TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 4, Pages 1585–1608
S 0002-9947(03)03427-5
Article electronically published on September 22, 2003

COPOLARITY OF ISOMETRIC ACTIONS

CLAUDIO GORODSKI, CARLOS OLMOS, AND RUY TOJEIRO

ABSTRACT. We introduce a new integral invariant for isometric actions of compact Lie groups, the *copolarity*. Roughly speaking, it measures how far from being polar the action is. We generalize some results about polar actions in this context. In particular, we develop some of the structural theory of copolarity k representations, we classify the irreducible representations of copolarity one, and we relate the copolarity of an isometric action to the concept of variational completeness in the sense of Bott and Samelson.

1. INTRODUCTION

An isometric action of a compact Lie group G on a complete Riemannian manifold M is called *polar* if there exists a connected, complete submanifold Σ of M which intersects all G-orbits and such that Σ is orthogonal to every G-orbit it meets. Such a submanifold is called a *section*. It is easy to see that a section is automatically totally geodesic. If the section is also flat in the induced metric, then the action is called *hyperpolar*. In the case of Euclidean spaces, there is clearly no difference between polar and hyperpolar representations, since totally geodesic submanifolds of a Euclidean space are affine subspaces. Polar representations were classified by Dadok [Dad85], and it follows from his work that a polar representation of a compact Lie group is orbit-equivalent to (i.e. has the same orbits as) the isotropy representation of a symmetric space.

In this paper we introduce a new invariant for isometric actions of compact Lie groups, the *copolarity*. Roughly speaking, it measures how far from being polar the action is. This is based on the idea of a *k-section*, which is a generalization of the concept of section. The *minimal k-section* passing through a regular point of the action is the smallest connected, complete, totally geodesic submanifold of the ambient space passing through that point which intersects all the orbits and such that, at any intersection point with a principal orbit, its tangent space contains the normal space of that orbit with codimension k. It is easy to see that this is a good definition and uniquely specifies an integer k, which we call the *copolarity* of the isometric action (see Section 2). It is also obvious that the $k = 0$ case precisely corresponds to the polar actions.

Received by the editors October 17, 2002 and, in revised form, March 3, 2003.
2000 *Mathematics Subject Classification.* Primary 57S15; Secondary 53C20.
The first author was supported in part by CNPq grant 300720/93-9 and by FAPESP grant 01/04793-8.
The second author was supported by Universidad Nacional de Córdoba and CONICET, and supported in part by CIEM, Secyt-UNC and ANPCYT.
The third author was supported in part by CNPq grant 300229/92-5 and FAPESP grant 01/05318-1.

It is apparent that for most isometric actions the minimal k-section coincides with the ambient space. Note that in this case k equals the dimension of a principal orbit. We say that such isometric actions have *trivial copolarity*. The obvious questions that emerge are:

What are the isometric actions with nontrivial copolarity?
What is the meaning of the integer k?

In this paper we examine this problem in the case of orthogonal representations. Examples of representations of nontrivial copolarity and minimal k-sections appear naturally in the framework of the reduction principle in compact transformation groups (see [GS00], [SS95], [Str94] for that principle). In fact, in [GT02] the reduction principle was used to describe the geometry of the irreducible representations in the table of Theorem 1.1 below, which have copolarity 1. In that paper one was motivated by the fact that the orbits of those representations are tautly embedded in Euclidean space; we call representations with this property *taut*. This work is mainly motivated by the desire to better understand and generalize that description.

We give a complete answer to the above questions for the extremal values of the invariant k. Namely, let (G, V) be an irreducible representation of a compact connected Lie group. Let n be the dimension of a principal orbit. We prove the following two theorems.

Theorem 1.1. *If $k = 1 < n$, then (G, V) is one of the following orthogonal representations ($m \geq 2$):*

$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	*(standard)* $\otimes_{\mathbf{R}}$ *(spin)*
$\mathbf{U}(2) \times \mathbf{Sp}(m)$	*(standard)* $\otimes_{\mathbf{C}}$ *(standard)*
$\mathbf{SU}(2) \times \mathbf{Sp}(m)$	*(standard)*^{3} $\otimes_{\mathbf{H}}$ *(standard)*

Theorem 1.2. *If $k = n - 1$ or $k = n - 2$, then $k = 0$.*

A few remarks are in order. Theorem 1.2 says that in the nonpolar case a nontrivial minimal k-section must have codimension at least 3. Also, the three representations listed in the table of Theorem 1.1 are precisely the irreducible representations of cohomogeneity 3 that are not polar (see [Yas86], [Dad85]). In fact, according to the main result of [GT03] (see also [GTa]), these three representations together with the polar ones precisely comprise all the taut irreducible representations. Hence, we have the following beautiful characterization of taut irreducible representations.

Theorem 1.3. *An irreducible representation of a compact Lie group is taut if and only if $k = 0$ or $k = 1$.*

Regarding Theorem 1.3, it is worth pointing out that the case $k = 1 = n$ is impossible, for such a representation would be orbit-equivalent to a linear circle action, and hence, by irreducibility, that would have to be the standard action of $\mathbf{SO}(2)$ on $\mathbf{R}^2$, which has $k = 0$. Notice that Theorems 1.1 and 1.3 cease to hold if the representation is not irreducible, as can be seen by taking the 7-dimensional representation of $\mathbf{U}(2)$ given by the direct sum of the vector representation on $\mathbf{C}^2$ and the adjoint representation on $\mathfrak{su}(2)$. In fact this representation has copolarity one (see [Str94]) and can easily be seen not to be taut by the methods of [GTa], [GT03]. (It is interesting to remark that this representation still has cohomogeneity

3.) We add that the main ingredients in the proofs of Theorems 1.1 and 1.2 are respectively the following two theorems that apply to more general situations.

Theorem 1.4. *Let (G, V) be an orthogonal representation with copolarity $k = 1$ and let N be a principal orbit. Then the submanifold N of V splits extrinsically as $N = N_0 \times N_1$, where N_0 is either a homogeneous isoparametric submanifold or a point, and N_1 is one of the following:*

(i) *a nonisoparametric homogeneous curve;*
(ii) *a focal manifold of a homogeneous irreducible isoparametric submanifold which is obtained by focalizing a one-dimensional distribution;*
(iii) *a codimension 3 nonisoparametric homogeneous submanifold.*

Theorem 1.5. *Let (G, V) be an irreducible representation with nontrivial copolarity k. Then the cohomogeneity of (G, V) is at most $\frac{l(l+1)}{2}$, where l is the codimension of a k-section in V.*

Another result we would like to explain here is the following. Let (G, V) be an orthogonal representation of nontrivial copolarity. It is easy to see that the G-translates of a nontrivial minimal k-section naturally determine a group-invariant foliation $\mathcal{F}$ on the G-regular set of V (in fact, here the k-section need not be minimal, but we do not go into details in this introduction). We prove the following theorem (see Theorem 5.23).

