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TRUNCATED SECOND MAIN THEOREM WITH MOVING TARGETS

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Abstract. We prove a truncated Second Main Theorem for holomorphic curves intersecting a finite set of moving or fixed hyperplanes. The set of hyperplanes is assumed to be non-degenerate. Previously only general position or subgeneral position was considered.

1. Introduction

In this paper, we prove a truncated Second Main Theorem for holomorphic curves intersecting a finite set of moving or fixed hyperplanes. The set of hyperplanes is assumed to be non-degenerate (see the definition below). Previously only general position or subgeneral position was considered. Applications to the uniqueness problem appeared elsewhere (see [Ru2]). This paper is partially motivated by Ru's result (see [Ru1]) that $\mathbb{P}^n - \mathcal{H}$ is Brody hyperbolic if and only $\mathcal{H}$ is non-degenerate, where $\mathcal{H}$ is a finite set of hyperplanes in $\mathbb{P}^n$.

To state our results, we first introduce some standard definitions in Nevanlinna theory. Let

$$f = [f_0 : \cdots : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$$

be a holomorphic map, where $f_0, ..., f_n$ are entire and without common zeros. Define $\mathbf{f} = (f_0, \ldots, f_n)$. $\mathbf{f}$ is called a **reduced representation** of f. The characteristic function $T_f(r)$ of f is defined by

$$T_f(r) = \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} - \log \|\mathbf{f}(0)\|. \tag{1.1}$$

Note that the characteristic function $T_f(r)$ is independent of the choice of the reduced representation of f. A moving hyperplane assigns, to every $z \in \mathbb{C}$, a hyperplane given by

$$H(z) = \left\{ [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C}) \mid \sum_{i=0}^{n} a_i(z) x_i = 0 \right\},$$

where $a_i, 0 \leq i \leq n$, are entire functions without common zeros. Denote by $\mathbf{a} = (a_0, \ldots, a_n)$ the vector associated with H. A moving hyperplane H gives a holomorphic map $\mathbb{P}(\mathbf{a}) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$. We define $T_H(r) = T_{\mathbb{P}(\mathbf{a})}(r)$.

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We now define the counting function. For a moving hyperplane H, we say (f, H) is **free** if $\mathbf{a} \cdot \mathbf{f} \not\equiv 0$, where $\mathbf{a}$ is the vector associated to H and $\cdot$ is the dot product in $\mathbb{C}^{n+1}$. Under the assumption that (f, H) is free, let $n_f(r, H)$ be the number of zeros of $\mathbf{a} \cdot \mathbf{f}$ in $|z| < r$. Let $n_f^{(n)}(r, H)$ be the number of zeros of $\mathbf{a} \cdot \mathbf{f}$ in $|z| < r$, where the multiplicity is counted only as n if the vanishing order of $\mathbf{a} \cdot \mathbf{f}$ at the point is greater than or equal to n. The counting function is defined by

$$N_f(r, H) = \int_0^r \frac{n_f(t, H) - n_f(0, H)}{t} dt + n_f(0, H) \log r,$$

and the truncated counting function is

$$N_f^{(n)}(r, H) = \int_0^r \frac{n_f^{(n)}(t, H) - n_f^{(n)}(0, H)}{t} dt + n_f^{(n)}(0, H) \log r.$$

Denote by $\mathcal{M}$ the field of meromorphic functions on $\mathbb{C}$.

Definition 1.1. We say the set of moving hyperplanes $\mathcal{H} = \{H_1, \ldots, H_q\}$ (or $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_q\}$) is **non-degenerate over** $\mathcal{M}$ if $\dim(\mathcal{A})_{\mathcal{M}} = n + 1$ and for each proper subset $\mathcal{A}_1$ of $\mathcal{A}$

$$(\mathcal{A}_1)_{\mathcal{M}} \cap (\mathcal{A} - \mathcal{A}_1)_{\mathcal{M}} \cap \mathcal{A} \neq \emptyset, \tag{1.2}$$

where $(\mathcal{A})_{\mathcal{M}}$ is the linear span of $\mathcal{A}$ over the field $\mathcal{M}$.

Our result is stated as follows:

Theorem 1.1. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a finite collection of moving hyperplanes. Assume that $\mathcal{H}$ is non-degenerate over $\mathcal{M}$, and (f, H) is free for every $H \in \mathcal{H}$. Then*

$$T_f(r) \leq \sum_{i=1}^{q} n(2n-1) N_f^{(n)}(r, H_i) + O\left(\max_{1 \leq i \leq q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right),$$

where O_{exc} means the estimate holds except for r in a set of finite Lebesgue measure.

Note that when $H_1, \ldots, H_q$ are (fixed) hyperplanes, we say that $\mathcal{H} = \{H_1, \ldots, H_q\}$ (or $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_q\}$) is **non-degenerate over** $\mathbb{C}$ if $\dim(\mathcal{A}) = n + 1$ and for each proper subset $\mathcal{A}_1$ of $\mathcal{A}$

$$(\mathcal{A}_1) \cap (\mathcal{A} - \mathcal{A}_1) \cap \mathcal{A} \neq \emptyset,$$

where $(\mathcal{A})$ is the linear span of $\mathcal{A}$ over $\mathbb{C}$. The proof of Theorem 1.1 implies that if $\mathcal{H} = \{H_1, \ldots, H_q\}$ is non-degenerate over $\mathbb{C}$, then every holomorphic map $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) - (\bigcup_{j=1}^q H_j)$ must be constant. In this case we say that $P^n(\mathbb{C}) - (\bigcup_{j=1}^q H_j)$ **is Brody hyperbolic**. We note that Min Ru (cf. [Ru1]) proved that $P^n(\mathbb{C}) - (\bigcup_{j=1}^q H_j)$ is Brody hyperbolic if and only if $\mathcal{H} = \{H_1, \ldots, H_q\}$ is non-degenerate over $\mathbb{C}$, so Theorem 1.1 can also be viewed as a quantitative extension of Ru's result.

