TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 3, Pages 853–880
S 0002-9947(03)03157-X
Article electronically published on October 21, 2003

ON THE WEYL TENSOR OF A SELF-DUAL COMPLEX 4-MANIFOLD

FLORIN ALEXANDRU BELGUN

ABSTRACT. We study complex 4-manifolds with holomorphic self-dual conformal structures, and we obtain an interpretation of the Weyl tensor of such a manifold as the projective curvature of a field of cones on the ambitwistor space. In particular, its vanishing is implied by the existence of some compact, simply-connected, null-geodesics. We also show that the projective structure of the β-surfaces of a self-dual manifold is flat. All these results are illustrated in detail in the case of the complexification of $\mathbb{CP}^2$.

1. INTRODUCTION

Twistor theory, created by Penrose [16], establishes a close relationship between conformal Riemannian geometry in dimension 4, and (almost) complex geometry in dimension 3. In particular, to a Riemannian manifold M for which the part W^- of the Weyl tensor vanishes identically (*self-dual*), one associates its *twistor space* Z, a complex 3-manifold containing rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (called *twistor lines*), and admitting a real structure with no fixed points [1], [6], [3].

The space of such curves is a complex 4-manifold $\mathbf{M}$ [10] with a holomorphic conformal structure and is, therefore, a conformal complexification of M [1], [6], [3].

As the conformal geometry of M is encoded by the complex geometry of Z, we ask ourselves what holomorphic object on Z corresponds to W^+, the Weyl tensor of the self-dual manifold M. It seems that this question, although natural, has not been considered in the literature, and maybe a reason for that is that the answer appears to be a highly non-linear object.

This object is more easily understood in the framework of complex-Riemannian geometry (see Section 2): For a self-dual (complex) 4-manifold $\mathbf{M}$, its (local) twistor space is then defined as the 3-manifold of β-surfaces (some totally geodesic isotropic surfaces; see Section 2). Following LeBrun [14], we further introduce the (locally-defined) space B of complex null-geodesics of $\mathbf{M}$ (*ambitwistor space*).

The ambitwistor space B and (in the self-dual case) the twistor space Z completely describe the conformal structure of $\mathbf{M}$. In particular, a null-geodesic γ in $\mathbf{M}$ corresponds to the set of twistor lines in Z tangent to a 2-plane [13]. The union of these curves, called the *integral α-cone* of γ (see Section 3), is lifted to a (linearized) *α-cone* in $T_\gamma B$. Our first result (Theorem 1) is that the Weyl tensor of $\mathbf{M}$ is equivalent to the projective curvature (see Section 4) of the field of α-cones

Received by the editors February 27, 2000.
2000 *Mathematics Subject Classification.* Primary 53C21, 53A30, 32Qxx.

on B. In particular, if such a cone is *flat*, then W^+ vanishes on certain isotropic planes in $\mathbf{M}$.

We use Theorem 1 to investigate global properties of a self-dual manifold $\mathbf{M}$: If the integral α-cone of γ is part of a smooth surface in Z, then the linearized α-cone is flat (Theorems 2, 2′). In particular, the space $\mathbf{M}_0$ of rational curves of Z with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ is compact iff $Z \simeq \mathbb{CP}^3$. On the other hand, it is known, from a theorem of Campana [4], that, for a compact twistor space Z, $\mathbf{M}_0$ can be compactified within the space of analytic cycles iff Z is Moishezon. It appears then that the conformal structure does not extend smoothly to the compactification.

A good illustration of what happens in the non-flat (self-dual) case is the Kähler-Einstein manifold $\mathbb{CP}^2$ whose twistor space is known to be the manifold of flags in $\mathbb{C}^3$ [1]; see Section 7.

A first application of Theorem 2 is that a *civilized* (a topological assumption on the conformal manifold, permitting the global construction of its (ambi-)twistor space—see [12], [15], and Section 2) self-dual manifold containing a compact null-geodesic is conformally flat, and the null-geodesic is simply-connected (Theorem 3). If we assume, in addition, that the compact null-geodesic is simply-connected, then the above result can be deduced, using Theorem 2′, for any self-dual 4-manifold, and also (using the *LeBrun correspondence* [12]) for the case of a conformal complex 3-manifold [2]; In fact, we have recently proven [2] that the existence of a compact, simply-connected, null-geodesic on an n-dimensional conformal complex manifold implies its conformal flatness, for any $n \geq 3$ (different methods are used for $n > 3$).

Another application of Theorem 2 is that the family of *twistor lines* on a *twistor space* never induces a projective structure on it, unless the twistor space is an open set in $\mathbb{CP}^3$ (Corollary 1).

The isotropic, totally geodesic surfaces (called β-surfaces) in a self-dual manifold $\mathbf{M}$ have a projective structure, given by the null-geodesics of $\mathbf{M}$ contained in it (Section 6). We show that it is *flat* (i.e. locally equivalent to $\mathbb{CP}^2$) (Corollary 2), and we obtain a classification of the compact β-surfaces of a self-dual 4-manifold (Theorem 4).

The paper is organized as follows. In Section 2 we recall the classical results of the twistor theory (especially for complex 4-manifolds), in Section 3 we introduce the α-cones on the (ambi-)twistor space, and, in Section 4, we prove the equivalence between the projective curvature of the latter and the Weyl tensor W^+ of $\mathbf{M}$. Section 5 is devoted to the proof of some results of the type "compactness implies conformal (projective) flatness": Theorems 2, 2′ and 3, mentioned above. We study the projective structure of β-surfaces in Section 6, and we illustrate the above results on the special case of the self-dual manifold $\mathbb{CP}^2$ in Section 7.

2. Preliminaries

The content of this paper makes use of *complex-Riemannian geometry*, which is obtained by analogy from Riemannian geometry by replacing the field $\mathbb{R}$ by $\mathbb{C}$ (e.g. a *complex metric* is a non-degenerate symmetric complex-bilinear form on the tangent space), and all classical results hold, naturally with the exception of those making use of partitions of unity. We will often omit the prefix *complex-* when referring to geometric objects, and we will always consider them, unless otherwise stated, in the framework of complex-Riemannian geometry.

2.1. **Conformal complex 4-manifolds.** Let $\mathbf{M}$ be a 4-dimensional complex manifold. A *conformal structure* is defined, as in the real case [5], by an everywhere non-degenerate section c of the complex bundle $S^2(T^*\mathbf{M}) \otimes L^2$, where L is a given line bundle of *scalars of weight 1*, and $L^4 \simeq \kappa^{-1}$, the anti-canonical bundle of $\mathbf{M}$. (While on an oriented real manifold such a line bundle always exists, being topologically trivial, in the complex case the existence of L^2, a square root of the anti-canonical bundle, is submitted to some topological restrictions.) From now on, only holomorphic conformal structures will be considered; thus L is a holomorphic bundle and c a holomorphic section of $S^2(T^*\mathbf{M}) \otimes L^2$. (In fact, all we need to define the conformal structure c on the 4-manifold $\mathbf{M}$ is just the holomorphic bundle L^2; in odd dimensions the situation is different; see [2].)

As in the real case, c is locally represented by symmetric bilinear forms on $T\mathbf{M}$, or local sections in L^2, but global representative metrics do not exist, in general.

For each point $x \in \mathbf{M}$, there is an isotropy cone C_x in the tangent space $T_x\mathbf{M}$, which uniquely determines the conformal structure c. In the associated projective space, $\mathbb{P}(T_x\mathbf{M}) \simeq \mathbb{CP}^3$, the cone C_x projects onto the non-degenerate quadratic surface $\mathbb{P}(C_x)$, which is actually a ruled surface isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. We thus get 2 families of complex projective lines contained in $\mathbb{P}(C)$, that is, 2 families of isotropic 2-planes in $C \subset T\mathbf{M}$, respectively called α-planes and β-planes. This choice corresponds to the choice of an *orientation* of $\mathbf{M}$. On a real 4-manifold an orientation is chosen by picking a class of "positive" volume forms (which is not possible in this complex framework) or by choosing one of the two possible Hodge operators compatible with the conformal structure $* : \Lambda^2\mathbf{M} \to \Lambda^2\mathbf{M}$ (which can also be done in our complex case, [17]). As $*$ is a symmetric involution, $\Lambda^2\mathbf{M}$ decomposes into $\Lambda^+\mathbf{M} \oplus \Lambda^-\mathbf{M}$, consisting in ± 1–eigenvectors of $*$, respectively called *self-dual* and *anti-self-dual* 2-forms; the isotropic vectors in $\Lambda^+\mathbf{M}$ and $\Lambda^-\mathbf{M}$ are then exactly the *decomposable* elements $u \wedge v \in \Lambda^\pm\mathbf{M}$, with $u, v \in \mathbf{M}$.

Definition 1. An *α-plane* F^α (resp. a *β-plane* F^β) in $T\mathbf{M}$ is a 2-plane such that $\Lambda^2 F^\alpha$ (resp. $\Lambda^2 F^\beta$) is a self-dual (resp. anti-self-dual) isotropic line in $\Lambda^2\mathbf{M}$.

Remark. The α- and β-planes can be interpreted in terms of spinors. The structure group of the tangent bundle $T\mathbf{M}$ is restricted to the conformal orthogonal complex group, $CO(4,\mathbb{C}) := (O(4,\mathbb{C}) \times \mathbb{C}^*)/\{\pm\mathbf{1}\}$, where $O(4,\mathbb{C}) := \{A \in GL(4,\mathbb{C}) | A\,{}^tA = \mathbf{1}\}$, by the given conformal structure of $\mathbf{M}$. The choice of an orientation is the further restriction of this group to the connected component of the identity, $CO_0(4,\mathbb{C}) := SO(4,\mathbb{C}) \times \mathbb{C}^*$, where $SO(4,\mathbb{C}) := O(4,\mathbb{C}) \cap SL(4,\mathbb{C})$. Consider a local metric g in the conformal class c. We have then locally defined *Spin* structures, and associated *Spin* bundles V_+, V_-, as in the real case [1],[18]. They are rank 2 complex vector bundles, and for each local section of L^2 (i.e. a metric in c), each of them is equipped with a (complex) symplectic structure $\omega_+ \in \Lambda^2 V_+, \omega_- \in \Lambda^2 V_-$, respectively. Then we locally have $T\mathbf{M} \simeq V_+ \otimes V_-$, and $g = \omega_+ \otimes \omega_-$, for the fixed metric $g \in c$. α- (resp. β-) planes are then nothing but the isotropic 2-planes obtained by fixing the first (resp. the second) factor in the above tensor product:

Proposition 1 ([17])**.** *An α-plane, resp. β-plane $F \subset T_x\mathbf{M}$ is a complex plane $\psi_+ \otimes V_-$, resp $V_+ \otimes \psi_-$, where $\psi_+ \in V_+ \setminus \{0\}$, resp. $\psi_- \in V_- \setminus \{0\}$.*

The α-planes in $T_x\mathbf{M}$ are thus indexed by $\mathbb{P}(V_+)_x$, and β-planes by $\mathbb{P}(V_-)_x$, and these projective bundles are globally well-defined on $\mathbf{M}$ [1].

Remark. It is obvious that a change of orientation interchanges the α- and β-planes; the same is true for self-duality and anti-self-duality, to be defined below.

For a local metric g in c, we denote by R^g its Riemannian curvature, and by W the Weyl tensor, i.e. the trace-free component of R^g, which is known to be independent of the chosen metric within the conformal class [5]. It splits into two components W^+, W^-, and the easiest way to see that is the spinorial decomposition of the space of the curvature tensors $\mathcal{R} \subset \Lambda^2 \otimes \Lambda^2$ ([1], [18], [19]), obtained from the relation $T\mathbf{M} = V_+ \otimes V_-$ and some of the Clebsch-Gordan identities [18]. We have

$$\mathcal{R} = \mathcal{S} \oplus \mathcal{B} \oplus \mathcal{W}^+ \oplus \mathcal{W}^-,$$

where $\mathcal{S}$ is the complex line of scalar curvature tensors, ("diagonally") included in $\Lambda^2 V_+ \oplus \Lambda^2 V_- \simeq \mathbb{C} \oplus \mathbb{C}$, $\mathcal{B} = S^2 V_+ \otimes S^2 V_-$ is the space of trace-free Ricci tensors, and $\mathcal{W}^+ = S^4 V_+$, $\mathcal{W}^- = S^4 V_-$ are the spaces of self-dual, resp. anti-self-dual Weyl tensors (where $S^p V_\pm$ denotes the p-symmetric power of $V_\pm$).

The curvature R^g restricted to any α-plane F yields a weighted bilinear symmetric form R^F on $\Lambda^2 F$, i.e. a section in $L^2 \otimes (\Lambda^2 F \otimes \Lambda^2 F)^*$:

$$(g, X \wedge Y) \overset{R^F}{\longmapsto} g(R^g(X,Y)X, Y).$$

Proposition 2. *The (weighted) bilinear form R^F depends only on the self-dual Weyl tensor, and this one is completely determined by the (weighted) values of R^F for all α-planes F.*

We have the same result for β-planes.

Proof. Let $F = \psi_+ \otimes V_-$ be an α-plane, let $X = \psi_+ \otimes \varphi_1, Y = \psi_+ \otimes \varphi_2 \in F$, and suppose, for simplicity, that $\omega_-(\varphi_1, \varphi_2) = 1$, so $X \wedge Y \in \Lambda^2 F$ is identified with the element $\psi_+ \otimes \psi_+ \in S^2 V_+$. Then it is easy to see that R^F, evaluated on $X \wedge Y$, is nothing but the evaluation of $R \in S^2(\Lambda^2 \mathbf{M}) \supset \mathcal{R}$ on $(X \wedge Y) \otimes (X \wedge Y) \simeq \psi_+ \otimes \psi_+ \otimes \psi_+ \otimes \psi_+ \in S^4 V_+$, which depends only on the positive (or self-dual) part of the Weyl tensor. To prove the second assertion, we remark that W^+, being a quadrilinear symmetric form on V_+, can be identified with a polynomial of degree 4 on V_+, which is determined by its values. □

Definition 2. A conformal structure c on a 4-manifold $\mathbf{M}$ is called *self-dual* (resp. *anti-self-dual*) iff $W^- = 0$ (resp. $W^+ = 0$).

