

## CHERN NUMBERS OF AMPLE VECTOR BUNDLES ON TORIC SURFACES

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ABSTRACT. This article shows a number of strong inequalities that hold for the Chern numbers  $c_1^2$ ,  $c_2$  of any ample vector bundle  $\mathcal{E}$  of rank  $r$  on a smooth toric projective surface,  $S$ , whose topological Euler characteristic is  $e(S)$ . One general lower bound for  $c_1^2$  proven in this article has leading term  $(4r + 2)e(S) \ln_2\left(\frac{e(S)}{12}\right)$ . Using Bogomolov instability, strong lower bounds for  $c_2$  are also given. Using the new inequalities, the exceptions to the lower bounds  $c_1^2 > 4e(S)$  and  $c_2 > e(S)$  are classified.

### INTRODUCTION

Let  $\mathcal{E}$  be an ample rank  $r$  bundle on a smooth toric projective surface  $S$ , whose topological Euler characteristic is  $e(S)$ . In this article, we prove a number of surprisingly strong lower bounds for  $c_1(\mathcal{E})^2$  and  $c_2(\mathcal{E})$ .

First, we show Corollary 3.2, which says that, given  $S$  and  $\mathcal{E}$  as above, if  $e(S) \geq 5$ , then  $c_1(\mathcal{E})^2 \geq r^2e(S)$ . Though simple, this is much stronger than the known lower bounds over not necessarily toric surfaces. For example, see [BSS94, Lemma 2.2], where it is shown that there are many rank two ample vector bundles with  $(c_1(\mathcal{E})^2, c_2(\mathcal{E})) = (4, 1)$  on products of two smooth curves, at least one of which has positive genus.

We then prove an estimate, Theorem 3.6, which is quite strong for large  $e(S)$  and  $r$ . As  $e(S)$  goes to  $\infty$  with  $r$  fixed, the leading term of this lower bound is  $(4r + 2)e(S) \ln_2(e(S)/12)$ , while if  $e$  is fixed and  $r$  goes to  $\infty$ , the leading term of this lower bound is  $3(e(S) - 4)r^2$ . For example,  $c_1^2(\mathcal{E}) \geq 3r^2e(S)$ , for  $r \leq 3$  if  $e(S) \geq 13$ , or for  $r \leq 6$  if  $e(S) \geq 19$ , or for  $r \leq 141$  if  $e(S) \geq 100$ . Or again,  $c_1^2(\mathcal{E}) \geq 5r^2e(S)$ , for  $r \leq 10$  if  $e(S) \geq 100$ . We include a three-line Maple program in Remark 3.7 for plotting the expression for the lower bound.

The strategy is to use the adjunction process to find lower bounds for  $c_1(\mathcal{E})^2$ . Toric geometry has two major implications for the adjunction process. First, given an ample rank  $r$  vector bundle  $\mathcal{E}$  on a smooth toric surface  $S$ , there is the inequality  $-\det \mathcal{E} \cdot K_S \geq e(S)(\text{rank } \mathcal{E})$ . Adjunction theory yields the lower bound for  $c_1(\mathcal{E})^2$  given in Corollary 3.2, which implies that  $c_1(\mathcal{E})^2 > r^2e(S)$  for  $e(S) \geq 7$ . The second important fact is that  $h^0(tK_S + \det \mathcal{E}) > 0$  for integers  $t$  between 0 and at least  $\text{rank } \mathcal{E} + \ln_2(e(S)/6)$ . Adjunction theory yields the strong lower bound given in Theorem 3.6 for  $c_1(\mathcal{E})^2$  when  $e(S) \geq 7$ .

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Using Bogomolov's instability theorem, we get the strong lower bound given in Theorem 3.9 for the second Chern class,  $c_2(\mathcal{E})$ , of a rank two ample vector bundle. Basically if  $c_2(\mathcal{E})$  is less than one fourth the lower bound already derived for  $c_1(\mathcal{E})^2$ , then we have an unstable bundle, and Bogomolov's instability theorem combined with the Hodge index theorem gives strong enough conditions to get a contradiction. The short list of exceptions to the bound  $c_2(\mathcal{E}) > e(S)$  are classified. Even assuming  $\mathcal{E}$  very ample on a nontoric surface, the best general result [BSS96] shows only that  $c_2(\mathcal{E}) \geq 1$  with equality for  $\mathbb{P}^2$ .

Inequalities derived from adjunction theory usually have the form, "some inequality is true if certain projective invariants are large enough." Typically, examples exist outside the range where the adjunction-theoretic method works. For rank two ample vector bundles  $\mathcal{E}$  we use a variety of special methods, including adjunction theory and Bogomolov's instability theorem, to enumerate the exceptions to either the inequality  $c_1(\mathcal{E})^2 \geq 4e(S)$  or the inequality  $c_2(\mathcal{E}) \geq e(S)$  holding. The exceptions are collected in Table 1.

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## 1. BACKGROUND MATERIAL

In this paper we work over  $\mathbb{C}$ . By a variety we mean a complex analytic space, which might be neither reduced or irreducible.

A rank 2 vector bundle  $\mathcal{E}$  on a nonsingular surface  $S$  is called *Bogomolov unstable* [R78], or *unstable* for short, if  $c_1(\mathcal{E})^2 > 4c_2(\mathcal{E})$ . When  $\mathcal{E}$  is unstable there exist a line bundle  $\mathcal{A}$  and a zero subscheme  $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$  fitting in the exact sequence

$$(1) \quad 0 \rightarrow \mathcal{A} \rightarrow \mathcal{E} \rightarrow (\det \mathcal{E} - \mathcal{A}) \otimes \mathcal{I}_{\mathcal{Z}} \rightarrow 0,$$

with the property that for all ample line bundles  $\mathcal{L}$  on  $S$ ,  $(2\mathcal{A} - \det \mathcal{E}) \cdot \mathcal{L} > 0$ . The standard consequences of this result that we will often use in this article are:

- (1)  $(2\mathcal{A} - \det \mathcal{E}) \cdot (2\mathcal{A} - \det \mathcal{E}) > 0$ , and  $2\mathcal{A} - \det \mathcal{E}$  is  $\mathbb{Q}$ -effective; and
- (2) for all nef and big line bundles  $\mathcal{L}$  on  $S$ ,  $(2\mathcal{A} - \det \mathcal{E}) \cdot \mathcal{L} > 0$ .