Recall that (fixed) hyperplanes $\{H_1, \ldots, H_q\}$ (or $\{\mathbf{a}_1, \ldots, \mathbf{a}_q\}$) are said to be in general position if $\mathbf{a}_{\mu(0)}, \ldots, \mathbf{a}_{\mu(n)}$ are linearly independent for any injective map $\mu : \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$. Moving hyperplanes $\{H_1, \ldots, H_q\}$ are said to be in general position if $\{H_1(z), \ldots, H_q(z)\}$ are in general position for some (and hence for almost all) $z \in \mathbb{C}$. A typical example of $\mathcal{H} = \{H_1, \ldots, H_q\}$ being non-degenerate over $\mathcal{M}$ is that $\mathcal{H}$ is in general position. In this case, we have a stronger result.

Theorem 1.2. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Let $\mathcal{H} = \{H_1, \dots, H_q\}$ be a finite set of moving hyperplanes in general position. Assume that (f, H) is free for every $H \in \mathcal{H}$. If $q \geq 2n + 1$, then*

$$\frac{q}{2n+1} T_f(r) \leq \sum_{i=1}^{q} n N_f^{(n)}(r, H_j) + O\left(\max_{1 \leq i \leq q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right),$$

where O_{exc} means the estimate holds except for r in a set of finite Lebesgue measure.

For the applications of the above truncated SMT with moving targets to the uniqueness problem, see [Ru2]. We also note that Ru and Stoll [R-S2] obtained the following inequality without truncation: Under the same assumptions as in Theorem 1.2, for every $\epsilon > 0$, the inequality

$$(q - 2n - \epsilon) T_f(r) \leq \sum_{i=1}^{q} N_f(r, H_j) + O\left(\max_{1 \leq i \leq q} T_{H_i}(r)\right)$$

holds for all r outside a set of finite Lebesgue measure.

2. A refinement of diagonal equations of holomorphic functions

In this section, we give the following refinement for holomorphic functions satisfying a diagonal equation, which generalizes the well-known Borel Lemma in Nevanlinna theory. We use the standard notation in Nevanlinna theory.

Theorem 2.1. *Let $f = [f_0 : \cdots : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, with $f_0, ..., f_n$ entire and no common zeros. Assume that f_{n+1} is a holomorphic function and $f_0 + \cdots + f_n + f_{n+1} = 0$. If $\sum_{i \in I} f_i \neq 0$ for any proper subset I of $\{0, ..., n+1\}$, then*

$$T_f(r) \leq \sum_{j=0}^{n+1} N_{f_j}^{(n)}(r, 0) + O_{exc}(\log^+ T_f(r))$$

for $r \to \infty$, where O_{exc} means the estimate holds except for r in a set of finite Lebesgue measure.

To prove Theorem 2.1, we recall the following lemma from [B-M].

Lemma 2.2. *Assume $\sum_{i=0}^{m} f_i = 0$ but no non-empty proper subsum vanishes. If some proper subset of $\{f_0, \dots, f_m\}$ is linearly dependent, then we can find an integer $l \geq 2$, a partition*

$$\{0, 1, \dots, m\} = I_1 \cup ... \cup I_l$$

into non-empty disjoint sets $I_1, ..., I_l$, and non-empty sets

$$J_1 \subseteq I_1, \ J_2 \subseteq I_1 \cup I_2, ... J_{l-1} \subseteq I_1 \cup ... \cup I_{l-1}$$

such that

$$I_1, \ I_2 \cup J_1, ..., I_l \cup J_{l-1}$$

are minimal. Here, we say an index set $I \subset \{0, 1, \dots, m\}$ is **minimal** *if the set $\{f_i | \ i \in I\}$ is linearly dependent, and for any proper subset I' of I the set $\{f_i | \ i \in I'\}$ is linearly independent.*

Proof. Throughout this proof, we use the term **linear forms**. Linear forms are the homogeneous polynomials of degree one in $m+1$ variables with coefficients in $\mathbb{C}$; that is, $L(X) = c_0x_0 + \cdots + c_mx_m$, where $c_0, \ldots, c_m \in \mathbb{C}$, $X = (x_0, \ldots, x_m)$. We denote by $\mathcal{L}$ the set of linear forms which vanish on $(f_0, \ldots, f_m)$, so $L(X) = c_0x_0 + \cdots + c_mx_m$ is in $\mathcal{L}$ if and only if $c_0f_0 + \cdots + c_mf_m = 0$. By the assumption $f_0 + \cdots + f_m = 0$, $\mathcal{L}$ is non-empty. We make the following claim.

Claim 1. *Every linear form L in $\mathcal{L}$ can be written as*

$$L = \sum c_J L_J \text{ with } L_J \in \mathcal{L}$$

for certain minimal sets J, where L_J is a linear combination of $\{x_j \mid j \in J\}$, and c_J is constant.

We prove Claim 1 by induction on the length t of L, i.e., the number of nonzero coefficients. The case $t = 1$ is trivial. So assume that for some $t > 1$ this holds for all elements of $\mathcal{L}$ of length strictly less than t. If $L \in \mathcal{L}$ has length exactly t, we may suppose that

$$L = c_0x_0 + \cdots + c_{t-1}x_{t-1}, \qquad c_i \neq 0, \text{ for } 0 \leq i \leq t-1.$$

If $I = \{0, 1, \ldots, t-1\}$ is minimal, we are done. Otherwise, there is a linear form L' in $\mathcal{L}$ with less length. Without loss of generality we can assume that

$$L' = c'_0x_0 + \cdots + c'_kx_k$$

lies in $\mathcal{L}$ for some k with $0 \leq k < t-1$ and $c'_0 \neq 0$. Then L' and $L'' = c'_0L - c_0L'$ are both of length strictly less than t, and so the induction hypothesis can be applied to both linear forms. Since

$$L = (c_0/c'_0)L' + (1/c'_0)L'',$$

L has the desired decomposition. So Claim 1 is proved.

We now prove Claim 2.