Remark. In general, geodesics on a conformal manifold depend on the chosen metric, with the exception of the isotropic ones (or *null-geodesics*). Therefore the existence of totally geodesic surfaces tangent to α- (resp. β-) planes is a property of the conformal structure alone.

2.2. **Twistor spaces.**

Definition 3. An *α-surface* (resp. *β-surface*) $\alpha \subset \mathbf{M}$ is a maximal, totally geodesic surface in $\mathbf{M}$ whose tangent space at any point is an α-plane (resp. β-plane).

On the other hand, any totally geodesic, isotropic surface in $\mathbf{M}$ is included in an α- or in a β-surface.

Definition 4 ([16], [17])**.** If, at any point $x \in \mathbf{M}$, and for any α- (resp. β-) plane $F \subset T_x\mathbf{M}$, there is an α- (resp. β-) surface tangent to F at x, we say that the family of α- (resp. β-) planes is *integrable.*

Theorem ([1], [17]). *The family of α- (resp. β-) planes of a conformal 4-manifold* $(\mathbf{M}, c)$ *is integrable if and only if the conformal structure c is anti-self-dual (resp. self-dual).*

The integrability of α-planes is equivalent to the integrability (in the sense of Frobenius) of a distribution H^α of 2-planes on the total space of the projective bundle $\mathbb{P}(V_+)$. Namely, let g be a local metric in the conformal class c, and let ∇ be its Levi-Civita connection. ∇ induces a connection in the bundle $\mathbb{P}(V_+)$, thus a horizontal distribution H, isomorphic to $T\mathbf{M}$ via the bundle projection. Let H^α be the 2-dimensional subspace of H_F—where $F \in \mathbb{P}(V_+)$ is an α-plane in $T_x\mathbf{M}$—which projects onto $F \subset T_x\mathbf{M}$. It can easily be shown (as in [17], see also [1]) that the *tautological* 2-plane distribution H^α is independent of the metric g. Then α-surfaces are canonically lifted as integrable manifolds of the distribution H^α. For a geodesically convex open set of $\mathbf{M}$, one can prove (see [15]) that the space of these integrable leaves is a complex 3-manifold. (This point of view is closely related to that of [1], about the integrability of the canonical almost complex structure of the real twistor space.)

The same remark can be made about β-surfaces.

Remark. The existence, for any point $x \in \mathbf{M}$, of an α-surface containing x does not imply, in general, the integrability of the whole family of α-planes: in the conformal self-dual (but not anti-self-dual) manifold $\mathbf{M} = \mathbb{CP}^2 \times (\mathbb{CP}^2)^* \smallsetminus \mathcal{F}$ (the complexification of $\mathbb{CP}^2$, [1]), the surfaces $(\{x\} \times (\mathbb{CP}^2)^*) \cap \mathbf{M}$ and $(\mathbb{CP}^2 \times \{y\}) \cap \mathbf{M}$ are all α-surfaces, see Section 7.

Remark. In the real framework, the twistor space of a real Riemannian 4-manifold $M^{\mathbb{R}}$ is the total space $Z^{\mathbb{R}}$ of the S^2-bundle of almost-complex structures on $TM^{\mathbb{R}}$, compatible with the conformal structure and the (opposite) orientation; it admits a natural almost-complex structure $\mathcal{J}$, equal, at the point $J \in Z^{\mathbb{R}}$, to the complex structure of the fibers on the *vertical* space $T_J^\vee Z^{\mathbb{R}}$, and to J itself on the horizontal space (induced by the Levi-Civita connection). Such a complex structure J is equivalent to an isotropic complex 2-plane in $TM \otimes \mathbb{C}$, thus to an α- or β-surface (depending on the conventions), which then becomes the space of vectors of type $(1,0)$ for J; as the integrability of the almost-complex structure $\mathcal{J}$ can be expressed as the Frobenius condition applied to $T^{(1,0)}Z^{\mathbb{R}}$, it is equivalent to the integrability of the family of α-, resp. β-planes.

The *Penrose construction* associates to an (anti-)self-dual manifold $\mathbf{M}$ the space Z of α- (resp. β-)surfaces of $\mathbf{M}$; we have seen above that Z admits complex-analytic maps, but it may be non-Hausdorff. This is why we need to introduce the following condition; see also [15]:

Definition 5. An (anti-)self-dual manifold $\mathbf{M}$ is called *civilized* iff the space Z^α (resp. Z^β) of integral leaves of the distribution H^α (resp. H^β) in $\mathbb{P}(V_+)$ (resp. $\mathbb{P}(V_-)$) is a complex 3-manifold, and the projection $p^+ : \mathbb{P}(V_+) \to Z^\alpha$ (resp. $p^- : \mathbb{P}(V_-) \to Z^\beta$) is a submersion.

In this case, the manifold Z^α (resp. Z^β)—which is the space of α-surfaces (resp. β-surface) of $\mathbf{M}$ — is called the α- (resp. the β-)*twistor space* of $\mathbf{M}$.

From now on, we suppose that $(\mathbf{M}, c)$ is a self-dual complex analytic 4-manifold. As any point $x \in \mathbf{M}$ has a geodesically convex neighborhood U [21] (which is, therefore, civilized [15]), we can construct Z^U, the β-twistor space (for short, twistor space) of U. As most results of this paper are infinitesimal, we will usually suppose (with no loss of generality) that $\mathbf{M}$ is civilized (for example, by replacing $\mathbf{M}$ by U).

We recall now the correspondence between differential geometric objects on $\mathbf{M}$ and complex analytic objects on its twistor space Z ([1], [17]; see also [12], [13], [18]).

β-surfaces $\beta \subset \mathbf{M}$ correspond to points $\bar{\beta} \in Z$, by definition, and the set of β-surfaces passing through a point $x \in \mathbf{M}$ is a complex projective line Z_x, with normal bundle isomorphic (non-canonically) to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (where $\mathcal{O}(1)$ is the dual of the *tautological* bundle $\mathcal{O}(-1)$ on $\mathbb{CP}^1$) ([1], [17]; see also [3]). Such a curve will be called a *twistor line*.

In fact, the family of twistor lines in Z permits us to recover $\mathbf{M}$ and its conformal structure, at least locally, by the *reverse Penrose construction*: The normal bundle N_x of a line Z_x in Z has the property $H^1(N_x, \mathcal{O}) = 0$; thus, by a theorem of Kodaira [10], the space $\mathbf{M}_0$ of projective lines in Z having the above normal bundle is a smooth complex manifold whose tangent space at a point $x \simeq Z_x \subset Z$ is canonically isomorphic to the space of global sections of the normal bundle N_x of Z_x (thus $\mathbf{M}_0$ has dimension 4). The conformal structure of $\mathbf{M}_0$ is described by its isotropy cone, which corresponds to the sections of N_x having at least one zero (as such a section decomposes as 2 sections of $\mathcal{O}(1)$, the vanishing condition means that they both vanish at the same point, which is a quadratic condition on the sections of N_x). We thus get a conformal diffeomorphism from $\mathbf{M}$ to an open set of $\mathbf{M}_0$.

2.3. **Ambitwistor spaces.** We remark that $\mathbb{P}(V_-)$ is an open set of the projective tangent bundle of Z, as Z is the space of leaves of $\mathbb{P}(V^-)$, but it is important to note that, in general, the reverse inclusion is not true (i.e. not every direction in Z is tangent to a line corresponding to a point in $\mathbf{M}$, or, equivalently, β-surfaces are not compact $\mathbb{CP}^2$'s, in general, see Section 5).

For example, if $\mathbf{M} = \mathbb{CP}^2 \times \mathbb{CP}^* \setminus \mathcal{F}$ (with the notation in Section 7), $\mathbb{P}(V_-)$ is an open subset in the $\mathbb{CP}^2$-bundle $\mathbb{P}(TZ) \to Z$, consisting of the set of directions transverse to the *contact structure* of Z (see Subsection 7.4). $\mathbb{P}(V_-)$ is thus, in this case, a rank 2 affine bundle over Z.

Another canonical $\mathbb{CP}^2$-bundle on Z, that is, $\mathbb{P}(T^*Z) \to Z$, leads to the *ambitwistor space* B, which is by definition the space of null-geodesics of $\mathbf{M}$ [13]. It is an open set of the projective cotangent bundle of Z (or, equivalently, the Grassmannian of 2-planes in TZ) [13]. More precisely, a plane $F \subset T_{\bar{\beta}}Z$ corresponds to a null-geodesic $\gamma \subset \mathbf{M}$ (contained in β) if it is tangent to at least one projective line Z_x, corresponding to a point $x \in \mathbf{M}$.

To see that, let x be a point in $\mathbf{M}$, β a β-surface passing through x, i.e. $\bar{\beta} \in Z$ and Z_x contains $\bar{\beta}$; let $F \subset T_{\bar{\beta}}Z$ be a plane tangent to Z_x. As small deformations of Z_x still correspond to points of $\mathbf{M}$, we consider the twistor lines which are tangent to F. They correspond to a (path-connected) set of points on a curve $\gamma \subset \beta$, which will turn out to be a null-geodesic. Indeed, all we have to prove is $\ddot{\gamma} = 0 \pmod{\dot{\gamma}}$, and $\dot{\gamma}_x$ corresponds to a section η of N_x, vanishing at $\bar{\beta} \in Z_x$; as $N_x \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$, η is determined by its derivative at $\bar{\beta}$, which is a linear map $T_{\bar{\beta}} \to F/T_{\bar{\beta}}$ (the infinitesimal deformation of the direction of Z_x within F). As the points of γ correspond to lines tangent to F, we have that $\ddot{\gamma}_x$ corresponds to a section of N_x collinear to η; thus γ satisfies the equation of a (non-parameterized) geodesic. See [12], [15], and Section 4 for details.

Example. The space of null-geodesics of $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$ is the total space of a $\mathbb{C} \times \mathbb{CP}^1$-bundle over $Z = \mathcal{F}$, the flag manifold (see Section 7); a 2-plane $F \subset T_{(L,l)}\mathcal{F}$ which corresponds to a null-geodesic in $\mathbf{M}$ is identified either with a projective diffeomorphism $\varphi : \mathbb{P}(l) \to \mathbb{P}(L^o)$ (Subsection 7.4, case 3), or with a point $A \subset l$, $A \neq L$, resp. a plane a containing L, and different from l (Subsection 7.4, cases **2** and **2′**).

3. The structure of the ambitwistor space and the field of α-cones

Conventions. Except for some results in Section 5, we will consider $\mathbf{M}$ to be a self-dual civilized 4-manifold, i.e. the (twistor) space Z of β-surfaces of $\mathbf{M}$ is a Hausdorff smooth complex 3-manifold, and the projection $\mathbb{P}(V_-) \to Z$ is a submersion (e.g. $\mathbf{M}$ is geodesically convex); see [15].

We will frequently identify, following the deformation theory of Kodaira (see [10]), the vectors in $T_x\mathbf{M}$ with sections in the normal bundle $N(Z_x)$ of the projective line Z_x in Z.

We also consider the space of null-geodesics B, as an open subset of $\mathbb{P}(T^*Z)$.

For a null-geodesic γ, resp. a β-surface $\beta \subset \mathbf{M}$, we denote by $\bar{\gamma}$, resp. $\bar{\beta}$, the corresponding point in B, resp. Z.

3.1. **α- and β-cones on the ambitwistor space.** The vectors on B can be expressed in terms of infinitesimal deformations of geodesics of $\mathbf{M}$ (Jacobi fields). More precisely,

$$T_{\bar{\gamma}}B \simeq \mathcal{J}_\gamma^\perp / \mathcal{J}_\gamma^\gamma,$$

where, for a null-geodesic γ, $\mathcal{J}_\gamma^\perp$ is the space of Jacobi fields J such that $\nabla_{\dot{\gamma}} J \perp \dot{\gamma}$, and $\mathcal{J}_\gamma^\gamma$ is its subspace of Jacobi fields "along" γ, i.e. $J \in \mathbb{C}\dot{\gamma}$ at any point of the geodesic.

Remark. A class in $\mathcal{J}_\gamma^\perp / \mathcal{J}_\gamma^\gamma$ is represented by Jacobi fields yielding the same local section of the normal bundle $N(\gamma)$ of γ in $\mathbf{M}$. This is equivalent to the following obvious fact:

Lemma 1. *The kernel of the natural application $\mathcal{J}_\gamma^\perp \to N(\gamma)$ is $\mathcal{J}_\gamma^\gamma$.*

As a consequence, Jacobi fields on γ induce particular local sections in $N(\gamma)$, which turn out to be (conformally invariant) solutions of a differential operator of order 2 on $N(\gamma)$; see [2].

The conformal geometry of $\mathbf{M}$ induces a particular structure on B: we describe it in order to obtain an expression of W^+ in terms of the geometry of the (ambitwistor) space.

We have a canonical hyperplane $V_{\bar{\gamma}}$ in $T_{\bar{\gamma}}B$, defined by

$$V_{\bar{\gamma}} := \mathcal{J}_\gamma^{\perp\perp} / \mathcal{J}_\gamma^\gamma,$$

where $\mathcal{J}_\gamma^{\perp\perp}$ is the set of Jacobi fields J everywhere orthogonal to $\dot{\gamma}$ (i.e. $\nabla_{\dot{\gamma}} J \perp \dot{\gamma}$ *and* $J \perp \dot{\gamma}$).