We define  $\mathcal{H} := \det \mathcal{E}$ . Note that

- $c_2(\mathcal{E}) = \mathcal{A} \cdot (\mathcal{H} - \mathcal{A}) + \deg(\mathcal{Z})$ , where  $\deg(\mathcal{Z}) = h^0(\mathcal{O}_{\mathcal{Z}})$ ; and
- the line bundle  $\mathcal{H} - \mathcal{A}$  is a quotient of  $\mathcal{E}$  off a codimension two subset, and therefore it is ample when  $\mathcal{E}$  is ample.

Using the Hodge inequality  $(\mathcal{H} - \mathcal{A})^2 (2\mathcal{A} - \mathcal{H})^2 \leq [(\mathcal{H} - \mathcal{A}) \cdot (2\mathcal{A} - \mathcal{H})]^2$ , we obtain the following:

$$(2) \quad \mathcal{A} \cdot (\mathcal{H} - \mathcal{A}) \geq (\mathcal{H} - \mathcal{A})^2 + \sqrt{(\mathcal{H} - \mathcal{A})^2}.$$

A toric surface  $S$  is a surface containing a two-dimensional torus as Zariski open subset and such that the action of the torus on itself extends to  $S$ . All toric surfaces are normal. In this article we consider surfaces polarized by an ample vector bundle; therefore  $S$  will always denote a normal projective toric surface. For basic definitions on toric varieties we refer to [O88].

We recall that if  $e := e(S)$  is the Euler characteristics of  $S$ , then  $\text{rank}(\text{Pic}(S)) = e - 2$  and  $K_S^2 = 12 - e$ .

We need the following useful lemmas, which are probably well known.

**Lemma 1.1.** *Let  $\mathcal{E}$  be a vector bundle over a normal  $n$ -dimensional toric variety. Assume  $\mathbb{P}(\mathcal{E})$  is toric. Then  $\mathcal{E} = \bigoplus L_i$ , where the  $L_i$  are equivariant line bundles.*

*Proof.* Consider the bundle map  $\mathbb{P}(\mathcal{E}) \rightarrow X$  with fiber  $F = \mathbb{P}^{r-1}$ , where  $r := \text{rank}(\mathcal{E})$ . Every fiber has  $r$  fixed points which define an unramified  $r$ -to-one cover of  $X$ ,  $p : Y \rightarrow X$ .  $X$  being a normal toric variety, and thus simply connected, implies  $Y = \bigcup X_i$  and  $\mathcal{E} = \bigoplus L_i$ . □

It is classical [L82], [RV] that a surjective morphism  $p : X \rightarrow Y$ , with connected fibers between normal projective varieties, induces a homomorphism from the connected component of the identity of the automorphism group of  $X$  to the connected component of the identity of the automorphism group of  $Y$ , with respect to which  $p$  is equivariant. Using this basic fact, we have the following lemma.

**Lemma 1.2.** *Let  $p : X \rightarrow Y$  be a surjective morphism with connected fibers from a normal toric variety  $X$  onto a normal variety  $Y$ . Then  $Y$  admits the structure of a toric variety such that  $p$  becomes a toric morphism.*

**Corollary 1.3.** *Let  $L$  be an ample line bundle on a smooth projective toric surface  $S$ . If  $f : S \rightarrow \mathbb{P}^1$  is a morphism with connected fibers, then the general fiber  $F$  is isomorphic to  $\mathbb{P}^1$ , there are at most two singular fibers, and  $e(S) \leq 2 + 2L \cdot F$ .*

*Proof.* Since the general fiber is toric, it is isomorphic to  $\mathbb{P}^1$ . From equivariance we see that any singular fiber must lie over the two fixed points of  $\mathbb{P}^1$ . Since there are at most  $L \cdot F$  irreducible components in a fiber, and there are at most two singular fibers, the inequality follows by considering the cases of no, one, or two singular fibers. □

**Corollary 1.4.** *Let  $f : S \rightarrow S'$  express a smooth toric surface  $S$  as the equivariant blowup of a smooth projective toric surface  $S'$  at a finite set  $B$ . Then  $e(S) \leq 2e(S')$ .*

*Proof.* Let  $b := e(B)$ , i.e.,  $b$  equals the cardinality of the finite set  $B$ . Then we have  $e(S) = e(S') + b$ . Since  $S'$  is toric and the elements of  $B$  are fixed points of the toric action, we conclude that  $e(B)$  is bounded by the cardinality of the set of toric fixed points on  $S'$ , which is equal the Euler characteristic of  $S'$ . Thus we have  $e(S) = e(S') + b \leq 2e(S')$ . □

Let  $S$  be an irreducible toric surface. Then under the prescribed torus action there are  $e := e(S)$  one-dimensional orbits. Denote their closures by  $D_i$  where  $1 \leq i \leq e$ . We have the fundamental fact that

$$(3) \quad -K_S = \sum_{i=1}^{e(S)} D_i.$$

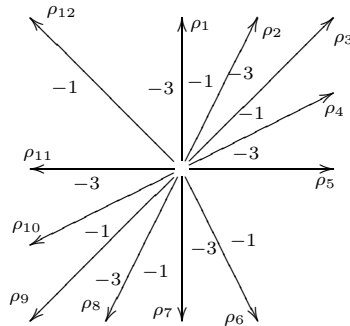
We begin with a very simple observation which is in fact an important tool in all our main results:

**Lemma 1.5.** *Let  $\mathcal{E}$  be an ample rank  $r$  vector bundle on a projective normal toric surface  $S$ , and let  $\mathcal{H}$  denote  $\det \mathcal{E}$ . Then  $-K_S \cdot \mathcal{H} \geq re(S)$ .*

*Proof.* Let  $\mathcal{H} := \det \mathcal{E} = \sum_1^e a_i D_i$ . By ampleness,  $\mathcal{H} \cdot D_i \geq r$  for all  $i = 1, \dots, e$ . Since  $K_S = \sum_1^e (-D_i)$ , we have  $-K_S \cdot \mathcal{H} = \sum_1^e \mathcal{H} \cdot D_i \geq er$ .  $\square$

*Remark 1.6.* In order to obtain the results in this paper we use the bound 1.5 for  $-KL$ . The following example shows that in general we cannot hope for a better bound.

Consider the toric surface given by the fan below, spanned by 12 edges  $\{\rho_i\}$  and with 12 2-cones, i.e., 12 fixed points. The number before each edge indicates the self-intersection of the associated invariant divisor  $D_i$ .



