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# THE TWO-BY-TWO SPECTRAL NEVANLINNA-PICK PROBLEM

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ABSTRACT. We give a criterion for the existence of an analytic $2 \times 2$ matrix-valued function on the disc satisfying a finite set of interpolation conditions and having spectral radius bounded by 1. We also give a realization theorem for analytic functions from the disc to the symmetrised bidisc.

## INTRODUCTION

The problem of the title is the following:

*Given distinct points $\lambda_1, \dots, \lambda_n$ in the open unit disc $\mathbb{D}$ and $2 \times 2$ matrices $W_1, \dots, W_n, n \geq 1$, find conditions for the existence of an analytic $2 \times 2$ matrix function $F$ on $\mathbb{D}$ such that*

$$F(\lambda_j) = W_j, \qquad j = 1, 2, \dots, n,$$

*and*

$$r(F(\lambda)) \leq 1 \quad \text{for all } \lambda \in \mathbb{D}.$$

Here $r(\ )$ denotes the spectral radius of a matrix. The problem has attracted much attention over the past 20 years [6, 7, 8, 16, 13, 15] partly because it is a challenging variant of a well-loved classical topic, but mainly because it is a test case of a fundamental question which arises in $H^\infty$ control, the *problem of $\mu$-synthesis* [10, 11]. A solution of the general problem would have applications to the design of automatic controllers that are robust with respect to structured uncertainty.

There are computational approaches to $\mu$-synthesis, for example, a Matlab toolbox [14], but not yet an analytic solution. There is accordingly good reason to analyse even very special cases of the problem, both to provide tests of existing software and to throw light on the complexities of the general problem. In this paper we establish a necessary and sufficient condition for the existence of a solution in the case of an arbitrary finite number of interpolation points. There are slight subtleties in the case that some of the target matrices $W_j$ are scalar (see the end of Section 2 below); the generic case is the following.

**Main Theorem 0.1.** *Let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$ for some $n \in \mathbb{N}$ and let $W_1, \dots, W_n$ be $2 \times 2$ matrices, none of them a scalar multiple of the identity. The following two statements are equivalent:*

(1) *there exists an analytic $(2 \times 2)$-matrix function $F$ on $\mathbb{D}$ such that $F(\lambda_j) = W_j,\ 1 \leq j \leq n$, and $r(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$;*

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(2) *there exist* $b_1, \dots, b_n, c_1, \dots, c_n \in \mathbb{C}$ *such that*

$$\left[\frac{I - \begin{bmatrix} \frac{1}{2}s_i & b_i \\ c_i & -\frac{1}{2}s_i \end{bmatrix}^* \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}}{1 - \overline{\lambda}_i \lambda_j}\right]_{i,j=1}^n \geq 0 \tag{0.1}$$

*where*

$$s_j = \text{tr } W_j, \quad p_j = \det W_j$$

*and*

$$b_j c_j = p_j - \frac{s_j^2}{4}, \quad 1 \leq j \leq n. \tag{0.2}$$

The proof of this theorem is in Section 2. The theorem also enables us to treat the nongeneric case, in which some of the $W_j$ are scalar matrices; see the end of Section 2 below.

Can the Main Theorem be regarded as a solution of the two-by-two spectral Nevanlinna-Pick problem? It gives a criterion for existence which is nontrivial to check, but which can be attacked by standard optimization techniques. Note that if $b_j, c_j$ do exist satisfying the Pick-type condition (0.1), then necessarily (from consideration of the block-diagonal entries)

$$\left\| \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix} \right\| \leq 1,$$

and so $|b_j| \leq 1,\ |c_j| \leq 1$. Hence, in view of equation (0.2),

$$\left| p_j - \frac{s_j^2}{4} \right| \leq |b_j|, |c_j| \leq 1.$$

Thus the checking of the criterion (0.1) involves in principle a search over a compact set in $\mathbb{C}^{2n}$. The Pick-type inequality (0.1) is equivalent to the relation $\|T(b,c)\| \leq 1$, where $T(b,c)$ is a certain linear operator on a $2n$-dimensional Hilbert space and is such that $T(b,c)$ depends *linearly* on $(b,c) = (b_1, \dots, b_n, c_1, \dots, c_n)$. The set $C$ of all $2n$-tuples $(b,c)$ such that the Pick inequality (0.1) holds is therefore convex; it is expresssible in the form

$$\{(b_1, \dots, b_n, c_1, \dots, c_n) : \|T(b,c)\| \leq 1\}.$$

Thus $C$ is a compact convex set in $\Delta^{2n}$, where $\Delta$ denotes the closed unit disc. The Main Theorem tells us that the given spectral Nevanlinna-Pick problem has a solution if and only if $C$ meets the set

$$\{(b_1, \dots, b_n, c_1, \dots, c_n) \in \mathbb{C}^{2n} : b_j c_j = p_j - \frac{s_j^2}{4}, j = 1, \dots, n\}.$$

This is an essentially non-convex problem, but in view of its simple structure there is hope that it may be amenable to numerical solution.