Now we define two fields of cones in TB, both contained in $V_{\bar{\gamma}}$:

Definition 6. Let γ be a null-geodesic in $\mathbf{M}$, and, for each point $x \in \gamma$, let F_x^β be the β-plane containing $\dot{\gamma}_x$. The (*infinitesimal*) *β-cone $V_{\bar{\gamma}}^\beta$ at $\bar{\gamma} \in B$* is defined as follows:

$$V_{\bar{\gamma}}^\beta := \mathcal{J}_\gamma^\beta / \mathcal{J}_\gamma^\gamma \subset \mathcal{J}_\gamma^{\perp\perp} / \mathcal{J}_\gamma^\gamma \simeq V_{\bar{\gamma}} \subset T_{\bar{\gamma}}B,$$

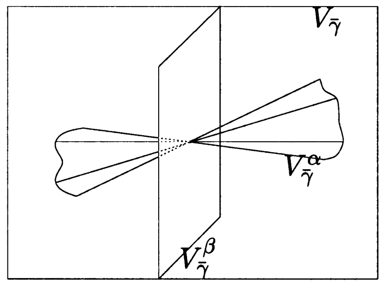

FIGURE 1.

where $\mathcal{J}_\gamma^\beta$ is the set of Jacobi fields J on γ satisfying the condition

$$\exists\, x \in \gamma \text{ such that } J_x = 0 \text{ and } (\nabla_{\dot\gamma} J)_x \in F_x^\beta.$$

Proposition 3. *The β-cone $V_{\bar\gamma}^\beta$ is* flat*, i.e. it is included in the 2-plane $F_{\bar\gamma}^\beta$ consisting of Jacobi fields contained in the β-plane defined by $\dot\gamma$ at each point of it.*

Proof. We have to prove that $\mathcal{J}_\gamma^\beta$ is included in $\bar{\mathcal{J}}_\gamma^\beta$, defined as follows:

$$\bar{\mathcal{J}}_\gamma^\beta := \{J \text{ a Jacobi field on } \gamma \mid J_x, \dot J_x \in F_x^\beta,\ \forall x \in \gamma\}.$$

We will prove that $\mathcal{J}_\gamma^\beta \subset \bar{\mathcal{J}}_\gamma^\beta$; therefore it will follow that the latter is non-empty, and is a linear space of dimension 2.

We denote by K^0 the parallel displacement, along γ, of a non-zero vector in F_x^β, transverse to $\dot\gamma$. Then $K^0 \in T\beta|_\gamma \smallsetminus T\gamma$, because γ is included in the totally geodesic surface β; thus we can characterize F_y^β as the set $\{X \in T_yM \mid X \perp \dot\gamma, X \perp K^0\}$, for any $y \in \gamma$. We then observe that

$$\dot\gamma.\langle \dot J, K^0\rangle = \langle R(\dot\gamma, J)\dot\gamma, K^0\rangle = \langle R(\dot\gamma, K^0)\dot\gamma, J\rangle = k\langle K^0, J\rangle,$$

because $R(\dot\gamma, K^0)\dot\gamma$ is in F^β, thus $R(\dot\gamma, K^0)\dot\gamma = h\dot\gamma + kK^0$. So the scalar function $\langle J, K^0\rangle$ satisfies a linear second order equation, and hence it it determined by its initial value and derivative. It follows then that it is identically zero; thus $J \in F^\beta$ everywhere, as claimed. □

Another subset in $T_{\bar\gamma}B$ is the α-cone $V_{\bar\gamma}^\alpha$, defined as follows:

Definition 7. Let γ be a null-geodesic in **M**, and, for each point $x \in \gamma$, let F_x^α be the α-plane containing $\dot\gamma_x$. The (*infinitesimal*) *α-cone $V_{\bar\gamma}^\alpha$ at $\bar\gamma \in B$* is defined as follows:

$$V_{\bar\gamma}^\alpha := \mathcal{J}_\gamma^\alpha / \mathcal{J}_\gamma^\gamma \subset \mathcal{J}_\gamma^{\perp\perp} / \mathcal{J}_\gamma^\gamma \simeq V_{\bar\gamma} \subset T_{\bar\gamma}B,$$

where $\mathcal{J}_\gamma^\alpha$ is the set of Jacobi fields J on γ satisfying the condition

$$\exists x \in \gamma \text{ such that } J_x = 0 \text{ and } (\nabla_{\dot\gamma} J)_x \in F_x^\alpha.$$

It is important to note that, in general, the projective curves $\mathbb{P}(V_{\bar\gamma}^\alpha)$ and $\mathbb{P}(V_{\bar\gamma}^\beta)$ are non-compact, as each of them corresponds to the set of points on γ, which is non-compact, in general. The field of α-cones on B is the object of main interest in this paper. We may already guess that its flatness (i.e. the situation when $V_{\bar\gamma}^\alpha$ is a subset in a 2-plane) can be related to some vanishing property of the self-dual Weyl tensor of **M**. See Figure 1.

Remark. We have seen that $V_{\bar\gamma}^{\beta}$ is included in the 2-plane $F_{\bar\gamma}^{\beta}$, i.e. the condition $J_x = 0, \dot J_x \in F_x$ can be generalized to the linear condition $J, \dot J \in F^\beta$, but there is no canonical way of supplying the "missing" points of γ with some appropriate Jacobi fields in order to "complete" $V_{\bar\gamma}^{\alpha}$ as in the β-cones case. This would be possible, for example, if $\mathbb{P}(V_{\bar\gamma}^{\alpha})$ were an open subset in a projective line. But the failure of $V_{\bar\gamma}^{\alpha}$ to be part of a 2-plane is measured by its *projective curvature*, and we will see in Section 4 that the vanishing of the latter implies the vanishing of W^+ (Theorem 1).

3.2. **Integral α-cones in Z and B.** Now we study the field of α-cones of B in relation with Z and the canonical projection $\pi : B \to Z$. First, we note that there are complex projective lines in B tangent to the directions in $V_{\bar\gamma}^{\alpha}$:

Definition 8. Let $\bar\gamma \in B$, and let $x \in \gamma$ be a point on the null-geodesic γ; let F_x^α be the α-plane tangent to $T_x\gamma$. The *rational curve* $B_{\bar\gamma,x}^{\alpha}$ *in* B (*containing* $\bar\gamma$) is by definition the set of null-geodesics passing through x and tangent to F_x^α.

The curves $B_{\bar\gamma,x}^{\alpha}$, $x \in \gamma$, are projected by π onto the complex lines Z_x through $\bar\beta$ (corresponding to the β-surface β containing γ) tangent to the 2-plane F^γ.

On the other hand, it is easy to see that the complex projective lines $B_{\bar\gamma,x}^{\beta}$ (defined analogously to $B_{\bar\gamma,x}^{\alpha}$), which are tangent to (an open set of the directions of) $V_{\bar\gamma}^{\beta}$, are contained in the fibers of π. In fact, they coincide with some of the projective lines passing through the point $\gamma \in \mathbb{P}(T_{\bar\beta}^*Z) \simeq \mathbb{CP}^2$.

Definition 9. The *integral α-cones* in B, resp. Z, are defined by:

$$B_{\bar\gamma}^{\alpha} := \bigcup_{x\in\gamma} B_{\bar\gamma,x}^{\alpha} \ (\beta\text{-cone in } B); \quad Z^\gamma := \bigcup_{x\in\gamma} Z_x \ (\beta\text{-cone in } Z).$$

We intend to prove that $B_{\bar\gamma}^{\alpha}$ is the *canonical lift* of Z^γ (see Proposition 5). We know that $\pi(B_{\bar\gamma}^{\alpha}) = Z^\gamma$. We have

Proposition 4. *Except for the vertices $\bar\gamma \in B_{\bar\gamma}^{\alpha}$ and $\bar\beta \in Z^\gamma$, the two integral cones $B_{\bar\gamma}^{\alpha}$ and Z^γ are smooth, immersed surfaces of B, resp. Z.*

Proof. The open set B of $\mathbb{P}(T^*Z)$ which is the space of null-geodesics of $\mathbf{M}$ can be viewed as the space of integral curves of the *geodesic distribution* G of lines in $\mathbb{P}(C)$, the total space of the fibre bundle of isotropic directions in $T\mathbf{M}$. G_v is defined as the horizontal lift (for the Levi-Civita connection on $\mathbf{M}$) of v, which is an isotropic line in $T_x\mathbf{M}$. This definition is independent of the chosen metric and connection [15], and, by integrating this distribution (as $\mathbf{M}$ is civilized), we get a holomorphic map $p : \mathbb{P}(C) \to B$, where B is the space of leaves of this foliation. This map can be used to compute the normal bundle of $B_{\bar\gamma,x}^{\alpha}$, $N(B_{\bar\gamma,x}^{\alpha})$; see [12], [13], [15].

Indeed, we have lines $C_{\gamma,x}^{\alpha} \in \mathbb{P}(C)_x$, such that $\dot\gamma_x \in C_{\gamma,x}^{\alpha}$, which project onto $B_{\bar\gamma,x}^{\alpha}$; thus we get the following exact sequence of normal bundles:

$$0 \to N(C_{\gamma,x}^{\alpha}; p^{-1}(B_{\bar\gamma,x}^{\alpha})) \to N(C_{\gamma,x}^{\alpha}; \mathbb{P}(C)) \to N(B_{\bar\gamma,x}^{\alpha}; B) \to 0,$$

where we have written the ambient spaces of the normal bundles on the second position. The central bundle is trivial ($C_{\gamma,x}^{\alpha}$ is trivially embedded in $\mathbb{P}(C)_x \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, which is trivially embedded in $\mathbb{P}(C)$ as a fibre), and it is easy to check that the left-hand bundle is isomorphic to the tautological bundle over $\mathbb{CP}^1$, $\mathcal{O}(-1)$. This proves that $N(B_{\bar\gamma,x}^{\alpha}; B) \simeq \mathcal{O}(0) \oplus \mathcal{O}(0) \oplus \mathcal{O}(1)$; in particular, the conditions in the completeness theorem of Kodaira [10] are satisfied. Thus the lines in the integral α-cone $B^\alpha_{\bar\gamma}$ form an analytic subfamily of the family $\{B^\alpha_{\bar\gamma,x}\}_{\bar\gamma\in B, x\in\gamma\subset M}$ that correspond to the sections of the normal bundle of $B^\alpha_{\bar\gamma,x}$ vanishing at $\bar\gamma\in B$, or, equivalently, to the points x of $\gamma\subset\mathbf{M}$.

But, in order to prove the smoothness of $B^\alpha_{\bar\gamma}\smallsetminus\{\bar\gamma\}$, we first remark that the surface $C^\alpha_\gamma\subset\mathbb{P}(C)$, defined as follows, is smooth:

$$C^\alpha_\gamma := \{v\in\mathbb{P}(C)_x | x\in\gamma,\ v\subset F^\alpha_\gamma\},$$

where F^α_γ is the α-plane containing $\dot\gamma$. C^α_γ is smooth, and $p(C^\alpha_\gamma)=B^\alpha_{\bar\gamma}$. We note now that C^α_γ is everywhere, except at the points of $p^{-1}(\bar\gamma)$, transverse to the fibers of the submersion $p:\mathbb{P}(C)\to B$. We may conclude that $B^\alpha_{\bar\gamma}\smallsetminus\{\gamma\}$ is a smooth analytic submanifold of B (not closed).

We can use similar methods to prove that $Z^\gamma\smallsetminus\{\bar\beta\}$ is an immersed submanifold of Z (by using the projection $\pi:B\to Z$). □

There is another argument for this latter claim, which gives the tangent space to Z^γ at any point.

We see Z^γ as the *trajectory* of a 1-parameter deformation of Z_x: we fix $\bar\beta$ and we "turn" Z_x around $\bar\beta$ by keeping it tangent to F^γ. The trajectory of this deformation is smooth in $\zeta\in Z^\gamma\smallsetminus\beta$ iff any non-identically-zero section ν of the normal bundle $N(Z_x)$ corresponding to this 1-parameter deformation does not vanish at ζ. In particular, the tangent space $T_\zeta Z^\gamma$ is spanned by $T_\zeta Z_x$ and $\nu(\zeta)$.

But the sections ν generating this deformation are the sections of $N(Z_x)$ vanishing at $\bar\beta$, and they vanish at only one point (and even there, only to order 0) unless they are identically zero, because $N(Z_x)\simeq\mathcal{O}(1)\oplus\mathcal{O}(1)$.

Remark. The values of these sections at the points of Z_x other than $\bar\beta$, plus their derivatives at $\bar\beta$ (well-defined as they all vanish at $\bar\beta$), define a 1-dimensional subbundle of $N(Z_x)$ which is isomorphic to $\mathcal{O}(1)$. In fact, we have a 1–1 correspondence between the subbundles of $N(Z_x)$ isomorphic to $\mathcal{O}(1)$ and the 2-planes in $T_{\bar\beta}Z$. Then, the space of holomorphic sections of such a bundle is a linear space of dimension 2, consisting of a family of sections of $N(Z_x)$ vanishing on *different* points of Z_x. Thus we get a 2-plane F^α of isotropic vectors in $T_x\mathbf{M}$, which is easily seen to be an α-plane, as the β-plane $F^\beta_x = T_x\beta$ consists of the set of all sections of $N(Z_x)$ vanishing at $\bar\beta$ (we have $F^\alpha_x\cap T_x\beta=T_x\gamma$). The tangent space to Z^γ at a point $\zeta\in Z_x$ is spanned by the subbundle of $N(Z_x)$ (isomorphic to $\mathcal{O}(1)$—see above) defined by the isotropic vectors $v\in F^\alpha_x$. If γ^ζ is the null-geodesic generated by v^ζ, we conclude that $T_\zeta Z^\gamma$ is the 2-plane determined by γ^ζ, and that $\zeta=\pi(\overline{\gamma^\zeta})$. See Figure 2.