This surface is the equivariant blowup of  $\mathbb{P}^2$  in 9 points, and thus the Euler characteristic  $e(S) = 12$ . Consider the line bundle

$$L = 3D_1 + 5D_2 + 3D_3 + 5D_4 + 3D_5 + 5D_6 + 3D_7 + 5D_8 + 3D_9 + 3D_{10} + 3D_{11} + 5D_{12}.$$

It is ample, since  $L \cdot D_i = 5 - 9 + 5 = 1$  for  $i = 1, 3, 5, 7, 9, 11$  and  $L \cdot D_i = 3 - 5 + 3 = 1$  for  $i = 2, 4, 6, 8, 10, 12$ . This also gives

$$-LK_S = \sum_1^{12} L \cdot D_i = 12 = e.$$

Clearly this example can be generalized to higher values of  $e$ .

We end with a simple corollary of Lemma 1.5.

**Corollary 1.7.** *Let  $\mathcal{E}$  be an ample rank  $r$  vector bundle on a smooth projective toric surface  $S$ , and let  $c_1^2 := c_1(\mathcal{E})^2$ . If  $c_1^2 \leq re(S)$ , then  $r \leq 3$  and either  $g(\det \mathcal{E}) = 0$ , and  $(S, \mathcal{E})$  is*

- (1)  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$  with  $(c_1^2, e) = (1, 3)$ ; or
- (2)  $(\mathbb{F}_0, aE + bf)$  with  $1 \leq ab \leq 2$  and  $(c_1^2, e) = (2ab, 4)$ ; or
- (3)  $(\mathbb{F}_1, E + 2f)$  with  $(c_1^2, e) = (3, 4)$ ; or
- (4)  $(\mathbb{F}_2, E + 3f)$  with  $(c_1^2, e) = (4, 4)$ ; or
- (5)  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$  with  $(c_1^2, e) = (4, 3)$ ;

or  $g(\mathcal{H}) = 1$ , and  $(S, \mathcal{E})$  is

- (1)  $(S, -K_S)$  with  $(c_1^2, e) = (6, 6)$ ; or
- (2)  $(\mathbb{F}_0, (E + f) \oplus (E + f))$  with  $(c_1^2, e) = (8, 4)$ ; or
- (3)  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$  with  $(c_1^2, e) = (9, 3)$ .

*Proof.* Let  $\mathcal{H} := \det \mathcal{E}$ . If  $\mathcal{H}^2 \leq re$ , then from  $K_S \cdot \mathcal{H} \leq -re$  we conclude that  $2g(\mathcal{H}) - 2 = \mathcal{H}^2 + K_S \cdot \mathcal{H} \leq 0$ , and thus that  $g(\mathcal{H}) \leq 1$ .

If  $g(\mathcal{H}) = 0$ , we know from classification theory, e.g., [BS95], [F90], that  $S$  is  $\mathbb{P}^2$  or  $\mathbb{F}_r$ . A simple calculation shows the listed examples are the only ones possible.

If  $g(\mathcal{H}) = 1$ , then from classification theory, e.g., [BS95], [F90], we know that  $(S, \mathcal{H})$  is either a scroll over an elliptic curve or a del Pezzo surface with  $\mathcal{H} = -K_S$ . Since  $S$  is toric and therefore rational,  $S$  is del Pezzo.  $\square$

2. VECTOR BUNDLES OVER  $\mathbb{P}^2$  AND  $\mathbb{F}_e$

In this section we describe all pairs  $(S, \mathcal{E})$  where  $\mathcal{E}$  is an ample rank two bundle on a  $\mathbb{P}^2$  or a Hirzebruch surface, with the property that either  $c_1(\mathcal{E})^2 \leq 4e(S)$  or  $c_2(\mathcal{E}) \leq e(S)$ . Later in the paper it will be shown that these are all of the examples of rank 2 ample vector bundles  $\mathcal{E}$  on smooth toric surfaces  $S$  with either  $c_1(\mathcal{E})^2 \leq 4e(s)$  or  $c_2(\mathcal{E}) \leq e(S)$ . The following table includes the various cases. We give the Chern classes and indicate whether the bundle is Bogomolov unstable ( $U$ ), stable ( $S$ ), or a boundary case, i.e.,  $c_1^2 = 4c_2$ , ( $B$ ).

TABLE 1. All pairs  $(S, \mathcal{E})$ , with  $\mathcal{E}$  an ample rank two vector bundle on a smooth toric projective surface  $S$ , and with either  $c_1(\mathcal{E})^2 \leq 4e(S)$  or  $c_2(\mathcal{E}) \leq e(S)$ . The only class where we do not know existence and uniqueness is listed on the last line of the table.

$S$	$e(S)$	$\mathcal{E}$	$c_1(\mathcal{E})^2$	$c_2(\mathcal{E})$	$U/S/B$
$\mathbb{P}^2$	3	$\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$	4	1	$B$
$\mathbb{P}^2$	3	$\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$	9	2	$U$
$\mathbb{P}^2$	3	$T_{\mathbb{P}^2}$	9	3	$S$
$\mathbb{P}^2$	3	$\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$	16	3	$U$
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \xi$	8	2	$B$
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \xi$	12	3	$B$
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3)) \otimes \xi$	16	4	$B$
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \xi$	16	4	$B$
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)$	18	4	$U$
$\mathbb{F}_1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \xi$	12	3	$B$
$\mathbb{F}_1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \xi$	16	4	$B$
$\mathbb{F}_2$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \xi$	16	4	$B$
del Pezzo	6	$(-K_S) \oplus (-K_S)$	24	6	$B$
del Pezzo	6	if any example exists, $\det \mathcal{E} = -2K_S$	24	$\geq 7$	$S$

Fix the notation  $c_2 := c_2(\mathcal{E})$ ,  $\mathcal{H} := c_1 = \det \mathcal{E}$ , and  $e := e(S)$ . The strategy that we follow is to first classify the pairs with  $c_1(\mathcal{E})^2 \leq 4e(S)$ . Then any pair  $(S, \mathcal{E})$  with  $c_2 \leq e$  has already been enumerated, or we have  $c_2 \leq e < 4c_1^2$ . In the latter case the bundle is unstable and we use the extra relations arising from Bogomolov’s instability theorem to classify the pair.