The spectral Nevanlinna-Pick problem is alluringly simple in its formulation, but far-reaching investigations by Bercovici, Foiaş, and Tannenbaum [6, 7, 8] reveal that it contains a great deal of subtlety. For example, one of the thrusts of their work is to show by some ingenious examples [6] that the idea of diagonalisation, or "interpolating the eigenvalues", does not resolve the problem. They obtain qualitative results on extremals, and they use a variant of commutant lifting theory to prove a theorem [6, Theorem 4] that contains the following result, and thereby they provide in principle a method of determining whether an interpolating function with the desired property exists.

*Let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $W_1, \dots, W_n$ be $N \times N$ matrices. There exists an analytic $N \times N$ matrix-valued function $F$ such that $F(\lambda_j) = W_j$, $j = 1, \dots, n$, and $r(F(\lambda)) \le 1$ for all $\lambda \in \mathbb{D}$ if and only if, for all $\varepsilon > 0$, there exist invertible $N \times N$ matrices $P_1, \dots, P_n$ such that*

$$\left[ \frac{(1+\varepsilon)P_i P_j^* - W_i P_i P_j^* W_j^*}{1 - \lambda_i \bar{\lambda}_j} \right]_{i,j=1}^{n} \ge 0.$$

This criterion is similar in some ways to that of the Main Theorem; it would be interesting to compare the effectiveness of practical algorithms based on the two criteria, since both require a nontrivial search process.

In this paper we address the $2 \times 2$ case of the problem by trying to interpolate the coefficients in the characteristic polynomials of the $W_j$ rather than their eigenvalues. We are accordingly led to study the set

$$\Gamma \overset{\text{def}}{=} \{(\text{tr } A, \det A) : A \text{ is } 2 \times 2,\ r(A) \le 1\}.$$

We have investigated the operator theory, function theory and geometry of this set in several papers (see references in [2, 3]), but the present paper is self-contained. Although operator theory led us to the present results (and, in particular, to the linear fractional functions described in Lemma 1.2 below), it barely appears explicitly in this paper.

An alternative viewpoint currently being investigated by Ransford and others (e.g., [15]) utilises the theory of algebroid analytic multifunctions. This approach is roughly equivalent to the formulation in terms of the symmetrised polydisc, but it suggests other questions and methods.

One way to view the interpolation problem (2) in the Main Theorem is as a variant of the much-studied bitangential matricial Nevanlinna-Pick problem [4, Chapter 18]. In addition to the usual bitangential interpolation conditions, the determinant of the matricial function $F$ is specified at each interpolation point.

A second result of the paper is a realization formula for analytic functions from $\mathbb{D}$ to the symmetrised bidisc (Corollary 1.4).

We denote the unit circle by $\mathbb{T}$. For $\alpha, \lambda \in \Delta$ we write

$$B_\alpha(\lambda) = \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda}.$$

If $\alpha \in \mathbb{T}$, then $B_\alpha$ is constant; otherwise $B_\alpha$ is an automorphism of $\mathbb{D}$ with inverse $B_{-\alpha}$. The *Schur class* of operator-valued or matricial functions (of a given type) is the set of analytic operator- or matrix-valued functions $F$ on $\mathbb{D}$ bounded by 1 in norm, that is, satisfying

$$||F(\lambda)|| \le 1 \qquad \text{for all } \lambda \in \mathbb{D},$$

where $||\cdot||$ denotes the operator norm. We shall need the Realization Theorem (e.g., [18, Chapter VI.3]) for functions of the Schur class. If, for some Hilbert spaces $H, U$ and $Y$,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : H \oplus U \to H \oplus Y \tag{0.3}$$

is a contractive operator, then, for any $z \in \mathbb{D}$,

$$\|D + Cz(1 - Az)^{-1}B\| \le 1.$$

Conversely, any function in the Schur class (of functions from $\mathbb{D}$ to the space of bounded linear operators from $U$ to $Y$) has such a representation, in which the block operator matrix (0.3) is a unitary operator from $H \oplus U$ to $H \oplus Y$.

By *Nevanlinna-Pick data* we mean a finite set $\lambda_1, \dots, \lambda_n$ of distinct points in $\mathbb{D}$, where $n \in \mathbb{N}$, and an equal number of "target matrices" $W_1, \dots, W_n$, of type (say) $m \times k$. We write these data

$$\lambda_j \mapsto W_j, \qquad 1 \le j \le n. \tag{0.4}$$

We say that these data are *solvable* if there exists a function in the Schur class such that $F(\lambda_j) = W_j,\ 1 \le j \le n$. By the classical theorem of Pick, or more precisely its extension to matricial data, e.g. [4], the Nevanlinna-Pick problem with data (0.4) is solvable if and only if the "Pick matrix"

$$\left[\frac{I_k - W_i^* W_j}{1 - \overline{\lambda}_i \lambda_j}\right]_{i,j=1}^n$$

is positive.

