $\mathbb{Z}[B_n(M)]$ generated by $\eta(S_d B_n(M))$. From the above considerations, we immediately get

Proposition 2.1. *Let J be the two-sided ideal of $\mathbb{Z}[B_n(M)]$ generated by $\{\sigma_i - \sigma_i^{-1};\ \ i = 1, \dots, n-1\}$. Then $V_d = J^d$ for all $d \in \mathbb{N}$.*

2.2. **From J^d to $I(K_n)^d$.** Recall that H_n denotes the semi-direct product $\pi_1(M)^n \rtimes \Sigma_n$. We define a homomorphism $\varphi :\ B_n(M) \to H_n$ as follows. We fix a disc D embedded in M which contains $\mathcal{P}$ and, for all $i, j \in \{1, \dots, n\}$, a path $\alpha_{i,j}$ in D going from P_i to P_j. Pick a braid $b = (b_1, \dots, b_n)$, $b_i :\ [0,1] \to M \times [0,1]$, based at $\mathcal{P}$. Let $s \in \Sigma_n$ be the permutation induced by b. Let $\overline{b}_i :\ [0,1] \to M$ be the projection of b_i on the first coordinate, and let μ_i be the loop based at P_i defined by $\mu_i = \overline{b}_i\, \alpha_{s(i),i}$. Then we set

$$\varphi(b) = (\mu_1, \dots, \mu_n) s \in \pi_1(M)^n \rtimes \Sigma_n = H_n.$$

One can easily verify that $\varphi :\ B_n(M) \to H_n$ is a well-defined homomorphism, and that its definition depends on the choice of D but not on the choice of the paths $\alpha_{i,j}$.

Let K_n denote the kernel of φ. It is a classical matter that a set-section $\sigma :\ H_n \to B_n(M)$ of φ determines a $\mathbb{Z}$-isomorphism $\Phi :\ \mathbb{Z}[B_n(M)] \to \mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n]$ defined by

$$\Phi(b) = b\, (\sigma \circ \varphi)(b)^{-1} \otimes \varphi(b).$$

Let us fix such a set-section.

Recall that the *augmentation ideal* of a group G is defined to be the two-sided ideal $I(G)$ of $\mathbb{Z}[G]$ generated by the set $\{1-g;\ g \in G\}$. In this subsection, we prove the following.

Proposition 2.2. *The isomorphism $\Phi :\ \mathbb{Z}[B_n(M)] \to \mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n]$ sends J^d isomorphically to $I(K_n)^d \otimes \mathbb{Z}[H_n]$ for all $d \in \mathbb{N}$.*

Note that Proposition 2.2 implies that, in order to prove Theorem 1.2, it will suffice to prove the following two conditions:

(1) $\bigcap_{d=0}^{\infty} I(K_n)^d = \{0\}$, and
(2) $I(K_n)^d / I(K_n)^{d+1}$ is a free $\mathbb{Z}$-module for all $d \geq 0$.

To prove Proposition 2.2, we will make use of some classical exact sequences involving braid groups (see [5]). The first one comes from the homomorphism π which maps a given braid to the permutation that it induces on $\mathcal{P}$. The kernel of this (clearly well-defined) homomorphism is a subgroup of $B_n(M)$ denoted by $PB_n(M)$, whose elements are called *pure braids*. Then one has

$$1 \longrightarrow PB_n(M) \longrightarrow B_n(M) \xrightarrow{\pi} \Sigma_n \longrightarrow 1.$$

On the other hand, there is a homomorphism $\varrho : PB_n(M) \to PB_{n-1}(M)$ which sends $(b_1, \dots, b_n)$ to $(b_2, \dots, b_n)$. If we set $\mathcal{P}_{n-1} = \{P_2, \dots, P_n\}$, then the kernel of ϱ can be seen as the group $\pi_1(M \backslash \mathcal{P}_{n-1})$. This gives

$$1 \longrightarrow \pi_1(M\backslash\mathcal{P}_{n-1}) \longrightarrow PB_n(M) \xrightarrow{\varrho} PB_{n-1}(M) \longrightarrow 1.$$

Finally, if b is a pure braid, the projection of each string b_i $(i \in \{1, \dots, n\})$ over M, denoted by $\overline{b_i}$, is a loop in M based at P_i, which determines an element $\mu_i \in \pi_1(M)$. This gives a homomorphism $\theta : PB_n(M) \to \pi_1(M)^n$, which sends $(b_1, \dots, b_n)$ to $(\mu_1, \dots, \mu_n)$. One can easily verify that $K_n = \ker\theta$, and that the exact sequence

$$1 \longrightarrow K_n \longrightarrow PB_n(M) \xrightarrow{\theta} \pi_1(M)^n \longrightarrow 1$$

extends to the exact sequence

$$1 \longrightarrow K_n \longrightarrow B_n(M) \xrightarrow{\varphi} H_n \longrightarrow 1.$$

Moreover, K_n is the normal closure in $PB_n(M)$ of the subgroup $PB_n(D)$, where D is a disc in M which contains $\mathcal{P}$ (see [5]).

In what follows, we write $I = I(K_n)$ and we consider $\mathbb{Z}[K_n]$ as a subring of $\mathbb{Z}[B_n(M)]$. The next lemma is a preliminary to the proof of Proposition 2.2.

Lemma 2.3. *Let $B = \mathbb{Z}[B_n(M)]$. For every $d \geq 1$, one has*

$$J^d = B\, I^d\, B = BI^d = I^d B.$$

Proof. Since K_n is a normal subgroup of $B_n(M)$, it is straightforward to prove that $B\, I^d\, B = BI^d = I^d B$. So, it suffices to prove that $J = B\, I\, B$.

The inclusion $J \subset B\, I\, B$ is obvious, once we notice that $\sigma_i^2 \in K_n$ and that $\sigma_i - \sigma_i^{-1} = \sigma_i^{-1}(\sigma_i^2 - 1) \in B\, I\, B$. For the other inclusion, we must prove that for all $p \in K_n$ one has $p - 1 \in J$. Suppose that $p = p_1\, p_2$, with $p_1, p_2 \in K_n$; then $p - 1 = p_1(p_2 - 1) + (p_1 - 1)$, so it suffices to show it for a set of generators of K_n. As we said before, K_n is the normal closure of $PB_n(D)$ in $PB_n(M)$, so a set of generators of K_n consists of elements of the form $\alpha\, b\, \alpha^{-1}$, where $\alpha \in PB_n(M)$ and $b \in PB_n(D)$.

Take an element $\alpha b \alpha^{-1}$ as above. One has: $\alpha b \alpha^{-1} - 1 = \alpha(b-1)\alpha^{-1}$, so we only have to show that $b - 1 \in J$ for $b \in PB_n(D)$. It is known ([16], Lemma 1.2) that $b-1$ belongs to the ideal of $\mathbb{Z}[B_n(D)]$ generated by $\{\sigma_i - \sigma_i^{-1};\ i = 1, \dots, n-1\}$. But the extension of this ideal to $\mathbb{Z}[B_n(M)] \supset \mathbb{Z}[B_n(D)]$ is precisely J, so $b - 1 \in J$. ☐

Proof of Proposition 2.2. First, we show that $\Phi(J^d) \subset I^d \otimes \mathbb{Z}[H_n]$ for all $d \geq 1$. By Lemma 2.3, we know that $J^d = I^d\, B$; thus J^d is generated as a $\mathbb{Z}$-module by the elements of the form $(k_1 - 1) \cdots (k_d - 1)b$, where $b \in B_n(M)$ and $k_i \in K_n$ for $i = 1, \dots, d$. Now, the image of such an element by Φ is $(k_1 - 1) \cdots (k_d - 1)\, b' \otimes \varphi(b)$, where $b' = b\,(\sigma \circ \varphi)(b)^{-1}$, which clearly belongs to $I^d \otimes \mathbb{Z}[H_n]$.

The inclusion $I^d \otimes \mathbb{Z}[H_n] \subset \Phi(J^d)$ follows from the facts that $I^d \otimes \mathbb{Z}[H_n]$ is generated as a $\mathbb{Z}$-module by the elements of the form $(k_1 - 1) \cdots (k_d - 1)k \otimes \beta$, where $k_1, \dots, k_d, k \in K_n$ and $\beta \in H_n$, and that such an element is the image by Φ of $(k_1 - 1) \cdots (k_d - 1)k\, \sigma(\beta) \in I^d\, B = J^d$. ☐

2.3. **The structure of K_n.** The goal of this subsection is to prove the following.

Proposition 2.4. *For $n \geq 2$, there exists a free group F_n such that $K_n = F_n \rtimes K_{n-1}$. Moreover, the action of K_{n-1} on the abelianization of F_n is trivial.*

Remarks. (i) The notation F_n may lead to some confusion; indeed, here F_n is not a free group of rank n. It is actually of infinite rank.

(ii) A direct consequence of Proposition 2.4 is that K_n can be expressed as an iterated semi-direct product of (infinitely generated) free groups

$$K_n = F_n \rtimes (F_{n-1} \rtimes (\cdots \rtimes (F_3 \rtimes F_2)\cdots)).$$

Recall the exact sequences defined in the previous subsection. Since K_n is a subgroup of $PB_n(M)$, we can consider the image by ϱ of K_n. By definition, it is equal to K_{n-1}. If we denote $F_n = \ker \varrho \cap K_n$, we obtain the following commutative diagram, where all rows and columns are exact:

$$\begin{array}{ccccccc}
 & 1 & & 1 & & 1 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
1 \to & \pi_1(M, P_1) & \to & \pi_1(M)^n & \to & \pi_1(M)^{n-1} & \to 1 \\
 & \uparrow & & \uparrow \theta & & \uparrow \theta & \\
1 \to & \pi_1(M \backslash \mathcal{P}_{n-1}) & \to & PB_n(M) & \xrightarrow{\varrho} & PB_{n-1}(M) & \to 1 \\
 & \uparrow & & \uparrow & & \uparrow & \\
1 \to & F_n & \to & K_n & \xrightarrow{\varrho} & K_{n-1} & \to 1 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 1 & & 1 & & 1 &
\end{array}$$

Notice that F_n is a free group, since it is a subgroup of $\pi_1(M\backslash\mathcal{P}_{n-1})$, which is a free group. We are especially interested in the lowest row of the diagram. In particular, in order to show Proposition 2.4, we will show that there exists a homomorphism $s: K_{n-1} \to K_n$ which is a section of ϱ, and that K_{n-1} acts trivially on the abelianization of F_n. We turn first to find a free set of generators for F_n.