Claim 2. *Suppose $\sum_{i=0}^m f_i = 0$ and $\sum_{i\in I} f_i \neq 0$ for some $I \subset N = \{0, 1, \ldots, m\}$. Then there is a minimal set J with $L_J \in \mathcal{L}$ such that $J \cap I \neq \emptyset$ and $J \cap I^c \neq \emptyset$, where I^c is the complement of I in N.*

In fact, the set $L = \sum_{i=0}^m x_i$ is in $\mathcal{L}$ because $\sum_{i=0}^m f_i = 0$. By Claim 1, we have

$$L = \sum c_J L_J \text{ with } L_J \in \mathcal{L}$$

for certain minimal sets J. If Claim 2 is false, then every such J is contained either in I or in I^c. So $\sum_{i=0}^m x_i = L(x_0, \ldots, x_m) = \sum_{J\subset I} c_JL_J(x_0, \ldots, x_m) + \sum_{J\subset I^c} c_JL_J(x_0, \ldots, x_m)$. However, for those $J \subset I$, $L_J(x_0, \ldots, x_m)$ involves only $\{x_i \mid i \in I\}$ while for those $J \subset I^c$, $L_J(x_0, \ldots, x_m)$ involves only $\{x_i \mid i \in I^c\}$. If we set $x_i = 0$ for $i \in I^c$, the above equation becomes

$$\sum_{i\in I} x_i = \sum_{J\subset I} c_JL_J(x_0, \ldots, x_m).$$

Since $L_J \in \mathcal{L}$, $L_J(f_0, \ldots, f_m) = 0$. Hence $\sum_{i\in I} f_i = 0$, which leads to a contradiction that proves Claim 2.

We now pick any minimal set I_1. By hypothesis $N = \{0, 1, \ldots, m\}$ is not minimal, so $I_1 \neq N$. Hence, $\sum_{i\in I_1} f_i \neq 0$. So Claim 2 implies that there exists a minimal set I'_2 with $L_{I'_2} \in \mathcal{L}$ such that $I'_2 \cap I_1 \neq \emptyset$ and $I'_2 \cap I_1^c \neq \emptyset$, where I_1^c is the complement of I_1 in N. Put $I_2 = I'_2 \cap I_1^c$ and $J_1 = I'_2 \cap I_1$. If $N = I_1 \cup I_2$, then we are done. Otherwise, let $I = I_1 \cup I_2$. Applying Claim 2 to I, there exists a minimal set I'_3 with $L_{I'_3} \in \mathcal{L}$, such that $I'_3 \cap I \neq \emptyset$ and $I'_3 \cap I^c \neq \emptyset$. Let $I_3 = I'_2 \cap (I_1 \cup I_2)^c$ and $J_2 = I'_3 \cap (I_1 \cup I_2)$. If $N = I_1 \cup I_2 \cup I_3$, then we are done. Otherwise, we repeat the same procedures until the union reaches N. This proves Lemma 2.2. $\square$

Proof of Theorem 2.1. If $f_0, ..., f_n$ are linearly independent, then this is a consequence of Cartan's truncated Second Main Theorem. If $f_0, ..., f_n$ are linearly dependent, then by Lemma 2.2, we can find an integer $l \geq 2$, a partition

$$\{0, 1, ..., n+1\} = I_1 \cup ... \cup I_l$$

into non-empty disjoint sets $I_1, ..., I_l$, and non-empty sets

$$J_1 \subseteq I_1, \ J_2 \subseteq I_1 \cup I_2, ..., J_{l-1} \subseteq I_1 \cup ... \cup I_{l-1}$$

such that

$$I_1, \ I_2 \cup J_1, ..., I_l \cup J_{l-1}$$

are minimal. Let $n_i = \#I_i$. Then $\sum_{i=1}^{l} n_i = n + 2$. Without loss of generality we may assume that

$$\{0, ..., n_1 - 1\} = I_1, \{n_1, \ldots, n_1 + n_2 - 1\} = I_2, ..., \{n + 2 - n_l, ..., n + 1\} = I_l.$$

We also write

$$\hat{n}_\lambda = \sum_{\nu=1}^{\lambda} n_\nu. \tag{2.1}$$

Since I_1 is minimal, there is a linear relation among $\{f_0, \ldots, f_{n_1-1}\}$. That is,

$$c_{0,1} f_0 + \cdots + c_{n_1-1,1} f_{n_1-1} = \sum_{j \in I_1} c_{j,1} f_j = 0.$$

Define $c_{j,1} = 0$ for all $j \geq n_1$. Then

$$\sum_{j=0}^{n+1} c_{j,1} f_j = 0.$$

Differentiation yields, for each positive integer ρ,

$$\sum_{j=0}^{n+1} c_{j,1} f_j^{(\rho)} = 0. \tag{2.2}$$

Take $2 \leq \lambda \leq l$. Since $I_\lambda \cup J_{\lambda-1}$ is minimal, there are non-zero complex numbers $c_{j,\lambda}$ such that

$$\sum_{j \in I_\lambda \cup J_{\lambda-1}} c_{j,\lambda} f_j = 0.$$

Put $c_{j,\lambda} = 0$ for all $j \notin (I_\lambda \cup J_{\lambda-1})$. Then

$$\sum_{j=0}^{n+1} c_{j,\lambda} f_j = 0.$$

Differentiation yields, for each positive integer ρ,

$$\sum_{j=0}^{n+1} c_{j,\lambda} f_j^{(\rho)} = 0. \tag{2.3}$$

We consider an $(n+1)\times(n+2)$ **master matrix** M given by

$$M=\begin{bmatrix} c_{0,1}f_0 & \cdots & c_{n+1,1}f_{n+1} \\ c_{0,1}f_0' & \cdots & c_{n+1,1}f_{n+1}' \\ \vdots & \ddots & \vdots \\ c_{0,1}f_0^{(n_1-2)} & \cdots & c_{n+1,1}f_{n+1}^{(n_1-2)} \\ c_{0,2}f_0 & \cdots & c_{n+1,2}f_{n+1} \\ \vdots & \ddots & \vdots \\ c_{0,2}f_0^{(n_2-1)} & \cdots & c_{n+1,2}f_{n+1}^{(n_2-1)} \\ c_{0,3}f_0 & \cdots & c_{n+1,3}f_{n+1} \\ \vdots & \ddots & \vdots \\ c_{0,3}f_0^{(n_3-1)} & \cdots & c_{n+1,3}f_{n+1}^{(n_3-1)} \\ \vdots & \ddots & \vdots \\ c_{0,l}f_0^{(n_l-1)} & \cdots & c_{n+1,l}f_{n+1}^{(n_l-1)} \end{bmatrix},$$