## 1. The symmetrised bidisc

The *symmetrised bidisc* is defined to be the set

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| \le 1, |z_2| \le 1\} \subset \mathbb{C}^2.$$

The relevance of this set to the spectral Nevanlinna-Pick problem lies in the fact that a $2 \times 2$ matrix $A$ has both eigenvalues in $\Delta$ if and only if $(\operatorname{tr} A, \det A) \in \Gamma$. The following simple result essentially reduces the original problem to one of interpolation from $\mathbb{D}$ to $\Gamma$.

**Theorem 1.1.** *Let $\lambda_1, \dots, \lambda_n$ be distinct in $\mathbb{D}$ and let $W_1, \dots, W_n$ be $2 \times 2$ matrices. Suppose that either all or none of $W_1, \dots, W_n$ are scalar matrices. The following statements are equivalent:*

(1) *there exists an analytic $2 \times 2$ matrix function $F$ in $\mathbb{D}$ such that $F(\lambda_j) = W_j, j = 1, 2, \dots, n$ and $r(F(\lambda)) \le 1$ for all $\lambda \in \mathbb{D}$;*
(2) *there exists an analytic function $f : \mathbb{D} \to \Gamma$ such that $f(\lambda_j) = (\operatorname{tr} W_j, \det W_j), j = 1, 2, \dots, n$.*

*Proof.* This is practically the same as that given in [3, Theorem 2.1] for the case $n = 2$. The forward implication follows by choice of

$$f(\lambda) = (\operatorname{tr} F(\lambda), \det F(\lambda)), \qquad \lambda \in \mathbb{D}.$$

For the converse, there are two cases. If none of the $W_j$ are scalar, there exist matrices $L_1, \dots, L_n$ such that

$$W_j = e^{-L_j} \begin{bmatrix} 0 & 1 \\ -p_j & s_j \end{bmatrix} e^{L_j}$$

where $s_j = \operatorname{tr} W_j,\ p_j = \det W_j$. If (2) holds, then we may define

$$F(\lambda) = e^{-L(\lambda)} \begin{bmatrix} 0 & 1 \\ -f_2(\lambda) & f_1(\lambda) \end{bmatrix} e^{L(\lambda)}$$

where $L$ is a polynomial matrix such that $L(\lambda_j) = L_j, 1 \le j \le n$. This $F$ has the desired properties, and so (2) implies (1). If all the $W_j$ are scalar matrices, we may simply let

$$F(\lambda) = \tfrac{1}{2} f_1(\lambda) I,$$

and again, (1) is satisfied. □

We need a characterization of points of $\Gamma$.

**Lemma 1.2.** *For $s, p \in \mathbb{C}$, the following conditions are equivalent:*

(1) $(s,p) \in \Gamma$;
(2) $|s| \le 2$ *and, for all* $z \in \mathbb{D}$,

$$\left| \frac{2zp - s}{2 - zs} \right| \le 1.$$

*Proof.* It may be verified by expansion that, for any $w_1, w_2 \in \mathbb{C}$,

$$\begin{aligned} &|2 - w_1 - w_2|^2 - |2w_1w_2 - w_1 - w_2|^2 \\ &\quad = \quad 2(1 - |w_1|^2)|1 - w_2|^2 + 2(1 - |w_2|^2)|1 - w_1|^2. \end{aligned}$$

Consider any $(s,p) \in \Gamma$, and write

$$s = \zeta_1 + \zeta_2, \qquad p = \zeta_1\zeta_2$$

with $\zeta_1, \zeta_2 \in \Delta$. Certainly $|s| \le 2$. If $\zeta_1 = \zeta_2 = \zeta \in \mathbb{T}$, then

$$\left| \frac{2zp - s}{2 - zs} \right| = |-\zeta| = 1$$

and so (2) holds. Otherwise $|s| < 2$. Pick $z \in \mathbb{T}$ and let $w_1 = z\zeta_1, w_2 = z\zeta_2$ in the above identity to obtain

$$\begin{aligned} &|2 - zs|^2 - |2z^2p - zs|^2 \\ &\quad = \quad 2(1 - |\zeta_1|^2)|1 - z\zeta_2|^2 + 2(1 - |\zeta_2|^2)|1 - z\zeta_1|^2 \ge 0. \end{aligned}$$

Thus,

$$\left| \frac{2zp - s}{2 - zs} \right| = \left| \frac{2z^2p - zs}{2 - zs} \right| \le 1$$

for all $z \in \mathbb{T}$ and hence also, by the Maximum Principle, for all $z \in \mathbb{D}$. Thus (1) implies (2).