Let $\Omega = \{\omega_1, \ldots, \omega_{2g}\}$ be a set of $2g$ letters. It is well known that a presentation for $\pi_1(M)$ is as follows:

$$\pi_1(M) = \left\langle \Omega;\ \ (\omega_1\omega_2\cdots\omega_{2g}\omega_1^{-1}\omega_2^{-1}\cdots\omega_{2g}^{-1}) = 1 \right\rangle.$$

For every element $\gamma \in \pi_1(M)$ we choose a unique word $\widetilde{\gamma}$ over $\Omega \cup \Omega^{-1}$ which represents γ. We call this word the *normal form of* γ. Normal forms are chosen in such a way that they are prefix-closed (namely, if $\omega_1\omega_2$ is a normal form, then ω_1 is also a normal form). For every word ω over $\Omega\cup\Omega^{-1}$, we will denote by $\omega_{(i)}$ the word over $\{a_{i,1}^{\pm1}, \ldots, a_{i,2g}^{\pm1}\}$ obtained from ω by replacing $\omega_j^{\pm1}$ by $a_{i,j}^{\pm1}$, for all $j = 1, \ldots, 2g$.

Let us consider, for $1 \le i < j \le n$, the braid $T_{i,j}$ drawn in Figure 6. All its strings are trivial except the i-th one, which goes around the points $P_{i+1}, \ldots, P_j$ and turns back to P_i.

Notice that in $\pi_1(M\backslash\mathcal{P}_{n-1})$, viewed as a subgroup of $PB_n(M)$, one has

$$T_{1,n} = a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1}.$$

Lemma 2.5. *The following set is a free system of generators for* F_n*.*

$$\mathcal{B} = \{\widetilde{\gamma}_{(1)}\, T_{1,j}\, \widetilde{\gamma}_{(1)}^{-1}\,;\ \ 2 \le j \le n \text{ and } \gamma \in \pi_1(M)\}.$$

Proof. Consider the Cayley graph of $\pi_1(M)$, which is defined as follows. Its vertices are the elements of $\pi_1(M)$, and its edges are labeled by Ω. For every vertex $\gamma \in \pi_1(M)$ and for every $i \in 1, \ldots, 2g$, there is exactly one edge labeled by ω_i, with source γ and target $\gamma\omega_i$.

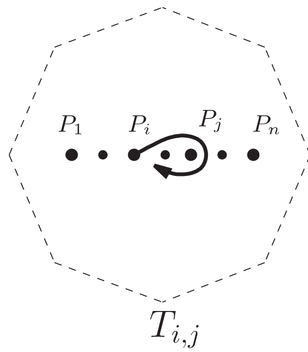


FIGURE 6. The path (or braid) $T_{i,j}$.

In this graph, the normal form of an element $\gamma \in \pi_1(M)$ corresponds to a unique path going from 1 to γ. By the prefix-closed condition mentioned above, the set of normal forms of $\pi_1(M)$ defines a maximal tree T of the Cayley graph.

The Cayley graph of $\pi_1(M)$ can be seen as the one-skeleton of a tiling of the (euclidean or hyperbolic) plane. For every vertex γ, the path which starts at γ and which is labeled by $\omega_1 \dots \omega_{2g}\omega_1^{-1} \dots \omega_{2g}^{-1}$ bounds a fundamental region R_γ of this tiling, and all fundamental regions are obtained in this way. Hence, there is a one-to-one correspondence between the vertices of the Cayley graph and its fundamental regions. Therefore, the fundamental group of the Cayley graph of $\pi_1(M)$ is the free group with free system of generators

$$\{\widetilde{\gamma}\,(\omega_1 \dots \omega_{2g}\omega_1^{-1} \dots \omega_{2g}^{-1})\,\widetilde{\gamma}^{-1}\,; \quad \gamma \in \pi_1(M, P_1)\}.$$

We now define a graph Γ as follows. Take the Cayley graph of $\pi_1(M)$ and replace the labels ω_i by $a_{1,i}$. Then, for every vertex γ, add $n-2$ edges with source and target γ, labeled by $T_{1,2}, \dots, T_{1,n-1}$, respectively. Notice that the fundamental group of Γ is the free group with free system of generators $\mathcal{B}$, where $T_{1,n} = a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1}$.

Recall the exact sequence

$$1 \longrightarrow F_n \longrightarrow \pi_1(M\backslash\mathcal{P}_{n-1}) \xrightarrow{\theta} \pi_1(M, P_1) \longrightarrow 1.$$

One can easily verify that $\pi_1(M\backslash\mathcal{P}_{n-1})$ is freely generated by $\{a_{1,1}, \dots, a_{1,2g}, T_{1,2}, \dots, T_{1,n-1}\}$, and that θ sends $a_{1,i}$ to ω_i for all $i = 1, \dots, 2g$, and sends $T_{1,j}$ to 1 for all $j = 2, \dots, n-1$. It follows from classical geometric methods (see [11], Chapter III, Proposition 3.2) that the group F_n is the fundamental group of the graph Γ, hence $\mathcal{B}$ is a free system of generators for F_n, as we wanted to prove. □

Lemma 2.6. *There is a homomorphism $\sigma : K_{n-1} \to K_n$ which is a section of $\varrho : K_n \to K_{n-1}$.*

Proof. The case $n = 2$ is trivial, since $K_1 = \ker\left(\pi_1(M) \xrightarrow{\theta} \pi_1(M)\right) = 1$. Hence $K_2 = F_2$ is a free group of infinite rank. Suppose now that $n > 2$. The image of a given braid by σ is obtained by doubling the first string (the one starting at P_2). This can be done for elements of K_n since we can first isotope this string to a straight line, and then double it, as follows.

It is well known that the kernel of the homomorphism $\theta_2 : PB_n(M) \to \pi_1(M, P_2)$ is $PB_{n-1}(M\backslash\{P_2\})$ (see [5]). Moreover, one can easily see that K_n lies in this kernel, namely $K_n \subset PB_{n-1}(M\backslash\{P_2\})$. Similarly, one has $K_{n-1} \subset PB_{n-2}(M\backslash\{P_2\})$. The homomorphism $\varrho_n : PB_{n-1}(M\backslash\{P_2\}) \longrightarrow PB_{n-2}(M\backslash\{P_2\})$ which sends $(b_1, b_3, \dots, b_n)$ to $(b_3, \dots, b_n)$ is the restriction of ϱ to $PB_{n-1}(M\backslash\{P_2\})$. In particular, it sends K_n onto K_{n-1}.

We consider an embedding $f : M\backslash\{P_2\} \to M\backslash\{P_2\}$ satisfying:

- $f(P_i) = P_i$ for $i = 3, \dots, n$;
- P_1 does not lie in the image of f;
- f is homotopy equivalent (relative to $\{P_3, \dots, P_n\}$) to the identity.

Then, f induces a homomorphism $\sigma : PB_{n-2}(M\backslash\{P_2\}) \to PB_{n-1}(M\backslash\{P_2\})$, which sends $(b_3, \dots, b_n)$ to $(1_{P_1}, (f \times \mathrm{id})b_3, \dots (f \times \mathrm{id})b_n)$. By the third condition, this homomorphism is a section of ϱ_n. It obviously sends K_{n-1} to K_n. □

Now, K_{n-1} acts on F_n in the following way: Given $b \in K_{n-1}$, the action induced by b sends $f \in F_n$ to $\sigma(b) f \sigma(b)^{-1}$. This action induces an action of K_{n-1} on the abelianization $F_n/[F_n, F_n]$ of F_n (here $[F_n, F_n]$ denotes the commutator subgroup of F_n). The proof of Proposition 2.4 is finally obtained from the following result.

Lemma 2.7. *The action of K_{n-1} on the abelianization of F_n is trivial.*

Proof. We only need to verify that the action of the generators of K_{n-1} on the generators of F_n is trivial after abelianization. Moreover, let us see that it suffices to show the result for the action defined by any set-map section s of ϱ.

Indeed, if s is a section of ϱ, then for every $b \in K_{n-1}$, there exists an element $\widehat{b} \in F_n$ such that $\sigma(b) = \widehat{b}\, s(b)$. Therefore, if K_{n-1} acts trivially on $F_n/[F_n, F_n]$ via s, we obtain, for every $f \in F_n$,

$$\sigma(b)\ f\ \sigma(b)^{-1} \equiv \widehat{b}\ \left(s(b)\ f\ s(b)^{-1}\right) \widehat{b}^{-1} \equiv \widehat{b}\ f\ \widehat{b}^{-1} \equiv f \qquad (\mathrm{mod}\ [F_n, F_n]).$$

As we said in Subsection 2.2, a set of generators for K_{n-1} consists of elements of the form $\alpha\, b\, \alpha^{-1}$, where $\alpha \in PB_{n-1}(M)$ and $b \in PB_{n-1}(D)$. On the other hand, it is known that $T = \{T_{i,j} \mid \ \ 2 \le i < j \le n\}$ is a set of generators for $PB_{n-1}(D)$, where $T_{i,j}$ denotes the braid defined in Subsection 2.1. Therefore, the following is a set of generators for K_{n-1}:

$$\left\{\alpha\, T_{i,j}\, \alpha^{-1};\ \ 2 \le i < j \le n, \text{ and } \alpha \text{ is a word over } \{a_{k,r}^{\pm 1};\ 2 \le k \le n,\ 1 \le r \le 2g\}\right\}.$$

We take s such that $s(\alpha\, T_{i,j}\, \alpha^{-1}) = \alpha\, T_{i,j}\, \alpha^{-1} \in K_n$. In other words, we just add a trivial string based at P_1 for any element of the set of generators of K_{n-1}. We remark that s is a set map section of ϱ, but it is not a homomorphism.

Now, F_n is by definition a normal subgroup of $PB_n(M)$. Therefore, if we show that each $T_{i,j}$ $(i \ge 2)$ acts trivially on $F_n/[F_n, F_n]$ by conjugation, then we will have finished the proof, since in that case

$$\begin{aligned}(\alpha\, T_{i,j}\, \alpha^{-1})\, f\, (\alpha\, T_{i,j}^{-1}\, \alpha^{-1}) &\equiv \alpha\, T_{i,j}\, (\alpha^{-1} f\, \alpha)\, T_{i,j}^{-1}\, \alpha^{-1} \\ &\equiv \alpha\, (\alpha^{-1} f\, \alpha)\, \alpha^{-1} \equiv f \quad (\mathrm{mod}\ [F_n, F_n]).\end{aligned}$$

Recall from Lemma 2.5 that

$$\mathcal{B} = \{\widetilde{\gamma}_{(1)}\, T_{1,k}\, \widetilde{\gamma}_{(1)}^{-1}\ ;\ \ 2 \le k \le n \text{ and } \ \gamma \in \pi_1(M)\}$$

is a free system of generators for F_n, where $\widetilde{\gamma}_{(1)}$ is a word over $\{a_{1,1}^{\pm 1}, \dots, a_{1,2g}^{\pm 1}\}$ for all $\gamma \in \pi_1(M)$. One can verify (just drawing the corresponding braids) that in $PB_n(M)$ one has the following relations:
$T_{i,j} a_{1,r} T_{i,j}^{-1} = a_{1,r}$,
$T_{i,j} T_{1,k} T_{i,j}^{-1} = T_{1,k}$ $\quad (k < i \text{ or } k \geq j)$,
$T_{i,j} T_{1,k} T_{i,j}^{-1} = T_{1,i-1} T_{1,i}^{-1} T_{1,k} T_{1,j}^{-1} T_{1,i} T_{1,i-1}^{-1} T_{1,j} \equiv T_{1,k} \pmod{[F_n, F_n]}$ $(i \leq k < j)$,
where $2 \leq i < j \leq n$, $r \in \{1, \dots, 2g\}$ and $k \in \{2, \dots, n\}$.