where we note that $n_1+\cdots+n_l=n+2$. We also note that, by (2.2) and (2.3), the sum of each row of M is zero. Let D_j be the determinant of the matrix obtained by deleting the j-th column of the master matrix M. Then, since the sum of each row of M is zero, we actually have

$$D_j=(-1)^jD_0. \tag{2.4}$$

We now show that

$$D_0\not\equiv 0. \tag{2.5}$$

For this, we first prove that

$$D_0=\gamma_1\gamma_2\cdots\gamma_l, \tag{2.6}$$

where

$$\gamma_1=\begin{vmatrix} c_{1,1}f_1 & \cdots & c_{n_1-1,1}f_{n_1-1} \\ \vdots & \ddots & \vdots \\ c_{1,1}f_1^{(n_1-2)} & \cdots & c_{n_1-1,1}f_{n_1-1}^{(n_1-2)} \end{vmatrix}$$

and, for $2\le\lambda\le l$,

$$\gamma_\lambda=\begin{vmatrix} c_{\hat n_{\lambda-1},\lambda}f_{\hat n_{\lambda-1}} & \cdots & c_{\hat n_\lambda-1,\lambda}f_{\hat n_\lambda-1} \\ \vdots & \ddots & \vdots \\ c_{\hat n_{\lambda-1},\lambda}f_{\hat n_{\lambda-1}}^{(n_\lambda-1)} & \cdots & c_{\hat n_\lambda-1,\lambda}f_{\hat n_\lambda-1}^{(n_\lambda-1)} \end{vmatrix},$$

where $\hat n_\lambda$ is defined in (2.1). (2.6) is true because of the definition of D_0 and the fact that $c_{j,1}=0$ for $j\ge n_1$ and $c_{j,\lambda}=0$ for $j\ge\hat n_\lambda$ for $\lambda=2,\cdots,l$. Now, since I_1 is minimal, $c_{i,1}\ne 0$ for $0\le i\le n_1-1$, and also $\{f_1,\dots,f_{n_1-1}\}$ is linearly independent, so $\gamma_1\not\equiv 0$ by the property of the Wronskian. Also, since $I_\lambda\cup J_{\lambda-1}$ is minimal, $c_{i,\lambda}\ne 0$ for $\hat n_{\lambda-1}\le i\le\hat n_\lambda-1$ and also $\{f_j, j\in I_\lambda\}$ is linearly independent. So $\gamma_\lambda\not\equiv 0$ for $2\le\lambda\le l$. Hence $D_0\not\equiv 0$ by (2.6). So (2.5) is verified. The rest of the proof is similar to the proof of Cartan's Second Main Theorem, replacing the Wronskian W by D_0. The following is the detail. Consider the coordinate hyperplanes $H_i=\{[x_0:\cdots:x_n]\mid x_{i-1}=0\}$ for $1\le i\le n+1$ and $H_{n+2}=\{[x_0:\cdots:x_n]\mid x_0+\cdots+x_n=0\}$, and notice that these hyperplanes are in general position. By the well-known "product to the sum formula"(see [Ru3], Lemma A3.1.6), we have

$$\sum_{j=1}^{n+2} m_f(r, H_j) \leq \int_0^{2\pi} \max_{0\leq j\leq n+1} \log \frac{\|\mathbf{f}(re^{i\theta})\|^{n+1}}{\prod_{t=0,t\neq j}^{n+1} |f_t(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1). \tag{2.7}$$

However, using (2.4),

$$\begin{aligned}
&\int_0^{2\pi} \max_{0\leq j\leq n+1} \log \frac{\|\mathbf{f}(re^{i\theta})\|^{n+1}}{\prod_{t=0,t\neq j}^{n+1} |f_t(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \max_{0\leq j\leq n+1} \log \frac{|D_0(re^{i\theta})|}{\prod_{t=0,t\neq j}^{n+1} |f_t(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&+ (n+1) \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} - \int_0^{2\pi} \log |D_0(re^{i\theta})| \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \max_{0\leq j\leq n+1} \log \frac{|D_j(re^{i\theta})|}{\prod_{t=0,t\neq j}^{n+1} |f_t(re^{i\theta})|} \frac{d\theta}{2\pi} + (n+1)T_f(r) + O(1) \\
&- \int_0^{2\pi} \log |D_0(re^{i\theta})| \frac{d\theta}{2\pi} \\
&\leq \sum_{j=0}^{n+1} \int_0^{2\pi} \log^+ \frac{|D_j(re^{i\theta})|}{\prod_{t=0,t\neq j}^{n+1} |f_t(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&+ (n+1)T_f(r) - N_{D_0}(r, 0) + O(1)
\end{aligned} \tag{2.8}$$

where, in the last step, we used the fact that, by Jensen's formula,

$$\int_0^{2\pi} \log |D_0(re^{i\theta})| \frac{d\theta}{2\pi} = N_{D_0}(r, 0).$$

For each fixed j with $0 \leq j \leq n+1$, we now estimate

$$\int_0^{2\pi} \log^+ \frac{|D_j(re^{i\theta})|}{\prod_{t=0,t\neq j}^{n+1} |f_t(re^{i\theta})|} \frac{d\theta}{2\pi}.$$

Note that D_j does not involve f_j, so we write

$$D_j = D(f_0, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{n+1}).$$

Write $g_i = f_i/f_j$ for $1 \leq i \leq n+1$ and the fixed j. It is easy to verify that

$$\begin{aligned}
&D(f_0, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{n+1}) \\
&= f_j^{n+1} D(f_0/f_j, \ldots, f_{j-1}/f_j, f_{j+1}/f_j, \ldots, f_{n+1}/f_j).
\end{aligned}$$