Example. If $\mathbf{M}=\mathbb{P}(E)\times\mathbb{P}(E)^*\smallsetminus\mathcal{F}$, then the integral α-cone Z^γ in Z, for $\gamma\equiv F^\gamma=F^\varphi\subset T_{(L,l)}Z$ (where $\varphi:\mathbb{P}(l)\to\mathbb{P}(L^o)$ is a projective diffeomorphism), is the (smooth away from the vertex (L,l)) surface $\{(S,\varphi(s))|S\neq L, s\neq l, \varphi(s\cap l)=S\}$. Its compactification (by adding the *special cycle* $\bar Z_{(L,l)}$) is singular (Subsection 7.4).

As any smooth surface in Z has a canonical lift in $B=\mathbb{P}(T^*Z)$, we get

Proposition 5. *The integral α-cone $B^\alpha_{\bar\gamma}$ is the canonical lift of the integral α-cone Z^γ on Z. See Figure* 3.

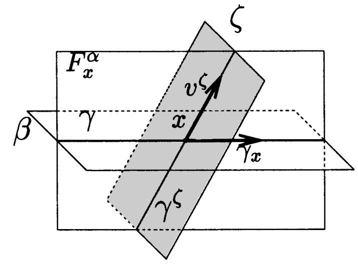

FIGURE 2.

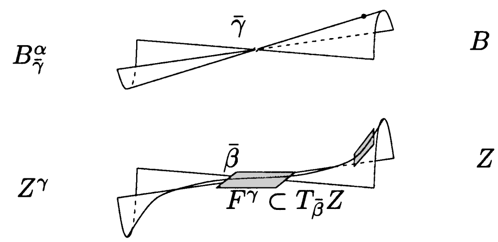

FIGURE 3.

Remark. Basically, this lift can only be defined for $Z^{\gamma} \smallsetminus \{\bar{\beta}\}$, but in this special case it can be extended by continuity to $\bar{\beta}$. Of course, the smoothness of the lifted surface can only be deduced away from the vertex $\bar{\gamma}$ (from the smoothness of $Z^{\gamma} \smallsetminus \{\bar{\beta}\}$).

4. THE PROJECTIVE CURVATURE OF THE α-CONE V^{α}_{γ} AND THE SELF-DUAL WEYL TENSOR W^{+} ON $\mathbf{M}$

As noted in Section 3, we intend to find a relation between the "curvature" of the α-cone $V^{\alpha}_{\bar{\gamma}}$ (its non-flatness) and the Weyl tensor W^{+} of $(\mathbf{M}, c)$. We begin by defining the *projective curvature* of V^{α}_{γ}: A projective structure on a manifold X is an equivalence class of linear connections yielding the same geodesics. In such a space, we can define the *projective curvature* of a curve S at a point σ as the linear application $k : T_{\sigma}S \otimes T_{\sigma}S \to N(S)_{\sigma} = T_{\sigma}X/T_{\sigma}S$, with $k(Y) := \nabla_Y Y$(modulo $T_{\sigma}S$), for ∇ any connection in the projective structure of X. In particular, we take for X the projective space $\mathbb{P}(T_{\bar{\gamma}}B)$, with its canonical projective structure, and for S we take $\mathbb{P}(V^{\alpha}_{\bar{\gamma}})$, the projectivized α-cone in $\bar{\gamma}$.

Definition 10. The *projective curvature of the α-cone $V^{\alpha}_{\bar{\gamma}}$ at the generating line* $\sigma \subset V^{\alpha}_{\bar{\gamma}}$ is the projective curvature of $S := \mathbb{P}(V^{\alpha}_{\bar{\gamma}})$ in σ, and is identified with a linear application

$$K^{\alpha}_{\gamma,x} : T_{\sigma}S \otimes T_{\sigma}S \to N(S)_{\sigma},$$

where σ is the tangent direction to $B^{\alpha}_{\bar{\gamma},x}$ in $\bar{\gamma}$.

In order to compute the projective curvature of $V^{\alpha}_{\bar{\gamma}}$, we first establish some canonical isomorphisms between the spaces appearing in the above definition and some linear subspaces of $T_x\mathbf{M}$. We fix the geodesic γ, the point $x \in \gamma$ (therefore also $\sigma = T_{\bar\gamma} B^\alpha_{\bar\gamma,x} \in \mathbb{P}(T_{\bar\gamma}B))$, and, thus, the α-plane $F^\alpha_x \subset T_x\mathbf{M}$ containing $\dot\gamma_x$, as well as $\dot\gamma_x^\perp \subset T_x\mathbf{M}$, the space orthogonal to $\dot\gamma_x$.

For simplicity, in the following lemmas we will omit some indices referring to these fixed objects.

Lemma 2. *There is a canonical isomorphism τ between the tangent space $T_\sigma S$ to the projective cone $S = \mathbb{P}(V^\alpha_{\bar\gamma})$ and the tangent space $T_x\gamma$ to the geodesic γ at the point x corresponding to the direction $\sigma \in \mathbb{P}(T_{\bar\gamma}B)$.*

Proof. Let $Y \in T_x\gamma$. We define $\tau^{-1}(Y)$ as follows. Recall that $T_\sigma S \simeq \mathrm{Hom}(\sigma, E/\sigma)$, where $E\ (= E_x) := T_\sigma V^\alpha_{\bar\gamma}$ (the tangent space at a point to a cone is the same for all points on the line containing the point). We know that σ corresponds to $\mathcal{J}^\alpha_{\gamma,x}$, the space of Jacobi fields on γ vanishing at x and such that $\dot J_x \in F^\alpha$. It will be shown in the proof of the next theorem that E consists of classes of Jacobi vector fields such that $J_x, \dot J_x \in F^\alpha$, (4).

Then, on a representative Jacobi field $J \in \mathcal{J}^\alpha_{\gamma,x}$, we define $\tau^{-1}(Y)$ to be the class of Jacobi fields in E/σ represented by the Jacobi field J^Y on γ which is given by $J^Y_x := \nabla_Y J,\ \dot J^Y_x := 0$. We remark that $\nabla_Y J$ is what we usually denote $\dot J$, when the parameter on γ is understood.

It is straightforward to check that $J \mapsto J^Y$ induces an isomorphism $\tau^{-1}(Y) : \sigma \to E/\sigma$ for each non-zero $J \in \sigma = \mathcal{J}^\alpha_{\gamma,x}/\mathcal{J}^\gamma_\gamma$. □

We remark that $V^\alpha_{\bar\gamma} \subset V_{\bar\gamma}$, the 4-dimensional subspace represented by Jacobi fields J such that $J, \dot J \perp \dot\gamma$. We further introduce the subspace $H^\alpha_{\bar\gamma,x} \subset V_{\bar\gamma}$, represented by Jacobi fields J as before, with the additional condition $J_x \in F^\alpha_x$. It is a 3-dimensional subspace, and it contains E_x. The curvature of $V^\alpha_{\bar\gamma}$ will take values in $\mathrm{Hom}(TS \otimes TS, N^V(S))$, and we show ((6) in the proof of the next theorem) that it takes values in a smaller space, $\mathrm{Hom}(TS \otimes TS, N^H(S))$. $N^V_\sigma(S) \simeq \mathrm{Hom}(\sigma, V_{\bar\gamma})/T_\sigma S$ is just the normal space of S in $\mathbb{P}(V_\gamma)$ at σ, and $N^H(S)$ is the subspace of $N^V_\sigma(S)$ consisting of elements represented by $\xi \in \mathrm{Hom}(\sigma, H^\alpha_{\bar\gamma,x}) \subset \mathrm{Hom}(\sigma, V_{\bar\gamma})$.

Lemma 3. *There is a canonical isomorphism*

$$\rho : N^H(S) \to \mathrm{Hom}(F^\alpha/T\gamma, \gamma^\perp/F^\alpha).$$

Proof. As H is a subbundle of the normal bundle $N(S)$, $N^H(S)$ is isomorphic to $\mathrm{Hom}(\sigma, H/E)$. As in Lemma 2, we will construct the inverse isomorphism ρ^{-1}. Let $\xi : F^\alpha/T\gamma \to \gamma^\perp/F^\alpha$ be a linear application. Let $\xi_0 : F^\alpha \to \gamma^\perp$ be a representant of ξ (it involves a choice of a complementary space to F^α in $\gamma^\perp$). We define $\rho^{-1}(\xi) \in \mathrm{Hom}(\sigma, H/E)$ as being induced by the following linear application between spaces of Jacobi fields on γ.

$\rho^{-1}(\xi) : \mathcal{J}^\alpha_{\gamma,x} \to \mathcal{J}^{\alpha,\perp}_{\gamma,x}$, where the second space corresponds to H_x, i.e. it contains Jacobi fields J such that $J_x \in F^\alpha, \dot J_x \perp \dot\gamma_x$. Consider a parameterization of γ around x, and let $J \in \mathcal{J}^\alpha_{\gamma,x}$. We define $J^\xi := \rho^{-1}(\xi)(J)$ by $J^\xi_x := 0, \dot J^\xi_x := \xi_0(\dot J_x)$, and it is easy to check that the class of J^ξ in H/E is independent of the representant ξ_0 such that ρ^{-1} is well-defined. It is also obviously invertible. □

We are now in position to translate the projective curvature of V^α_γ in terms of conformal invariants of $(\mathbf{M}, c)$.

Theorem 1. *Let x be a point on a null-geodesic γ. Then the projective curvature K of the α-cone $V^{\alpha}_{\tilde{\gamma}}$ at σ (corresponding to x, see Definition 10), which is a linear map*

$$K : T_\sigma S \otimes T_\sigma S \to N^V(S)_\sigma,$$

takes values in $N^H(S)_\sigma$ (see above), and is canonically identified with the linear map

$$K' : T_x\gamma \otimes T_x\gamma \to \mathrm{Hom}(F^{\alpha}_x/T_x\gamma, \gamma^{\perp}_x/F^{\alpha}_x)$$

defined by the self-dual Weyl tensor of $\mathbf{M}$:

$$K'(Y,Y)(X) = W^+(Y,X)Y, \quad Y \in T_x\gamma,\ X \in F^{\alpha}_x.$$

Proof. Consider the following analytic map, which parameterizes, locally around $x \in \gamma$, the deformations of the geodesic γ that correspond to points contained in the integral α-cone $B^{\alpha}_{\tilde{\gamma}}$:

$$f : U \to \mathbf{M},\ f(t,s,u) = \gamma^{t,s}(u),$$

where U is a neighborhood of the origin in $\mathbb{C}^3$, and $\gamma^{t,s}$ is a deformation of the null-geodesic γ, such that

$$\gamma^{t,s}(t) = \gamma(t),\ \dot{\gamma}^{t,s}(t) \in F^{\alpha}_{\gamma(t)},$$

where the parameterization of the geodesic γ satisfies $\gamma(0) = x$, and $F^{\alpha}_{\gamma(u)}$ is the α-plane in $T_{\gamma(u)}\mathbf{M}$ containing $\dot{\gamma}(u)$.

Convention. We know that f is defined around the origin in $\mathbb{C}^3$, so there exists a polydisc centered at the origin included in U, and so all the relations that we will use are true for values of the variables t, s, u sufficiently close to 0. For simplicity, we will not mention these domains.

The geodesics $\gamma^{t,s}$ correspond to points in $B^{\alpha}_{\tilde{\gamma},\gamma(t)}$, and the Jacobi fields J^t on γ, defined as

$$J^t(u) := \partial_s f(t,0,u) \in T_{\gamma(u)}\mathbf{M},$$

correspond to vectors in $V^{\alpha}_{\tilde{\gamma}}$ tangent to the above-mentioned lines. We suppose that the deformation f is *effective*, i.e. $\partial_u \gamma^{t,s}(u) \neq 0$ and $J^t \notin \mathcal{J}^{\gamma}_{\gamma}$, which is equivalent to $\dot{J}^t(t) \notin T_{\gamma(t)}\gamma$. In order to compute the projective curvature of $V^{\alpha}_{\tilde{\gamma}}$, we thus need to study the (second order) infinitesimal variation of these Jacobi fields on γ. As they are determined by their value and first order derivative in $\gamma(0) = x$, we need to evaluate $\partial_t J^t(0)|_{t=o}, \partial_t \dot{J}^t(0)|_{t=0}$ for the first derivative of J^t at $t = 0$, and $\partial^2_t J^t(0)|_{t=0}, \partial^2_t \nabla \dot{J}^t(0)|_{t=0}$ for the second. Dots mean, as before, covariant differentiation with respect to the "speed" vector $\dot{\gamma}$; thus they correspond to the operator ∂_u.

As the covariant derivation ∇ has no torsion, we can apply the usual commutativity relations between the operators $\partial_t, \partial_s, \partial_t$ and use them to differentiate the following equation, which follows directly from the definition of f and J^t:

$$J^t(t) = 0\ \forall t. \tag{1}$$

We get then

$$\partial_t J^t(t) + \dot{J}^t(t) = 0, \tag{2}$$

We recall now that, besides (1), we have $\dot{J}^t(t) \in F^\alpha_{\gamma(t)}$; thus $\dot{J}^t(t)$ is isotropic, which implies that

$$\langle \partial_t \dot{J}^t(t), \dot{J}^t(t) \rangle = 0, \tag{3}$$

as $\ddot{J}^t(t) = R(\dot{\gamma}(t), J^t(t))\dot{\gamma}(t) = 0$. Equations (2) and (3) prove that

$$\partial_t J^t|_{t=0} \in \mathcal{J}^\alpha_{\gamma,x}, \tag{4}$$

which completes the proof of Lemma 2. From (3), it equally follows that $\partial_t \dot{J}^t(t)$ is isotropic, and, by differentiating (3), we get

$$\langle \partial_t^2 \dot{J}^t(t), \dot{J}^t(t) \rangle = -\langle \partial_t \ddot{J}^t(t), \dot{J}^t(t) \rangle. \tag{5}$$

From (2) we have that $\partial_t J^t(t)$ is isotropic, and also

$$\partial_t^2 J^t(t) + 2\partial_t \dot{J}^t(t) = 0,$$

which, together with (3), implies that $\partial_t^2 J^t(0)|_{t=0} \in F^\alpha_x$. Then we have

$$\partial_t^2 J^t|_{t=0} \in \mathcal{J}^{\alpha,\perp}_{\gamma,x}, \tag{6}$$

which proves that the curvature K of the α-cone takes values in $N^H(S)$, as it is represented by $\partial_t^2 J^t|_{t=0}$.