Therefore, in $F_n/[F_n, F_n]$, we have

$$T_{i,j}\, (\widetilde{\gamma}_{(1)}\, T_{1,k}\, \widetilde{\gamma}_{(1)}^{-1})\, T_{i,j}^{-1} = \widetilde{\gamma}_{(1)}\, (T_{i,j}\, T_{1,k}\, T_{i,j}^{-1}) \widetilde{\gamma}_{(1)}^{-1} \equiv \widetilde{\gamma}_{(1)}\, T_{1,k}\, \widetilde{\gamma}_{(1)}^{-1} \pmod{[F_n, F_n]},$$

as we wanted to show. □

2.4. **Proof of Theorem 1.2.** Let A and C be two groups such that C acts on A. For $a \in A$ and $c \in C$, we denote by a^c the action of c on a. Then, the $\mathbb{Z}$-module $\mathbb{Z}[A] \otimes \mathbb{Z}[C]$ carries a natural structure of $\mathbb{Z}$-algebra, where the multiplication is defined by

$$(a_1 \otimes c_1) \cdot (a_2 \otimes c_2) = (a_1\, a_2^{c_1}) \otimes (c_1 c_2).$$

Moreover, this algebra is naturally isomorphic to $\mathbb{Z}[A \rtimes C]$ via an isomorphism which sends $a \otimes c$ to ac for all $a \in A$ and all $c \in C$.

Recall that the augmentation ideal of a group G is denoted by $I(G)$. The following lemma will be used to prove Theorem 1.2. Its proof can be found in [13, Lemma 3.1].

Lemma 2.8. *Let A and C be two groups. Assume that an action of C on A is given, and that this action induces the trivial action on the abelianization of A. Then one has*

$$I(A \rtimes C)^m = \sum_{k=0}^{m} I(A)^k \otimes I(C)^{m-k}$$

for all $m \geq 0$. □

Proof of Theorem 1.2. As pointed out in Subsection 2.2, it suffices to prove the following two conditions.

(1) $\bigcap_{d=0}^{\infty} I(K_n)^d = \{0\}$, and
(2) $I(K_n)^d / I(K_n)^{d+1}$ is a free $\mathbb{Z}$-module for all $d \geq 0$.

We argue by induction on n. The case $n = 1$ is trivial, since $I(K_1) = I(\{1\}) = 0$. So, we assume that $n \geq 2$ and that conditions 1 and 2 hold for K_p, where $p < n$.

The group F_n is free; thus, by [8], one has

(1) $\bigcap_{d=0}^{\infty} I(F_n)^d = \{0\}$, and
(2) $I(F_n)^d / I(F_n)^{d+1}$ is a free $\mathbb{Z}$-module for all $d \geq 0$.

These two properties imply that $I(F_n)^d$ is a free $\mathbb{Z}$-module and that

$$I(F_n)^d \simeq \bigoplus_{k=d}^{\infty} I(F_n)^k / I(F_n)^{k+1}$$

for all $d \geq 0$. We choose a $\mathbb{Z}$-basis $\mathcal{B}_d$ of $I(F_n)^d / I(F_n)^{d+1}$ for all $d \geq 0$. From the above isomorphism one has that $\mathcal{B}_{\geq d} = \coprod_{k=d}^{\infty} \mathcal{B}_k$ is a $\mathbb{Z}$-basis of $I(F_n)^d$.

From the induction hypothesis, one also has

(1) $\bigcap_{d=0}^{\infty} I(K_{n-1})^d = \{0\}$, and
(2) $I(K_{n-1})^d / I(K_{n-1})^{d+1}$ is a free $\mathbb{Z}$-module for all $d \geq 0$.

This implies that $I(K_{n-1})^d$ is a free $\mathbb{Z}$-module and that

$$I(K_{n-1})^d \simeq \bigoplus_{k=d}^{\infty} I(K_{n-1})^k/I(K_{n-1})^{k+1}$$

for all $d \geq 0$. We choose a $\mathbb{Z}$-basis $\mathcal{C}_d$ of $I(K_{n-1})^d/I(K_{n-1})^{d+1}$ for all $d \geq 0$. Thus $\mathcal{C}_{\geq d} = \coprod_{k=d}^{\infty} \mathcal{C}_k$ is a $\mathbb{Z}$-basis of $I(K_{n-1})^d$.

Now, by Proposition 2.4, F_n and K_{n-1} satisfy the hypothesis of Lemma 2.8. Hence, we have the equality

$$I(K_n)^m = \sum_{d=0}^{m} I(F_n)^d \otimes I(K_{n-1})^{m-d}$$

for all $m \geq 0$. From this equality, one can easily verify that the set

$$\mathcal{D}_{\geq m} = \{b \otimes c \in \mathbb{Z}[F_n] \otimes \mathbb{Z}[K_{n-1}]\ ; \quad b \in \mathcal{B}_i,\ c \in \mathcal{C}_j,\ i+j \geq m\}$$

is a generating set for $I(K_n)^m$. Since this set is linearly independent, $I(K_n)^m$ is a free $\mathbb{Z}$-module whose basis is $\mathcal{D}_{\geq m}$. It follows that $I(K_n)^m/I(K_n)^{m+1}$ is a free $\mathbb{Z}$-module with basis

$$\mathcal{D}_m = \mathcal{D}_{\geq m} \backslash \mathcal{D}_{\geq m+1} = \{b \otimes c \in \mathbb{Z}[F_n] \otimes \mathbb{Z}[K_{n-1}]\ ; \quad b \in \mathcal{B}_i,\ c \in \mathcal{C}_j,\ i+j = m\},$$

and that $\bigcap_{d=0}^{\infty} I(K_n)^d = \{0\}$, since $\mathcal{D}_{\geq 0}$ is a basis for $\mathbb{Z}[K_n]$. □

3. The universal Vassiliev invariant

The proof of Theorem 1.3 is divided into five steps. In what follows each subsection will correspond to one of them.

The first subsection is dedicated to the definition of a linear map

$$u:\ \mathbb{Z}[B_n(M)] \longrightarrow \widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n].$$

Recall that $\mathbb{Z}[B_n(M)]$ is isomorphic to $\mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n]$ as a $\mathbb{Z}$-module (see Subsection 2.2), and notice that $\widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n]$ is equal as a $\mathbb{Z}$-module to $\widehat{\mathcal{A}}_n \otimes \mathbb{Z}[H_n]$. Hence, we will only need to define a linear map $v : \mathbb{Z}[K_n] \to \widehat{\mathcal{A}}_n$.

Recall that the subgroups G_i of the *lower central series* of a group G are defined recursively by $G_1 = G$ and $G_{i+1} = [G, G_i]$ for $i \geq 1$. The *associated graded Lie algebra of* G is defined by $\mathrm{gr}(G) = \bigoplus_{i\geq 1} G_i/G_{i+1}$. It is a graded Lie algebra over $\mathbb{Z}$ whose enveloping algebra is denoted by $\mathcal{U}\mathrm{gr}(G)$. In Subsection 3.2 we construct a homomorphism $\chi_1 :\ \mathcal{A}_n \to \mathcal{U}\mathrm{gr}(K_n)$ of $\mathbb{Z}$-algebras, and we prove that this homomorphism is actually an isomorphism.

Write $I = I(K_n)$. In Subsection 3.3 we construct an isomorphism of $\mathbb{Z}$-algebras $\chi_2 : \mathcal{U}\mathrm{gr}(K_n) \to \mathrm{gr}_I\mathbb{Z}[K_n]$, using a result due to Quillen.

In Subsection 3.4 we consider the isomorphism $\chi = \chi_2 \circ \chi_1 :\ \mathcal{A}_n \to \mathrm{gr}_I\mathbb{Z}[K_n]$ and we prove that $\mathrm{gr}v$ is the inverse of χ. It will immediately follow that $\mathrm{gr}v$: $\mathrm{gr}_I\mathbb{Z}[K_n] \to \mathcal{A}_n$ is an isomorphism of $\mathbb{Z}$-algebras, and that $\mathrm{gr}u :\ \mathrm{gr}_V\mathbb{Z}[B_n(M)] \to \mathcal{A}_n \rtimes \mathbb{Z}[H_n]$ is an isomorphism of $\mathbb{Z}$-modules.

Finally, we prove in Subsection 3.5 that $\mathrm{gr}u$ is a homomorphism of $\mathbb{Z}$-algebras. This will finish the proof of Theorem 1.3.

3.1. **Construction of** $u: \mathbb{Z}[B_n(M)] \longrightarrow \widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n]$**.** From now on we fix a set-section $\sigma : H_n \to B_n(M)$ of $\varphi : B_n(M) \to H_n$. As we pointed out in Subsection 2.2, this set-section leads to an isomorphism $\Phi : \mathbb{Z}[B_n(M)] \longrightarrow \mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n]$ of $\mathbb{Z}$-modules.

We turn now to define a linear homomorphism $v : \mathbb{Z}[K_n] \to \widehat{\mathcal{A}}_n$. Then we will set

$$u = (v \otimes \mathrm{id}) \circ \Phi : \mathbb{Z}[B_n(M)] \longrightarrow \widehat{\mathcal{A}}_n \rtimes \mathbb{Z}[H_n] = \widehat{\mathcal{A}}_n \otimes \mathbb{Z}[H_n].$$

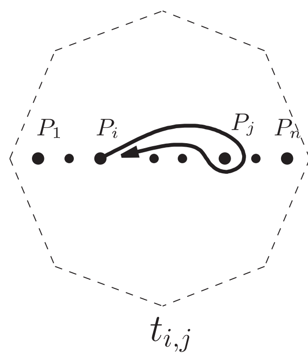


FIGURE 7. The braid $t_{i,j}$.