Conversely, suppose that (2) holds. For all $z \in \mathbb{D}$ we have

$$\left| \frac{2z^2p - zs}{2 - zs} \right| < 1. \tag{1.1}$$

Choose $r \in (0,1)$ and let $(s_1, p_1) = (rs, r^2p)$: we shall show that $(s_1, p_1)$ lies in the interior of $\Gamma$ using a criterion of Schur. For any $\omega \in \mathbb{T}$ we may substitute $z = r\omega$ in the inequality (1.1) to obtain

$$\left| \frac{2\omega^2 p_1 - \omega s_1}{2 - \omega s_1} \right| < 1,$$

which implies

$$|2 - \omega s_1|^2 - |2\omega^2 p_1 - \omega s_1|^2 > 0,$$

which expands to give

$$4\{1 - |p_1|^2 - \mathrm{Re}(\omega(s_1 - \overline{s}_1 p_1))\} > 0.$$

Since this holds for all $\omega \in \mathbb{T}$ we have

$$|s_1 - \overline{s}_1 p_1| < 1 - |p_1|^2,$$

which relation can also be expressed as

$$\begin{bmatrix} 1-|p_1|^2 & s_1 - \overline{s}_1 p_1 \\ \overline{s}_1 - s_1\overline{p}_1 & 1-|p_1|^2 \end{bmatrix} > 0.$$

A well-known result of Schur [17] asserts that this is precisely the condition that all zeros of the polynomial $x^2 - s_1 x + p_1$ lie in $\mathbb{D}$ or, in other words, that

$$s_1 = z_1 + z_2, \qquad p_1 = z_1 z_2$$

for some $z_1, z_2 \in \mathbb{D}$. Hence $(rs, r^2 p) \in \Gamma$ for $r \in (0,1)$, and since $\Gamma$ is closed, we find on letting $r \to 1$ that $(s,p) \in \Gamma$. Hence (2) implies (1). □

The next result relates the property of mapping $\mathbb{D}$ analytically to $\Gamma$ and membership of the Schur class.

**Theorem 1.3.** *For any function*

$$\varphi = (s,p) : \mathbb{D} \to \mathbb{C}^2,$$

*the following three statements are equivalent:*

(1) *$\varphi$ is analytic and maps $\mathbb{D}$ into $\Gamma$;*
(2) *there exists an analytic $(2\times 2)$-matrix function $\psi = [\psi_{ij}]$ on $\mathbb{D}$ such that $\|\psi\|_\infty \le 1$, $\operatorname{tr}\,\psi = 0$ identically on $\mathbb{D}$ and $\varphi = (2\psi_{11}, -\det\psi)$;*
(3) *there exists an analytic $(2\times 2)$-matrix function $\chi = [\chi_{ij}]$ on $\mathbb{D}$ such that $\|\chi\|_\infty \le 1$ and $\varphi = (\chi_{11} - \chi_{22}, \frac{1}{4}\operatorname{tr}^2\,\chi - \det\chi)$.*

*Proof.* $(2 \Rightarrow 1)$ Suppose $\psi$ satisfies (2). Consider a fixed $\lambda \in \mathbb{D}$. Since $[\psi_{ij}]$ is contractive, the Realization Theorem tells us that

$$|\psi_{22}(\lambda) + \psi_{21}(\lambda) z (1 - \psi_{11}(\lambda) z)^{-1} \psi_{12}(\lambda)| \le 1$$

for all $z \in \mathbb{D}$. That is,

$$\Big| - \frac{s(\lambda)}{2} + \psi_{21}(\lambda)\psi_{12}(\lambda) \frac{z}{1 - \frac{s(\lambda)}{2} z} \Big| \le 1.$$

Note that

$$\begin{aligned} \psi_{21}\psi_{12} &= -\det\psi + \psi_{11}\psi_{22} \\ &= p - \frac{s^2}{4}, \end{aligned}$$

so that the last inequality becomes

$$\Big| - \frac{s(\lambda)}{2} + \frac{(4p(\lambda) - s(\lambda)^2) z}{4(1 - \frac{s(\lambda)}{2} z)} \Big| \le 1,$$

or equivalently,

$$\Big| \frac{2p(\lambda) z - s(\lambda)}{2 - s(\lambda) z} \Big| \le 1.$$

Since this holds for all $z \in \mathbb{D}$, and clearly $|s(\lambda)| \le 2$, it follows from Lemma 1.2 that $(s(\lambda), p(\lambda)) \in \Gamma$. Thus $\varphi$ maps $\mathbb{D}$ into $\Gamma$. It is immediate from the relations

$$s = 2\psi_{11}, \qquad p = -\det\psi$$

that $\varphi$ is analytic in $\mathbb{D}$.