Theorem 1.6. *If the distribution orthogonal to $\mathcal{F}$ is integrable, then (G, V) is orbit-equivalent to a direct product representation $(G_1 \times G_2, V_1 \oplus V_2)$, where G_1, G_2 are subgroups of G, (G_1, V_1) is a polar representation and (G_2, V_2) is any representation; here the leaves of the distribution orthogonal to $\mathcal{F}$ correspond to the G_1-orbits. In particular, if (G, V) is nonpolar, then it cannot be irreducible.*

In this paper we also relate the copolarity of an isometric action to the concept of variationally complete actions which was introduced by Bott in [Bot56] (see also [BS58]). Roughly speaking, an isometric action of a compact Lie group on a complete Riemannian manifold is *variationally complete* if it produces enough Jacobi fields along geodesics to determine the multiplicities of focal points to the orbits (see Section 4 for the precise definition). Conlon proved in [Con71] that a hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete. On the other hand, it is known that a variationally complete representation is polar [DO01], [GTa], and that a variationally complete action on a compact symmetric space is hyperpolar [GTb]. This implies that the converse to Conlon's theorem is true for actions on Euclidean spaces or compact symmetric spaces. In this paper we introduce the notion of variational co-completeness of an isometric action and prove that it does not exceed k for an action that admits a flat k-section (Theorem 4.1). This reduces to Conlon's theorem for $k = 0$. We also prove a weak converse of this result in the case of representations (Theorem 4.8). As a consequence of the discussion in Section 4, we can establish the following generalization of the result that says that an orthogonal representation is polar if and only if there exists a principal orbit with the property that equivariant normal vector fields are parallel in the normal connection (see Corollaries 4.6 and 4.10).

Theorem 1.7. *The copolarity of an orthogonal representation (G, V) is at most k if and only if there exist a principal orbit N and a G-invariant, k-dimensional* *distribution $\mathcal{D}$ on N that is autoparallel in N and invariant under the second fundamental form of N, such that every equivariant normal vector field on N is parallel in the normal connection along $\mathcal{E}$, where $\mathcal{E}$ is the orthogonal complement distribution of $\mathcal{D}$ in TN.*

The paper is organized as follows. We first define k-sections and the copolarity of an isometric action (Section 2) and present some examples (Section 3). Then we introduce the concept of variational co-completeness and prove the extension of Conlon's theorem for copolarity k actions (Theorem 4.1) and its weak converse in the case of representations (Theorem 4.8). After that, we go on to develop some of the structural theory of copolarity k representations. In particular, we show that the copolarity of an orthogonal representation behaves well with respect to taking slice representations (Theorem 5.6) and forming direct sums (Theorem 5.13), and we obtain a reduction principle in terms of k-sections (Theorem 5.9). We also establish a bound on the cohomogeneity of an irreducible representation with nontrivial copolarity (Theorem 5.21), show that the codimension of a nontrivial minimal k-section of an irreducible representation is at least 3 (Theorem 5.22), and characterize the orthogonal representations admitting a minimal k-section whose orthogonal distribution is integrable (Theorem 5.23). We finally describe the geometry of a principal orbit of a representation of copolarity one (Theorem 6.3) and classify the irreducible representations of copolarity one (Theorem 6.4).

As a final note, we recall that the principal orbits of polar representations can be characterized as being the only compact homogeneous isoparametric submanifolds of Euclidean space [PT87]. An open problem in the area is to similarly characterize the principal orbits of more general orthogonal representations in terms of their submanifold geometry and topology. We believe that orthogonal representations of low copolarity may serve as testing cases for this problem.

The first author wishes to thank Prof. Gudlaugur Thorbergsson for very useful conversations. Part of this work was completed while the third author was visiting the University of São Paulo (USP), for which he wishes to thank Prof. Antônio Carlos Asperti and the other colleagues from USP for their hospitality, and FAPESP for financial support.

2. Actions admitting k-sections

Let (G, M) be an isometric action of the compact Lie group G on the complete Riemannian manifold M. A *k-section* for (G, M), where k is a nonnegative integer, is a connected, complete submanifold Σ of M such that the following hold:

- (C1) Σ is totally geodesic in M;
- (C2) Σ intersects all G-orbits;
- (C3) for every G-regular point $p \in \Sigma$ we have that $T_p\Sigma$ contains the normal space $\nu_p(Gp)$ as a subspace of codimension k;
- (C4) for every G-regular point $p \in \Sigma$ we have that if $gp \in \Sigma$ for some $g \in G$, then $g\Sigma = \Sigma$.

If Σ is a k-section through p, then $g\Sigma$ is a k-section through gp for any $g \in G$. We also remark that, since a k-section Σ is connected, complete and totally geodesic, for every $p \in \Sigma$ we have that $\Sigma = \exp_p T_p\Sigma$; and, since the G-orbits are compact, for every $p \in M$ we have that the set $\exp_p \nu(Gp)$ intersects all G-orbits. Using these remarks, it is easy to see that, given a G-regular $p \in M$, the connected component containing p of the intersection of a k_1-section and a k_2-section passing through p is a smooth submanifold, and it is a k-section passing through p with $k \leq \min\{k_1, k_2\}$. It is also clear that the ambient M is a trivial k-section for k equal to the dimension of a principal orbit. It follows that the set of k-sections, $k = 0, 1, 2, \ldots$, passing through a fixed regular point admits a unique minimal element. We say that the *copolarity* of (G, M) is k_0, and we write $\mathrm{copol}(G, M) = k_0$, if that minimal element is a k_0-section. In this way, the copolarity is well defined for any isometric action (G, M) as being an integer k_0 between zero and the dimension of a principal orbit, and then a k_0-section is uniquely determined through any given regular point. We say that (G, M) has *nontrivial copolarity* if k_0 is strictly less than the dimension of a principal orbit (or, equivalently, if a k_0-section is properly contained in M). Since a 0-section is simply a section, we have that $\mathrm{copol}(G, M) = 0$ if and only if (G, M) is polar. Note that the set of connected, complete submanifolds of M passing through a fixed G-regular point and satisfying only conditions (C1), (C2) and (C3) is also closed under connected intersection, and that a minimal element in this set automatically satisfies condition (C4), so that it represents the same minimal k_0-section. *It follows from this observation that, in order to show that an isometric action has copolarity at most k, it is enough to construct a connected, complete submanifold of codimension k satisfying conditions (C1), (C2) and (C3).* On the other hand, note that many of our applications will not depend on the fact that a k-section is minimal, but rather will depend on the fact that it satisfies condition (C4).

Next we discuss the conditions in the definition of a k-section. Note that if $k = 0$ condition (C4) is unnecessary and also (C1) follows rather easily from (C2) and (C3), but in general we cannot dispense with them. A standard argument also shows that in the general case condition (C2) follows from (C3), if the latter is not empty, namely if we assume that Σ contains a regular point. Condition (C4) (combined with (C2)) is equivalent to the fact that the G-translates of Σ define a foliation in the regular set of M. Even more interesting is the following rephrasing of condition (C4). (Note that condition (C3) implies that the intersection of a k-section with a principal orbit is a smooth manifold.)

Proposition 2.1. *Let (G, M) be an isometric action of the compact Lie group G on the complete Riemannian manifold M. Suppose Σ be a connected, complete submanifold of M satisfying conditions (C1), (C2) and (C3). For every regular $p \in \Sigma$, define a k-dimensional subspace $\mathcal{D}_p$ of $T_p(Gp)$ by $\mathcal{D}_p = T_p\Sigma \cap T_p(Gp)$. Then condition (C4) is equivalent to any one of the following assertions:*

(a) *the subspaces $\{\mathcal{D}_p\}$ extend to a k-dimensional, G-invariant distribution $\mathcal{D}$ on the regular set of M;*
(b) *for every G-regular point $p \in \Sigma$ we have that if $gp \in \Sigma$ for some $g \in G$, then $g_*\mathcal{D}_p = \mathcal{D}_{gp}$;*
(c) *there exists a principal orbit ξ such that if p, $gp \in \Sigma \cap \xi$ for some $g \in G$, then $g_*\mathcal{D}_p = \mathcal{D}_{gp}$.*

Moreover, any one of the preceding conditions implies that the stabilizer G_Σ of Σ acts transitively on the intersection of Σ with any principal orbit.