Recall that, for all $\gamma \in \pi_1(M)$ and all $i \in \{1, \ldots, n\}$, we denote by $\widetilde{\gamma}_{(i)}$ the normal form of γ over the generators $\{a_{i,1}^{\pm 1}, \ldots, a_{i,2g}^{\pm 1}\}$ of $\pi_1(M, P_i)$. For $1 \le i < j \le n$, we write $t_{i,j} = t_{j,i} = T_{i,j}T_{i,j-1}^{-1}$, which is the braid drawn in Figure 7. These braids are the classical generators of $PB_n(D)$. Then, for $i \neq j$, we denote by $f_{i,j,\gamma}$ the element $\widetilde{\gamma}_{(i)} t_{i,j} \widetilde{\gamma}_{(i)}^{-1}$ of $B_n(M)$. From Lemma 2.5 it follows that $F_{(n+1)-i}$ is the free group freely generated by

$$\mathcal{F}_{i,n} = \{f_{i,j,\gamma}\ ;\ \ j = i+1, \ldots, n,\ \ \gamma \in \pi_1(M)\}.$$

Moreover, it is shown in Subsection 2.3 that $K_n = F_n \rtimes (F_{n-1} \rtimes (\cdots \rtimes (F_3 \rtimes F_2) \cdots))$. So, every element $k \in K_n$ can be uniquely written in the form $k = k_1 \cdots k_{n-1}$, where k_i is a reduced word over $\mathcal{F}_{i,n} \cup \mathcal{F}_{i,n}^{-1}$. Now, for $i \in \{1, \ldots, n-1\}$, there is an injective multiplicative homomorphism $u_i : F_{(n+1)-i} \to \mathbb{Z}[\![t_{i,j,\gamma}]\!]$, where $\mathbb{Z}[\![t_{i,j,\gamma}]\!]$ denotes the ring of non-commutative formal power series over non-commutative variables $t_{i,j,\gamma}$, defined by

$$\begin{aligned} u_i(f_{i,j,\gamma}) &= 1 + t_{i,j,\gamma}, \\ u_i(f_{i,j,\gamma}^{-1}) &= 1 - t_{i,j,\gamma} + t_{i,j,\gamma}^2 - \cdots . \end{aligned}$$

This well-known homomorphism is called the *Magnus expansion* of $F_{(n+1)-i}$ (see [12]). We denote by v_i the composition of u_i with the canonical projection $\mathbb{Z}[\![t_{i,j,\gamma}]\!] \to \widehat{\mathcal{A}}_n$, and we finally define the linear map $v : \mathbb{Z}[K_n] \to \widehat{\mathcal{A}}_n$ by

$$v(k) = v_1(k_1) v_2(k_2) \cdots v_{n-1}(k_{n-1}),$$

where $k = k_1 k_2 \cdots k_{n-1}$ is the decomposition of $k \in K_n$ defined above.

3.2. **The isomorphism** $\chi_1 : \mathcal{A}_n \to \mathcal{U}\mathbf{gr}(K_n)$**.** The goal of this subsection is to prove the following.

Proposition 3.1. *There is a well-defined isomorphism* $\chi_1 : \mathcal{A}_n \to \mathcal{U}\mathrm{gr}(K_n)$ *of* $\mathbb{Z}$*-algebras which sends* $t_{i,j,\gamma}$ *to* $f_{i,j,\gamma}$ *for all* $i, j \in \{1, \dots, n\}$*,* $i \neq j$*, and all* $\gamma \in \pi_1(M)$*.*

Consider the graded Lie algebra L_n given by the following presentation:

- **Generators:** $\{t_{i,j,\gamma};\ 1 \leq i, j \leq n,\ i \neq j,\ \gamma \in \pi_1(M)\}$.
- **Relations:**
  (L1) $t_{i,j,\gamma} = t_{j,i,\gamma^{-1}}$, for all $i, j \in \{1, \dots, n\}$, $i \neq j$, and all $\gamma \in \pi_1(M)$,
  (L2) $[t_{i,j,\gamma}\ ,\ t_{k,l,\delta}] = 0$, for all distinct $i, j, k, l \in \{1, \dots, n\}$ and all $\gamma, \delta \in \pi_1(M)$,
  (L3) $[t_{i,j,\gamma}\ ,\ t_{j,k,\delta} + t_{i,k,(\gamma\delta)}] = 0$, for all distinct $i, j, k \in \{1, \dots, n\}$ and all $\gamma, \delta \in \pi_1(M)$,

where $[\_\,,\_]$ denotes the Lie bracket.

One has $\mathcal{U}L_n = \mathcal{A}_n$, so, in order to prove Proposition 3.1, it suffices to prove the following.

Proposition 3.2. *There is a well-defined Lie algebra isomorphism* $\psi_n : L_n \longrightarrow \mathrm{gr}(K_n)$ *which sends* $t_{i,j,\gamma}$ *to* $f_{i,j,\gamma}$ *for all* $i, j \in \{1, \dots, n\}$*,* $i \neq j$*, and all* $\gamma \in \pi_1(M)$*.*

The following Lemmas 3.3 to 3.7 are preliminary results to the proof of Proposition 3.2.

Lemma 3.3. *Let* ω *be a word over* $\Omega^{\pm 1}$*. Then there exists* $W_\omega \in (K_n)_2 = [K_n, K_n]$ *such that*

$$\omega_{(j)}\, t_{i,j}\, \omega_{(j)}^{-1} = (\omega_{(i)}^{-1}\, t_{i,j}\, \omega_{(i)})\, W_\omega.$$

Proof. We can suppose, without loss of generality, that $i < j$. Suppose first that ω is a single letter. If $\omega_{(j)} = a_{j,r}$ and r is odd, then one can easily show by drawing the braids that the following equality holds in $PB_n(M)$:

$$a_{j,r}\, t_{i,j}\, a_{j,r}^{-1} = (t_{i,j-1} \cdots t_{i,i+1})\, a_{i,r}^{-1}\, t_{i,j}\, a_{i,r}\, (t_{i,i+1}^{-1} \cdots t_{i,j-1}^{-1}).$$

Hence

$$a_{j,r}\, t_{i,j}\, a_{j,r}^{-1} = (a_{i,r}^{-1}\, t_{i,j}\, a_{i,r})\, W_\omega,$$

where

$$W_\omega = \left[a_{i,r}^{-1}\, t_{i,j}^{-1}\, a_{i,r}\ ,\ t_{i,j-1} \cdots t_{i,i+1}\right] \in (K_n)_2.$$

If $\omega_{(j)} = a_{j,r}$ and r is even, then one has

$$a_{j,r}\, t_{i,j}\, a_{j,r}^{-1} = a_{i,r}^{-1}\, (t_{i,i+1}^{-1} \cdots t_{i,j-1}^{-1}\, t_{i,j} \cdots t_{i,i+1})\, a_{i,r}.$$

Therefore,

$$a_{j,r}\, t_{i,j}\, a_{j,r}^{-1} = (a_{i,r}^{-1}\, t_{i,j}\, a_{i,r})\, W_\omega,$$

where

$$W_\omega = \left[a_{i,r}^{-1}\, t_{i,j}^{-1}\, a_{i,r}\ ,\ a_{i,r}^{-1}\, (t_{i,i+1}^{-1} \cdots t_{i,j-1}^{-1})\, a_{i,r}\right] \in (K_n)_2.$$

The computations for $\omega_{(j)} = a_{j,r}^{-1}$ are the same as for $\omega_{(j)} = a_{j,r}$, interchanging the case r odd with the case r even.

Suppose now that ω is a word of length $k > 1$, and that the result is true for words of length less than k. We write $\omega = \alpha\,\beta$, with $|\alpha|, |\beta| < k$. Consider

$$\begin{aligned}
&W' = \alpha_{(j)}^{-1}\, W_\beta\, \alpha_{(j)},\\
&W'' = \left[\beta_{(i)}^{-1}\, \alpha_{(j)}\, t_{i,j}^{-1}\, \alpha_{(j)}^{-1}\, \beta_{(i)}\ ,\ [\alpha_{(j)}\ ,\ \beta_{(i)}^{-1}]\right]\, W',\\
&W = \beta_{(i)}^{-1}\, W_\alpha\, \beta_{(i)}\, W''.
\end{aligned}$$

The hypothesis $i \neq j$ implies that $[\alpha_{(j)}\ ,\ \beta_{(i)}^{-1}] \in K_n$. Furthermore, both K_n and $(K_n)_2$ are normal subgroups of $PB_n(M)$; thus $W \in (K_n)_2$. Finally, a direct calculation shows that:

$$\omega_{(j)}\, t_{i,j}\, \omega_{(j)}^{-1} = (\omega_{(i)}^{-1}\, t_{i,j}\, \omega_{(i)})\, W,$$

as we wanted to show. □

Lemma 3.4. *Let $i, j \in \{2, \dots, n\}$, $i \neq j$, let $\gamma \in \pi_1(M)$, and let ω be a word over $\Omega^{\pm 1}$. Then there exists $W \in (K_n)_2$ such that*

$$f_{i,j,\gamma}\, \omega_{(1)}\, f_{i,j,\gamma}^{-1} = W\, \omega_{(1)}.$$

Proof. Recall the epimorphism $\varrho\ :\ PB_n(M) \longrightarrow PB_{n-1}(M)$. Since one has $\varrho(\widetilde{\gamma}_{(i)}^{-1}\, \omega_{(1)}\, \widetilde{\gamma}_{(i)}) = 1$, then one can write $\widetilde{\gamma}_{(i)}^{-1}\, \omega_{(1)}\, \widetilde{\gamma}_{(i)} = b$, where b is a word over $\mathcal{B} = \{a_{1,1}^{\pm 1}, \dots, a_{1,2g}^{\pm 1}, t_{1,2}^{\pm 1}, \dots, t_{1,n}^{\pm 1}\}$.