$(1 \Rightarrow 2)$ Suppose that $\varphi : \mathbb{D} \to \Gamma$ is analytic. Then $p - s^2/4 \in H^\infty$, and so by a theorem of F. Riesz, which follows easily from inner-outer factorization [12], there exist functions $f_1, f_2 \in H^\infty$ such that $f_1 f_2 = p - s^2/4$ and

$$|f_1(\lambda)| = |f_2(\lambda)| \qquad \text{a.e. on } \mathbb{T}.$$

For $\lambda \in \mathbb{D}$ let

$$\psi(\lambda) = \begin{bmatrix} \frac{1}{2}s(\lambda) & f_1(\lambda) \\ f_2(\lambda) & -\frac{1}{2}s(\lambda) \end{bmatrix}.$$

Then $\psi$ is analytic on $\mathbb{D}$ and

$$\mathrm{tr}\ \psi = 0, \qquad s = 2\psi_{11}, \qquad p = -\det \psi.$$

By Fatou's Theorem [12], we may extend $\psi$ to almost every point of $\mathbb{T}$ by taking radial limits. Then, at almost every point $\lambda \in \mathbb{T}$, we have

$$|f_1(\lambda)|^2 = |f_2(\lambda)|^2 = |f_1(\lambda) f_2(\lambda)| = |p(\lambda) - s(\lambda)^2/4|$$

and

$$\begin{aligned} 1 - \psi^*\psi &= 1 - \begin{bmatrix} \frac{1}{2}\overline{s} & \overline{f}_2 \\ \overline{f}_1 & -\frac{1}{2}\overline{s} \end{bmatrix} \begin{bmatrix} \frac{1}{2}s & f_1 \\ f_2 & -\frac{1}{2}s \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 4 - |s|^2 - |s^2 - 4p| & -2\overline{s}f_1 + 2s\overline{f}_2 \\ -2s\overline{f}_1 + 2\overline{s}f_2 & 4 - |s|^2 - |s^2 - 4p| \end{bmatrix}. \end{aligned}$$

The diagonal entries of the matrix on the right-hand side are nonnegative, for suppose $s(\lambda) = z_1 + z_2, p(\lambda) = z_1 z_2$ where $z_1, z_2 \in \Delta$. Then

$$\begin{aligned} 4 - |s|^2 - |s^2 - 4p| &= 4 - |z_1 + z_2|^2 - |z_1 - z_2|^2 \\ &= 4 - 2|z_1|^2 - 2|z_2|^2 \\ &\geq 0. \end{aligned} \tag{1.2}$$

Furthermore, for almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned} |-2\overline{s}f_1 + 2s\overline{f}_2|^2 &= 8\{|s|^2|p - \frac{s^2}{4}| - \mathrm{Re}(\overline{s}^2 f_1 f_2)\} \\ &= 2\{|s|^2|s^2 - 4p| + \mathrm{Re}(\overline{s}^2(s^2 - 4p))\}. \end{aligned}$$

Hence, for almost all $\lambda \in \mathbb{T}$, we have

$$\begin{aligned} 16\det(1 - \psi^*\psi) &= (4 - |s|^2 - |s^2 - 4p|)^2 - 2|s|^2|s^2 - 4p| - 2\mathrm{Re}(\overline{s}^2(s^2 - 4p)) \\ &= 16 + |s|^4 + |s^2 - 4p|^2 - 8|s|^2 - 8|s^2 - 4p| - 2\mathrm{Re}(\overline{s}^2(s^2 - 4p)). \end{aligned}$$

Note that

$$|s|^4 + |s^2 - 4p|^2 - 2\mathrm{Re}(\overline{s}^2(s^2 - 4p)) = |s^2 - (s^2 - 4p)|^2 = 16|p|^2.$$

Thus, for almost all $\lambda \in \mathbb{T}$,

$$16\det(1 - \psi^*\psi) = 16 + 16|p|^2 - 8|s|^2 - 8|s^2 - 4p|. \tag{1.3}$$

Since $(s, p)$ maps $\mathbb{D}$ into $\Gamma$, by continuity, $(s(\lambda), p(\lambda))$ can be written as $(z_1 + z_2, z_1 z_2)$ for some $z_1, z_2 \in \Delta$, and we have

$$\begin{aligned} 16\det(1 - \psi^*\psi) &= 16 + 16|z_1 z_2|^2 - 8|z_1 + z_2|^2 - 8|z_1 - z_2|^2 \\ &= 16(1 + |z_1 z_2|^2 - |z_1|^2 - |z_2|^2) \\ &= 16(1 - |z_1|^2)(1 - |z_2|^2) \geq 0. \end{aligned} \tag{1.4}$$

The inequalities (1.2) and (1.4) show that

$$1 - \psi(\lambda)^*\psi(\lambda) \geq 0$$

and hence

$$\|\psi(\lambda)\| \leq 1$$

for almost all $\lambda \in \mathbb{T}$, and hence also for all $\lambda \in \mathbb{D}$.

($2 \Rightarrow 3$) This implication is trivial, since we may take $\chi = \psi$.