Proof. Since Σ is connected, complete and totally geodesic in M, for any G-regular point $p \in \Sigma$ we have that $\Sigma = \exp_p T_p\Sigma$, where $T_p\Sigma = \mathcal{D}_p \oplus \nu_p(Gp)$. We now show that (c) implies (C4). Assume that (c) holds with respect to the principal orbit

ξ. Let $q \in \Sigma$ be a G-regular point. There exists a minimal geodesic from q to ξ. Therefore we can write $p = \exp_q v$ for some $p \in \xi$ and $v \in \nu_q(Gq)$. Conditions (C3) and (C1) imply that $p \in \Sigma$. Now if $gq \in \Sigma$ for some $g \in G$, then $gp = \exp_{gq} g_* v \in \exp_{gq}(\nu_{gq}(Gq)) \subset \exp_{gq}(T_{gq}\Sigma) = \Sigma$. Since $p,\ gp \in \Sigma \cap \xi$, by (c) we deduce that $g_*\mathcal{D}_p = \mathcal{D}_{gp}$. It follows that $g\Sigma = g\exp_p(\mathcal{D}_p \oplus \nu_p(Gp)) = \exp_{gp}(g_*\mathcal{D}_p \oplus g_*\nu_p(Gp)) = \exp_{gp}(\mathcal{D}_{gp} \oplus \nu_{gp}(Gp)) = \Sigma$, and this gives (C4).

Notice that (a) is just a reformulation of (b), and the fact that G_Σ is transitive on the intersection of Σ with any principal orbit follows immediately from (C4). Also, the equivalence between (C4) and (b) follows from $\Sigma = \exp_p(\mathcal{D}_p \oplus \nu_p(Gp))$. Since (b) trivially implies (c), this completes the proof of the proposition. □

Let Σ be a k-section of (G, M) and consider the distribution $\mathcal{D}$ as in Proposition 2.1. Define $\mathcal{E}$ to be the distribution on the regular set of M such that $\mathcal{E}_p$ is the orthogonal complement of $\mathcal{D}_p$ in $T_p(Gp)$. If p is G-regular, then it follows from the fact that Σ is totally geodesic in M that the distribution $\mathcal{D}|_{Gp}$ is autoparallel in Gp and invariant under the second fundamental form α of Gp, in the sense that $\alpha(\mathcal{D}, \mathcal{E}) = 0$ on Gp.

The following proposition will be used later to show that some orthogonal representations admit k-sections.

Proposition 2.2. *Let (G, V) be an orthogonal representation of a compact Lie group on a Euclidean space V. Let Gp be a principal orbit. Suppose there is a G-invariant, autoparallel, k-dimensional distribution $\mathcal{D}$ on Gp such that $\alpha(\mathcal{D}, \mathcal{E}) = 0$, where α is the second fundamental form of Gp and $\mathcal{E}$ is the distribution on Gp orthogonal to $\mathcal{D}$. Define $\Sigma = \mathcal{D}_p \oplus \nu_p(Gp)$.*

(a) *Suppose that for every $v \in \nu_p(Gp)$ with $q = p + v$ a G-regular point we have that $\mathcal{E}_p \subset T_q(Gq)$. Then Σ satisfies conditions (C1), (C2) and (C3) in the definition of a k-section.*
(b) *Suppose that, in addition to the hypothesis in (a), for every $v \in \nu_p(Gp)$ with $q = p + v \in Gp$ we have that $\mathcal{E}_p = \mathcal{E}_q$. Then Σ is a k-section for (G, V).*

Proof. Let β be a the maximal integral submanifold of $\mathcal{D}$ through p. Since β is totally geodesic in Gp and $\alpha(\mathcal{D}, \mathcal{E}) = 0$, we have that the covariant derivative in V of an $\mathcal{E}$-section along β is in $\mathcal{E}$, which implies that $\mathcal{E}$ is constant in V along β. Therefore β is contained in the affine subspace of V orthogonal to $\mathcal{E}_p$, which is Σ, and, in fact, β is the connected component of $\Sigma \cap Gp$ containing p.

Now if $gp \in \beta$ then $\mathcal{E}_p = \mathcal{E}_{gp}$. Taking orthogonal complements in V, we get that $\Sigma = \mathcal{D}_{gp} \oplus \nu_{gp}(Gp)$; but the right hand side in turn equals $g\Sigma$, as $\mathcal{D}$ is G-invariant. This shows that conditions (C3) and (C4) are already satisfied for points in β.

Next let $\gamma \neq \beta$ be a connected component of the intersection of Σ with a principal orbit (which possibly could be Gp). Let $q \in \gamma$ and consider the minimal geodesic in V from q to β. Then we can write $q = gp + v$, where $gp \in \beta$ for some $g \in G$ and $v \in \nu_{gp}(Gp)$. Since $g^{-1}q = p + g^{-1}v$, we have $T_q(Gq) = gT_{g^{-1}q}(Gq) \supset g\mathcal{E}_p = \mathcal{E}_{gp} = \mathcal{E}_p$ (where the inclusion follows from the hypothesis in (a)). Taking orthogonal complements in V, we get that $\nu_q(Gq) \subset \Sigma$. This shows that condition (C3) is fully satisfied. If, in addition, γ is a connected component of $\Sigma \cap Gp$ and $q = hp$ for some $h \in G$, then we can write $h\mathcal{E}_p = \mathcal{E}_q = g\mathcal{E}_{g^{-1}q} = g\mathcal{E}_p = \mathcal{E}_{gp} = \mathcal{E}_p$ (where the equality $\mathcal{E}_{g^{-1}q} = \mathcal{E}_p$ follows from the hypothesis in (b)). Taking orthogonal complements, we get $h\Sigma = \Sigma$, which, by Proposition 2.1(c), finally implies condition (C4). □

3. Examples

In this section we exhibit some examples of actions admitting k-sections.

3.1. **Product actions.** Let (G_1, M_1) be a polar action with a section Σ_1, and let (G_2, M_2) be any isometric action. Let $G = G_1 \times G_2$, $M = M_1 \times M_2$, and consider the product action (G, M). Then it is immediate to see that $\Sigma = \Sigma_1 \times M_2$ is a k-section of (G, M), where k is the dimension of a principal orbit of (G_2, M_2). Note that in this example the distribution $\mathcal{E}$ is integrable and its leaves are the G_2-orbits in M_2. We will show later that, in the case of orthogonal representations, this example is essentially the only one whose distribution $\mathcal{E}$ is integrable.

3.2. **The reduction principle in compact transformation groups.** Consider an isometric action (G, M) and fix a principal isotropy subgroup H. Then the H-fixed point set M^H is a totally geodesic submanifold of M. Let ${}_cM$ be the closure in M of the subset of regular points of M^H. Then it can be shown that ${}_cM$ consists of those components of M^H which contain regular points of M, and that any component of ${}_cM$ intersects all the orbits (see [SS95], or [GS00], where ${}_cM$ is called the *core* of M). Now any connected component Σ of ${}_cM$ is a k-section for (G, M), where k is the difference between the dimension of Σ and the cohomogeneity of (G, M). In fact, Σ is a totally geodesic submanifold of M, being a connected component of the common fixed point set of a set of isometries of M. This is condition (C1) in the definition of a k-section. Condition (C3) follows from the fact that the slice representation at a regular point is trivial. Moreover, if p, $gp \in \Sigma$ are regular points for some $g \in G$, then both the isotropy subgroups at p and gp are H, so g normalizes H and therefore fixes Σ. This is condition (C4). The rest follows.