By drawing the braids, one sees that $t_{i,j}\, a_{1,r}^{\pm 1}\, t_{i,j}^{-1} = a_{1,r}^{\pm 1}$, for $r = 1, \dots, 2g$. Moreover, since $t_{i,j}, t_{1,k} \in K_n$, one has $t_{i,j}\, t_{1,k}^{\pm 1}\, t_{i,j}^{-1} = W\, t_{1,k}^{\pm 1}$, where $W \in (K_n)_2$, for all $k = 2, \dots, n$. Therefore, since b is a word over $\mathcal{B}$, one has: $t_{i,j}\, b\, t_{i,j}^{-1} = W_b\, b$, where $W_b \in (K_n)_2$. Hence,

$$\begin{aligned}
f_{i,j,\gamma}\, \omega_{(1)}\, f_{i,j,\gamma}^{-1} &= \left(\widetilde{\gamma}_{(i)}\, t_{i,j}\, \widetilde{\gamma}_{(i)}^{-1}\right)\, \omega_{(1)}\, \left(\widetilde{\gamma}_{(i)}\, t_{i,j}^{-1}\, \widetilde{\gamma}_{(i)}^{-1}\right)\\
&= \widetilde{\gamma}_{(i)}\, t_{i,j}\, b\, t_{i,j}^{-1}\, \widetilde{\gamma}_{(i)}^{-1}\\
&= \widetilde{\gamma}_{(i)}\, W_b\, b\, \widetilde{\gamma}_{(i)}^{-1}\\
&= W\, \widetilde{\gamma}_{(i)}\, b\, \widetilde{\gamma}_{(i)}^{-1}\\
&= W\, \omega_{(1)},
\end{aligned}$$

where $W = \widetilde{\gamma}_{(i)}\, W_b\, \widetilde{\gamma}_{(i)}^{-1} \in (K_n)_2$, as we wanted to show. □

Lemma 3.5. *Let ω be a word over $\Omega^{\pm 1}$, and let $i, j, k \in \{1, \dots, n\}$, all distinct. Then there exists $W \in (K_n)_2$ such that*

$$\omega_{(i)}\, t_{j,k}\, \omega_{(i)}^{-1} = W\, t_{j,k}.$$

Proof. Clearly, it suffices to show the lemma when ω is a single letter. Besides, we can suppose that $j < k$. Then the result is a consequence of the following relations in $PB_n(M)$:

$$\begin{array}{lll}
a_{i,r}\, t_{j,k}\, a_{i,r}^{-1} = t_{j,k}, & a_{i,r}^{-1}\, t_{j,k}\, a_{i,r} = t_{j,k}, & \text{if } i < j \text{ or } i > k,\\[2ex]
a_{i,r}\, t_{j,k}\, a_{i,r}^{-1} = t_{j,i}^{-1}\, t_{j,k}\, t_{j,i}, & a_{i,r}^{-1}\, t_{j,k}\, a_{i,r} = \alpha\, t_{j,k}\, \alpha^{-1}, & \text{if } j < i < k,\ r \text{ odd},\\[2ex]
a_{i,r}\, t_{j,k}\, a_{i,r}^{-1} = \alpha\, t_{j,k}\, \alpha^{-1}, & a_{i,r}^{-1}\, t_{j,k}\, a_{i,r} = t_{j,i}^{-1}\, t_{j,k}\, t_{j,i}, & \text{if } j < i < k,\ r \text{ even},
\end{array}$$

where $\alpha = \left(a_{j,r}^{-1}\, t_{j,j+1}^{-1} \cdots t_{j,i-1}^{-1}\, t_{j,i} \cdots t_{j,j+1}\, a_{j,r}\right) \in K_n$. These relations can be easily verified by drawing pictures. □

Lemma 3.6. *Let $i,j,k,l \in \{1,\dots,n\}$, all distinct. Then*

$$[t_{i,j}, t_{k,l}] \equiv 0 \qquad \pmod{(K_n)_3},$$

$$[t_{i,j}, t_{i,k}] \equiv [t_{j,k}, t_{i,j}] \qquad \pmod{(K_n)_3}.$$

Proof. This lemma follows from the well-known congruences

$$[t_{i,j}, t_{k,l}] \equiv 0 \qquad \pmod{(PB_n(D))_3},$$

$$[t_{i,j}, t_{i,k}] \equiv [t_{j,k}, t_{i,j}] \qquad \pmod{(PB_n(D))_3}$$

(see, for example, [4]), together with the inclusion $(PB_n(D))_3 \subset (K_n)_3$. □

Lemma 3.7. *There is a well-defined Lie algebra homomorphism $\psi_n : L_n \longrightarrow \mathrm{gr}(K_n)$ which sends $t_{i,j,\gamma}$ to $f_{i,j,\gamma}$ for all $i,j \in \{1,\dots,n\}$, $i \neq j$, and all $\gamma \in \pi_1(M)$.*

Proof. We have to show that the following congruences hold:

(R1) $f_{i,j,\gamma} \equiv f_{j,i,\gamma^{-1}} \pmod{(K_n)_2}$, for all $i,j \in \{1,\dots,n\}$, $i \neq j$, and all $\gamma \in \pi_1(M)$;

(R2) $[f_{i,j,\gamma}\ ,\ f_{k,l,\delta}] \equiv 0 \pmod{(K_n)_3}$, for all distinct $i,j,k,l \in \{1,\dots,n\}$ and all $\gamma,\delta \in \pi_1(M)$;

(R3) $[f_{i,j,\gamma}\ ,\ f_{j,k,\delta}] \equiv [f_{i,k,(\gamma\delta)}\ ,\ f_{i,j,\gamma}] \pmod{(K_n)_3}$, for all distinct $i,j,k \in \{1,\dots,n\}$ and all $\gamma,\delta \in \pi_1(M)$.

Notice that (R1) follows from Lemma 3.3. So, it remains to prove (R2) and (R3). We argue by induction on n. The conditions (R2) and (R3) being empty if $n = 2$, we may assume that $n > 2$, that (R2) holds if $i,j,k,l \in \{2,\dots,n\}$ (by induction), and that (R3) holds if $i,j,k \in \{2,\dots,n\}$ (by induction).

We now prove (R2) for $k = 1$. By Lemma 3.4, there exists $W_1 \in (K_n)_2$ such that

$$f_{i,j,\gamma}\, \widetilde{\delta}_{(1)}\, f_{i,j,\gamma}^{-1} = W_1\, \widetilde{\delta}_{(1)}.$$

Also, by Lemma 3.5, there exists $W_2 \in (K_n)_2$ such that

$$\widetilde{\gamma}_{(i)}^{-1}\, t_{1,l}\, \widetilde{\gamma}_{(i)} = W_2\, t_{1,l}.$$

Then

$$\begin{aligned}
f_{i,j,\gamma}\, f_{1,l,\delta}\, f_{i,j,\gamma}^{-1} &= \left(f_{i,j,\gamma}\, \widetilde{\delta}_{(1)}\, f_{i,j,\gamma}^{-1}\right)\, f_{i,j,\gamma}\, t_{1,l}\, f_{i,j,\gamma}^{-1}\, \left(f_{i,j,\gamma}\, \widetilde{\delta}_{(1)}^{-1}\, f_{i,j,\gamma}^{-1}\right) \\
&= W_1\, \left(\widetilde{\delta}_{(1)}\, f_{i,j,\gamma}\, t_{1,l}\, f_{i,j,\gamma}^{-1}\, \widetilde{\delta}_{(1)}^{-1}\right)\, W_1^{-1} \\
&\equiv \widetilde{\delta}_{(1)}\, f_{i,j,\gamma}\, t_{1,l}\, f_{i,j,\gamma}^{-1}\, \widetilde{\delta}_{(1)}^{-1} \qquad (\mathrm{mod}\ (K_n)_3) \\
&= \widetilde{\delta}_{(1)}\, \left(\widetilde{\gamma}_{(i)}\, t_{i,j}\, \widetilde{\gamma}_{(i)}^{-1}\right)\, t_{1,l}\, \left(\widetilde{\gamma}_{(i)}\, t_{i,j}^{-1}\, \widetilde{\gamma}_{(i)}^{-1}\right)\, \widetilde{\delta}_{(1)}^{-1} \\
&= \widetilde{\delta}_{(1)}\, \widetilde{\gamma}_{(i)}\, t_{i,j}\, W_2\, t_{1,l}\, t_{i,j}^{-1}\, \widetilde{\gamma}_{(i)}^{-1}\, \widetilde{\delta}_{(1)}^{-1} \\
&\equiv \widetilde{\delta}_{(1)}\, \widetilde{\gamma}_{(i)}\, W_2\, t_{i,j}\, t_{1,l}\, t_{i,j}^{-1}\, \widetilde{\gamma}_{(i)}^{-1}\, \widetilde{\delta}_{(1)}^{-1} \qquad (\mathrm{mod}\ (K_n)_3) \\
&\equiv \widetilde{\delta}_{(1)}\, \widetilde{\gamma}_{(i)}\, W_2\, t_{1,l}\, \widetilde{\gamma}_{(i)}^{-1}\, \widetilde{\delta}_{(1)}^{-1} \qquad (\mathrm{mod}\ (K_n)_3) \qquad (\text{by Lemma 3.6}) \\
&= \widetilde{\delta}_{(1)}\, t_{1,l}\, \widetilde{\delta}_{(1)}^{-1} = f_{1,l,\delta}.
\end{aligned}$$

Therefore, (R2) holds for $k = 1$.

The congruence (R2) holds for either $i = 1$, or $j = 1$, or $l = 1$, because of the above case and the relation (R1).

We now prove (R3) for $i = 1$. By Lemma 3.4, there exists $W_1 \in (K_n)_2$ such that

$$f_{j,k,\delta}\, \widetilde{\gamma}_{(1)}\, f_{j,k,\delta}^{-1} = W_1\, \widetilde{\gamma}_{(1)}.$$

Also, by Lemma 3.3, there exists $W_2 \in (K_n)_2$ such that

$$\widetilde{\delta}_{(j)}^{-1}\, t_{1,j}\, \widetilde{\delta}_{(j)} = \widetilde{\delta}_{(1)}\, t_{1,j}\, \widetilde{\delta}_{(1)}^{-1}\, W_2.$$