In view of Lemmas 2 and 3, it is clear now that the projective curvature K is represented by the following application:

$$(\dot{\gamma}, \dot{\gamma}, \dot{J})_x \longmapsto \partial_t^2 J^t(0)|_{t=0}.$$

From (5), as $\partial_t \ddot{J}^t(t) = R(\dot{\gamma}, \partial_t J^t)\dot{\gamma}$ and $\partial_t J^t(t) = -\dot{J}^t(t)$, we get

$$\langle K(\dot{\gamma}, \dot{\gamma})(\dot{J}), \dot{J} \rangle = \langle R(\dot{\gamma}, \dot{J})\dot{\gamma}, \dot{J} \rangle.$$

The right-hand side actually involves only W^+, as the other components of the Riemannian curvature vanish on this combination of vectors. Thus we can replace R by W^+ in the above relation. On the other hand, the class of $W^+(\dot{\gamma}, \dot{J})\dot{\gamma}$ modulo F^α is determined by its scalar product with $\dot{J}$, which represents a non-zero generator of $F^\alpha/T\gamma$.

The proof of the theorem is now complete. □

Remark. We may ask whether the projective lines in Z are the geodesics of some projective structure. Indeed, in the conformally flat case, when $\mathbf{M}$ is the Grassmannian of 2-planes in $\mathbb{C}^4$ (the complexification of the Möbius 4-sphere), $Z \simeq \mathbb{CP}^3$, and the complex lines are given by the standard (flat) projective structure. But there are two reasons (related to each other, as we will soon see) why Z cannot carry, in general, a canonical projective structure. First, we do not necessarily have projective lines $Z_x \ni \bar{\beta}$ in every direction of $T_{\bar{\beta}}Z$ (this would mean that $\beta \simeq \mathbb{CP}^2$, see the next section for a treatment of this problem), and second, the lift of a 2-plane $F^\gamma \subset T_{\bar{\beta}}Z$ would be a 2-plane in $T_{\bar{\gamma}}B$, so $V^\alpha_{\bar{\gamma}}$ would be a flat cone.

Corollary 1. *The projective lines Z_x in the twistor space Z are geodesics of a projective structure iff it is projectively flat and $\mathbf{M}$ is conformally flat.*

Proof. If Z admits a projective structure, some of whose geodesics are the lines Z_x, then we have, for a fixed $\bar{\beta} \in Z$, a linear connection around $\bar{\beta}$, whose geodesics in the directions of Z_x, $\bar{\beta} \in Z_x$ ($\Leftrightarrow x \in \beta \subset \mathbf{M}$) coincide, locally, with Z_x. This means that the integral α-cone Z^γ, for $\gamma \subset \beta$ a null-geodesic, is part of a complex surface (namely $\exp(F^\gamma)$, where $F^\gamma \subset T_{\bar{\beta}}Z$ is the 2-plane corresponding to γ). Then the integral α-cone $B^\alpha_{\bar\gamma}$, the lift to B of Z^γ, is also a complex surface, and so $V^\alpha_{\bar\gamma}$ is a subset of the tangent space $T_{\bar\gamma}B^\alpha_{\bar\gamma}$, thus a flat cone. As this is true for all points of Z and for all null-geodesics γ, Theorem 1 implies that $\mathbf{M}$ is flat.

On the other hand, it is well-known that the twistor space of a conformally flat manifold admits a flat projective structure, for which the projective lines Z_x are geodesics, [1]. □

5. Compactness of null-geodesics and conformal flatness

5.1. **Complete α-cones in Z.** We have given, in the preceding section, a way to measure the projective curvature of the α-cone in B; we shall see now what happens in the special case when this cone is *complete* at a point $\bar\gamma$, i.e. when $\mathbb{P}(V^\alpha_{\bar\gamma})$ is a compact submanifold in $\mathbb{P}(T_\gamma B)$.

This situation appears for example if, for every direction in $F^\gamma \subset T_{\bar\beta}Z$, there are projective lines in Z tangent to it.

Theorem 2. *Let Z be the twistor space of the connected civilized self-dual 4-manifold $(\mathbf{M}, c)$, and suppose that, for a point $\beta \in Z$ and for a 2-plane $F^\alpha \subset T_{\bar\beta}Z$, there are projective lines Z_x tangent to each direction of F^α. Then $(\mathbf{M}, c)$ is conformally flat.*

Proof. The idea is to prove that the integral α-cone Z^γ is a smooth surface. We know that this holds at all its points except the vertex $\bar\beta$ (Proposition 4). The fact that all directions in F^γ admit a tangent line is a necessary condition for this cone to be a smooth surface, as it needs to be well-defined around $\bar\beta$.

We choose an auxiliary Hermitian (real) metric h on Z. Its restrictions h_x to the lines $Z_x \subset Z^\gamma$ yield Kählerian metrics on these lines; in fact these metrics are deformations of one another, just like the lines Z_x are. This means that the metrics h_x depend continuously on $x \in \mathbb{P}(F^\alpha)$, a parameter in a compact set. We can therefore find a lower bound $r_0 > 0$ for the injectivity radius of all (Z_x, h_x) at $\bar\beta$, and a finite upper bound R for the norm of all the second fundamental forms $H_x : TZ_x \otimes TZ_x \to (TZ_x)^\perp\,(\subset TZ)$. We can also suppose that r_0 is smaller than the injectivity radius of (Z, h) at $\bar\beta$.

The first step is to prove that Z^γ is a submanifold of class $\mathcal{C}^1$. As its tangent space is everywhere a complex subspace of TZ, it will follow that it is a complex analytic submanifold.

Consider now the exponential map $\exp_{\bar\beta} : T_{\bar\beta}Z \to Z$, defined for the metric h. If we restrict it to a ball of radius less than r_0, it is a diffeomorphism into Z. The image of the complex plane F^α is then a smooth 4-dimensional real submanifold S of Z, and there exists a positive number r_1 such that the exponential map in the directions normal to S,

$$\exp_S : TS^\perp \to Z,\, \exp(Y) := \exp_y(Y),\ y \in S,\, Y \in T_yS^\perp,$$

restricted to the vectors of length less than r_1, is a diffeomorphism.

The image of this diffeomorphism is a tubular neighborhood of S, and we will denote by $N(S, r)$ such a tubular neighborhood of "width" r, for $r < r_1$.

The existence of an upper bound R for the second fundamental forms of $Z_x, \forall x \in \gamma$, implies the following fact.

Lemma 4. *For any $r < r_1$, there is a neighborhood $U \subset T_{\bar\beta}Z$ of the origin such that $\exp(U) \cap Z^\gamma$ is contained in $N(S,r)$ and is transverse to the fibers of the orthogonal projection $p^S : N(S,r) \to S$, $p^S(\exp(Y)) := y$, where $Y \in T_yS$.*

This is standard if Z^γ is a submanifold; but it is also true in our case, where Z^γ is a union of submanifolds Z_x.

Now it is easy to prove that Z^γ is a $\mathcal{C}^1$ submanifold of Z (the projection p^S yields a local $\mathcal{C}^1$ diffeomorphism from a neighborhood of $\bar\beta$ in S to a neighborhood of $\bar\beta$ in Z^γ; it is $\mathcal{C}^1$ at $\bar\beta$ because S is tangent to Z^γ at $\bar\beta$).

So Z^γ is a $\mathcal{C}^1$ submanifold of Z. Its tangent space is complex at each point, and so Z^γ is a complex-analytic surface immersed in Z.

Then $B^\alpha_{\tilde\gamma} \subset B = \mathbb{P}(T^*Z)$, being the lift of Z^γ, is a smooth analytic surface immersed in B; in particular, the α-cone $V^\alpha_{\tilde\gamma}$ is a complex plane.

Theorem 1 implies that W^+ vanishes on the α-plane $F^\alpha_x \subset T_x\mathbf{M}$ which contains $\dot\gamma_x$, for every point $x \in \gamma$. Now, the plane $F^\gamma \subset T_{\bar\beta}Z$ is not the only one admitting projective lines Z_x tangent to any of its directions: all planes "close" to F^γ have the same property. Then W^+ vanishes on a neighborhood of γ, hence on the whole connected manifold $\mathbf{M}$. □

Remark. There is a more general situation where the integral α-cone Z^γ through β is smooth in β.

Theorem 2′. *Suppose that, for each direction $\sigma \in \mathbb{P}(T_\beta Z)$, there is a smooth (not necessarily compact) curve Z_σ, tangent to σ, such that*

(i) if σ is tangent to a projective line Z_x, then $Z_\sigma = Z_x$, and

(ii) Z_σ varies smoothly with $\sigma \in \mathbb{P}(F^\gamma)$.

Then

$$\bar Z^\gamma_\beta := \bigcup_{\sigma \in \mathbb{P}(F^\gamma)} Z_\sigma$$

is a smooth surface around β containing the α-cone Z^γ, and $W^+(F^\gamma_x) = 0$, $\forall x \in \gamma$, where $F^\gamma_x \subset T_x\mathbf{M}$ is the α-plane containing $\dot\gamma$.

The proof is similar to that of the previous theorem. Note that, if there is a direction σ which is not tangent to a projective line Z_x, we cannot apply the deformation argument in Theorem 2 to conclude that W^+ vanishes everywhere.

Example. If $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$, then $Z = \mathcal{F}$ and there are some particular planes for which the conditions in Theorem 2′ are satisfied, although Theorem 2 never applies to Z: for a generic 2-plane F^γ, the α-cone V^α_γ is not flat. These particular planes in TZ correspond to the vanishing of W^+ on some particular α-planes, but $\mathbf{M}$ is not anti-self-dual (see Section 7.3, and also 7.7, 7.8).

The following result is a direct consequence of Theorem 2:

Theorem 3. *If a civilized self-dual complex 4-manifold $(\mathbf{M}, c)$ admits a compact null-geodesic, then the conformal structure of $\mathbf{M}$ is flat, and the null-geodesic is simply-connected.*

We simply have to use the fact that a null-geodesic γ of a civized self-dual manifold identifies with an open set of $\mathbb{P}(F^\gamma)$, where F^γ is the associated 2-plane in $T_{\bar\beta}Z$, where $\beta \supset \gamma$.

The condition that $\mathbf{M}$ be civilized is not essential, if we assume that γ is simply-connected (and compact); in order to prove that, we need to cover γ with civilized (e.g. geodesically connected) open sets U_i, and relate the local twistor spaces $Z_i := Z(U_i)$; the key point is that, if γ is diffeomeorphic to $\mathbb{CP}^1$, it turns out that a neighbourhood of $\bar{\beta} \in Z_i$—for β the β-surface containing γ—can be identified with the space of *deformations of* γ *as a compact curve* ([2], Proposition 5). Then we conclude, using the criterion from Theorem 2′ and a deformation argument, that $\mathbf{M}$ is conformally flat.

This method is used in [2] to prove the same thing starting from a conformal complex 3-manifold (using the *LeBrun correspondence*, i.e. the local realization of a conformal 3-manifold as the conformal infinity of a (germ-unique) self-dual manifold [12]), but we also show there, by different methods, that, in all generality, a conformal complex n-manifold ($n \geq 4$) containing a compact, simply-connected null-geodesic is conformally flat ([2], Theorem 4).

6. The projective structure of β-surfaces in a self-dual manifold

The null-geodesics contained in a β-surface β define a *projective structure* on the totally-geodesic surface β, which is also given by any connection on β induced by a Levi-Civita connection on $\mathbf{M}$. We claim that this projective structure is *flat*, i.e. locally equivalent to $\mathbb{CP}^2$.

Example. If $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$, then a β-surface indexed by $(L, l) \in \mathcal{F}$ is $\beta^{(L,l)} = \{(A, a) | A \subset l, L \subset a, A \not\subset a\} \simeq \mathbb{C}^2$, and the null-geodesics in $\beta^{(L,l)}$ are identified with the affine lines in $\mathbb{C}^2$ (see Section 7.5).

To prove the projective flatness of a 2-dimensional manifold β, we need to prove that the *Thomas tensor* T vanishes identically [20]. This tensor is an analog of the *Cotton-York tensor* in conformal geometry (there is also a *Weyl tensor* of a projective structure, but it only appears in dimensions greater than 2).

For a connection ∇ in the projective class of β, the Thomas tensor is defined as follows [20]: For $X, Y, Z \in T\beta$,

$$\begin{aligned} T(X, Y, Z) := & -2(\nabla_Z K)(Y)X + 2(\nabla_Y K)(Z)X \\ & - (\nabla_Z K)(X)Y + (\nabla_Y K)(X)Z, \end{aligned} \tag{7}$$

where the derivation involves only the curvature term K, which is defined by $K(Y)X := \operatorname{tr} R(Y, \cdot)X$, the trace of the endomorphism $R(Y, \cdot)X \in \operatorname{End}(T\beta)$.

The Thomas tensor is independent of the connection ∇. Therefore we will consider that ∇ is induced by a Levi-Civita connection on $\mathbf{M}$.

Proposition 6. *The Thomas tensor of a β-surface can be expressed in terms of the anti-self-dual Cotton-York tensor of* $\mathbf{M}$. *Thus it is identically zero.*

Proof. First we need to define the *anti-self-dual* Cotton-York tensor as an irreducible component of the Cotton-York tensor of $\mathbf{M}$.