This example is very important in the sense it shows that if (G, M) is an arbitrary isometric action, and Σ is a k-section that is *minimal* (so that $\mathrm{copol}(G, M) = k$), then we always have that $\Sigma \subset M^{G_p}$, for any G-regular point $p \in \Sigma$. Therefore G_p acts trivially on $\mathcal{D}_p = T_p\Sigma \cap T_p(Gp)$, and this implies that the isotropy subgroup of G_Σ at p is G_p. Since G_p acts trivially on Σ, using Lemma 5.8 below, we get that $Gp \cap \Sigma = G_\Sigma p = G_\Sigma / G_p$ is a group manifold.

It is interesting to notice that sometimes the group G can be enlarged to another group $\hat{G}$ that has the same orbits in M as G but has a larger principal isotropy subgroup $\hat{H}$, and then the $\hat{H}$-fixed point set is contained in the H-fixed point set (see [Str94]). On the other hand, the concept of copolarity is purely geometrical, and so it coincides for orbit equivalent actions. We do not know of any example of an isometric action such that, *after enlarging the group to the maximal group with the same orbits*, the minimal k-section is strictly contained in a connected component of the fixed point set of a principal isotropy subgroup.

3.3. **Some orthogonal representations of low copolarity.** Some calculations in [GT03], [GTa] involving the reduction principle in transformation groups produced some examples of irreducible orthogonal representations of low copolarity.

(i) The three nonpolar irreducible representations of cohomogeneity 3 have copolarity 1. This follows from results in Section 5 in [Str94] or from the computation in Proposition 7.12 in [GTa]. (These representations are listed in the table of Theorem 1.1; for the classification of cohomogeneity 3 representations, see [Yas86] for the irreducible case and [Uch80] for the reducible case; the nonpolar cohomogeneity 3 representations are listed in Table II in [Str94] and are essentially proved to have copolarity 1 in Theorem 5.1 of the same paper.)

(ii) The tensor product of the vector representation of $\mathbf{SO}(3)$ and the 7-dimensional representation of $\mathbf{G}_2$ is an irreducible representation of real dimension 21, cohomogeneity 4 and copolarity 2 (see Lemma 6.11 in [GT03]).

(iii) Let $(S^1 \cdot K, V)$ be a polar irreducible representation which is *Hermitian*; namely, it leaves a complex structure on V invariant. Assume that the restricted representation (K, V) is not orbit equivalent to $(S^1 \cdot K, V)$. Then (K, V) is nonpolar, has nontrivial copolarity, and a k-section is given by the complexification of a 0-section of $(S^1 \cdot K, V)$, where k is equal to the cohomogeneity of $(S^1 \cdot K, V)$ minus one. (There are four families of such Hermitian polar irreducible representations, and the simplest example is possibly the action of the unitary group $\mathbf{U}(3)$ on the space of complex symmetric bilinear forms in three variables. The induced representation of $\mathbf{SU}(3)$ has real dimension 12, cohomogeneity 4 and copolarity 2.)

3.4. **Examples derived from polar actions.** Let (G, V) be a polar representation of a compact Lie group G on a Euclidean space. Then, of course, $\mathrm{copol}(G, M) = 0$. Nevertheless, in general there are interesting examples of k-sections for (G, V) with $k > 0$.

In fact, let Gp be a principal orbit. It is known that equivariant normal vector fields along Gp are parallel in the normal connection, and therefore Gp is an *isoparametric submanifold* of V; namely, the principal curvatures of Gp along a parallel normal vector field are constant and the normal bundle of Gp in V is globally flat. Let $v \in \nu_p(Gp)$, A_v the Weingarten operator in the direction of v, and λ a nonzero principal curvature in the direction of v with multiplicity k. Since the subspace $\ker(A_v - \lambda \mathrm{id}_{T_p(Gp)})$ of $T_p(Gp)$ is G_p and A_v-invariant, it extends to a G-invariant distribution $\mathcal{D}$ on Gp which is invariant under the second fundamental form of Gp, and one can see from the Codazzi equation that $\mathcal{D}$ is also autoparallel. Moreover, it is very easy to deduce from the theory of isoparametric submanifolds that the conditions on $\mathcal{E}$ from Proposition 2.2 are satisfied, so $\Sigma = \mathcal{D}_p \oplus \nu_p(Gp)$ is a k-section.

4. Variational co-completeness

Let N be a submanifold of a complete Riemannian manifold M. Let $\eta : \nu(N) \to M$ denote the *endpoint map* of N, that is, the restriction of the exponential map of M to the normal bundle of N. A point $q = \eta(v)$ is a *focal point of N in the direction of $v \in \nu(N)$ of multiplicity $m > 0$* if $d\eta_v : T_v\nu(N) \to T_qM$ is not injective and the dimension of its kernel is m. Let $v \in \nu_p(N)$, and let γ_v denote the geodesic $t \mapsto \exp_p(tv)$. A Jacobi field along γ_v is called an *N-Jacobi field* if it is the variational vector field of a variation through geodesics that are at time zero orthogonal to N. We will denote the space of N-Jacobi fields along γ_v by $\mathcal{J}^N(\gamma_v)$. It is not difficult to see that J is an N-Jacobi field along γ_v if and only if $J(0) \in T_pN$ and $J'(0) + A_v u \in \nu_p(N)$, where p is the footpoint of v, $u = J(0)$ and A_v is the Weingarten map in direction v. The point q is a focal point of N in the direction v if there is an N-Jacobi field along γ_v that vanishes at q. We will denote the space of N-Jacobi fields along γ_v that vanish at q by $\mathcal{J}^N_q(\gamma_v)$.

Now let (G, M) be an isometric action of a compact Lie group G on the complete Riemannian manifold M. The action (G, M) is called *variationally complete* if every Jacobi field $J \in \mathcal{J}_q^N(\gamma_v)$, where N is a G-orbit and q is a focal point of N in the direction of v, is the restriction along γ_v of a Killing field on M induced by the action of G.

More generally, let N be a fixed principal orbit of an isometric action (G, M) and let $p \in N$. For each $v \in \nu_p(N)$, we have an isomorphism $\mathcal{J}^N(\gamma_v) \to T_pN \oplus \nu_pN = T_pM$ given by $J \mapsto (J(0), J'(0) + A_vJ(0))$. Let U_p be a subspace of T_pM with the following property:

(P) For each $v \in \nu_p(N)$, if the field $J \in \mathcal{J}^N(\gamma_v)$ vanishes for some $t_0 > 0$ and $(J(0), J'(0) + A_vJ(0))$ is orthogonal to U_p, then J is the restriction along γ_v of a G-Killing field.