Then

$$\begin{aligned}
f_{j,k,\delta}\, f_{1,j,\gamma}\, f_{j,k,\delta}^{-1} &= \left(f_{j,k,\delta}\, \widetilde{\gamma}_{(1)}\, f_{j,k,\delta}^{-1}\right)\left(f_{j,k,\delta}\, t_{1,j}\, f_{j,k,\delta}^{-1}\right)\left(f_{j,k,\delta}\, \widetilde{\gamma}_{(1)}^{-1}\, f_{j,k,\delta}^{-1}\right) \\
&= W_1\, \left(\widetilde{\gamma}_{(1)}\, f_{j,k,\delta}\, t_{1,j}\, f_{j,k,\delta}^{-1}\, \widetilde{\gamma}_{(1)}^{-1}\right)\, W_1^{-1} \\
&\equiv \widetilde{\gamma}_{(1)}\, f_{j,k,\delta}\, t_{1,j}\, f_{j,k,\delta}^{-1}\, \widetilde{\gamma}_{(1)}^{-1} \quad (\mathrm{mod}\ (K_n)_3) \\
&= \widetilde{\gamma}_{(1)}\, \left(\widetilde{\delta}_{(j)}\, t_{j,k}\, \widetilde{\delta}_{(j)}^{-1}\right)\, t_{1,j}\, \left(\widetilde{\delta}_{(j)}\, t_{j,k}^{-1}\, \widetilde{\delta}_{(j)}^{-1}\right)\, \widetilde{\gamma}_{(1)}^{-1} \\
&= \widetilde{\gamma}_{(1)}\, \widetilde{\delta}_{(j)}\, t_{j,k}\, \widetilde{\delta}_{(1)}\, t_{1,j}\, \widetilde{\delta}_{(1)}^{-1}\, W_2\, t_{j,k}^{-1}\, \widetilde{\delta}_{(j)}^{-1}\, \widetilde{\gamma}_{(1)}^{-1} \\
&\equiv \widetilde{\gamma}_{(1)}\, \widetilde{\delta}_{(j)}\, \left(t_{j,k}\, \widetilde{\delta}_{(1)}\, t_{1,j}\, \widetilde{\delta}_{(1)}^{-1}\, t_{j,k}^{-1}\right)\, W_2\, \widetilde{\delta}_{(j)}^{-1}\, \widetilde{\gamma}_{(1)}^{-1} \quad (\mathrm{mod}\ (K_n)_3) \\
&= \widetilde{\gamma}_{(1)}\, \widetilde{\delta}_{(j)}\, \left[t_{j,k}\ ,\ \widetilde{\delta}_{(1)}\, t_{1,j}\, \widetilde{\delta}_{(1)}^{-1}\right]\left(\widetilde{\delta}_{(1)}\, t_{1,j}\, \widetilde{\delta}_{(1)}^{-1}\right)\, W_2\, \widetilde{\delta}_{(j)}^{-1}\, \widetilde{\gamma}_{(1)}^{-1} \\
&= \widetilde{\gamma}_{(1)}\, \widetilde{\delta}_{(j)}\, \left[t_{j,k}\ ,\ \widetilde{\delta}_{(1)}\, t_{1,j}\, \widetilde{\delta}_{(1)}^{-1}\right] \widetilde{\delta}_{(j)}^{-1}\, t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1}.
\end{aligned}$$

Since $\widetilde{\delta}_{(1)}$ commutes with $t_{j,k}$, it follows that

$$\begin{aligned}
f_{j,k,\delta}\, f_{1,j,\gamma}\, f_{j,k,\delta}^{-1} &\equiv \widetilde{\gamma}_{(1)}\, \widetilde{\delta}_{(j)}\, \widetilde{\delta}_{(1)}\, [t_{j,k}\ ,\ t_{1,j}]\, \widetilde{\delta}_{(1)}^{-1}\, \widetilde{\delta}_{(j)}^{-1}\, t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1} \qquad (\mathrm{mod}\ (K_n)_3) \\
&\equiv \widetilde{\gamma}_{(1)}\, \widetilde{\delta}_{(j)}\, \widetilde{\delta}_{(1)}\, [t_{1,j}\ ,\ t_{1,k}]\, \widetilde{\delta}_{(1)}^{-1}\, \widetilde{\delta}_{(j)}^{-1}\, t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1} \qquad (\mathrm{mod}\ (K_n)_3) \\
&\qquad\qquad (\text{by Lemma 3.6}) \\
&= \widetilde{\gamma}_{(1)}\, [\widetilde{\delta}_{(j)}, \widetilde{\delta}_{(1)}]\left(\widetilde{\delta}_{(1)}\widetilde{\delta}_{(j)}\ [t_{1,j}, t_{1,k}]\ \widetilde{\delta}_{(j)}^{-1}\widetilde{\delta}_{(1)}^{-1}\right)[\widetilde{\delta}_{(j)}, \widetilde{\delta}_{(1)}]^{-1}\, t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1}.
\end{aligned}$$

Notice that $[\widetilde{\delta}_{(j)}, \widetilde{\delta}_{(1)}] \in K_n$ and $[t_{1,j}, t_{1,k}] \in (K_n)_2$; thus

$$\begin{aligned} f_{j,k,\delta}\, f_{1,j,\gamma}\, f_{j,k,\delta}^{-1} &\equiv \widetilde{\gamma}_{(1)} \left(\widetilde{\delta}_{(1)}\widetilde{\delta}_{(j)}\ [t_{1,j}, t_{1,k}]\ \widetilde{\delta}_{(j)}^{-1}\widetilde{\delta}_{(1)}^{-1}\right) t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1} \qquad (\mathrm{mod}\ (K_n)_3) \\ &\equiv \widetilde{\gamma}_{(1)} \left[t_{1,j}\ ,\ \widetilde{\delta}_{(1)}\, t_{1,k}\, \widetilde{\delta}_{(1)}^{-1}\right] t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1} \qquad (\mathrm{mod}\ (K_n)_3) \\ &\qquad\qquad \text{(by Lemma 3.3 and Lemma 3.5)} \\ &= \left[\widetilde{\gamma}_{(1)}\, t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1}\ ,\ \widetilde{\gamma}_{(1)}\, \widetilde{\delta}_{(1)}\, t_{1,k}\, \widetilde{\delta}_{(1)}^{-1}\, \widetilde{\gamma}_{(1)}^{-1}\right] \left(\widetilde{\gamma}_{(1)}\, t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1}\right). \end{aligned}$$

Let $h = \widetilde{\gamma}_{(1)}\, \widetilde{\delta}_{(1)}\, \widetilde{\gamma\delta}_{(1)}^{-1}$. One has $h \in K_n$; thus

$$\begin{aligned} f_{j,k,\delta}\, f_{1,j,\gamma}\, f_{j,k,\delta}^{-1} &\equiv \left[\widetilde{\gamma}_{(1)}\, t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1}\ ,\ h\, \widetilde{(\gamma\delta)}_{(1)}\, t_{1,k}\, \widetilde{(\gamma\delta)}_{(1)}^{-1}\, h^{-1}\right] \left(\widetilde{\gamma}_{(1)}\, t_{1,j}\, \widetilde{\gamma}_{(1)}^{-1}\right) \\ &\qquad\qquad (\mathrm{mod}\ (K_n)_3) \\ &= \left[f_{1,j,\gamma}\ ,\ k\, f_{1,k,(\gamma\delta)}\, k^{-1}\right] f_{1,j,\gamma} \\ &\equiv \left[f_{1,j,\gamma}\ ,\ f_{1,k,(\gamma\delta)}\right] f_{1,j,\gamma} \qquad (\mathrm{mod}\ (K_n)_3). \end{aligned}$$

This proves that (R3) holds for $i = 1$.

For $j = 1$, (R3) holds because of the above case and (R1), since one has

$$[f_{i,1,\gamma}\ ,\ f_{1,k,\delta}] \equiv [f_{1,i,\gamma^{-1}}\ ,\ f_{1,k,\delta}] \equiv [f_{i,k,(\gamma\delta)}\ ,\ f_{1,i,\gamma^{-1}}] \equiv [f_{i,k,(\gamma\delta)}\ ,\ f_{i,1,\gamma}].$$

Finally, (R3) also holds for $k = 1$, since in this case

$$[f_{i,j,\gamma}\ ,\ f_{j,1,\delta}] \equiv [f_{j,i,\gamma^{-1}}\ ,\ f_{j,1,\delta}] \equiv [f_{i,1,(\gamma\delta)}\ ,\ f_{j,i,\gamma^{-1}}] \equiv [f_{i,1,(\gamma\delta)}\ ,\ f_{i,j,\gamma}]. \qquad \square$$

Proof of Proposition 3.2. It suffices to prove that the homomorphism $\psi_n : L_n \to \mathrm{gr}(K_n)$ of Lemma 3.7 is an isomorphism. We argue by induction on n.

For $n = 2$, $K_2 = F_2$ is a free group freely generated by $\mathcal{F}_{1,2} = \{f_{1,2,\gamma};\ \gamma \in \pi_1(M)\}$, so $\mathrm{gr}(K_2)$ is the free Lie algebra generated by $\mathcal{F}_{1,2}$. On the other hand, L_2 is by definition the free Lie algebra generated by $\{t_{1,2,\gamma};\ \gamma \in \pi_1(M)\}$. Therefore, ψ_2 is a Lie algebra isomorphism.

Suppose now that ψ_m is an isomorphism for $m < n$. Recall that $K_n = F_n \rtimes K_{n-1}$, and that we have the exact sequence

$$\begin{array}{ccccccccc} 1 \longrightarrow & F_n & \longrightarrow & K_n & \xrightarrow{\varrho} & K_{n-1} & \longrightarrow 1, \\ & f_{1,j,\gamma} & \longmapsto & f_{1,j,\gamma} & \longmapsto & 1, & \\ & & & f_{i+1,j+1,\gamma} & \longmapsto & f_{i,j,\gamma}. & \end{array}$$

Since K_{n-1} acts trivially on the abelianization of F_n, we can apply the result in [7] which claims that the associated graded sequence of Lie algebras is exact, that is,

$$1 \longrightarrow \mathrm{gr}(F_n) \xrightarrow{i} \mathrm{gr}(K_n) \xrightarrow{\mathrm{gr}\varrho} \mathrm{gr}(K_{n-1}) \longrightarrow 1,$$

where i is the natural inclusion. Besides, since $\mathrm{gr}(F_m)$ is a free Lie algebra for all $m \geq 2$, the above sequence shows, by induction, that $\mathrm{gr}(K_n)$ is a free $\mathbb{Z}$-module. This fact will be used later on.

Let us now define the following Lie algebra homomorphism:

$$\begin{array}{rccc} \widetilde{\varrho}_n : & L_n & \longrightarrow & L_{n-1}, \\ & t_{1,j,\gamma} & \longmapsto & 0, \\ & t_{i+1,j+1,\gamma} & \longmapsto & t_{i,j,\gamma}. \end{array}$$

Looking at the relations (L1), (L2) and (L3), we see that $\widetilde{\varrho}_n$ is a well-defined epimorphism of Lie algebras. We will denote $Q_n = \ker \widetilde{\varrho}_n$. In this way, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 1 \longrightarrow & \mathrm{gr}(F_n) & \stackrel{i}{\longrightarrow} & \mathrm{gr}(K_n) & \stackrel{\mathrm{gr}(\varrho)}{\longrightarrow} & \mathrm{gr}(K_{n-1}) & \longrightarrow 1 \\ & \uparrow \eta_n & & \uparrow \psi_n & & \uparrow \psi_{n-1} & \\ 1 \longrightarrow & Q_n & \stackrel{\widetilde{i}}{\longrightarrow} & L_n & \stackrel{\widetilde{\varrho}_n}{\longrightarrow} & L_{n-1} & \longrightarrow 1, \end{array}$$

where η_n is the restriction of ψ_n to Q_n.