2.1.  $\mathbb{P}^2$ . Let  $\mathcal{E}$  be a rank two ample vector bundle over  $\mathbb{P}^2$ . Since  $\mathcal{H}$  is the determinant bundle of a rank two bundle,  $\deg(\mathcal{H}|_\ell) \geq 2$  for every line  $\ell \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ . It follows that  $\mathcal{H} = \mathcal{O}_{\mathbb{P}^2}(a)$  with  $a \geq 2$ . If  $\mathcal{H}^2 \leq 4e = 12$ , then  $a = 2, 3$ . In case  $a = 2$ , the restriction of  $\mathcal{E}$  to each line  $\ell$  is  $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ , and thus by the classical results on uniform bundles [OSS80],  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ . In case  $a = 3$ , the restriction

of  $\mathcal{E}$  to each line  $\ell$  is  $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ , and thus by the classical results on uniform bundles [OSS80],  $\mathcal{E} = \boxed{\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)}$  or  $\mathcal{E} = \boxed{T_{\mathbb{P}^2}}$ , the tangent bundle of  $\mathbb{P}^2$ .

Now assume that  $c_2(\mathcal{E}) \leq 3$ , but  $c_1^2 > 4e = 12$ . Thus it follows that  $\mathcal{H} = \mathcal{O}_{\mathbb{P}^2}(a)$  with  $a \geq 4$ . Since  $\mathcal{E}$  is unstable, we have a sequence as in (1) where  $\mathcal{H} - \mathcal{A} = \mathcal{O}_{\mathbb{P}^2}(x)$  and  $\mathcal{A} = \mathcal{O}_{\mathbb{P}^2}(x + b)$  for  $x, b > 0$ . The inequalities  $3 \geq c_2(\mathcal{E}) = x(x + b) + \deg(\mathcal{Z})$  and  $a = 2x + b \geq 4$  yield the only numerical possibility:  $(x, b + x) = (1, 3)$  and  $\deg(\mathcal{Z}) = 0$ . Since  $H^1(\mathbb{P}^2, 2\mathcal{A} - \mathcal{H}) = 0$ , we conclude that the exact sequence splits, and it follows that  $\mathcal{E} = \boxed{\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3)}$ .

**2.2. The Hirzebruch surfaces  $\mathbb{F}_\epsilon$ .** Let  $\mathbb{F}_\epsilon = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\epsilon))$  be the Hirzebruch surface of degree  $\epsilon$ . Denote by  $p : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\epsilon)) \rightarrow \mathbb{P}^1$  the projection map, and let  $F$  denote a fiber of  $p$ . Let  $\xi_{\mathcal{E}}$  denote the tautological line bundle on  $\mathbb{F}_\epsilon$ , so that  $p_*\xi_{\mathcal{E}} \cong \mathcal{E}$ . Recall that  $\text{Pic}(\mathbb{F}_\epsilon) = \mathbb{Z}F \oplus \mathbb{Z}E$ , where  $E$  is the section corresponding to the surjection  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\epsilon) \rightarrow \mathcal{O}_{\mathbb{P}^1}$ . Note that  $E^2 = -\epsilon$ .

The following is useful.

**Lemma 2.1.** *Let  $\mathcal{E}$  be a rank  $r$  ample vector bundle on  $\mathbb{F}_\epsilon$ . Then  $\det \mathcal{E} \cdot F \geq r$ , with equality if and only if  $\mathcal{E} \cong p^*V \otimes \xi_{\mathcal{E}}$ , where  $V \cong \mathcal{E}_E$ . In particular, in this case*

$$c_1(\mathcal{E})^2 = r^2\epsilon + 2r \det \mathcal{E} \cdot E \geq r^2(2 + \epsilon),$$

and

$$c_2(\mathcal{E}) = \binom{r}{2}\epsilon + (r - 1) \det \mathcal{E} \cdot E \geq \binom{r}{2}(2 + \epsilon).$$

*Proof.* Since  $\mathcal{E}$  is a rank  $r$  ample vector bundle, and  $F$  is a smooth rational curve, we conclude that  $\det \mathcal{E} \cdot F \geq r$ , with equality if and only if  $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ . In this case we have that  $\mathcal{E} \otimes \xi_{\mathcal{E}}^*$  is trivial on every fiber, and thus  $\mathcal{E} \otimes \xi_{\mathcal{E}}^* \cong p^*V$  for some rank  $r$  vector bundle on  $\mathbb{P}^1$ . Finally, note that  $V \cong (p^*V)_E \cong \mathcal{E}_E$ . The rest of the lemma is a straightforward calculation.  $\square$

We record one simple corollary of the above lemma.

**Corollary 2.2.** *Let  $\mathcal{E}$  be a rank  $r$  ample vector bundle on  $\mathbb{F}_\epsilon$ . If  $\epsilon \geq 2$  and  $c_1(\mathcal{E})^2 \leq 4r^2$ , then  $\epsilon = 2$  and  $\mathcal{E} \cong p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \xi_{\mathcal{E}}$ . In this case  $c_1(\mathcal{E})^2 = 4r^2$  and  $c_2(\mathcal{E}) = 2r(r - 1)$ .*

*Proof.* Let  $\mathcal{H} := \det \mathcal{E} = aE + bF$ . Using Lemma 2.1, we only need to show that  $a = \mathcal{H} \cdot F = r$ . Assume therefore that  $a \geq r + 1$ . Then we have  $\mathcal{H}^2 \geq a(2b - a\epsilon) \geq (r + 1)(2r + (r + 1)\epsilon) > 4r^2$ .

Now assume that  $c_2 \leq e = 4$  or  $c_1^2 \leq 4e = 16$ , and  $\mathcal{E}|_F = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$  with  $a, b > 0$ .

**Case I:** First consider the case when  $(a, b) = (1, 1)$ . We are in the situation of Lemma 2.1. Letting  $V = \mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$ , we get

$$4 \geq c_2(\mathcal{E}) = c_2(p^*(V) \otimes \xi) = \xi^2 + \alpha + \beta = \epsilon + \alpha + \beta,$$

or

$$12 \geq c_1^2 = c_1(p^*(V) \otimes \xi)^2 = 4\xi^2 + 4\alpha + 4\beta = 4(\epsilon + \alpha + \beta);$$

The only possible numerical possibilities are  $\mathcal{E} = \boxed{p^*(\mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)) \otimes \xi}$  with  $(\epsilon, \alpha, \beta) = (0, 1, 1), (0, 1, 2), (0, 1, 3), (0, 2, 2), (1, 1, 1), (1, 1, 2), (2, 1, 1)$ .