Of course, U_{gp} can always be taken to be g_*U_p, where $g \in G$, and in particular, U_p can always be taken to be G_p-invariant. Moreover, U_p can always be taken to be all of T_pM. In any case we write $\operatorname{covar}_N(G, M) \leq \dim U_p$. We say that the *variational co-completeness* of (G, M) is less than or equal to k, where k is an integer between 0 and $\dim M$, and we write $\operatorname{covar}(G, M) \leq k$, if $\operatorname{covar}_N(G, M) \leq k$ for all principal G-orbits N. Clearly, $\operatorname{covar}(G, M) \leq 0$ (or, in this case, $\operatorname{covar}(G, M) = 0$) if and only if (G, M) is variationally complete, and one cannot do better than $\operatorname{covar}(G, M) \leq \dim M$ for a generic isometric action. In the next section we will describe a situation where the intermediate values occur.

Observe that the intersection of two subspaces of T_pM with property (P) does not need to have that property, so we cannot speak of a minimal subspace with property (P). Nonetheless, given a general isometric action (G, M) and a G-regular $p \in M$, we next show how to construct a canonical subspace U_p^0 of T_pM with property (P). Let $N = Gp$. For each $v \in \nu_p(Gp)$ and q a focal point of N in the direction v, consider the subspace $\tilde{U}_p^{v,q}$ of T_pM spanned by the initial conditions $(J(0), J'(0) + A_vJ(0))$ for all $J \in \mathcal{J}_q^N(\gamma_v)$. Now take the subspace of $\tilde{U}_p^{v,q}$ spanned by the initial conditions of all G-Killing fields in $\mathcal{J}_q^N(\gamma_v)$, and let $U_p^{v,q}$ denote its orthogonal complement in $\tilde{U}_p^{v,q}$. Finally, define U_p^0 as the sum over v, q of the subspaces $U_p^{v,q}$. It is clear that U_p^0 has property (P). Note also that these U_p^0, for $p \in N$, define a G-invariant distribution on N.

4.1. **The theorem of Conlon for actions admitting k-sections.**

In this section we prove a version of Conlon's theorem [Con71].

Theorem 4.1. *If (G, M) is an isometric action admitting a k-section that is flat in the induced metric, then* $\operatorname{covar}(G, M) \leq k$.

Let $N = Gp$ be a principal orbit, $v \in \nu_p(N)$, and choose a flat k-section Σ through p.

Lemma 4.2. *Let J be an N-Jacobi field along γ_v such that $J(0) \in \mathcal{E}_p$. If $J(t_0) = 0$ for some $t_0 > 0$, then J is always orthogonal to Σ.*

Proof. Decompose $J = J_1 + J_2$, where $J_1(t)$ and $J_2(t)$ are respectively the tangent and normal components of $J(t)$ relative to $T_{\gamma_v(t)}\Sigma$. Since Σ is totally geodesic, J_1 and J_2 are Jacobi fields along γ_v. Now J_1 is a Jacobi field in Σ with $J_1(0) = J_1(t_0) = 0$. Since Σ is flat, we have that J_1 vanishes identically, and hence $J = J_2$. □

Lemma 4.3. *Let J be an N-Jacobi field along γ_v such that $J(0) \in \mathcal{E}_p$. If J is the restriction along γ_v of a G-Killing field on M, then J satisfies $J'(0) + A_v J(0) = 0$.*

Proof. Let X be a G-Killing field on M that restricts to J along γ_v. Note that $X_p = J(0) \in \mathcal{E}_p$. Denote by ∇ the Levi-Civita connection of M. Now $J'(0) = (\nabla_v X)_p$. Let $E(t)$ be any vector field along $\gamma_v(t)$ that is normal to $G\gamma_v(t)$. Then $E(t)$ is tangent to Σ for small t. We have $\langle J'(0), E\rangle_p = \langle \nabla_v X, E\rangle_p = -\langle X, \nabla_v E\rangle_p = 0$. Therefore $J'(0) \in T_pN$, and this implies $J'(0) + A_v J(0) \in T_pN$. Since we already have $J'(0) + A_v J(0) \in \nu_p N$, we get that $J'(0) + A_v J(0) = 0$. □

Lemma 4.4. *Let J be an N-Jacobi field along γ_v such that $J(0) \in \mathcal{E}_p$. Then J is always orthogonal to Σ if and only if J satisfies $J'(0) + A_v J(0) = 0$.*

Proof. If J is always orthogonal to Σ, then $J'(0) + A_v J(0)$ is also orthogonal to Σ, as Σ is totally geodesic. But as an N-Jacobi field, J satisfies $J'(0) + A_v J(0) \in \nu_p(N)$. Now we have $\nu_p(N) \subset T_p\Sigma$; hence $J'(0) + A_v J(0) = 0$. Conversely, if $J'(0) + A_v J(0) = 0$, then $J'(0) = -A_v J(0) \in \mathcal{E}_p$, since $\mathcal{E}_p$ is A_v-invariant and $J(0) \in \mathcal{E}_p$. Since J and J' are both orthogonal to Σ at time zero and Σ is totally geodesic, we deduce that J is always orthogonal to Σ. □

We finish the proof of Theorem 4.1 by observing that $\mathcal{D}_p$ has property (P). In fact, if an N-Jacobi field J along γ_v for some $v \in \nu_p N$ vanishes for some $t_0 > 0$ and $(J(0), J'(0) + A_v J(0))$ is orthogonal to $\mathcal{D}_p$, then $J(0) \in \mathcal{E}_p$. By Lemmas 4.2 and 4.4, $J'(0) + A_v J(0) = 0$. Let X be a G-Killing field on M such that $X_p = J(0)$. Then X restricts to a Jacobi field along γ_v, which by Lemma 4.3 must be J. □

Since Lemmas 4.2, 4.3 and 4.4 do not depend on condition (C4) in the definition of a k-section, we have the following corollary of the proof.

Corollary 4.5. *Let (G, M) be an isometric action. Suppose there is a flat, connected, complete submanifold Σ of M satisfying conditions (C1), (C2) and (C3) in the definition of k-section. Let N be a principal orbit, $p \in N \cap \Sigma$ and $v \in \nu_p N$. Then $T_p(N \cap \Sigma)$ has property (P).*

Let N be a principal orbit of an isometric action (G, M) admitting a k-section Σ. Let ξ be a normal vector field parallel along a curve in N that is everywhere tangent to the distribution $\mathcal{E}$. The next corollary implies that the principal curvatures of N along ξ are constant.

Corollary 4.6. *Let (G, M) be an isometric action admitting a k-section Σ (not necessarily flat). Let N be a principal orbit, $p \in N \cap \Sigma$ and $v \in \nu_p N$. Extend v to an equivariant normal vector field $\hat{v}$ along N. Then $\hat{v}$ is parallel along $\mathcal{E}$.*

Proof. By homogeneity, it is enough to show that $\nabla^{\perp}_u \hat{v} = 0$ for all $u \in \mathcal{E}_p$. Let X be a G-Killing field such that $X_p = u \in \mathcal{E}_p$. Let J be the N-Jacobi field along γ_v which is the restriction of X. Then $J(0) = u \in \mathcal{E}_p$. By Lemma 4.3, we have $J'(0) + A_v u = 0$. On the other hand, since $(L_X \hat{v})_p = 0$, $J'(0) = (\nabla_v X)_p = (\nabla_X \hat{v})_p = -A_v u + \nabla^{\perp}_u \hat{v}$. Hence, $\nabla^{\perp}_u \hat{v} = 0$. □