In fact, from (2.6) we see that D_j in fact is the product of several "small" Wronskians. So the above identity is true by the property of Wronskians. So

$$D_j = f_j^{n+1} D(g_0, \ldots, g_{j-1}, g_{j+1}, \ldots, g_{n+1}).$$

Hence, by Theorem A1.2.5 in [Ru3] (the lemma of logarithmic derivatives),

$$\begin{aligned}
&\int_0^{2\pi} \log^+ \frac{|D_j|}{|f_0 \cdots f_{j-1} f_{j+1} \cdots f_{n+1}|} \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log^+ \frac{|D(g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_{n+1})|}{|g_0 \cdots g_{j-1} g_{j+1} \cdots g_{n+1}|} \frac{d\theta}{2\pi} \\
&\leq O\left(\sum_{i=0}^{n+1} \log T_{g_i}(r)\right),
\end{aligned}$$

where the inequality holds for all r outside a set $E \subset (0, +\infty)$ with finite Lebesgue measure. Using the fact that $f_0 + \cdots + f_n + f_{n+1} = 0$, we get

$$\sum_{i=0}^{n+1} \log T_{g_i}(r) \leq O(\log^+ T_f(r)).$$

Hence

$$\int_0^{2\pi} \log^+ \frac{|D_j(re^{i\theta})|}{\prod_{t=0, t\neq j}^{n+1} |f_t(re^{i\theta})|} \frac{d\theta}{2\pi} \leq O(\log^+ T_f(r)), \tag{2.9}$$

where the inequality holds for all r outside a set $E \subset (0, +\infty)$ with finite Lebesgue measure. Hence, combining (2.7), (2.8) and (2.9),

$$\sum_{j=1}^{n+2} m_f(r, H_j) + N_{D_0}(r, 0) \leq (n+1) T_f(r) + O(\log^+ T_f(r)),$$

or we can write, by the First Main Theorem, the above inequality as

$$T_f(r) \leq \sum_{j=1}^{n+2} N_f(r, H_j) - N_{D_0}(r, 0) + O(\log^+ T_f(r));$$

here the inequality holds for all r outside a set $E \subset (0, +\infty)$ with finite Lebesgue measure. However, by the definition of H_j, we have

$$N_f(r, H_j) = N_{f_{j-1}}(r, 0).$$

So the inequality

$$T_f(r) \leq \sum_{j=0}^{n+1} N_{f_j}(r, 0) - N_{D_0}(r, 0) + O(\log^+ T_f(r))$$

holds for all r outside a set $E \subset (0, +\infty)$ with finite Lebesgue measure. It remains to verify that

$$\sum_{j=0}^{n+1} N_{f_j}(r, 0) - N_{D_0}(r, 0) \leq \sum_{j=0}^{n+1} N_{f_j}^{(n)}(r, 0).$$

Let $z_0 \in \mathbb{C}$. Since $D_j = (-1)^j D_0$ and $f_0, \dots, f_n$ have no common zeros, we may assume that $f_0(z_0) \neq 0$, f_j vanishes at z_0 for $1 \leq j \leq q_1$ and f_j does not vanish at z_0 for $j > q_1$. There are integers $k_j \geq 0$ and nowhere vanishing holomorphic functions g_j in a neighborhood U of z_0 such that

$$f_j = (z - z_0)^{k_j} g_j \text{ for } j = 1, \dots, n+1.$$

Here $k_j = 0$ if $q_1 < j \leq n+1$. Also we can assume that $k_j \geq n$ if $1 \leq j \leq q_0$ and $1 \leq k_j < n$, where $0 \leq q_0 \leq q_1$. By the definition of D_0, we have

$$D_0 = \prod_{j=1}^{q_0} (z - z_0)^{k_j - n} h(z),$$

where $h(z)$ is a holomorphic function defined on U. Thus D_0 vanishes at z_0 with order at least $\sum_{j=1}^{q_0}(k_j - n) = \sum_{j=1}^{q_0} k_j - q_0 n$. Hence, we have

$$\sum_{j=0}^{n+1} N_{f_j}(r, 0) - N_{D_0}(r, 0) \leq \sum_{j=0}^{n+1} N_{f_j}^{(n)}(r, 0). \qquad \square$$

3. Proof of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a finite set of moving hyperplanes. Assume that $\mathcal{H}$ is non-degenerate over $\mathcal{M}$; that is, (1.2) holds. Let

$$H_j = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C}) \mid \sum_{i=0}^{n} a_{ij} x_i = 0\},$$

where $a_{ij}, 0 \leq i \leq n$, are entire functions without common zeros for each $1 \leq j \leq q$. For each j, there exists j_0 such that $a_{j_0,j} \not\equiv 0$. Let $b_{ij} = a_{ij}/a_{j_0,j}$. Then b_{ij} are meromorphic function with the property that, for $1 \leq j \leq q$, $T_{b_{ij}}(r) \leq T_{H_j}(r)$. Let $\mathbf{a}_j = (b_{0j}, \ldots, b_{nj})$ and let $\mathcal{A} = \{\mathbf{a}_1, ..., \mathbf{a}_q\}$. Let $\{\mathbf{a}_{i_1}, ..., \mathbf{a}_{i_m}\}$ be a subset of $\mathcal{A}$. Suppose that this set is linearly dependent over $\mathcal{M}$ and no proper subset is linearly dependent over $\mathcal{M}$. Then we have a linear equation

$$c_{i_1}\mathbf{a}_{i_1} + \cdots + c_{i_m}\mathbf{a}_{i_m} = 0, \tag{3.1}$$

where c_{i_j}, $1 \leq j \leq m$, are nonzero meromorphic functions. By clearing the denominators, we can assume that c_{i_j} are entire functions. We call (3.1) a **minimal relation**. Since $c_{i_1}, \ldots, c_{i_m}$ are determined by solving the system of linear equations $c_{i_1} b_{j,i_1} + \cdots + c_{i_m} b_{j,i_m} = 0,\ 0 \leq j \leq n$, they can be chosen as non-vanishing minors of the matrix with entries b_{j,i_α}, $1 \leq \alpha \leq m$, $0 \leq j \leq n$, up to a sign. For such a choice of $c_{i_\alpha}, 1 \leq \alpha \leq m$, since $T_{b_{ij}}(r) \leq T_{H_j}(r)$, we have the following estimate:

$$T_{c_{i_\alpha}}(r) \leq O\left(\max_{1 \leq i \leq q} T_{H_i}(r)\right). \tag{3.2}$$

Let $\mathcal{R}$ be the collection of all minimal relations associated to $\mathcal{A}$ arising in this way. We also note that $\mathbf{a}_i$'s are pairwise linearly independent, because these hyperplanes are distinct. So we have $3 \leq m \leq n+2$.