Notice that $t_{1,j,\gamma} \in Q_n$ for all $j = 2, \dots, n$ and all $\gamma \in \pi_1(M)$. Notice as well that $\eta_n(t_{1,j,\gamma}) = f_{1,j,\gamma}$, and that $\mathrm{gr}(F_n)$ is the free Lie algebra generated by $\mathcal{F}_{1,n}$. Therefore, if we show that Q_n is generated (as a Lie algebra) by $\mathcal{B}_{1,n} = \{t_{1,j,\gamma};\ j = 2, \dots, n; \gamma \in \pi_1(M)\}$, then Q_n will be the free Lie algebra generated by $\mathcal{B}_{1,n}$, and η_n will be an isomorphism. In this case, since ψ_{n-1} is an isomorphism by the induction hypothesis, ψ_n will also be a Lie algebra isomorphism, as we want to show.

Let $l = \sum_{i=1}^k l_i$ be an element of Q_n, where each l_i is a Lie bracket over the generators of L_n. We can decompose $l = (\sum_{i=1}^r l_i) + (\sum_{i=r+1}^k l_i)$, where $\{l_1, \dots, l_r\}$ are the Lie brackets in which some $t_{1,j,\gamma}$ appears, and $\{l_{r+1}, \dots, l_k\}$ are Lie brackets over $\{t_{i,j,\gamma};\ 2 \le i < j \le n,\ \gamma \in \pi_1(M)\}$.

For all $i = 1, \dots, r$, $l_i \in Q_n$; hence $\sum_{i=r+1}^k l_i \in Q_n$. But if $\widetilde{\varrho}_n(\sum_{i=r+1}^k l_i) = 0$ in L_{n-1}, then $\sum_{i=r+1}^k l_i = 0$ in L_n, since the relations in L_{n-1} are the images by $\widetilde{\varrho}_n$ of the same relations in L_n which involve no $t_{1,j,\gamma}$. Therefore, $l = \sum_{i=1}^r l_i$, where each l_i contains some $t_{1,j,\gamma}$. We must then show that each l_i may be written as a sum of brackets over $\mathcal{B}_{1,n}$.

If l_i is a bracket of length 2, the result is a direct consequence of (L1), (L2) and (L3). Suppose that the result is true for brackets of length $d-1$, and consider $l_i = [a, b]$, a bracket of length $d > 2$. We can suppose that a contains some $t_{1,j,\gamma}$, and by induction, that it is a bracket over $\mathcal{B}_{1,n}$.

If $\mathrm{length}(a) \ge 2$, then $a = [a_1, a_2]$, where a_1, a_2 are brackets over $\mathcal{B}_{1,n}$. By the Jacoby identity,

$$l_i = [[a_1, a_2], b] = -[[a_2, b], a_1] - [[b, a_1], a_2],$$

where $[a_2, b]$ and $[b, a_1]$ can be written, by the induction hypothesis, as a sum of brackets over $\mathcal{B}_{1,n}$, so the result follows.

If $\mathrm{length}(a) = 1$, then $\mathrm{length}(b) \ge 2$, so $b = [b_1, b_2]$. Hence,

$$[a, [b_1, b_2]] = [b_1, [b_2, a]] - [b_2, [a, b_1]],$$

and we reduce to the previous case. Therefore, Q_n is generated by $\mathcal{B}_{1,n}$, and hence ψ_n is a Lie algebra isomorphism. □

3.3. **The isomorphism $\chi_2 : \mathcal{U}\mathbf{gr}(K_n) \to \mathbf{gr}_I\mathbb{Z}[K_n]$.** We start this subsection by stating a result due to Quillen.

Theorem 3.8 (Quillen [14]). *Let G be a group. Let $I = I(G)$ be the augmentation ideal of $\mathbb{Z}[G]$, let $\mathrm{gr}_I\mathbb{Z}[G]$ be the graded ring associated with the I-adic filtration, and let $G = G_1 \supset G_2 \supset \cdots \supset G_i \supset \cdots$ be the lower central series of G. Then the maps $\kappa_i : G_i \to I^i$, $g \mapsto g-1$, induce a surjective homomorphism $\kappa : \mathcal{U}\mathrm{gr}(G) \to \mathrm{gr}_I\mathbb{Z}[G]$ of $\mathbb{Z}$-algebras. Moreover, $\kappa \otimes \mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-algebras.* □

Notice that, if $\mathrm{gr}(G)$ is a free $\mathbb{Z}$-module, so is $\mathcal{U}\mathrm{gr}(G)$. Thus we have

Corollary 3.9. *If $\mathrm{gr}(G)$ is a free $\mathbb{Z}$-module, then the maps $\kappa_i : G_i \to I^i$, $g \mapsto g-1$, induce an isomorphism $\kappa : \mathcal{U}\mathrm{gr}(G) \to \mathrm{gr}_I\mathbb{Z}[G]$ of $\mathbb{Z}$-algebras.* □

Now, it is shown in the proof of Proposition 3.2 that $\mathrm{gr}(K_n)$ is a free $\mathbb{Z}$-module. So:

Proposition 3.10. *There is a well-defined isomorphism $\chi_2 : \mathcal{U}\mathrm{gr}(K_n) \to \mathrm{gr}_I\mathbb{Z}[K_n]$ which sends $f_{i,j,\gamma}$ to $f_{i,j,\gamma} - 1$, for all $i, j \in \{1, \dots, n\}$, $i \neq j$, and all $\gamma \in \pi_1(M)$.*

3.4. **$\mathbf{gr}v$ is the inverse of $\chi = \chi_2 \circ \chi_1$.** We have shown in Subsections 3.2 and 3.3 that there is a well-defined isomorphism $\chi = \chi_2 \circ \chi_1 : \mathcal{A}_n \longrightarrow \mathrm{gr}_I\mathbb{Z}[K_n]$ which sends $t_{i,j,\gamma}$ to $f_{i,j,\gamma} - 1$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$, and all $\gamma \in \pi_1(M)$. We now prove the following.

Proposition 3.11. *The homomorphism $\mathrm{gr}v$ is the inverse of χ. Hence it is an isomorphism of graded $\mathbb{Z}$-algebras.*

Proof. We only need to prove that $\mathrm{gr}v$ is the inverse of χ as a homomorphism of $\mathbb{Z}$-modules. For $d \geq 1$, let $\mathcal{A}_n^{(d)} = \widehat{\mathcal{A}}_n^{(\geq d)} / \widehat{\mathcal{A}}_n^{(\geq d+1)}$ be the submodule of $\mathcal{A}_n$ consisting of the homogeneous polynomials of degree d. Consider also $i, j, k, l \in \{1, \dots, n\}$, where $i < j$, $k < l$ and $i < k$. By relations (L1), (L2) and (L3), seen as relations in the enveloping algebra $\mathcal{A}_n$ of L_n, one has

$$t_{k,l,\delta} t_{i,j,\gamma} = \begin{cases} t_{i,j,\gamma}\, t_{k,l,\delta} & \text{if } i,j,k,l \text{ all distinct,} \\ t_{i,j,\gamma}\, t_{k,l,\delta} + t_{i,j,\gamma}\, t_{i,l,(\gamma\delta)} - t_{i,l,(\gamma\delta)}\, t_{i,j,\gamma} & \text{if } j = k, \\ t_{i,j,\gamma}\, t_{k,l,\delta} + t_{i,j,\gamma}\, t_{i,k,(\gamma\delta^{-1})} - t_{i,k,(\gamma\delta^{-1})}\, t_{i,j,\gamma} & \text{if } j = l. \end{cases}$$

Therefore, a set of generators for $\mathcal{A}_n^{(d)}$ as a $\mathbb{Z}$-module consists on the elements of the form

$$R = t_{i_1,j_1,\gamma_1}\, t_{i_2,j_2,\gamma_2} \cdots t_{i_d,j_d,\gamma_d},$$

where $i_1 \leq i_2 \leq \cdots \leq i_d$ and $i_k < j_k$ for all $k = 1, \dots, d$. But

$$\chi(R) = (f_{i_1,j_1,\gamma_1} - 1)(f_{i_2,j_2,\gamma_2} - 1) \cdots (f_{i_d,j_d,\gamma_d} - 1),$$

so, by definition of $\mathrm{gr}v$, and since $i_1 \leq i_2 \leq \cdots \leq i_d$, one has $\mathrm{gr}v(\chi(R)) = R$. This is true for all $d \geq 1$, so it follows that $\mathrm{gr}v \circ \chi = \mathrm{id}_{\mathcal{A}_n}$. Hence, since χ is an isomorphism, $\mathrm{gr}v$ is its inverse, as we wanted to show. □

This result implies the following.

Theorem 3.12. gr$u$ *is an isomorphism of* $\mathbb{Z}$*-modules.*

Proof. Recall that, by Proposition 2.2, the ideal $V_d = J^d$ of $\mathbb{Z}[B_n(M)]$ is isomorphic to $I_n^d \otimes \mathbb{Z}[H_n]$ via Φ, for all $d \geq 0$. Moreover, since $\mathbb{Z}[H_n]$ is a free $\mathbb{Z}$-module, one has

$$V_d/V_{d+1} \simeq \left(I_n^d/I_n^{d+1}\right) \otimes \mathbb{Z}[H_n].$$

Hence, $\mathrm{gr}_V \mathbb{Z}[B_n(M)] \simeq (\mathrm{gr}_I \mathbb{Z}[K_n]) \otimes \mathbb{Z}[H_n]$ via $\mathrm{gr}\Phi$. Now, $\mathrm{gr}u = (\mathrm{gr}v \otimes \mathrm{id}) \circ \mathrm{gr}\Phi$, and both $\mathrm{gr}\Phi$ and $\mathrm{gr}v \otimes \mathrm{id}$ are isomorphisms of $\mathbb{Z}$-modules. Thus $\mathrm{gr}u$ is an isomorphism of $\mathbb{Z}$-modules. □

3.5. **gr$u$ is a homomorphism.** In this subsection, we finish the proof of Theorem 1.3 by showing that $\mathrm{gr}u$ is a homomorphism.