**Case II:** Assume now that  $(a, b) \neq (1, 1)$ . First, let us consider the case  $\epsilon = 0$ . Then  $\mathcal{H}_F = \det(\mathcal{E})|_F = \mathcal{O}_{\mathbb{P}^1}(a + b)$  implies  $c_1^2 \geq 18 > 4e(S)$ . Thus if  $c_2 \leq e = 4$ ,

then  $c_1^2 \geq 4c_2(\mathcal{E})$ , which means  $\mathcal{E}$  is unstable. Consider the exact sequence (1). We have that  $\mathcal{H} - \mathcal{A} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(x, y)$  for some  $x > 0, y > 0$ , and  $\mathcal{A} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(x + t, y + l)$  for some  $t > 0, l > 0$ . The inequality  $4 \geq c_2(\mathcal{E}) = x(y + l) + y(x + t) + \deg(\mathcal{Z})$  yields  $\deg(\mathcal{Z}) = 0$  and  $(x, y, x + t, y + l) = (1, 1, 2, 2)$ . Since  $\deg(\mathcal{Z}) = 0$  and  $H^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(t, l)) = 0$ , we conclude that  $\mathcal{E} = \boxed{\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)}$ .

Now assume that  $\epsilon \geq 1$ , and let  $\mathcal{H} = yF + xE$  with  $x = a + b \geq 3$  and  $\mathcal{H} \cdot E = -x\epsilon + y \geq 2$  (since  $\mathcal{H}$  is the determinant of a rank 2 ample vector bundle). It follows that  $\mathcal{H}^2 = a(2b - a\epsilon) \geq a(4 + a\epsilon) \geq 3(4 + a\epsilon) \geq 21 > 4e(S)$ . Thus, if  $c_2 \leq e = 4$ , then  $c_1^2 > 4c_2(\mathcal{E})$ , and thus  $\mathcal{E}$  is unstable. Let  $\mathcal{A} := \alpha E_0 + \beta F$  be the line bundle in the sequence (1). We have the following straightforward inequalities:

- (1)  $x \geq 3, \epsilon \geq 1, y \geq x\epsilon + 2 \geq 5$ ;
- (2)  $x - \alpha > 0, y - \beta > 0, y \geq \beta + (x - \alpha)\epsilon + 1$ ;
- (3)  $2\alpha > x, 2\beta > y$ ;
- (4)  $0 < (2\mathcal{A} - \mathcal{H})^2 = (2\alpha - x)(4\beta - 2y - (2\alpha - x)\epsilon) > 0$ , and in particular  $4\beta + x\epsilon > 2y + 2\alpha\epsilon$ ; and
- (5)  $\mathcal{A} \cdot (\mathcal{H} - \mathcal{A}) \leq c_2(\mathcal{E}) \leq 4$ , which gives  $-\alpha(x - \alpha)\epsilon + \beta(x - \alpha) + \alpha(y - \beta) \leq 4$ .

Note that inequality (5) of the list can be written as

$$\alpha(\alpha\epsilon - x - \beta + y) + \beta(x - \alpha) \leq 4.$$

Using inequality (2) from the list,  $y - \beta \geq (x - \alpha)\epsilon + 1$ , we get

$$\begin{aligned} 4 &\geq \alpha(\alpha\epsilon - x + (x - \alpha)\epsilon + 1) + \beta(x - \alpha) \\ &\geq \alpha x(\epsilon - 1) + \alpha + \beta(x - \alpha). \end{aligned}$$

Now using equations (3) and (2) from the list we get the absurdity

$$4 \geq \alpha x(\epsilon - 1) + \frac{x + 1}{2} + \frac{y + 1}{2} \geq 0 + \frac{4}{2} + \frac{5 + 1}{2} \geq 5.$$

□

### 3. LOWER BOUNDS FOR THE CHERN NUMBERS OF $\mathcal{E}$

In this section we obtain a number of lower bounds for  $c_1(\mathcal{E})^2$  for a rank  $r$  ample vector bundle on a smooth toric surface. Our main tool is adjunction theory: good references for the standard adjunction results that we use are [BS95, Ch. 10, 11] and [F90]. The following is a restatement, taking into account the geometry of toric surfaces, of the main result for the adjunction theory for surfaces. Recall that on a toric surface, a line bundle is ample if and only if it is very ample.

**Theorem 3.1.** *Let  $L$  be an ample line bundle on a smooth projective toric surface  $S$ .*

- (1) *If  $e = e(S) \geq 5$ , then  $K_S + L$  is spanned by global sections.*
- (2) *If  $e = e(S) \geq 7$ , then  $S$  is the equivariant blowup  $\pi : S \rightarrow S_1$  of a smooth toric projective surface  $S_1$  at a finite set  $B$ , such that  $L = \pi^*L' - \pi^{-1}(B)$ , where  $K_S + L \cong \pi^*(K_{S_1} + L')$ , and both  $L'$  and  $L_1 := K_{S_1} + L'$  are very ample.*

*Proof.* Using [BS95, 9.2.2], note that the exceptions to  $K_S + L$  being spanned by global sections are all ruled out by  $e(S) \geq 5$ . The associated map  $p_{K_S+L}$  has a Remmert-Stein factorization  $p = s \circ \pi$ , where  $\pi : S \rightarrow S_1$  has connected fibers. By

Lemma 1.5, we see that  $e \geq 7$  rules out  $\dim S_1 = 0$ . If  $\dim S_1 = 1$ , then we have that  $L \cdot F = 2$  for a general fiber of  $r$ , but this and  $e \geq 7$  contradict Corollary 1.3.

Since  $\dim S_1 = 2$ , it follows from adjunction theory that  $\pi : S \rightarrow S_1$  is the blowup of a smooth toric projective surface  $S_1$  at a finite set  $B$ , such that  $L = \pi^*L' - \pi^{-1}(B)$ , where  $K_S + L \cong \pi^*(K_{S_1} + L')$ , and both  $L'$  and  $L_1 := K_{S_1} + L'$  are ample. The very ampleness of the last two bundles follows from the fact that ample line bundles are very ample on toric varieties.  $\square$

**Corollary 3.2.** *Let  $\mathcal{E}$  be an ample rank  $r$  vector bundle on a nonsingular toric surface  $S$ . If  $e(S) \geq 5$ , then*

$$c_1(\mathcal{E})^2 \geq r^2 e(S)$$

with equality only if  $\det \mathcal{E} = -rK_S$  and  $e(S) = 6$ .