4.2. **A weak converse for 4.1 in the Euclidean case.** In this section we prove a sort of converse to Theorem 4.1 in the Euclidean case (Theorem 4.8) and obtain, as a corollary, a generalization of a result about polar representations (Corollary 4.10). First, observe that if (G, V) is an orthogonal representation and $N = Gp$ is a principal orbit, then an N-Jacobi field J along γ_v which vanishes for some $t_0 > 0$ is necessarily of the form $J(t) = (1 - \frac{t}{t_0})J(0)$ (since the Jacobi equation in Euclidean space is $J'' = 0$). Therefore the vector $J'(0) + A_v J(0) = (A_v - \frac{1}{t_0}\mathrm{id}_{T_pN})J(0)$ is simultaneously normal and tangent to N, so that it must vanish. In this way we see that an element J of $\mathcal{J}_q^N(\gamma_v)$ is completely determined by the value of $J(0)$, so that it is enough to consider property (P) for subspaces of T_pN. Note also that the span in T_pN of the $J(0)$, where $J \in \mathcal{J}_q^N(\gamma_v)$ and $J = X \circ \gamma_v$ for some $X \in \mathfrak{g}$, is $T_p(G_q p)$. We conclude that in the Euclidean case property (P) can be rewritten in the following form:

($\mathrm{P_{Euc}}$) For each $v \in \nu_p(N)$, if $A_v u = \lambda u$ for some $\lambda \neq 0$ and u is orthogonal to U_p, then $u \in T_p(G_q p)$, where $q = p + \frac{1}{\lambda}v$.

Moreover, it follows that the canonical subspace U_p^0 is the sum, over $v \in \nu_p N$ and q a focal point of N in the direction v, of the orthogonal complement of $T_p(G_q p)$ in $\{J(0) : J \in \mathcal{J}_q^N(\gamma_v)\}$. Recall that U_p^0 is G_p-invariant, but it is not clear that U_p^0 is invariant under the second fundamental form of N.

Proposition 4.7. *Every subspace $\mathcal{D}_p$ of T_pN which satisfies property (P_{Euc}) and is invariant under the second fundamental form of N must contain U_p^0.*

Proof. Suppose, on the contrary, that $U_p^0 \not\subset \mathcal{D}_p$. Then, by the definition of U_p^0, there exist $v \in \nu_p N$ and an eigenvector u of A_v with eigenvalue $\lambda \neq 0$ such that $u \notin \mathcal{D}_p$ and u is in the orthogonal complement of $T_p(G_q p)$ in $\{J(0) : J \in \mathcal{J}_q^N(\gamma_v)\}$, where $q = p + \frac{1}{\lambda}v$.

Write $u = u_1 + u_2$, where $u_1 \in \mathcal{D}_p$ and $u_2 \perp \mathcal{D}_p$. Then $u_2 \neq 0$. Since $\mathcal{D}_p$ is A_v-invariant, we get that $A_v u_2 = \lambda u_2$. By ($\mathrm{P_{Euc}}$) for $\mathcal{D}_p$, we have that $u_2 \in T_q(G_q p)$. Therefore $u_2 \perp u$, but this is a contradiction to the fact that $u_2 \neq 0$. □

Theorem 4.8. *Let (G, V) be an orthogonal representation, N be a principal orbit and suppose there is a G-invariant, k-dimensional distribution $\mathcal{D}$ in N which is autoparallel in N and invariant under the second fundamental form of N, and satisfies property (P_{Euc}). Then* $\mathrm{copol}(G, V) \leq k$ *(and hence, by Theorem 4.1, we have that* $\mathrm{covar}(G, V) \leq k$*).*

Proof. Fix $p \in N$. We will prove that $\Sigma = \mathcal{D}_p \oplus \nu_p(Gp)$ satisfies conditions (C1), (C2) and (C3) for (G, V). For that purpose, we will use Proposition 2.2(a). Let $v \in \nu_p N$ be such that the principal curvatures of A_v are all nonzero (note that the subset of all such v in $\nu_p N$ is open and dense). Suppose that $q = p + v$ is a G-regular point. Let $\mathcal{E}$ be the orthogonal complement distribution of $\mathcal{D}$ in N. Let $u \in \mathcal{E}_p$. Since $\mathcal{D}$ is A_v-invariant, we may assume that u is an eigenvector of A_v, and we know that the corresponding eigenvalue λ is not zero. Now the N-Jacobi field J along $\gamma_v(t) = p + tv$ with initial conditions $J(0) = u$, $J'(0) + A_v u = 0$ is given by $J(t) = (1 - t\lambda)u$. By property ($\mathrm{P_{Euc}}$), J is the restriction of a G-Killing field along γ_v. In particular, $J(1)$ is tangent to $T_q(Gq)$. Since Gq is a principal orbit, the slice representation at q is trivial, so the one-parameter subgroup of G that induces J cannot fix q, and thus $J(1) \neq 0$. This implies that $u \in T_q(Gq)$. We have shown that $\mathcal{E}_p \subset T_q(Gq)$ in the case $q = p + v$ is a G-regular point and $v \in \nu_p N$ is such that the principal curvatures of A_v are all nonzero. The case of an arbitrary $v \in \nu_p N$ follows from a limiting argument. This implies that Σ satisfies conditions (C1), (C2) and (C3) by Proposition 2.2(a). □

Remark 4.9. In Theorem 4.8, if it happens that the distribution $\mathcal{D}$ coincides with the distribution defined by the canonical subspaces, namely $\mathcal{D}_p = U_p^0$ for all $p \in N$, then we can show that Σ satisfies condition (C4), so that it is already a k-section (not necessarily minimal). In fact, following the notation of the proof, suppose that $v \in \nu_p N$ is arbitrary and $q = p + v$ is a G-regular point such that $q = gp$ for some $g \in G$. We want to show that $\mathcal{E}_p = \mathcal{E}_q$. Note that $T_q(\Sigma \cap N)$ is invariant under the second fundamental form of N at q, since Σ is totally geodesic. Moreover, $T_q(\Sigma \cap N)$ has property ($\mathrm{P_{Euc}}$) by Corollary 4.5. Now Proposition 4.7 implies that $T_q(\Sigma \cap N) \supset \mathcal{D}_q$. Since $\mathcal{E}_p$ is the orthogonal complement of $T_q(\Sigma \cap N)$ in T_qN, we deduce that $\mathcal{E}_p \subset \mathcal{E}_q$, and thus, by dimensional reasons, $\mathcal{E}_p = \mathcal{E}_q$. It follows from Proposition 2.2(b) that Σ is a k-section.

It is known that if a principal orbit of an orthogonal representation has the property that equivariant normal vector fields are parallel in the normal connection, then the representation is polar. The following corollary is a generalization of this result.

Corollary 4.10. *Let (G, V) be an orthogonal representation, let N be a principal orbit, and suppose there is a G-invariant, k-dimensional distribution $\mathcal{D}$ in N which is autoparallel in N and invariant under the second fundamental form of N. Let $\mathcal{E}$ be the orthogonal complement distribution of $\mathcal{D}$ in N. Assume that every equivariant normal vector field on N is parallel along $\mathcal{E}$. Then* $\mathrm{copol}(G, V) \leq k$.