($3 \Rightarrow 2$) Suppose that $\chi$ with the stated properties exists. Let

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and let

$$\psi = \tfrac{1}{2}\{\chi + J\chi^T J\}$$

where the superscript $T$ denotes transposition. Observe that if $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\tfrac{1}{2}\{X + JX^T J\} = \begin{bmatrix} \frac{1}{2}(a-d) & b \\ c & \frac{1}{2}(d-a) \end{bmatrix}.$$

Thus tr $\psi = 0$ and

$$2\psi_{11} = \chi_{11} - \chi_{22} = s.$$

Since $\chi$ belongs to the Schur class, the same is true of $\psi$. Furthermore,

$$\begin{aligned} \det \psi &= -\tfrac{1}{4}(\chi_{11} - \chi_{22})^2 - \chi_{12}\chi_{21} \\ &= -\tfrac{1}{4}(\chi_{11} + \chi_{22})^2 + \chi_{11}\chi_{22} - \chi_{12}\chi_{21} \\ &= -\tfrac{1}{4}\mathrm{tr}^2\ \chi + \det \chi \\ &= -p. \end{aligned}$$

□

The foregoing theorem allows us to give a solution to the two-dimensional case of a problem raised in [5]: to find a realization formula for analytic functions from $\mathbb{D}$ to the symmetrised polydisc. The 2-by-2 matrix functions $\psi$ and $\chi$ appearing in conditions (2) and (3) of Theorem 1.3 belong to the Schur class, and can therefore be realised, as in the Realization Theorem. It will be convenient to use some standard engineering notation. If $H, U$ and $Y$ are Hilbert spaces and

$$A : H \to H, \qquad B : U \to H,$$

$$C : H \to Y, \qquad D : U \to Y$$

are bounded linear operators, then we define the operator

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](z) = D + Cz(1 - Az)^{-1}B : U \to Y$$

whenever $1 - Az$ is invertible.

**Corollary 1.4.** *A function*

$$\varphi = (s, p) : \mathbb{D} \to \mathbb{C}^2$$

*maps* $\mathbb{D}$ *analytically into* $\Gamma$ *if and only if there exist a Hilbert space* $H$ *and a unitary operator*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : H \oplus \mathbb{C}^2 \to H \oplus \mathbb{C}^2 \tag{1.5}$$

*such that*

$$s = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right] - \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right] \tag{1.6}$$

*and*

$$p = \left(\frac{1}{4}\mathrm{tr}^2 \ - \det\right) \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{1.7}$$

*where* $B = [B_1 \ \ B_2] : \mathbb{C}^2 \to H$*,*

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} : H \to \mathbb{C}^2$$

*and* $D = [D_{ij}]_{i,j=1}^2$*. Furthermore, for any analytic function* $\varphi : \mathbb{D} \to \Gamma$*, one may choose* $H$*,* $A$*,* $B$*,* $C$ *and* $D$ *in the realization* (1.5) *so that*

$$\mathrm{tr} \ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = 0,$$

*and hence*

$$s = 2\left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right] = -2\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right] \quad \textit{and} \quad p = -\det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

*Proof.* Given the analytic function $\varphi : \mathbb{D} \to \Gamma$, choose $\psi$ as in Theorem 1.3, so that $\psi$ is in the Schur class, tr $\psi = 0$ and

$$\varphi = (2\psi_{11}, -\det\psi) = (\psi_{11} - \psi_{22}, -\det\psi).$$

By the Realization Theorem, there exist a Hilbert space $H$ and a unitary operator $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ on $H \oplus \mathbb{C}^2$ such that, for all $\lambda \in \mathbb{D}$,

$$\begin{aligned} \psi(\lambda) &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\lambda) \\ &= D + C\lambda(1 - A\lambda)^{-1}B \\ &= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \lambda(1 - A\lambda)^{-1}[B_1 \ B_2]. \end{aligned}$$

Thus,

$$\psi_{11} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right], \qquad \psi_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$$

and so equation (1.6) holds. Equation (1.7) holds by choice of $\psi$ since tr $\psi = 0$.

Conversely, if $H, A, B, C$ and $D$ are as described, let

$$\chi = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = [\chi_{ij}],$$

so that $\|\chi\|_\infty \le 1$ by the Realization Theorem. We have

$$\chi_{jj} = \left[\begin{array}{c|c} A & B_j \\ \hline C_j & D_{jj} \end{array}\right], \qquad j = 1, 2,$$

and so

$$s = \chi_{11} - \chi_{22}.$$

Furthermore, by hypothesis,

$$p = (\tfrac{1}{4}\mathrm{tr}^2 \ - \det)\chi.$$

Hence, by Theorem 1.3, $\varphi = (s,p)$ maps $\mathbb{D}$ analytically into $\Gamma$. □

Realizations of analytic functions from $\mathbb{D}$ to $\Gamma$ are explored further in a paper to be published [1].