We start by defining an algebra structure on $\mathrm{gr}_I \mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n]$. Consider the action of $B_n(M)$ on K_n by conjugation: an element $b \in B_n(M)$ sends $k \in K_n$ to $bkb^{-1} \in K_n$. This action extends naturally to $\mathbb{Z}[K_n]$ and preserves the I-adic filtration, so it defines an action of $B_n(M)$ on $\mathrm{gr}_I \mathbb{Z}[K_n]$. This action restricted to K_n becomes trivial, since if $k, k' \in K_n$, then

$$k(k'-1)k^{-1} = k\,k'k^{-1} - 1 = [k,k']\,k' - 1,$$

so, in $\mathrm{gr}_I \mathbb{Z}[K_n]$,

$$k(k'-1)k^{-1} \equiv ([k,k']-1)k' + (k'-1) \equiv (k'-1).$$

Therefore, the action induced on $\mathrm{gr}_I \mathbb{Z}[K_n]$ by an element $b \in B_n(M)$ depends only on $\varphi(b) \in H_n$. Recall the set map section $\sigma : H_n \to B_n(M)$. Now, define the product in $\mathrm{gr}_I \mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n]$ by

$$(k_1 \otimes \beta_1)(k_2 \otimes \beta_2) = (k_1\,\sigma(\beta_1)\,k_2\,\sigma(\beta_1)^{-1}) \otimes \beta_1\beta_2.$$

By the above discussion, this product does not depend on σ, and it endows $\mathrm{gr}_I \mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n]$ with a $\mathbb{Z}$-algebra structure.

Now, in order to prove that $\mathrm{gr}u = (\mathrm{gr}v \otimes \mathrm{id}) \circ \mathrm{gr}\Phi$ is a homomorphism of graded $\mathbb{Z}$-algebras, we turn to prove that both $\mathrm{gr}\Phi$ and $(\mathrm{gr}v \otimes \mathrm{id})$ are homomorphisms of graded $\mathbb{Z}$-algebras.

Lemma 3.13. $\mathrm{gr}\Phi : \mathrm{gr}_V \mathbb{Z}[B_n(M)] \to \mathrm{gr}_I \mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n]$ *is a homomorphism of graded* $\mathbb{Z}$*-algebras.*

Proof. Let $b_1, b_2 \in B_n(M)$. Write $\beta_i = \varphi(b_i)$ and $k_i = b_i(\sigma \circ \varphi)(b_i)^{-1}$ for $i = 1, 2$. Then

$$\mathrm{gr}\Phi(b_1)\,\mathrm{gr}\Phi(b_2) = (k_1 \otimes \beta_1)(k_2 \otimes \beta_2) = (k_1\,\sigma(\beta_1)\,k_2\,\sigma(\beta_1)^{-1}) \otimes \beta_1\beta_2,$$
$$\mathrm{gr}\Phi(b_1b_2) = (k_1\,\sigma(\beta_1)\,k_2\,\sigma(\beta_2)\,\sigma(\beta_1\beta_2)^{-1}) \otimes \beta_1\beta_2.$$

So, in order to prove that $\mathrm{gr}\Phi(b_1b_2) = \mathrm{gr}\Phi(b_1)\,\mathrm{gr}\Phi(b_2)$, it suffices to show that

$$\sigma(\beta_1)\,\sigma(\beta_2) \equiv \sigma(\beta_1\beta_2) \pmod{V_1}.$$

But $\varphi(\sigma(\beta_1)\,\sigma(\beta_2)) = \beta_1\beta_2 = \varphi(\sigma(\beta_1\beta_2))$, and thus there exists $k \in K_n$ such that $\sigma(\beta_1)\,\sigma(\beta_2) = k\,\sigma(\beta_1\beta_2)$ with $k \in K_n$. Hence, in $\mathbb{Z}[B_n(M)]$,

$$\sigma(\beta_1)\,\sigma(\beta_2) - \sigma(\beta_1\beta_2) = (k-1)\,\sigma(\beta_1\beta_2) \in V_1,$$

since $k - 1 \in V_1$, as we wanted to show. □

Lemma 3.14. $\mathrm{gr}v \otimes \mathrm{id} : \mathrm{gr}_I\mathbb{Z}[K_n] \otimes \mathbb{Z}[H_n] \to \mathcal{A}_n \rtimes \mathbb{Z}[H_n]$ *is a homomorphism of graded $\mathbb{Z}$-algebras.*

Proof. Write $g = \mathrm{gr}v$ and $g' = \mathrm{gr}v \otimes \mathrm{id} = g \otimes \mathrm{id}$, to simplify notation. Also, write $\beta_1' = \sigma(\beta_1)$. We know that g is a $\mathbb{Z}$-algebra isomorphism, so

$$\begin{aligned} g'((k_1 \otimes \beta_1)(k_2 \otimes \beta_2)) &= g'((k_1\,\beta_1'\,k_2\,\beta_1'^{-1}) \otimes \beta_1\beta_2) \\ &= g(k_1\,\beta_1'\,k_2\,\beta_1'^{-1}) \otimes \beta_1\beta_2 \\ &= g(k_1)\,g(\beta_1'\,k_2\,\beta_1'^{-1}) \otimes \beta_1\beta_2. \end{aligned}$$

On the other hand:

$$\begin{aligned} g'(k_1 \otimes \beta_1)\,g'(k_2 \otimes \beta_2) &= (g(k_1) \otimes \beta_1)\,(g(k_2) \otimes \beta_2) \\ &= g(k_1)\,(\beta_1\,g(k_2)\,\beta_1^{-1}) \otimes \beta_1\beta_2. \end{aligned}$$

Therefore, we need to show that, in $\mathcal{A}_n$,

$$g(\sigma(\beta_1)\,k_2\,\sigma(\beta_1)^{-1}) = \beta_1\,g(k_2)\,\beta_1^{-1}.$$

Since the action by conjugation does not depend on σ, we can assume that β_1 is a generator of H_n. Moreover, since g is a homomorphism of $\mathbb{Z}$-algebras, it suffices to verify the above formula when k_2 is a generator of $\mathrm{gr}_I\mathbb{Z}[K_n]$ as a $\mathbb{Z}$-algebra, that is, when $k_2 = f_{i,j,\gamma} - 1,\ i < j$. Hence, it suffices to prove Lemma 3.15 below. □

Lemma 3.15. *In $\mathrm{gr}_I\mathbb{Z}[K_n]$ the following relations hold, for all $i, j, k \in \{1, \ldots, n\}$ and all $\gamma \in \pi_1(M)$:*

- $\sigma_k\,f_{i,j,\gamma}\,\sigma_k^{-1} = f_{s_k(i),s_k(j),\gamma}$, *where s_k is the transposition $(k\ \ k+1)$,*
- $a_{k,r}\,f_{i,j,\gamma}\,a_{k,r}^{-1} = f_{i,j,\gamma}$, *if $k \neq i, j$,*
- $a_{i,r}\,f_{i,j,\gamma}\,a_{i,r}^{-1} = f_{i,j,(\omega_r\gamma)}$,

where $\{\sigma_1, \ldots, \sigma_{n-1}\}$ and $\{a_{i,r};\ 1 \le i \le n$ and $1 \le r \le 2g\}$ are the braids described in Subsection 2.1.

Proof. The first equation is a consequence of the following relations in $B_n(M)$, which are easily verified:

$$\sigma_k\,a_{i,r}\,\sigma_k^{-1} = \begin{cases} a_{i,r} & \text{if } k \neq i-1, i, \\ a_{i+1,r}\,t_{i,i+1}^{-1} & \text{if } k = i \quad \text{and } r \text{ is even}, \\ t_{i,i+1}\,a_{i+1,r} & \text{if } k = i \quad \text{and } r \text{ is odd}, \\ t_{i-1,i}\,a_{i-1,r} & \text{if } k = i-1 \quad \text{and } r \text{ is even}, \\ a_{i-1,r}\,t_{i-1,i}^{-1} & \text{if } k = i-1 \quad \text{and } r \text{ is odd}, \end{cases}$$

$$\sigma_k\,t_{i,j}\,\sigma_k^{-1} = \begin{cases} t_{i-1,j} & \text{if } k = i-1, \\ t_{i,i+1}\,t_{i+1,j}\,t_{i,i+1}^{-1} & \text{if } k = i, \\ t_{i,j-1} & \text{if } k = j-1, \\ t_{i,j}^{-1}\,t_{i,j+1}\,t_{i,j} & \text{if } k = j, \\ t_{i,j} & \text{otherwise.} \end{cases}$$

The second equation comes from Lemma 3.5, and from the following relations, where $i \neq k$ and we denote $b_{l,m} = a_{l,m}$ if m is odd, and $b_{l,m} = a_{l,m}^{-1}$ if m is even:

$$b_{k,r}\, b_{i,s}\, b_{k,r}^{-1} = \begin{cases} t_{i,k}^{-1}\, b_{i,s} & \text{if } s < r \quad \text{and } i < k, \\ b_{i,s}\, (b_{i,r}^{-1}\, t_{i,k}\, b_{i,r}) & \text{if } s > r \quad \text{and } i < k, \\ b_{i,s}\, (b_{i,r}^{-1}\, t_{k,i}^{-1}\, b_{i,r}) & \text{if } s < r \quad \text{and } i > k, \\ t_{k,i}\, b_{i,s} & \text{if } s > r \quad \text{and } i > k, \\ b_{i,s} & \text{if } s = r. \end{cases}$$

Indeed, in this case,

$$b_{k,r}\, f_{i,j,\gamma}\, b_{k,r}^{-1} \equiv b_{k,r}\, \widetilde{\gamma}_{(i)}\, t_{i,j}\, \widetilde{\gamma}_{(i)}^{-1}\, b_{k,r}^{-1} \equiv \widetilde{\gamma}_{(i)}\, b_{k,r}\, t_{i,j}\, b_{k,r}^{-1}\, \widetilde{\gamma}_{(i)}^{-1},$$

and by Lemma 3.5, this is equivalent to $f_{i,j,\gamma}$.

Finally, the third equation is verified as follows:

$$a_{i,r}\, f_{i,j,\gamma}\, a_{i,r}^{-1} \equiv a_{i,r}\, \widetilde{\gamma}_{(i)}\, t_{i,j}\, \widetilde{\gamma}_{(i)}^{-1}\, a_{i,r}^{-1} \equiv k\, \widetilde{(\omega_r \gamma)}_{(i)}\, t_{i,j}\, \widetilde{(\omega_r \gamma)}_{(i)}^{-1}\, k^{-1},$$

where $k \in K_n$, so this is equivalent to $f_{i,j,(\omega_r \gamma)}$. □

Acknowledgement

We are grateful to Ştefan Papadima for stimulating conversations and suggestions which were the starting point of this work.

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Departamento de Álgebra, Facultad de Matemáticas, Universidad de Sevilla, c/ Tarfia s/n, 41012 Sevilla, Spain
E-mail address: meneses@us.es

Université de Bourgogne, Laboratoire de Topologie, UMR 5584 du CNRS, B.P. 47870, 21078 - Dijon Cedex, France
E-mail address: lparis@u-bourgogne.fr