Convention. We denote by C the Cotton-York tensor of $(\mathbf{M}, c)$; we will not use this letter for the isotropic cone in this section.

The Cotton-York tensor is not conformally invariant; its definition depends on a (local) metric g in the conformal structure, which is supposed to be fixed [5]:

$$C(X, Y)(Z) := (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \ \forall X, Y, Z \in T\mathbf{M}, \tag{8}$$

where h is the normalized Ricci tensor of $\mathbf{M}$,

$$h = \frac{1}{2n(n-1)} \operatorname{Scal} \cdot g + \frac{1}{n-2} \operatorname{Ric}_0, \tag{9}$$

$\mathrm{Ric}_0, \mathrm{Scal}$ being the trace-free Ricci tensor, resp. the scalar curvature of the metric g, and $n := \dim \mathbf{M}$. In our case $n = 4$, but the formula applies in all dimensions greater than 2 [5].

Remark. The Cotton-York tensor C of $\mathbf{M}$ is a 2-form with values in $T^*\mathbf{M}$; thus it has two components, $C^+ \in T^*\mathbf{M} \otimes \Lambda^+\mathbf{M}$, and $C^- \in T^*\mathbf{M} \otimes \Lambda^-\mathbf{M}$. C satisfies a *first* Bianchi identity, due to the fact that h is a symmetric tensor, and also a *contracted* (second) Bianchi identity, which comes from the second Bianchi identity in Riemannian geometry, [5]:

$$\sum C(X,Y)(Z) = 0, \quad \text{circular sum;} \tag{10}$$

$$\sum C(X,e_i)(e_i) = 0, \quad \text{trace over an orthonormal basis.} \tag{11}$$

That means that $C \in \Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M}$, and is orthogonal on $\Lambda^3\mathbf{M} \subset \Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M}$ and on $\Lambda^1\mathbf{M}$, which is identified with the image in $\Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M}$ by the metric adjoint of the contraction (11).

Now, the Hodge operator $* : \Lambda^2\mathbf{M} \to \Lambda^2\mathbf{M}$ induces a symmetric endomorphism of $\Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M}$, which maps the two above spaces isomorphically into each other. This implies that C^+ and C^- satisfy (10) and (11) (note that these two relations are equivalent in their case) and are, therefore, $SO(4,\mathbb{C})$-irreducible.

The Cotton-York tensor is related to the Weyl tensor of $\mathbf{M}$ by the formula [5]

$$\delta W = C, \tag{12}$$

where $\delta : \Gamma(\Lambda^2\mathbf{M} \otimes \Lambda^2\mathbf{M}) \to \Gamma(\Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M})$ is induced by the codifferential on the second factor, and by the Levi-Civita connection ∇ on the first. Then, C^+ has to be the component of δW in $\Lambda^+\mathbf{M} \otimes \Lambda^1\mathbf{M}$, and we know that the restriction of W^- to $\Lambda^+\mathbf{M} \otimes \Lambda^2\mathbf{M}$ is identically zero. This means that

$$\delta W^+ = C^+, \text{ and also} \tag{13}$$

$$\delta W^- = C^-. \tag{14}$$

Hence, as $\mathbf{M}$ is self-dual, C^- vanishes identically.

We can prove now that the Thomas tensor of a β-surface β is identically zero. First we show that

$$K(Y)X = \mathrm{tr}|_{T\beta} R(Y,\cdot)X = h(X,Y), \ \forall X,Y \in T\beta. \tag{15}$$

We recall from [5] that the *suspension* $h \wedge \mathbf{I}$ of h by the identity, viewed as an endomorphism of $\Lambda^2\mathbf{M}$, is defined by

$$(h \wedge \mathbf{I})(X,Y) := h(X) \wedge Y - h(Y) \wedge X, \ \ X,Y \in T\mathbf{M}, \tag{16}$$

where h is identified with a symmetric endomorphism of $T\mathbf{M}$.

We have then the following decomposition of the Riemannian curvature [5]:

$$R = h \wedge \mathbf{I} + W^+ + W^-.$$

Of course, if $\mathbf{M}$ is self-dual, then $W^- = 0$ and $W^+(X,Y) = 0$ if $X, Y \in T\beta$ (in fact, the elements in $\Lambda^2 F^\beta$, for any β-plane $F^\beta \subset T_x\mathbf{M}$, correspond to the isotropic vectors in $\Lambda^-\mathbf{M}$), because $W^+|_{\Lambda^- M} = 0$. Then, if we choose the basis $\{X, Y\}$ in

$T\beta$, we get

$$\begin{aligned} K(Y)X &= \operatorname{tr}|_{T\beta}(h \wedge \mathbf{I})(Y,\cdot)X \\ &= \text{the component along } X \text{ of } (h\wedge\mathbf{I})(Y,X)X \\ &= h(Y,X), \end{aligned}$$

which proves (15). The Thomas tensor of the projective structure of β has the following expression (see (7)):

$$T(X,Y,Z) = -3(\nabla_Z h)(Y,X) + 3(\nabla_Y h)(Z,X) = 3C(Y,Z)(X),\ \forall X,Y,Z \in T\beta,$$

and, as $C^+(\cdot,\cdot)(X)$ vanishes on the anti-self-dual 2-form $Y \wedge Z$, we conclude that

(17) $$T(X,Y,Z) = C^-(Y,Z)(X) = 0.$$

□

As the flatness of the projective structure on a 2-dimensional manifold is equivalent to the vanishing of its Thomas tensor [20], we get

Corollary 2. *The projective structure of the β-surfaces of a self-dual complex manifold $\mathbf{M}$ is flat.*

From the classification of projectively flat compact complex surfaces ([9], see also [7]), we then get a classification of compact β-surfaces in $\mathbf{M}$:

Theorem 4. *A compact β-surface of a self-dual complex 4-manifold belongs (up to finite covering) to one of the following classes:*

(1) $\mathbb{CP}^2$;
(2) *a compact quotient of the complex-hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}/\Gamma$;*
(3) *a compact complex surface admitting a (flat) affine structure:*
 (i) *a Kodaira surface;*
 (ii) *a properly elliptic surface with b_1 odd;*
 (iii) *an affine Hopf surface;*
 (iv) *an Inoue surface;*
 (v) *a complex torus.*

See [7], [9], [11] for details.

7. Examples

7.1. **The flat case.** The first example is the "flat" case: $Z = \mathbb{CP}^3 = \mathbb{P}(\mathbb{C}^4)$, with its canonical projective structure and its space of projective lines $\mathbf{M} = \mathrm{Gr}(2,\mathbb{C}^4)$. ($Z$ is equally the twistor space of the Riemannian round 4-sphere, which is, therefore, a real part of $\mathrm{Gr}(2,\mathbb{C}^4)$.) If $\beta \in Z$, then the β-surface associated to it is the set $\{x \in \mathrm{Gr}(2,\mathbb{C}^4) | \beta \subset x \subset \mathbb{C}^4\}$. In this flat case, we can equally define the α-twistor space Z^*, which is the dual projective 3-space $(\mathbb{CP}^3)^* := \mathbb{P}((\mathbb{C}^4)^*) = \mathrm{Gr}(3,\mathbb{C}^4)$, and an α-surface $\alpha \in Z^*$ is the set $\{x \in \mathrm{Gr}(2,\mathbb{C}^4) | x \subset \alpha \subset \mathbb{C}^4\} \subset \mathbf{M}$. A null-geodesic γ is then determined by a pair of *incident* isotropic surfaces α and β such that $\alpha \cap \beta = \gamma$, where α is an α-surface and β is a β-surface; *incident* means (see above) that β, seen as a line in $\mathbb{C}^4$, is *included* in α, seen as a 3-plane in $\mathbb{C}^4$. γ is then the following set of points in $\mathbf{M}$:

$$\gamma = \{x \in \mathrm{Gr}(2,\mathbb{C}^4) | \beta \subset x \subset \alpha\}.$$

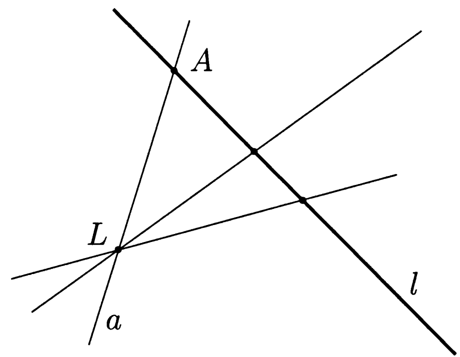

FIGURE 4.

α-surfaces and β-surfaces are diffeomorphic to $\mathbb{CP}^2$, null-geodesics to $\mathbb{CP}^1$, and the ambitwistor space B is the "partial flag" manifold

$$B = \{(\alpha, \beta) \in (\mathbb{CP}^3)^* \times \mathbb{CP}^3 | \beta \subset \alpha\}.$$

The flag manifold, of dimension 7, is isomorphic to the total space $\mathbb{P}(C)$ of the projective cone bundle over $\mathbf{M}$.

7.2. $\mathbb{CP}^2$**.** Another example is when Z is the twistor space of the real Riemannian manifold $\mathbb{CP}^2$, with the Fubini-Study metric. Then Z is the manifold of flags in $E = \mathbb{C}^3$, $\mathcal{F} := \{(L, l) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | L \subset l\}$ ($\mathbb{P}(E)$, resp. $\mathbb{P}(E)^*$ are viewed as the space of lines, resp. 2-planes, in E) [1]. A projective line Z_x in Z is a set

$$Z_x = \{(L, l) \in \mathcal{F} | L \subset a^x, \ A^x \subset l\}$$

(see Figure 4), where (A^x, a^x) belongs to $\mathbb{P}(E) \times \mathbb{P}(E)^* \smallsetminus \mathcal{F}$, which is, therefore, the space $\mathbf{M}$ of such lines, and a conformal self-dual 4-manifold. It can be naturally compactified within the space of analytic cycles of Z to $\overline{\mathbf{M}} = \mathbb{P}(E) \times \mathbb{P}(E)^*$, which is obviously a smooth manifold, but it carries no global conformal structure, as its canonical bundle has no square root. This means that the conformal structure on $\overline{\mathbf{M}}$ is smooth on $\mathbf{M}$, and *singular* on $\mathcal{F} = \overline{\mathbf{M}} \smallsetminus \mathbf{M}$. The cycles of Z corresponding to a point $\bar{x} = (A, a)$ in this subset are pairs of complex projective lines in Z:

$$Z_{\bar{x}} = \{(A, l) \in Z = \mathcal{F}\} \cup \{(L, a) \in Z = \mathcal{F}\}.$$

A β-surface in $\mathbf{M}$, corresponding to a point $\beta = (L, l) \in Z$, is the set

$$\beta = \{(A, a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | A \subset l, \ L \subset a, \ A \neq L, \ a \neq l\},$$

and can be naturally compactified to

$$\bar{\beta} = \{(A, l^\beta) \in \mathcal{F}\} \times \{(L^\beta, a) \in \mathcal{F}\} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1.$$

7.3. **The tangent space to $\mathcal{F}$.** In order to describe the null-geodesics of $\mathbf{M}$ as 2-planes in Z, we first study the tangent space of $Z = \mathcal{F}$ at $\beta = (L, l)$.

A vector in $T_{(L,l)}\mathcal{F}$ is a pair of vectors (V, v), with $V \in T_L\mathbb{P}(E)$ and $v \in T_l\mathbb{P}(E)^*$, which satisfy a linear condition (as $\mathcal{F} \subset \mathbb{P}(E) \times \mathbb{P}(E)^*$). Actually, there is a duality

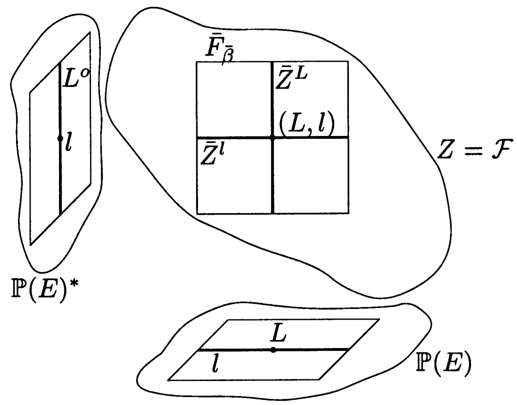

FIGURE 5.

between $\mathbb{P}(E)^*$, the Grassmannian of 2-planes in E, and $\mathbb{P}(E^*)$, the projective space of $E^* := \mathrm{Hom}(E, \mathbb{C})$, and an analogous one between $\mathbb{P}(E)$ and $\mathbb{P}(E^*)^*$:

$$\mathbb{P}(E)^* \ni l \quad \overset{\simeq}{\longmapsto} \quad l^o \in \mathbb{P}(E^*),$$
$$\mathbb{P}(E) \ni L \quad \overset{\simeq}{\longmapsto} \quad L^o \in \mathbb{P}(E^*)^*.$$

Then the flag manifold $\mathcal{F}$ is defined, as a submanifold of $\mathbb{P}(E) \times \mathbb{P}(E)^*$, by the equation

$$y(Y) = 0, \ \forall y \in l^o, \forall Y \in L.$$

The vector $V \in T_L\mathbb{P}(E)$ is an element in $\mathrm{Hom}(L, E/L)$. By duality, $v \simeq v^o \in \mathrm{Hom}(l^o, E^*/l^o)$. Then the vector $(V, v) \in T_{(L,l)}\mathbb{P}(E) \times \mathbb{P}(E)^*$ lies in $\mathcal{F}$ iff

$$v^o(y^o)(Y) + y^o(V(Y)) = 0, \ \forall Y \in L, \forall y^o \in l^o, \tag{18}$$

or, equivalently,

$$v|_L = \pi_l \circ V, \tag{19}$$

where $\pi_l : E/L \to E/l$ is the projection (as $L \subset l$).