We conclude this section with an observation about $\Gamma$-inner functions; this is not essential to the remainder of the paper. A $\Gamma$-*inner function* is an analytic function $\varphi : \mathbb{D} \to \Gamma$ for which almost all radial limits $\varphi(\mathrm{e}^{\mathrm{i}\theta})$, $\theta \in \mathbb{R}$, lie in the distinguished boundary $b\Gamma$ of $\Gamma$ (defined to be the Šilov boundary of the algebra of continuous functions on $\Gamma$ that are analytic on the interior of $\Gamma$). $b\Gamma$ is the symmetrisation of the 2-torus:

$$b\Gamma = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{T}\} = \{(s,p) \in \mathbb{C}^2 : |s| \le 2, |p| = 1, s = \bar{s}p\}.$$

Examples of $\Gamma$-inner functions are a) the symmetrisation of a pair of inner functions, $s = h_1 + h_2$, $p = h_1 h_2$ for some inner $h_1$, $h_2$, and b) the function

$$(s,p)(z) = (r(1+z), z), \qquad z \in \mathbb{D}$$

for any $r \in (0,1)$. In Example b) we have

$$(s^2 - 4p)(z) = r^2 z^2 + 2(r^2 - 2)z + r^2,$$

a quadratic that has a simple zero in $\mathbb{D}$, so that $s^2 - 4p$ does not have an analytic square root in $\mathbb{D}$. It follows that $(s,p)$ is not the symmetrisation of a pair of inner functions (so that Examples a) and b) are mutually exclusive).

**Theorem 1.5.** *A function* $\varphi : \mathbb{D} \to \mathbb{C}^2$ *is* $\Gamma$-*inner if and only if there exists a* $2 \times 2$ *matricial inner function* $\psi$ *such that* $\mathrm{tr}\ \psi = 0$ *and*

$$\varphi = (2\psi_{11}, -\det\psi).$$

*Proof.* Suppose there is such an inner function $\psi$. For almost all $\lambda \in \mathbb{T}$, $\psi(\lambda)$ is unitary, and hence the same is true of the matrix

$$U = \psi(\lambda)\mathrm{diag}\{1, -1\} = \begin{bmatrix} \psi_{11}(\lambda) & -\psi_{12}(\lambda) \\ \psi_{21}(\lambda) & -\psi_{22}(\lambda) \end{bmatrix}.$$

It follows that the two eigenvalues of $U$ lie in $\mathbb{T}$, and so $(\mathrm{tr}\ U, \det U) \in b\Gamma$. Hence, since $\psi_{11} + \psi_{22} = 0$,

$$\varphi(\lambda) = (2\psi_{11}(\lambda), -\det\psi(\lambda)) = (\mathrm{tr}\ U, \det U) \in b\Gamma.$$

By Theorem 1.3, $\varphi$ maps $\mathbb{D}$ analytically into $\Gamma$. Thus $\varphi$ is $\Gamma$-inner.

Conversely, suppose that $\varphi$ is a $\Gamma$-inner function. Construct the $2 \times 2$ Schur function $\psi$ exactly as in the proof of Theorem 1.3. Then $\varphi = (2\psi_{11}, -\det\psi)$, $\mathrm{tr}\ \psi = 0$ and it follows from the relations (1.2), (1.3) and (1.4) that

$$1 - \psi(\lambda)^*\psi(\lambda) = 0 \text{ for almost all } \lambda \in \mathbb{T}.$$

Thus $\psi$ is inner. □

## 2. The Nevanlinna-Pick problem for $\Gamma$

The effect of Theorem 1.3 is to reduce the problem of analytic interpolation $\mathbb{D} \to \Gamma$ to a standard classical matricial Nevanlinna-Pick problem.

**Theorem 2.1.** *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ for some $n \in \mathbb{N}$ and let $(s_j, p_j) \in \Gamma$ for $j = 1, \ldots, n$. There exists an analytic function $\varphi : \mathbb{D} \to \Gamma$ such that*

$$\varphi(\lambda_j) = (s_j, p_j), \qquad 1 \le j \le n,$$

*if and only if there exist $b_j, c_j \in \mathbb{C}$ such that*

$$b_j c_j = p_j - \frac{s_j^2}{4}, \qquad 1 \le j \le n, \tag{2.1}$$

*and the conditions*

$$\lambda_j \mapsto \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}, \qquad 1 \le j \le n, \tag{2.2}$$

*comprise solvable matricial Nevanlinna-Pick data.*

*Proof.* $(\Rightarrow)$ Suppose $\varphi$ as described exists. By Theorem 1.3, there is an analytic $2\times 2$ matrix function $\psi$ on $\mathbb{D}$ such that $\|\psi\|_\infty \le 1$, $\operatorname{tr} \psi \equiv 0, s = 2\psi_{11}$ and $p = -\det \psi$ (where $\varphi = (s, p)$). Choose

$$b_j = \psi_{12}(\lambda_j), \quad c_j = \psi_{21}(\lambda_j), \quad 1 \le j \le n.$$