*Proof.* Let  $\mathcal{H} := \det \mathcal{E}$ . Let  $t$  be the smallest positive integer for which  $tK_S + \mathcal{H}$  is not ample. Since  $e(S) \geq 5$ , we have  $E \cdot (tK_S + \mathcal{H}) = 0$  for a smooth rational curve  $E$  with self-intersection  $-1$ . Thus we have

$$-t + E \cdot \mathcal{H} = E \cdot (tK_S + \mathcal{H}) = 0.$$

Since  $\mathcal{E}$  has rank  $r$ , we have that  $r \leq \mathcal{H} \cdot E = t$ . Thus  $rK_S + \mathcal{H}$  is spanned. Using Lemma 1.5, we have

$$\mathcal{H}^2 \geq -\mathcal{H} \cdot rK_S \geq r^2 e(S).$$

Moreover, since  $\mathcal{H}$  is ample, we have equality only if  $\mathcal{H} \cong -rK_S$ . In this case we have  $r^2 K_S^2 = \mathcal{H}^2 = r^2 e(S)$ , or  $K_S^2 = e(S)$ . Since  $K_S^2 + e(S) = 12$ , we conclude that  $K_S^2 = 6$ .  $\square$

**Lemma 3.3.** *Let  $\mathcal{E}$  be an ample rank two vector bundle on a nonsingular toric surface  $S$ . If  $\det \mathcal{E} = -2K_S$ ,  $e(S) = 6$ , and  $c_2(\mathcal{E}) \leq 6$ , then  $\mathcal{E} := -K_S \oplus -K_S$ .*

*Proof.* A simple computation shows that the Chern character of  $\mathcal{E} \otimes K_S$  is  $2 + (K_S^2 - c_2(\mathcal{E})) = 2$ . Thus  $\chi(\mathcal{E} \otimes K_S) = 2$ . Since  $H^2(\mathcal{E} \otimes K_S) = H^0(\mathcal{E}^*) = 0$ , we conclude that  $\dim H^0(\mathcal{E} \otimes K_S) \geq 2$ . Choose linearly independent  $s_1, s_2 \in H^0(\mathcal{E} \otimes K_S)$ .

If  $s_1 \wedge s_2 \neq 0$ , then, since  $\det(\mathcal{E} \otimes K_S) = \mathcal{O}_S$ , we conclude that  $\mathcal{E} \otimes K_S = \mathcal{O}_S \oplus \mathcal{O}_S$ , i.e.,  $\mathcal{E} \cong -K_S \oplus -K_S$ .

Thus we can assume without loss of generality that  $s_1 \wedge s_2 = 0$ . The saturation  $\mathcal{A}$  of the images of  $\mathcal{O}_S$  in  $\mathcal{E}$ , under the two maps  $g \rightarrow g \cdot s_i$ , are equal.  $\mathcal{A}$  is invertible, and tensoring with  $-K_S$  we have an exact sequence

$$0 \rightarrow \mathcal{A} - K_S \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \otimes \mathcal{I}_{\mathcal{Z}} \rightarrow 0,$$

with  $\mathcal{Z}$  a 0-dimensional subscheme of  $S$ . Note that  $\mathcal{Q}$  is ample, and therefore, since  $S$  is toric, very ample. Since  $e(S) = 6$ , we know that  $S$  is not  $\mathbb{P}^2$  or a quadric, and thus

$$(4) \quad \mathcal{Q}^2 \geq 3.$$

Thus the Hodge index theorem gives  $(\mathcal{Q} \cdot (-K_S))^2 \geq \mathcal{Q}^2 (-K_S)^2 \geq 18$ , which implies that

$$(5) \quad \mathcal{Q} \cdot (-K_S) \geq 5.$$

Since  $h^0(\mathcal{A}) \geq 2$ , we have  $\mathcal{Q} \cdot \mathcal{A} \geq 1$ . Using this, and equations (4) and (5), we have

$$6 = c_2(\mathcal{E}) = (\mathcal{A} - K_S) \cdot \mathcal{Q} + \deg \mathcal{Z} \geq 1 + 5 + \deg \mathcal{Z}.$$

Thus  $\text{deg } \mathcal{Z} = 0$  and  $\mathcal{A} \cdot \mathcal{Q} = 1$ . The exact sequence

$$0 \rightarrow \mathcal{A} - K_S \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$$

gives  $-2K_S = c_1(\mathcal{E}) = \mathcal{A} + \mathcal{Q} - K_S$  and  $K_S + \mathcal{A} + \mathcal{Q} = \mathcal{O}$ . Thus  $(K_S + \mathcal{Q}) \cdot \mathcal{Q} = -\mathcal{A} \cdot \mathcal{Q} = -1$ . This is absurd, since on any smooth surface  $S$ , the parity of  $(K_S + L) \cdot L$  is even for any line bundle  $L$ .  $\square$

*Remark 3.4.* We do not know if there are any examples of  $\mathcal{E}$  satisfying all the hypotheses of Lemma 3.3, except that  $c_2(\mathcal{E}) > 6$ .

*Remark 3.5.* The only smooth toric surfaces  $S$  with  $e(S) \leq 4$  are  $\mathbb{P}^2$  or Hirzebruch surfaces. Corollary 1.7 classifies the exceptions to  $c_1^2(S) > r^2 e(S)$  for  $r = 1$ , and §2 classifies the exceptions for  $r = 2$  and  $e(S) \leq 4$ . They are contained in Table 1. For  $\mathbb{P}^2$  it seems difficult to classify the exceptions when  $r \geq 3$ . For the Hirzebruch surfaces  $\mathbb{F}_\epsilon$ , Corollary 2.2 classifies the exceptions if  $\epsilon \geq 2$ .

If  $e(S_1) \geq 7$ , we can repeat the procedure in Theorem 3.1, using  $L_1$  on  $S_1$  in the same way we used  $L$  on  $S$ , and get  $(S_2, L_2)$ . We say the procedure has terminated when we reach the first integer  $b$  with  $e(S_b) \leq 6$ . (See [BL89] for a further study of the adjunction process.) We call the sequence  $(S, L), \dots, (S_b, L_b)$  the iterated adjunction sequence, and  $b$  the adjunction length of  $S$ .

Notice that in the iterated adjunction sequence, at every step we contract down  $(-1)$ -lines in  $S_i$  with respect to the polarization  $K_{S_i} + L_i$ . This implies by Corollary 1.4 that  $e(S_{i+1}) \geq \lfloor \frac{e(S_i)}{2} \rfloor$ . If we assume, to start with, that the surface  $S$  has  $e(S) \geq 2^{b-1} \cdot 6 + 1$ , then the adjunction length is at least  $b$ .

We have the following strong bound.