The geometry of $\mathcal{F}$, as a subset of $\mathbb{P}(E) \times \mathbb{P}(E)^*$, can be described as in Figure 5.

7.4. **The 2-planes in $\mathcal{F}$.** Let us consider now a 2-plane F in $T_{(L,l)}\mathcal{F}$, and the cycles (corresponding to points in $\overline{\mathbf{M}}$) tangent to it. We have three cases:

1. $F = \bar{F}_\beta$ is the "degenerate" 2-plane tangent to the 2 special curves $\bar{Z}_L$, $\bar{Z}_l$ whose union is the special cycle $\bar{F}_(L, l)$ corresponding to $(L, l) \in \overline{\mathbf{M}} \setminus \mathbf{M}$. There are no projective lines Z_x, $x \in \mathbf{M}$, tangent to it; only the special cycles $\bar{Z}_{(L,a)}$, $L \subset a$, and $\bar{Z}_{(A,l)}$, $A \subset l$, are tangent to $\bar{F}_{(L,l)}$, actually only to the two privileged directions of $\bar{Z}_L$, resp. $\bar{Z}_l$.

Remark. The special curves $\bar{Z}_L$, $\bar{Z}_l$ have trivial normal bundle, being fibers of the projections from $\mathcal{F}$ to $\mathbb{P}(E)$, resp. $\mathbb{P}(E)^*$, so these special curves form two complete families of analytic cycles in $\mathcal{F}$, isomorphic to $\mathbb{P}(E)$, resp. $\mathbb{P}(E)^*$. Two such curves are incident iff they are of different types ($\bar{Z}_L$ is *of type* E, $\bar{Z}_l$ is *of type* E^*), so they can only form "polygons" with an even number of edges. But there are no quadrilaterals, as one can easily check, using the fact that $\bar{Z}_L$ and $\bar{Z}_l$ are incident

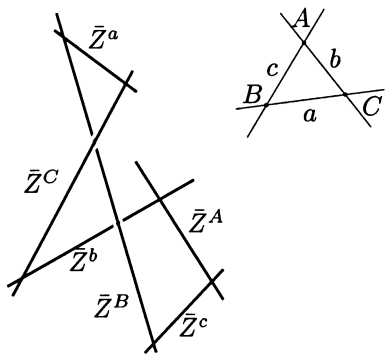

FIGURE 6.

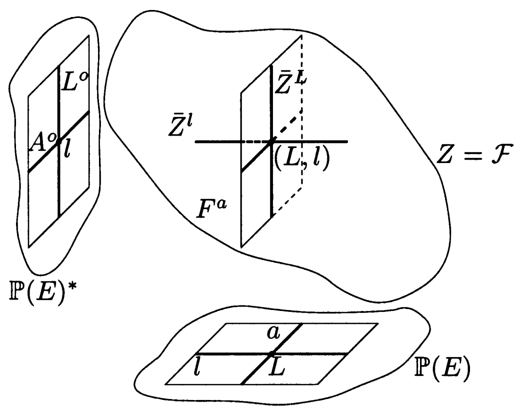

FIGURE 7.

iff $L \subset l$, thus iff l is a line in $\mathbb{P}(E)$ containing L. On the other hand, there are hexagons, corresponding to the 3 vertices and 3 sides of a triangle in $\mathbb{P}(E) \simeq \mathbb{CP}^2$ (see Figure 6).

The above hexagon is not "flat", i.e. there is no canonical submanifold of $\mathcal{F}$ containing it. This, and the fact that there are no quadrilaterals made of $\bar{Z}$-type curves, is just a consequence of the fact that the distribution $\bar{F}$ on $Z = \mathcal{F}$ is *not integrable*; in fact it is the holomorphic *contact structure* induced by the Fubini-Study *Einstein metric* on $\mathbb{CP}^2$ ([3]; see also Section 7.6).

2. $F = F^a$, for $a \supset L$, $a \neq l$. This is a 2-plane that is tangent to only one of the special curves $\bar{Z}_L$. The projective lines tangent to F^a at $\beta = (L, l)$ are $Z_{(A,a)}$, $\forall A \subset l$, $A \neq L$; hence the corresponding null-geodesic is

$$\gamma^a = \{(A, a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | A \subset l,\ A \neq l\}, \tag{20}$$

thus it is diffeomorphic to $\mathbb{C}$, and its closure is

$$\bar{\gamma}^a = \{(A, a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | A \subset l\} \simeq \mathbb{CP}^1$$

(see Figure 7).

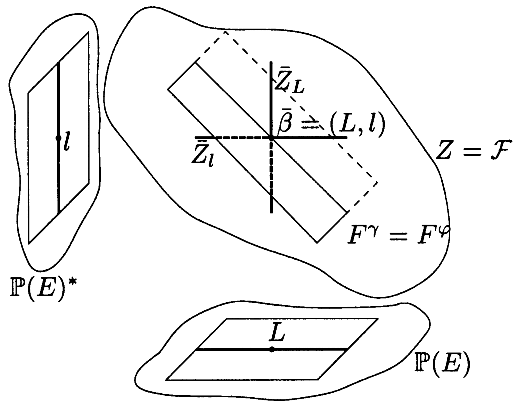

FIGURE 8.

Remark. The "limit" curve is $\bar{Z}_{(L,a)}$, so it is non-singular at (L,l). Actually, the points of $Z_{(A,a)}$ *close* to (L,l) converge, when $A \to L$, to some points in $\bar{Z}_L$, which is tangent to F^a. We can then apply the same method as in Theorem 2 to conclude that the integral α-cone associated to F^a is a smooth manifold *around* (L,l); thus, from Theorem 1, the Weyl tensor W^+ of $\mathbf{M}$ vanishes on the β-planes generated along γ^a by its own direction. We will see that the vanishing of W^+ on these α-planes leads to the existence of some α-surfaces, see below. Of course, the deformation argument in Theorem 2 does not hold in the present case, as the normal bundle of $\bar{Z}_L$ is trivial, thus different from that of the rest of the rational curves $Z_{(A,a)}$ (as we will see below, generic 2-planes through (L,l) do not admit projective lines tangent to all their directions).

2′. We have a similar situation for planes $F = F^A$ — $A \subset l$, $A \neq L$, tangent to the other special curve $\bar{Z}_l$.

3. This is the generic case: $F = F^\varphi$, where $\varphi : \mathbb{P}(l) \to \mathbb{P}(L^o)$ is a projective diffeomorphism such that $\varphi(L) = l^o$. Indeed, the tangent spaces $T_L\mathbb{P}(E)$ and $T_l\mathbb{P}(E)^*$ are isomorphic to $\mathrm{Hom}(L^o, E^*/L^o)$, resp. to $\mathrm{Hom}(l, E/l)$, and a generic 2-plane F in $T_{(L,l)}\mathcal{F}$ is the graph of a linear isomorphism $\phi : T_L\mathbb{P}(E) \to T_l\mathbb{P}(E)^*$ satisfying a linear condition (18) or (19). Actually, the graph is determined by the projective application φ induced by ϕ from $\mathbb{P}(T_L\mathbb{P}(E)) \simeq \mathbb{P}(L^o)$ to $\mathbb{P}(T_l\mathbb{P}(E)^*) \simeq \mathbb{P}(l)$ (see Figure 8).

The condition $\varphi(L) = l^o$ is implied by (19). The null-geodesic associated to the 2-plane F^φ is

$$\gamma^\varphi = \{(A,a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F} | A \subset l,\, a^o \subset L^o\, a^o = \varphi(A)\}, \tag{21}$$

and its closure in $\overline{\mathbf{M}}$ is

$$\bar{\gamma}^\varphi = \{(A,a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | A \subset l,\, a^o \subset L^o\}. \tag{22}$$

Hence the "limit" point is $(L,l) \in \overline{\mathbf{M}}$, corresponding to the special cycle $\bar{Z}_{(L,l)}$, none of whose components is tangent to F^φ. The integral α-cone associated to F^φ looks like what is shown in Figure 9.

7.5. **The null-geodesics of the complexification of $\mathbb{CP}^2$.** The application φ has the following interpretation in terms of projective geometry on $\mathbb{CP}^2 = \mathbb{P}(E)$: a

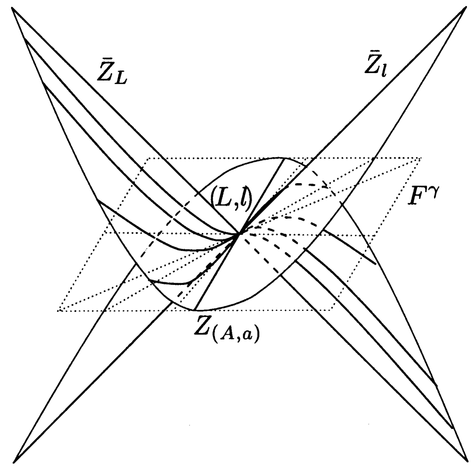

FIGURE 9.

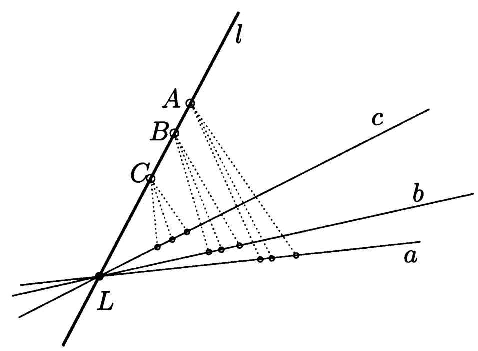

FIGURE 10.

direction $\mathbb{C}v$ in $T_l\mathbb{P}(E)^*$ is identified with the point $\ker v \equiv A \in l/L \subset \mathbb{P}(E)$ and a direction $\mathbb{C}V \subset T_L\mathbb{P}(E)$ is identified with a direction (thus a projective line a) through $L \in \mathbb{P}(E)$. φ is, thus, a *homography* that associates to $A \in l$ (we identify l with the projective line $l/L \subset \mathbb{P}(E)$) the line $a \ni L$. As $\varphi(L) = l$, we have, then, that three points $(A,a),(B,b),(C,c) \in \beta^{(L,l)}$ belong to the same null-geodesic iff

$$(A, B : C, L) = (a, b : c, l), \tag{23}$$

i.e. the cross-ratio of the points $A, B, C, L \in l$ equals the cross-ratio of the lines a, b, c, l through L (the dotted lines, together with their intersections with the lines a, b, c, correspond to the points in the integral α-cone; see Figure 10).

We can now describe the null-geodesics passing through a point $(A,a) \in \mathbf{M}$ and contained in a β-surface $\beta^(L,l)$ whose closure $\bar{\beta}$ is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$: they coincide with the rational curves in $\bar{\beta}$ containing (A,a); except the "horizontal" $(\bar{\gamma}^A)$ and "vertical" $(\bar{\gamma}^a)$ ones, all these curves contain (L,l); see Figure 11.

We remark that, in the usual affine coordinates on

$$\beta \simeq (\mathbb{CP}^1 \smallsetminus \{L\}) \times (\mathbb{CP}^1 \smallsetminus \{l\}) \simeq \mathbb{C}^2,$$

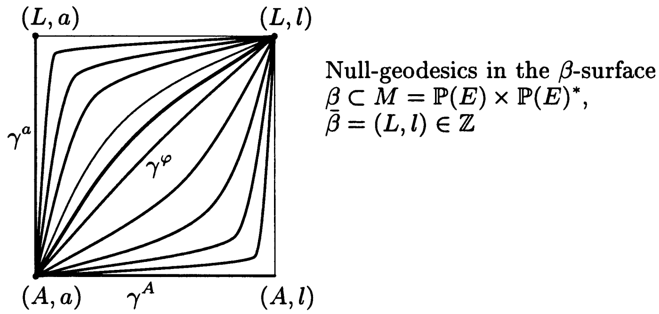

FIGURE 11.

these null-geodesics are the affine lines containing (A, a); thus the *projective structure* on β is (locally) isomorphic to a flat affine structure. We have seen, in Section 6 (Corollary 2), that this is true for all β-surfaces of a self-dual manifold.

7.6. **The conformal structure of the complexification of $\mathbb{CP}^2$.** Now let us study the conformal structure of $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$ directly; actually $\mathbf{M}$ has a complex metric g. Let $(A, a) \in \mathbf{M}$; then A is transverse to a, and so we have the isomorphisms $E/a \simeq A$ and $E/A \simeq a$. Then, a vector $(V, v) \in T_{(A,a)}\mathbf{M}$ is identified with a pair of homomorphisms $V : A \to a$ and $v : a \to A$, and the metric g is given by

$$g((V, v), (W, w)) := \operatorname{tr}(v \circ W + w \circ V), \quad \forall (V, v), (W, w) \in T_{(A,a)}\mathbf{M}. \tag{24}$$

Remark (The real part). Let h be an Hermitian metric on E. Then we have a real-analytic embedding of $\mathbf{M}_0 \simeq \mathbb{P}(E)$ into $\mathbf{M}$, given by

$$\mathbb{P}(E) \ni A \longmapsto (A, A^{\perp}) \in \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}.$$

A vector $(V, v) \in T_{(A,A^{\perp})}\mathbf{M}$ is tangent to $\mathbf{M}_0$ iff

$$h(x, v(y)) + h(V(x), y) = 0, \ \forall x \in A, \ \forall y \in A^{\perp}.$$

Then one easily checks that

$$g((V, v), (W, w)) = -2h(V, W), \ \forall (V, v), (W, w) \in T_{(A,A^{\perp})}\mathbf{M}_0.$$

Hence, up to a constant, the restriction of g to $\mathbf{M}_0 \simeq \mathbb{CP}^2$ is the Fubini-Study metric of $\mathbb{CP}^2 \simeq S^5/S^1$.