Let $f = [f_0 : \cdots : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Let $\mathbf{f} = (f_0, \ldots, f_n)$ be a reduced representation of f. We make the following claim:

Claim. *There exist $n+1$ linearly independent vectors $\mathbf{a}_{i_1}, ..., \mathbf{a}_{i_{n+1}}$ in $\mathcal{A}$ such that*

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_{i_\alpha}}{\mathbf{f}\cdot\mathbf{a}_{i_1}}}(r) \leq \sum_{j=1}^{q}(2n-1)N_f^{(n)}(r, H_j) + O\left(\max_{1 \leq i \leq q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right) \tag{3.3}$$

for $2 \leq \alpha \leq n+1$.

To prove the claim, we first find a minimal relation in $\mathcal{R}$ containing $\mathbf{a}_1$. Without loss of generality, we assume that this minimal relation is

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_m\mathbf{a}_m = 0.$$

Then $c_1\mathbf{f}\cdot\mathbf{a}_1 + \cdots + c_m\mathbf{f}\cdot\mathbf{a}_m = 0$. After rearranging the index again, we obtain an equation with no vanishing subsum:

$$c_1\mathbf{f}\cdot\mathbf{a}_1 + \cdots + c_u\mathbf{f}\cdot\mathbf{a}_u = 0, \tag{3.4}$$

$2 \le u \le n+2$. Theorem 2.1 and (3.2) thus imply

$$\begin{aligned} T_{\frac{c_j\mathbf{f}\cdot\mathbf{a}_j}{c_1\mathbf{f}\cdot\mathbf{a}_1}}(r) &\le T_{[c_1\mathbf{f}\cdot\mathbf{a}_1:\ldots:c_u\mathbf{f}\cdot\mathbf{a}_u]}(r) \\ &\le \sum_{t=1}^{u} N^{(u-2)}_{c_t\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O_{exc}(\log^+ T_{[c_1\mathbf{f}\cdot\mathbf{a}_1:\ldots:c_u\cdot\mathbf{f}\cdot\mathbf{a}_u]}(r)) \\ &\le \sum_{t=1}^{u} N^{(u-2)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\sum_{t=1}^{u} T_{c_t}(r)\right) + O_{exc}(\log^+ T_f(r)) \\ &\le \sum_{t=1}^{u} N^{(u-2)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right), \end{aligned}$$

for $2 \le j \le u$.

From the definition of characteristic function,

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_j}{\mathbf{f}\cdot\mathbf{a}_1}}(r) \le T_{\frac{c_j\mathbf{f}\cdot\mathbf{a}_j}{c_1\mathbf{f}\cdot\mathbf{a}_1}}(r) + T_{\frac{c_j}{c_1}}(r).$$

Therefore the above inequalities and (3.2) imply that

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_j}{\mathbf{f}\cdot\mathbf{a}_1}}(r) \le \sum_{t=1}^{q} N^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right), \tag{3.5}$$

for $2 \le j \le u$.

If the dimension of the vector space spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_u$ over $\mathcal{M}$ is $n+1$, then we are done. Otherwise we assume that the dimension of the vector space spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_u$ over $\mathcal{M}$ is less than $n+1$. Let $\mathcal{A}_1 = \{\mathbf{a}_i \in \mathcal{A} \mid \mathbf{a}_i \in (\mathbf{a}_1, \ldots, \mathbf{a}_u)_{\mathcal{M}}\}$. Suppose that $\mathcal{A}_1 = \{\mathbf{a}_1, \ldots, \mathbf{a}_{u_1}\}$. We now prove that

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_j}{\mathbf{f}\cdot\mathbf{a}_1}}(r) \le \sum_{t=1}^{q} 2N^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right), \tag{3.6}$$

for $2 \le j \le u_1$. If $u_1 = u$, then this is done already. Otherwise for each $u+1 \le j \le u_1$ we have a minimal relation $c_j\mathbf{a}_j + c_{i_1}\mathbf{a}_{i_1} + \cdots + c_{i_w}\mathbf{a}_{i_w} = 0$, where $\{i_1, \ldots, i_w\}$ is an index subset of $\{1, \ldots, u\}$. Repeating the procedure of deriving (3.5), we can show that

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_{i_j}}{\mathbf{f}\cdot\mathbf{a}_j}}(r) \le \sum_{t=1}^{q} N^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right), \tag{3.7}$$

for some $1 \le i_j \le u$. Since

$$\frac{\mathbf{f}\cdot\mathbf{a}_j}{\mathbf{f}\cdot\mathbf{a}_1} = \frac{\mathbf{f}\cdot\mathbf{a}_j}{\mathbf{f}\cdot\mathbf{a}_{i_j}}\frac{\mathbf{f}\cdot\mathbf{a}_{i_j}}{\mathbf{f}\cdot\mathbf{a}_1},$$

its characteristic function satisfies (3.6).

We now move to the second step of the proof. Since $\mathcal{H}$ is non-degenerate, $(\mathcal{A}_1)_{\mathcal{M}} \cap (\mathcal{A}-\mathcal{A}_1)_{\mathcal{M}} \cap \mathcal{A} \neq \emptyset$. We can find an $\mathbf{a}_i \in (\mathcal{A}_1)_{\mathcal{M}} \cap (\mathcal{A}-\mathcal{A}_1)_{\mathcal{M}}$. From the definition of $\mathcal{A}_1$, we have $1 \leq i \leq u_1$. Since $\mathbf{a}_i \in (\mathcal{A}-\mathcal{A}_1)_{\mathcal{M}}$, after rearranging the linear forms we have a minimal relation $c_i\mathbf{a}_i + c_{u_1+1}\mathbf{a}_{u_1+1} + \cdots + c_w\mathbf{a}_w = 0$. Similarly, after rearranging the index, we have an equation with no vanishing proper subsum

$$c_i\mathbf{a}_i + c_{u_1+1}\mathbf{a}_{u_1+1} + \cdots + c_\nu\mathbf{a}_\nu = 0, \quad \nu \leq w.$$

Therefore, similarly to the derivation of (3.5),

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_{u_1+1}}{\mathbf{f}\cdot\mathbf{a}_i}}(r) \leq \sum_{t=1}^{q} N^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\leq i\leq q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right). \tag{3.8}$$