Proof. It is enough to see that $\mathcal{D}$ has property ($\mathrm{P_{Euc}}$) and use Theorem 4.8. Let $p \in N$, $v \in \nu_p(N)$, and suppose that $A_v u = \lambda u$, where $\lambda \neq 0$ and u is orthogonal to $\mathcal{D}_p$. Set $q = p + \frac{1}{\lambda} v$. Consider the equivariant normal vector field $\hat{v}$ that extends v. Then $\nabla^\perp_u \hat{v} = 0$ by hypothesis. Let $X \in \mathfrak{g}$ be such that $X_p = u$. Now

$$\begin{aligned} X_q &= \frac{d}{ds}\Big|_{s=0} (\exp sX) q \\ &= \frac{d}{ds}\Big|_{s=0} (\exp sX) p + \frac{1}{\lambda} \frac{d}{ds}\Big|_{s=0} \hat{v}(s) = u + \frac{1}{\lambda}(-A_v u + \nabla^\perp_u \hat{v}) = 0. \end{aligned}$$

Hence $u \in T_q(G_q p)$. □

5. Structural theory of actions admitting k-sections

5.1. **Slice representations.** In this section we will prove that the copolarity of a slice representation is not bigger than the copolarity of the original representation (Theorem 5.6). Let (G, M) be an isometric action of the compact Lie group G on the complete Riemannian manifold M. Let $q \in M$. Then G_q acts on T_qM via the differential, and $T_qM = T_q(Gq) \oplus \nu_q(Gq)$ is an invariant decomposition. The orthogonal representation $(G_q, \nu_q(Gq))$ is called the *slice representation at q*.

Lemma 5.1. *The following assertions are equivalent:*

(a) *$v \in \nu_q(Gq)$ is G_q-regular.*
(b) *There exists $\epsilon > 0$ such that $\exp_q(tv)$ is G-regular for $0 < t < \epsilon$.*
(c) *$\exp_q(t_0 v)$ is G-regular for some $t_0 > 0$.*

Proof. (c) implies (b): Let $p = \exp_q(t_0 v)$ be G-regular. Then there exists $\epsilon > 0$ such that

$$G_{\exp_q(tv)} = (G_q)_{tv} = (G_q)_v \subset G_p \subset G_q,$$

for $0 < t < \epsilon$, where the first equality follows from the fact that the exponential map is an equivariant diffeomorphism of a small normal disk of radius ϵ onto the image, and the last inclusion follows from the fact that the slice representation at p is trivial, as p is G-regular. Again by the G-regularity of p, we have that $G_{\exp_q(tv)} = G_p$ for $0 < t < \epsilon$, and hence $\exp_q(tv)$ is G-regular for $0 < t < \epsilon$.

Clearly (b) implies (c), and the equivalence of (a) with (b) follows from the equality $G_{\exp_q(tv)} = (G_q)_v$ for $0 < t < \epsilon$ and the slice theorem. □

In the following we consider the case of an orthogonal representation (G, V). Let $q \in V$, choose a k-section Σ through q, and consider the slice representation $(G_q, \nu_q(Gq))$.

Lemma 5.2. *There exists $v \in \Sigma \cap \nu_q(Gq)$ which is a G_q-regular point.*

Proof. Let ξ be a principal G-orbit and choose a connected component β of $\Sigma \cap \xi$. Let $c(t) = q + tv$, $0 \le t \le l$, be a minimal geodesic in Σ from q to β. Then $\dot{c}(l) \in \Sigma = T_{c(l)}\beta \oplus \nu_{c(l)}(Gc(l))$. But $\dot{c}(l)$ must be orthogonal to β, by minimality. Therefore $\dot{c}(l) \in \nu_{c(l)}(Gc(l))$. Since a geodesic orthogonal to an orbit must be orthogonal to every orbit it meets, $v = \dot{c}(0) \in \Sigma \cap \nu_q(Gq)$. It follows from Lemma 5.1 that v is G_q-regular. □

For the G-regular $p = q + tv \in V$, $0 < t < \epsilon$, we then have that $\Sigma = \nu_p(Gp) \oplus \mathcal{D}_p$, where $\mathcal{D}$ has rank k.

Lemma 5.3. *There is an orthogonal decomposition*

$$T_p(G_q p) = T_p(G_q p) \cap \mathcal{D}_p \oplus T_p(G_q p) \cap \mathcal{E}_p.$$

Proof. Note that $T_p(Gp) = \mathcal{D}_p \oplus \mathcal{E}_p$ is an A_v-invariant decomposition, where A_v denotes the Weingarten operator of Gp with respect to the, say, unit vector $v \in \nu_p(Gp)$. We first claim that $T_p(G_q p)$ is contained in the eigenspace of A_v corresponding to the eigenvalue $-1/t$. In fact, let $u = \frac{d}{ds}|_{s=0}\varphi_s(p) \in T_p(G_q p)$ for $\varphi_s \in G_q$, $\varphi_0 = 1$. Define the normal vector field $\hat{v}(s) = \varphi_s v$ along $s \mapsto \varphi_s p$. Then $\hat{v}(s) = \frac{1}{t}(\varphi_s p - q)$ and

$$-A_v u + \nabla^{\perp}_u \hat{v} = \frac{d}{ds}\Big|_{s=0} \hat{v}(s) = \frac{1}{t}u,$$

so, by taking tangent components, we get that $A_v u = -\frac{1}{t}u$.

Next write $u = u' + u''$, where $u' \in \mathcal{D}_p$ and $u'' \in \mathcal{E}_p$. Then u' and u'' are eigenvectors of A_v with eigenvalue $-1/t$. Now $J(s) = (1 + \frac{s}{t})u''$ is a Jacobi field along $s \mapsto p + sv$ such that $J(-t) = 0$, $J'(0) + A_v J(0) = 0$, and $J(0) = u''$ is orthogonal to $\mathcal{D}_p$. Since $\mathcal{D}_p$ has property (P) by Corollary 4.5, we have that J is induced by a G-Killing field. Hence $u'' \in T_p(G_q p)$. □

Let $\mathcal{D}_{1p} = T_p G_q(p) \cap \mathcal{D}_p$ and $\mathcal{E}_{1p} = T_p G_q(p) \cap \mathcal{E}_p$. Let $\mathcal{D}_{2p}$ be the orthogonal complement of $\mathcal{D}_{1p}$ in $\mathcal{D}_p$ and let $\mathcal{E}_{2p}$ be the orthogonal complement of $\mathcal{E}_{1p}$ in $\mathcal{E}_p$.

Lemma 5.4. $\mathcal{E}_{2p} \subset T_q(Gq)$.

Proof. Let $u = \frac{d}{ds}|_{s=0}\varphi_s p \in \mathcal{E}_{2p}$ with $\varphi_s \in G$, $\varphi_0 = 1$. Without loss of generality we may assume that $A_v u = \lambda u$, and then $\lambda \neq -1/t$ because the $-1/t$ eigenvectors in

$\mathcal{E}_p$ belong to $T_p(G_qp)$ (Corollary 4.5). Let $r : Gp \to Gq$ be the canonical equivariant submersion. We compute

$$r_*(u) = \frac{d}{ds}\Big|_{s=0} \underbrace{r\varphi_s}_{=\varphi_s r}(p) = \frac{d}{ds}\Big|_{s=0} \underbrace{\varphi_s(q)}_{=\varphi_s(p)-t\varphi_s(v)} = u - t\frac{d}{ds}\Big|_{s=0}\hat{v}(s),$$

so

$$u = r_*(u) + t(-A_v u + \nabla^\perp_u \hat{v}).$$

Now $\nabla^\perp_u \hat{v} = 0$ because $u \in \mathcal{E}_p$ (Corollary 4.6). Hence $u = (1+\lambda t)^{-1} r_*(u) \in T_q(Gq)$. □

Notice that each one of $\mathcal{D}_{1p}$, $\mathcal{E}_{1p}$, $\nu_p(Gp) \oplus \mathcal{D}_{2p}$ and $\mathcal{E}_{2p}$ is a constant subspace of V with respect to $t \in (0, \epsilon)$, where $p = q + tv$.