An isotropic vector in $\mathbf{M}$ is $(V, v) \in T_{(A,a)}\mathbf{M}$, with $v \circ V = 0$, viewed as an endomorphism of A (see above), or, equivalently, with

$$\dim(A + V(A) \cap \ker v) > 0. \tag{25}$$

Let us see which is the limit of the isotropic cone in the points of $\mathcal{F}$: from the relation above, it follows that the isotropic cone at a point $x \in \mathcal{F}$ is

$$C_x = \{(0, v) \in T_x\mathcal{F}\} \cup \{(V, 0) \in T_x\mathcal{F}\},$$

so the conformal structure of $\mathbf{M}$ is *singular* at the "infinity" $\mathcal{F}$.

Remark. The situation $\mathcal{F} \subset \mathbb{P}(E) \times \mathbb{P}(E)^*$ is very similar to the one treated in [2]; see also [12]: $\mathbb{P}(E) \times \mathbb{P}(E)^*$ has an Einstein self-dual metric g, singular at the "infinity", and this Einstein structure yields a contact structure on the twistor space $Z = \mathcal{F}$; the field of 2-planes determined by this contact structure corresponds to the "infinity" $\mathcal{F} \subset \mathbb{P}(E) \times \mathbb{P}(E)^*$. But these planes do not admit tangent rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$: the conformal structure does not extend to the "infinity" (which is, therefore, not a *conformal infinity*).

7.7. **α-planes and β-planes.** We consider the isotropic planes in $T_{(A,a)}\mathbf{M}$ ($A \not\subset a$): For a fixed isotropic direction, represented by a generic vector $(V, v) \in T_{(A,a)}\mathbf{M}$, the line $\ker v \subset a$ and the plane $V(A) + A \supset A$ are fixed. The linear space of all vectors $(W, v) \in T_{(A,a)}\mathbf{M}$ satisfying

$$W(A) \subset A + V(A), \quad w|_{\ker v} = 0,$$

is isotropic and orthogonal to (V, v): they form a β-plane. The α-plane F^α containing (V, v) corresponds to the isotropic vectors (W, w) orthogonal to (V, v) with $\ker w \neq \ker v$. As a plane transverse to all the β-planes (whose projection onto $T_A\mathbb{P}(E)$ or $T_a\mathbb{P}(E)^*$ is never injective), F^α is determined by a linear isomorphism $\varphi : T_A\mathbb{P}(E) \to T_a\mathbb{P}(E)^*$, whose graph in $T_{(A,a)}\mathbb{P}(E) \times \mathbb{P}(E)^*$ is F^α; φ induces the application $\mathbb{P}\varphi : \mathbb{P}(a) \to \mathbb{P}(E/A)$ between the projective spaces of $T_A\mathbb{P}(E)$, resp. $T_a\mathbb{P}(E)^*$. The plane $F^\alpha = F^\varphi$, the graph of φ, is isotropic iff $V \subset \mathbb{P}\varphi(V)$, $\forall V \in \mathbb{P}(a)$, i.e. $\mathbb{P}\varphi$ is the homography that sends a point X in a into the projective line through A and X. We can extend φ to a projective isomorphism $\varphi' : \mathbb{P}(\mathbb{C} \oplus T_A\mathbb{P}(E)) \to \mathbb{P}(\mathbb{C} \oplus T_a\mathbb{P}(E)^*)$: for example, $\mathbb{P}(\mathbb{C} \oplus T_A\mathbb{P}(E))$ contains $T_A\mathbb{P}(E)$ as an affine open set. Then φ' is defined as follows:

$$\begin{aligned} \varphi'|_{T_A\mathbb{P}(E)} &:= \varphi, \\ \varphi'|_{\mathbb{P}(T_A\mathbb{P}(E))} &:= \mathbb{P}\varphi. \end{aligned}$$

Actually $\mathbb{P}(\mathbb{C} \oplus T_A\mathbb{P}(E)) \simeq \mathbb{P}(E)$ and $\mathbb{P}(\mathbb{C} \oplus T_a\mathbb{P}(E)^*) \simeq \mathbb{P}(E)^*$. We then have

Proposition 7. *A generic α-plane $F^\alpha = F^\varphi$ in $T_{(A,a)}\mathbf{M}$ is the graph of a linear isomorphism $\varphi : T_A\mathbb{P}(E) \to T_a\mathbb{P}(E)^*$, which is determined by a projective isomorphism*

$$\varphi' : \mathbb{P}(E) \to \mathbb{P}(E)^*$$

such that $\varphi'(A) = a$ and $\varphi'(l) = l \cap a$, for all $l \supset A$.

7.8. **Exponentials of α-planes.** The exponential $\exp(F^\varphi)$ has an interpretation in terms of projective geometry. Each direction $\mathbb{C}(V, v) \subset F^\varphi$ is determined by the point $\ker v$ in $a \subset \mathbb{P}(E)$ and the line through A and $\ker v$, and a homography $\phi^{(V,v)}$ from the points B of the projective line $A + \ker v$ to the space of lines b through $\ker v$ (see Figure 12 and the convention below). As this homography is the restriction of φ' to the appropriate spaces, it follows that it is related to the homography $\phi^{(W,w)}$, where $\mathbb{C}(W, w)$ is another direction in F^φ: the points $D := b \cap c$, $P := a \cap (B + C)$ and A are collinear (see Figure 12).

Of course, this implies that P determines a homography ψ^P between the lines $A + \ker v$ and $A + \ker w$, such that $\psi^P(A) = A$ and $\psi^P(\ker v) = \ker w$. Then, for any other points $B' \in (A + \ker v)$, $C' = \psi^P(B) \in (A + \ker w)$, the lines $b' = \phi^{(V,v)}(B')$, $c' = \phi^{(W,w)}(C')$ intersect on the line $(A + P)$ (see the right-hand side of Figure 12).

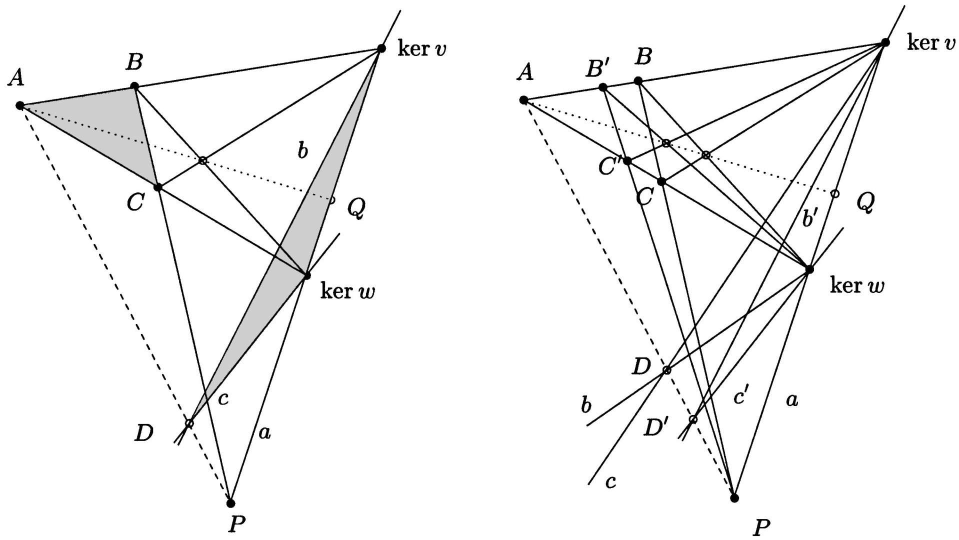

FIGURE 12.

Convention. In the framework of plane projective geometry, we identify a point in $\mathbb{P}(E)^*$ with a line in $\mathbb{P}(E)$ (we denote, for example, $\ker v \in a$). The lines determined by the distinct points B and C will be denoted by $(B+C)$ (thus $B, C \in (B+C)$).

The null-geodesic tangent to (V,v) at (A,a) is the set $\{(B,b) | B \in (A + \ker v),\ b = \phi^{(V,v)}(B)\}$, and the null-geodesic tangent to (W,w) is the analogous set of the pairs (C,c). Thus

$$\exp_{(A,a)}(F^{\varphi}) = \exp_{(A,a)}(F^{\alpha}) = \{(C,c) | C \in \mathbb{P}(E),\ C \neq A,\\ c = ((C+A) \cap a) + ((A+P) \cap b^C)\} \cup \{(A,a)\},$$

where $B^C := (A + \ker v) \cap (P + C)$, and $b^C := \phi^{(V,v)}(B^C)$, as in Figure 12 (where $B = B^C$, $b = b^C$). This gives the exponential of the α-plane determined by the isotropic vector (V,v). We remark that the point (A,a) has a privileged position in $\exp_{(A,a)}(F^{\alpha})$: $(a \cap b) \in (A+B)\ \forall (B,b) \in \exp_{(A,a)}(F^{\alpha})$; on the other hand, $(b \cap c) \notin (B+C)$ in general (see Figure 12), which means that the points (B,b) and (C,c) are not *null-separated* (i.e. they do not belong to the same null-geodesic). That means that $\exp_{(A,a)}(F^{\alpha})$ is not totally isotropic; thus there is no α-surface tangent to a generic α-plane—not surprising, as the corresponding α-cone is not flat (see Section 7.4).

But there are α-surfaces tangent to the two α-planes $\{(V,0) | V \in T_A\mathbb{P}(E)\}$ and $\{(0,v) | v \in T_a\mathbb{P}(E)^*\}$: the "slices" $\{A\} \times \mathbb{P}(E)^*$ and $\mathbb{P}(E) \times \{a\}$. (It is easy to see that these planes are isotropic, and that they are not β-planes, as these project on *lines* in $T_A\mathbb{P}(E)$, resp. $T_a\mathbb{P}(E)^*$.)

Thus $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$ is a conformal self-dual manifold, not anti-self-dual, that admits α-surfaces passing through any point.

Acknowledgements

The author is deeply indebted to Paul Gauduchon, for his care in reading the manuscript and for his constant help during the research and redaction.

References

[1] M.F. Atiyah, N.J. Hitchin, I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proceedings of the Royal Society of London, Series A **362** (1978), 425-461. MR **80d:**53023

[2] F.A. Belgun, *Null-geodesics on complex conformal mmanifolds and the LeBrun correspondence*, J. Reine Angew Math. **536** (2001), 43–63. MR **2002g:**53080

[3] A.L. Besse, *Einstein manifolds*, Springer Verlag (1987). MR **88f:**53087

[4] F. Campana, *On twistor spaces of the class $\mathcal{C}$*, J. Diff. Geom. **33** (1991), 541-549. MR **92g:**32059

[5] P. Gauduchon, *Connexion canonique et structures de Weyl en géométrie conforme*, Preprint, juin 1990.

[6] N.J. Hitchin, *Complex manifolds and Einstein's equations*, in *Twistor Geometry and Nonlinear Systems*, Lecture Notes in Mathematics **970** (1982) Springer-Verlag, 73-99. MR **84i:**32041

[7] B. Klingler, *Structures affines et projectives sur les surfaces complexes*, Ann. Inst. Fourier **48** (1998), 441-477. MR **99c:**32038

[8] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry I*, Interscience, John Wiley (1963). MR **27:**2945

[9] S. Kobayashi, T. Ochiai, *Holomorphic projective structures on compact complex surfaces*, Math. Annalen, **249** (1980), 75-94. MR **81g:**32021

[10] K. Kodaira, *A theorem of completeness of characteristic systems for analytic families of compact submanifols of complex manifolds*, Ann. Math., II Ser.,**75** (1962), 146-162. MR **24:**A3665b

[11] K. Kodaira, *On compact complex analytic spaces* I, Ann. Math. **71** (1960), 111-152; II, ibid. **77** (1963), 563-626; III, ibid. **78** (1963), 1-40. MR **24:**A2396; MR **32:**1730

[12] C.R. LeBrun, *$\mathcal{H}$-Space with a cosmological constant*, Proceedings of the Royal Society of London, Series A **380** (1982), 171-185. MR **83d:**83019

[13] C.R. LeBrun, *Twistors, Ambitwistors, and Conformal Gravity*, in *Twistors in mathematics and physics*, London Math. Soc. Lecture Note Ser., **156** (1990), Cambridge Univ. Press, Cambridge, 71-86. MR **92a:**83042

[14] C.R. LeBrun, *Foliated CR Manifolds*, Journal of Differential Geometry **22** (1985), 81-96. MR **88e:**32028

[15] C.R. LeBrun, *Spaces of Complex Null Geodesics in Complex-Riemannian Geometry*, Trans. Am. Math. Soc. **278** (1983), 209-231. MR **84e:**32023

[16] R. Penrose, *The Structure of Space-Time*, in *Battelle Rencontres* (ed. C. DeWitt & J. Wheeler), New York: Benjamin (1968), 121-235. MR **38:**955

[17] R. Penrose, *Nonlinear Gravitons and Curved Twistor Theory*, General Relativity and Gravitation **7** (1976), 31-52. MR **55:**11905

[18] S. Salamon, *Topics in four-dimensional Riemannian geometry*, in *Geometry Seminar "Luigi Bianchi"*, Lecture Notes in Mathematics **1022** (1982) Springer-Verlag, 36-124. MR **85i:**53002

[19] I.M. Singer, J.A. Thorpe, *The curvature of 4-dimensional Einstein spaces*, in *Global Analysis, in honor of Kodaira*, Princeton Math. Series **29** (1969), Princeton Univ. Press, 355-365. MR **41:**959

[20] T.Y. Thomas, *A projective theory of affinely connected manifolds*, Math. Zeit. **25** (1926), 723-733.

[21] J.H.C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. Math. **3** (1932), 33-42.

Centre de Mathématiques, UMR 7640 CNRS, Ecole Polytechnique, 91128 Palaiseau cedex, France

E-mail address: `belgun@math.polytechnique.fr`

Current address: Mathematisches Institut, Augustusplatz 10/11, 04109 Leipzig, Germany

E-mail address: `Florin.Belgun@math.uni-leipzig.de`