Then

$$\psi(\lambda_j) = \begin{bmatrix} \frac{1}{2}s(\lambda_j) & \psi_{12}(\lambda_j) \\ \psi_{21}(\lambda_j) & -\frac{1}{2}s(\lambda_j) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}$$

and

$$p_j = p(\lambda_j) = -\det \psi(\lambda_j) = \frac{s_j^2}{4} + b_j c_j.$$

Thus the equations (2.1) are satisfied, and for this choice of $b_j, c_j$, the matricial Nevanlinna-Pick problem with data (2.2) is indeed solvable, since $\psi$ is a solution of it.

$(\Leftarrow)$ Suppose $b_j, c_j$ can be found such that the equations (2.1) hold and the matricial Nevanlinna-Pick data (2.2) are solvable, with solution $\chi = [\chi_{ij}]$.

Thus $\chi$ is a $2 \times 2$ Schur function, and

$$\chi(\lambda_j) = \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}.$$

Define functions $s, p$ by

$$\begin{aligned} s &= \chi_{11} - \chi_{22}, \\ p &= (\tfrac{1}{4}\operatorname{tr}^2 - \det)\chi, \end{aligned}$$

and let $\varphi = (s, p)$. By Theorem 1.3, $\varphi$ maps $\mathbb{D}$ analytically to $\Gamma$, and we have

$$\begin{aligned} s(\lambda_j) &= \chi_{11}(\lambda_j) - \chi_{22}(\lambda_j) = s_j, \\ p(\lambda_j) &= \tfrac{1}{4}s_j^2 + b_j c_j = p_j. \end{aligned}$$

□

*Proof of the Main Theorem.* Let $\lambda_j, W_j, 1 \le j \le n$, be given as described, so that no $W_j$ is a scalar matrix. By Theorem 1.1, statement (1) (the existence of the desired analytic matrix function $F$) is equivalent to the existence of an analytic function $\varphi : \mathbb{D} \to \Gamma$ such that $\varphi(\lambda_j) = (s_j, p_j), 1 \le j \le n$, where $s_j = \operatorname{tr} W_j, p_j = \det W_j$. By Theorem 2.1, this in turn is equivalent to the solvability of the matricial Nevanlinna-Pick problem with data

$$\lambda_j \mapsto \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}, \qquad 1 \le j \le n,$$

for some $b_j, c_j \in \mathbb{C}$ such that

$$b_j c_j = p_j - \frac{s_j^2}{4}, \qquad 1 \le j \le n.$$

The matricial version of Pick's Theorem [4, Chapter 18] tells us that the last problem is solvable if and only if the Pick condition (0.1) is satisfied. □

The proof indicates a way of constructing solutions of a spectral Nevanlinna-Pick problem. Suppose we are given $\lambda_1, \dots, \lambda_n, W_1, \dots, W_n$ such that condition (2) of the Main Theorem is satisfied, and we can somehow find suitable numbers $b_1, \dots, b_n, c_1, \dots, c_n$ to make the Pick matrix positive. There are various ways of constructing Schur functions $\chi$ such that

$$\chi(\lambda_j) = \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}, \qquad j = 1, \dots, n$$

(see, for example, [4]). The function

$$\varphi = (\chi_{11} - \chi_{22}, \tfrac{1}{4}\operatorname{tr}^2 \chi - \det \chi)$$

is then an analytic function from $\mathbb{D}$ to $\Gamma$ such that

$$\varphi(\lambda_j) = (\operatorname{tr} W_j, \det W_j), \qquad j = 1, \dots, n,$$

and the proof of Theorem 1.1 shows how we can use $\varphi$ to construct an analytic $2 \times 2$ matrix-valued function $F$ in $\mathbb{D}$ such that $F(\lambda_j) = W_j$ and $r(F(\lambda)) \le 1$ for all $\lambda \in \mathbb{D}$.

What if some of the target data $W_j$ happen to be scalar multiples of the identity matrix? The Main Theorem will not then apply, but the problem may still be readily solved by a combination of the present results and a well-known reduction technique due to Schur (e.g., [19] for the scalar case). Schur reduction permits the successive removal of the interpolation conditions corresponding to scalar target matrices, leading either to an obvious inconsistency (in which case there is no solution) or to an interpolation problem to which the Main Theorem applies.

The problem with mixed (scalar and non-scalar) target matrices can also be related to interpolation from $\mathbb{D}$ to $\Gamma$ with conditions on a derivative [3].

## Added in proof

The genericity hypothesis in the Main Theorem is removed by H. Bercovici in his preprint "Spectral versus classical Nevanlinna-Pick interpolation in dimension two".

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