Since

$$\frac{\mathbf{f}\cdot\mathbf{a}_{u_1+1}}{\mathbf{f}\cdot\mathbf{a}_1} = \frac{\mathbf{f}\cdot\mathbf{a}_{u_1+1}}{\mathbf{f}\cdot\mathbf{a}_i}\frac{\mathbf{f}\cdot\mathbf{a}_i}{\mathbf{f}\cdot\mathbf{a}_1},$$

from (3.6) and (3.8) we have

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_{u_1+1}}{\mathbf{f}\cdot\mathbf{a}_1}}(r) \leq \sum_{t=1}^{q} 3N^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\leq i\leq q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right), \tag{3.9}$$

If $\dim(L_1, ..., L_{u_1+1})_{\mathcal{M}} = n+1$, then we are done. Otherwise, we can repeat the same argument. Then we will obtain a sequence of collections of linear forms $\mathcal{A}_1, ..., \mathcal{A}_r$ such that $\dim(\mathcal{A}_1)_{\mathcal{M}} < \dim(\mathcal{A}_2)_{\mathcal{M}} < \cdots < \dim(\mathcal{A}_r)_{\mathcal{M}} = n+1$ and

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}}{\mathbf{f}\cdot\mathbf{a}_1}}(r) \leq \sum_{t=1}^{q} 2iN^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\leq i\leq q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right), \tag{3.10}$$

for $\mathbf{a} \in \mathcal{A}_i$. Since $\dim(\mathcal{A})_{\mathcal{M}} = n+1$ and the cardinality of $\mathcal{A}_1$ is at least 2, $r \leq n$. It's also clear from the proof that to show the claim we only need to show (3.10) up to $r-1$, and show an inequality similar to (3.8) for one element in $\mathcal{A}_r - \mathcal{A}_{r-1}$. Hence we arrive at (3.3). Thus the claim is proved.

By the claim, there are $n+1$ linearly independent vectors $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{n+1}} \in \mathcal{A}$ such that (3.3) holds. Hence

$$\begin{aligned} T_f(r) &\leq \sum_{j=2}^{n+1} T_{\frac{\mathbf{f}\cdot\mathbf{a}_{i_j}}{\mathbf{f}\cdot\mathbf{a}_{i_1}}}(r) + O\left(\max_{1\leq i\leq q} T_{H_i}(r)\right) \\ &\leq \sum_{t=1}^{q} n(2n-1)N^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\leq i\leq q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right). \end{aligned} \tag{3.11}$$

This finishes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let

$$H_j = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C}) \mid \sum_{i=0}^{n} a_{ij}x_i = 0\},$$

where $a_{ij}, 0 \leq i \leq n$, are entire functions without common zeros for each $1 \leq j \leq q$. For each j, there exists j_0 such that $a_{j_0,j} \not\equiv 0$. Let $b_{ij} = a_{ij}/a_{j_0,j}$. Then b_{ij} are meromorphic functions with the property that, for $1 \leq j \leq q$, $T_{b_{ij}}(r) \leq T_{H_j}(r)$.

Let $\mathbf{a}_j = (b_{0j}, \dots, b_{nj})$. Let $I \subset \{2, \dots, q\}$ be the index set with the property that $i \in I$ if and only if

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_i}{\mathbf{f}\cdot\mathbf{a}_1}}(r) \le \sum_{i=1}^{q} N^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right).$$

We first show that $\#I \ge n+1$. After rearranging the index, we assume that $I = \{2, \dots, u\}$, and $u \le n$. For dimensional reasons, $\{\mathbf{a}_1, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{2n+1}\}$ is always linearly dependent over $\mathcal{M}$, i.e.

$$c_1\mathbf{a}_1 + c_{n+1}\mathbf{a}_{n+1} + \cdots + c_{2n+1}\mathbf{a}_{2n+1} = 0.$$

Moreover, since these linear forms are in general position, we can solve for $c_1, c_{n+1}, \dots, c_{n+2}$ explicitly. In fact, let

$$A = \begin{pmatrix} a_{10} & \dots & a_{1n} \\ a_{n+1,0} & \dots & a_{n+1,n} \\ \vdots & \ddots & \vdots \\ a_{2n+1,0} & \dots & a_{2n+1,n} \end{pmatrix},$$

and let $(-1)^{i-1}A_i$ be the determinant of the matrix obtained by deleting the i-th row, $1 \le i \le n+2$, from A; then $c_1 = A_1, c_{n+1} = A_2, \dots, c_{2n+1} = A_{n+2}$. For such $c_1, c_{n+1}, \dots, c_{2n+1}$, since $T_{b_{ij}}(r) \le T_{H_j}(r)$, we have

$$T_{c_1}(r) \le O\left(\max_{1\le i\le q} T_{H_i}(r)\right),$$

and, for $n+1 \le j \le 2n+1$,

$$T_{c_j}(r) \le O\left(\max_{1\le i\le q} T_{H_i}(r)\right).$$

After rearranging the index we will have an equation

$$c_1\mathbf{f}\cdot\mathbf{a}_1 + c_{n+1}\mathbf{f}\cdot\mathbf{a}_{n+1} + \cdots + c_w\mathbf{f}\cdot\mathbf{a}_w = 0,$$

with no proper subsum vanishing. Therefore, similarly to (3.5), we conclude that

$$T_{\frac{\mathbf{f}\cdot\mathbf{a}_{n+1}}{\mathbf{f}\cdot\mathbf{a}_1}}(r) \le \sum_{t=1}^{q} N^{(n)}_{\mathbf{f}\cdot\mathbf{a}_t}(r,0) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right).$$

This contradicts the fact that $n+1$ is not in I. Thus $\#I \ge n+1$. By the "in general position" assumption, any $n+1$ hyperplanes in $\mathcal{H}$ are linearly independent. Therefore, similarly to (3.11), we can derive the following inequality:

$$T_f(r) \le \sum_{j=1}^{q} nN^{(n)}_f(r, H_j) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right). \tag{3.12}$$