**Theorem 3.6.** *Let  $S$  be a nonsingular toric surface with  $2^b \cdot 12 \geq e(S) \geq 2^b \cdot 6 + 1$  for some integer  $b \geq 0$  and  $e := e(S)$ . Let  $\mathcal{E}$  be an ample rank  $r$  vector bundle on  $S$ . Then*

$$c_1(\mathcal{E})^2 \geq e(3r^2 + 2r + 4br + 2b - 2) - 12(b + 1)(b + 2r) - 12r(r - 1) + \frac{e}{2^{b-1}} - 2.$$

*Proof.* Since  $\mathcal{H} := \det(\mathcal{E})$  is the determinant of a rank  $r$  ample vector bundle, there are no smooth rational curves  $C$  on the polarized surface  $(S, \mathcal{H})$  with  $\mathcal{H} \cdot C \leq r - 1$ . Therefore by Theorem 3.1,  $L := K + (r - 1)\mathcal{H}$  ample. Using Lemma 3, we have the bound

$$(6) \quad -K\mathcal{H} \geq re.$$

The assumption  $e(S) \geq 2^b \cdot 6 + 1$  implies that we have the adjunction sequence  $(S, L), \dots, (S_b, L_b), (S_{b+1}, L_{b+1})$  with  $L_{b+1}$  very ample. It follows that the sectional genus  $g(L_{b+1}) = g(K_{S_b} + L_b) \geq 0$ , i.e.,  $(K_{S_b} + L_b) \cdot (K_{S_b} + K_{S_b} + L_b) \geq -2$ .

Let  $S \rightarrow S_1 \rightarrow \dots \rightarrow S_b$  be the sequence of contractions and let  $\pi_i$  denote the  $i$ -th contraction map. For simplicity let us set  $K_i := (\pi \circ \pi_1 \dots \circ \pi_i)^*(K_{S_i})$ ,  $K_0 := K_S$ , and  $S := S_0$ . We have

$$\begin{aligned} & (K_{S_b} + L_b) \cdot (K_{S_b} + K_{S_b} + L_b) \\ &= (K_b + K_{b-1} + \dots + K_1 + K_0 + L) \cdot (K_b + K_b + K_{b-1} + \dots + K_1 + K_0 + L). \end{aligned}$$

We can further decompose

$$\begin{aligned}
 & K_b \cdot (K_b + K_{b-1} + \dots + K_1 + K_0 + L) \\
 &= K_b^2 + K_b \cdot (K_{b-1} + K_{b-2} + \dots + K_1 + K_0 + L) \\
 &= K_b^2 + K_{b-1} \cdot (K_{b-1} + K_{b-2} + \dots + K_1 + K_0 + L) \\
 &= K_b^2 + K_{b-1}^2 + K_{b-1} \cdot (K_{b-2} + \dots + K_1 + K_0 + L) \\
 &\quad \vdots \\
 &= K_b^2 + K_{b-1}^2 + K_{b-2}^2 + \dots + K_1^2 + K_0^2 + K_0 \cdot L \\
 &(K_b + K_{b-1} + K_{b-2} + \dots + K_1 + K_0 + L)^2 \\
 &= K_b^2 + 2K_b \cdot (K_{b-1} + \dots + K_1 + K_0 + L) \\
 &\quad + (K_{b-1} + \dots + K_1 + K_0 + L)^2 \\
 &= K_b^2 + 2(K_{b-1}^2 + K_{b-2}^2 + \dots + K_1^2 + K_0^2 + K_0 \cdot L) \\
 &\quad + K_{b-1}^2 + 2K_{b-1} \cdot (K_{b-2} + \dots + K_1 + K_0 + L) \\
 &\quad + (K_{b-2} + \dots + K_1 + K_0 + L)^2 \\
 &= K_b^2 + 3K_{b-1}^2 + 5K_{b-2}^2 + 7K_{b-3}^2 \\
 &\quad + \dots + (2b - 1)K_1^2 + (2b + 1)K_0^2 + (2b + 2)K_0 \cdot L + L^2
 \end{aligned}$$

Then

$$\begin{aligned}
 (7) \quad & (K_{S_b} + L_b) \cdot (K_{S_b} + K_{S_b} + L_b) \\
 &= 2K_b^2 + 4K_{b-1}^2 + 6K_{b-2}^2 + \dots + 2bK_1^2 + (2b + 2)K_0^2 + (2b + 3)K_0 \cdot L + L^2 \\
 &\geq -2.
 \end{aligned}$$

Recall that  $K_i^2 = 12 - e(S_i)$  and  $e(S_i) \geq (\frac{e}{2^i})$ . Then

$$\begin{aligned}
 L^2 + (2b + 3)K_0 \cdot L &\geq -2 - 2 \left(12 - \frac{e}{2^b}\right) - 4 \left(12 - \frac{e}{2^{b-1}}\right) \\
 &\quad - \dots - (2b + 2)(12 - e) + (2b + 3)e \\
 &\geq -2 - 12(b + 1)(b + 2) + \frac{2e}{2^b} \sum_{j=0}^b ((j + 1)2^j).
 \end{aligned}$$

Using  $\sum_{j=0}^b ((j + 1)2^j) = 2^{b+1}b + 1$ , we have

$$L^2 + (2b + 3)K_0 \cdot L \geq -2 - 12(b + 1)(b + 2) + 4eb + \frac{e}{2^{b-1}}.$$

Recalling equation (6) and the fact that  $L = (r - 1)K_0 + \mathcal{H}$ , we get

$$\begin{aligned}
 \mathcal{H}^2 &\geq -2 - 12(b + 1)(b + 2) + 4eb + \frac{e}{2^{b-1}} + 2(r - 1)re + (r - 1)^2(e - 12) \\
 &\quad + (2b + 3)re + (2b + 3)(r - 1)(e - 12) \\
 &= e(3r^2 + 2r + 4br + 2b - 2) - 12(b + 1)(b + 2r) - 12r(r - 1) + \frac{e}{2^{b-1}} - 2. \square
 \end{aligned}$$

*Remark 3.7.* To get a global feel for the bound, we have found it helpful to graph the expression. We include a short Maple V Release 5.1 program to plot the expression divided by part of the leading term. Varying the range of the rank  $r$  and the Euler

characteristic  $e$ , and of the exact variant of `lowerBound`, the scaled expression for the lower bound is useful.

```

b := floor(ln[2]((e-1)/6));
lowerBound := (r,e) -> e*(3*r^2+2*r+4*b*r+2*b-2)-12*(b+1)*(b+2*r)
                -12*r*(r-1)+e/2^(b-1)-2;
plot3d(lowerBound(r,e)/(r*e*(3*r+4*b)),r=1..20,
        e=13..100,style=PATCH,axes=BOXED);
    
```

*Remark 3.8.* It is easily checked that the expression in  $e$  and  $r$  occurring in the lower bound is an increasing function of  $e$  and  $r$  for  $e \geq 7$ ,  $r \geq 1$ . It is also easy to check using the above bound that  $c_1(\mathcal{E})^2 \geq 2r^2e(S)$  if  $e(S) \geq 12$ , and  $c_1(\mathcal{E})^2 \geq 3r^2e(S)$  if  $e(S) \geq 6r + 7$ .