We now deduce the inequality of the theorem by induction on q. Let $\mathcal{H}_\gamma$ be a subset of $\mathcal{H}$ consisting of $\gamma \ge 2n+1$ elements. When $\gamma = 2n+1$, this is done by

(3.12). By the induction assumption

$$\frac{\gamma}{2n+1}T_f(r) \le \sum_{H\in\mathcal{H}_\gamma} nN_f^{(n)}(r,H) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right) \tag{3.13}$$

for any subset $\mathcal{H}_\gamma$ of $\mathcal{H}$ consisting of $\gamma \ge 2n+1$ elements. For $\mathcal{H}_{\gamma+1}$, we can choose γ linear forms at a time and apply (3.13). This gives $\gamma+1$ inequalities like (3.13). Summing up these $\gamma+1$ inequalities, we have

$$\frac{\gamma+1}{2n+1}T_f(r) \le \sum_{j=1}^{q} nN_f^{(n)}(r,H_j) + O\left(\max_{1\le i\le q} T_{H_i}(r)\right) + O_{exc}\left(\log^+ T_f(r)\right).$$

This completes the proof of Theorem 1.2. □

4. Some results on ABC variety

Motivated by Theorem 1.1, we introduce the concept of ABC variety. The definition is similar to the concept introduced by Buium (cf. [Bu]) in the function field case. For more discussions in this direction, see [W]. Let V be a smooth complex projective variety. Let A be an ample divisor on X. The characteristic (or height) function of f with respect to A is defined by

$$T_A(r,f) = \int_0^r \int_{\triangle_t} f^* c_1(A)\frac{dt}{t},$$

where $c_1(A)$ is the first Chern form of the line bundle $[A]$ associated with A. Since, with respect to different ample divisors, the characteristic functions of f differ only by a constant multiple plus a bounded term, we denote $T_A(r,f)$ simply by $T_f(r)$ for some ample divisor A. Let D be an effective divisor. The proximity function $m_f(r,D)$ is defined by

$$m_f(r,D) = \int_0^{2\pi} \log\frac{1}{\|s(f(re^{i\theta}))\|}\frac{d\theta}{2\pi},$$

where s is the canonical section of $[D]$. The truncated counting function $N_f^{(n)}(r,D)$ of f is the same as what we defined in Section 1.

Definition. Let V be a smooth projective variety defined over $\mathbb{C}$ of dimension n. Let D be an effective divisor over V. The pair (V,D) is called an **ABC-variety** if there is a positive constant C such that for all holomorphic map $f:\mathbb{C}\to V$,

$$T_f(r) \le CN_f^{(n)}(r,D) + O_{exc}(\log^+ T_f(r)). \tag{4.1}$$

Theorem 1.1 implies that $(\mathbb{P}^n(\mathbb{C}), \bigcup_{H\in\mathcal{H}} H)$ is an *ABC*-variety if $\mathcal{H}$ is non-degenerate. We now prove that they are in fact equivalent.

Theorem 4.1. *Let $\mathcal{H}$ be a finite set of hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Then*

$$(\mathbb{P}^n(\mathbb{C}), \bigcup_{H\in\mathcal{H}} H)$$

is an ABC-variety if and only if $\mathcal{H}$ is non-degenerate over $\mathbb{C}$. Or equivalently, $(\mathbb{P}^n(\mathbb{C}), \bigcup_{H\in\mathcal{H}} H)$ is an ABC-variety if and only if $\mathbb{P}^n(\mathbb{C}) - \bigcup_{H\in\mathcal{H}} H$ is Brody hyperbolic.

In fact, if $(\mathbb{P}^n(\mathbb{C}), \bigcup_{H\in\mathcal{H}} H)$ is an ABC-variety, then (4.1) holds, so it implies that $\mathbb{P}^n(\mathbb{C}) - \bigcup_{H\in\mathcal{H}} H$ is Brody hyperbolic. By the result of [Ru1], $\mathbb{P}^n(\mathbb{C}) - \bigcup_{H\in\mathcal{H}} H$ is Brody hyperbolic if and only if $\mathcal{H}$ is non-degenerate over $\mathbb{C}$. This, together with Theorem 1.1, implies Theorem 4.1. Below, for completeness, we include a proof which contains the step that explains why $\mathbb{P}^n(\mathbb{C}) - \bigcup_{H\in\mathcal{H}} H$ Brody hyperbolic implies $\mathcal{H}$ non-degenerate.

Proof of Theorem 4.1*.* As we indicated, we only need to prove that if $\mathbb{P}^n(\mathbb{C}) - \bigcup_{H\in\mathcal{H}} H$ is an ABC-variety, then $\mathcal{H}$ is non-degenerate. To prove this, we first recall a result from [Ru1].

Proposition (Ru)**.** *$\mathcal{H}$ is non-degenerate over $\mathbb{C}$ if and only if for every $\mathcal{H}$-admissible subspace V of $\mathbb{P}^n(\mathbb{C})$ of projective dimension greater than or equal to one, $\mathcal{H}\cap V$ contains at least three distinct hyperplanes which are linearly dependent over F, where V is called $\mathcal{H}$-admissible if V is not contained in any hyperplane in $\mathcal{H}$.*

Assume that $\mathcal{H}$ is degenerate over $\mathbb{C}$. Then the above proposition implies that there exists an $\mathcal{H}$-admissible subspace V of $\mathbb{P}^n$ of projective dimension greater than or equal to 1 such that $\mathcal{H}\cap V$ does not contain at least three distinct hyperplanes which are linearly dependent over $\mathbb{C}$. After a linear change of basis we may assume that $V = \mathbb{P}^m, m \le n$. Then $\mathcal{H}\cap V$ contains exactly q distinct hyperplanes which are linearly independent over $\mathbb{C}$, and $q \le n+1$. Obviously there is a non-constant holomorphic map $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ which omits these coordinate hyperplanes. On the other hand, by our assumption, $(\mathbb{P}^n(\mathbb{C}), \bigcup_{H\in\mathcal{H}} H)$ is an ABC-variety; thus (4.1) holds for $D = \bigcup_{H\in\mathcal{H}} H$. This implies that f must be constant. This leads to a contradiction. ☐

Finally, we conjecture that (V, D) is an ABC-variety if and only if $V - D$ is Kobayashi hyperbolic.

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