Theorem 3.6 gives a strong asymptotic lower bound for  $c_1^2$  as  $e$  goes to  $\infty$ . For any fixed  $c > 0$ , there will only be a finite number of possible pairs  $(c_1^2, e)$  of numerical invariants for ample vector bundles  $\mathcal{E}$  on smooth toric surfaces  $S$  with  $L^2 \leq ce$ . For example,  $c_1^2 \geq 2r^2e(S)$  as soon as  $e(S) \geq 13$ . This suggests that enumerating the pairs  $(S, \mathcal{E})$  with  $\mathcal{H}^2 \leq cre(S)$ , where  $\mathcal{E}$  is an ample vector bundle on a smooth toric surface  $S$ , and small  $c > 1$  should be a tractable classification problem with a nice answer.

**Theorem 3.9.** *Let  $\mathcal{E}$  be an ample rank two vector bundle on a nonsingular toric variety  $S$  with  $2^b \cdot 12 \geq e(S) \geq 2^b \cdot 6 + 1$  for some integer  $b \geq 0$  and  $e := e(S)$ . Then*

$$c_2(\mathcal{E}) \geq -3(b+2)(b+3) + \frac{5b+7}{2}e + \frac{e}{2^{b+1}} - \frac{1}{2}.$$

*Proof.* If the inequality is not satisfied, then, using Theorem 3.6,  $c_1(\mathcal{E})^2 > 4c_2(\mathcal{E})$ , and thus the bundle would be unstable. The exact sequence (1) and the inequality (2) give

$$c_2(\mathcal{E}) \geq (\mathcal{H} - \mathcal{A})^2 + \sqrt{(\mathcal{H} - \mathcal{A})^2};$$

the divisor  $\mathcal{H} - \mathcal{A}$  is ample, and thus by Theorem (3.6)

$$\begin{aligned}
 & -3(b+2)(b+3) + \frac{(5b+7)}{2}e + \frac{e}{2^{b+1}} - \frac{1}{2} \\
 & > c_2(\mathcal{E}) \geq e(6b+3) - 12(b+1)(b+2) + \frac{e}{2^{b-1}} - 2 + 1,
 \end{aligned}$$

which is equivalent to  $18b^2 + 42b + 13 - 7eb + e - 3e/2^b > 0$ , which is impossible.  $\square$

*Remark 3.10.* We expect that a generalization of Theorem 3.9 to ample vector bundles of arbitrary rank  $r$  is true. Based on a strong dose of optimism, we conjecture that if  $\mathcal{E}$  is an ample rank  $r$  vector bundle on a smooth toric projective surface  $S$  with  $2^b \cdot 12 \geq e(S) \geq 2^b \cdot 6 + 1$  for some integer  $b \geq 0$ , then

$$\begin{aligned}
 c_2(\mathcal{E}) \geq & \frac{r-1}{2r} [e(S)(3r^2 + 2r + 4br + 2b - 2) \\
 & - 12(b+1)(b+2r) - 12r(r-1) + \frac{e}{2^{b-1}} - 2].
 \end{aligned}$$

We now turn to the special case of rank two bundles where the inequality  $c_2(\mathcal{E}) > e(S)$  fails to be true.

**Lemma 3.11.** *Let  $\mathcal{E}$  be an unstable ample rank two vector bundle on a smooth toric projective surface  $S$ . If  $\mathcal{E}$  is Bogomolov unstable and  $c_2(\mathcal{E}) \leq e(S) + \sqrt{e(S)}$ , then  $S$  is either  $\mathbb{P}^2$  or  $\mathbb{F}_\epsilon$  with  $\epsilon \leq 2$ .*

*Proof.* Assume that  $\mathcal{E}$  is Bogomolov unstable. Consider the sequence (1) and the inequality

$$e(S) + \sqrt{e(S)} \geq c_2(\mathcal{E}) = \mathcal{A} \cdot (\mathcal{H} - \mathcal{A}) + \deg(\mathcal{Z}) \geq (\mathcal{H} - \mathcal{A})^2 + \sqrt{(\mathcal{H} - \mathcal{A})^2}.$$

We can then assume  $(\mathcal{H} - \mathcal{A})^2 \leq e$ . We now apply Theorem 1.7 to the ample line bundle  $\mathcal{H} - \mathcal{A}$ .  $\square$

*Remark 3.12.* Let  $\delta := \min\{L^2 \mid L \text{ an ample line bundle on } S\}$ . The above argument implies that any ample vector bundle  $\mathcal{E}$  with  $c_2(\mathcal{E}) < \delta + \sqrt{\delta}$  is Bogomolov stable.

**Corollary 3.13.** *Let  $\mathcal{E}$  be an ample rank two vector bundle on a smooth toric projective surface  $S$ . Assume that  $c_2(\mathcal{E}) \leq e(S)$ . If  $\mathcal{E}$  is not Bogomolov stable, then  $(S, \mathcal{E})$  is contained in Table 1.*

*Proof.* Simply use Lemma 3.11 and the results for  $\mathbb{P}^2$  and the Hirzebruch surfaces from §2.  $\square$

**Proposition 3.14.** *Let  $\mathcal{E}$  be an ample rank two vector bundle on a smooth projective toric surface  $S$ . If either  $c_1(\mathcal{E})^2 \leq 4e(S)$  or  $c_2(\mathcal{E}) \leq e(S)$ , then  $(S, \mathcal{E})$  is in Table 1.*

*Proof.* We can also assume that  $S$  is neither  $\mathbb{P}^2$  or a Hirzebruch surface by using the results of §2. Thus  $e(S) \geq 4$ . Using Corollary 3.2 and Lemma 3.3, we can assume without loss of generality that  $c_1(\mathcal{E})^2 > 4e(S)$ . If  $c_2(\mathcal{E}) \leq e$ , then we are in the situation of Lemma 3.13.  $\square$

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