Lemma 5.5. *$\Sigma \cap \nu_q(Gq)$ intersects all G_q-orbits.*

Proof. It follows from Lemma 5.4 that $\nu_q(Gq) \subset \Sigma \oplus \mathcal{E}_{1p}$. Let N_v be the normal space to $T_p(G_qp)$ in $\nu_q(Gq)$. Then N_v is orthogonal to $\mathcal{E}_{2p}$, so that

$$\nu_q(Gq) = \underbrace{T_p(G_qp)}_{=\mathcal{D}_{1p}\oplus\mathcal{E}_{1p}} \oplus \underbrace{N_v}_{\subset \nu_p(Gp)\oplus\mathcal{D}_{2p}}.$$

Hence $\nu_q(Gq) \cap \Sigma = N_v \oplus \mathcal{D}_{1p}$. Since $\nu_q(Gq) \cap \Sigma$ contains N_v and v is G_q-regular, we get that $\nu_q(Gq) \cap \Sigma$ intersects all G_q-orbits. □

For a G_q-regular $w \in \Sigma \cap \nu_q(Gq)$, we have that $q + tw$ is G-regular for $t > 0$ small, and thus it makes sense to define $\mathcal{D}_{1w} = \mathcal{D}_{q+tw} \cap T_{q+tw} G_q(q+tw)$. It is clear that $\mathcal{D}_1$ is a G_q-invariant distribution on the G_q-regular set of $\nu_q(Gq)$. Moreover, $\Sigma \cap \nu_q(Gq) = N_w \oplus \mathcal{D}_{1w}$, as above. It follows from Proposition 2.1 that $\Sigma \cap \nu_q(Gq)$ is a k_1-section for $(G_q, \nu_q(Gq))$, where k_1 is the dimension of $\mathcal{D}_1$. Since the dimension of $\mathcal{D}_1$ is not bigger than the dimension of $\mathcal{D}$, we finally conclude:

Theorem 5.6. *If* $\operatorname{copol}((,G), V) \le k$, *then* $\operatorname{copol}((,G)_q, \nu_q(Gq)) \le k$.

5.2. **The reduction.** In this section we establish a reduction principle for orthogonal representations in terms of k-sections (Theorem 5.9). Let (G, V) be an orthogonal representation admitting a k-section Σ.

Lemma 5.7. *Let $q \in V$. Then the isotropy subgroup G_q is transitive on the set $\mathcal{F}$ of k-sections through q which are G-translates of Σ, namely $\mathcal{F} = \{g\Sigma : g \in G,\ q \in g\Sigma\}$.*

Proof. The result is trivial for a G-regular point q, since in this case there is a unique element in $\mathcal{F}$ by condition (C4). Assume q is not a G-regular point, and let Σ_1, $\Sigma_2 \in \mathcal{F}$. Let S be a normal slice at q and choose a G_q-regular $p \in S$. By Lemma 5.5, $\Sigma_i \cap \nu_q(Gq)$ intersects all G_q-orbits on S, $i = 1, 2$. Therefore we can select $h_i \in G_q$ such that $h_i p \in \Sigma_i$. Now $p \in h_1^{-1}\Sigma_1 \cap h_2^{-1}\Sigma_2$, where p is G-regular, so $h_2 h_1^{-1}\Sigma_1 = \Sigma_2$, where $h_2 h_1^{-1} \in G_q$. □

Lemma 5.8. *Let G_Σ be the stabilizer of Σ. Then $G_\Sigma q = Gq \cap \Sigma$, for all $q \in \Sigma$.*

Proof. Let p, $q \in \Sigma$ be in the same G-orbit. We need to show that they are in the same G_Σ-orbit. If they belong to a principal G-orbit, then the result follows from Proposition 2.1. Suppose p and q are not G-regular points and $q = gp$ for some $g \in G$. Then $q \in \Sigma \cap g\Sigma$. By Lemma 5.7, there is $h \in G_q$ such that $h\Sigma = g\Sigma$. Then $h^{-1}g \in G_\Sigma$ and $(h^{-1}g)p = q$. □

Theorem 5.9. *The inclusion $\Sigma \to V$ induces a homeomorphism $\Sigma/G_\Sigma \to V/G$.*

Proof. The surjectivity is implied by the fact that Σ intersects *all* G-orbits. The injectivity is precisely Lemma 5.8. Since the inclusion $\Sigma \to V$ is continuous, the induced map $\Sigma/G_\Sigma \to V/G$ is continuous. The restriction $S(\Sigma)/G_\Sigma \to S(V)/G$, where $S(\Sigma) \subset \Sigma$, $S(V) \subset V$ are the unit spheres, is also a continuous bijection, and thus a homeomorphism, as $S(\Sigma)/G_\Sigma$ is compact. Since $\Sigma/G_\Sigma \to V/G$ is the cone over $S(\Sigma)/G_\Sigma \to S(V)/G$, it follows that $\Sigma/G_\Sigma \to V/G$ is a homeomorphism. □

In the remaining part of this section we study the intersection of the k-section with a nonprincipal orbit. Let $q \in \Sigma$ be a singular point.

Lemma 5.10. *We have an orthogonal decomposition $\Sigma = \Sigma \cap T_q(Gq) \oplus \Sigma \cap \nu_q(Gq)$.*

Proof. Write $\Sigma = T_q(G_\Sigma q) \oplus \nu_q^\Sigma(G_\Sigma q)$, where $\nu_q^\Sigma(G_\Sigma q)$ denotes the normal space to $G_\Sigma q$ at q in Σ. Lemma 5.8 implies that $T_q(G_\Sigma q) = \Sigma \cap T_q(Gq)$. We still need to show that $\nu_q^\Sigma(G_\Sigma q) = \Sigma \cap \nu_q(Gq)$ in order to complete the proof.

It is clear that $\nu_q^\Sigma(G_\Sigma q) \supset \Sigma \cap \nu_q(Gq)$. In order to prove the reverse inclusion, suppose first that $v \in \nu_q^\Sigma(G_\Sigma q)$ is such that $p = q + v$ is G-regular. We use the remark that a geodesic orthogonal to an orbit is orthogonal to every orbit it meets; we must have $v \in \nu_p^\Sigma(G_\Sigma p)$. Since p is G-regular, this implies that $v \in \nu_p(Gp)$. Again, by the same remark, $v \in \nu_q(Gq)$.

Since $\Sigma \cap \nu_q(Gq) \subset \nu_q^\Sigma(G_\Sigma q)$ and $\Sigma \cap \nu_q(Gq)$ intersects all $(G_q, \nu_q(Gq))$-orbits by Lemma 5.5, there is $w \in \nu_q^\Sigma(G_\Sigma q)$ such that $r = q + w$ is G-regular by Lemma 5.1. Now there is a neighborhood U of w in $\nu_q^\Sigma(G_\Sigma q)$ such that $p = q + v$ is G-regular for all $v \in U$. Since any open subset of a vector space contains a basis, the argument in the previous paragraph implies that $\nu_q^\Sigma(G_\Sigma q) \subset \Sigma \cap \nu_q(Gq)$, and this completes the proof of the lemma. □

Lemma 5.11. *Let $r : Gp \to Gq$ be the canonical equivariant submersion, where p is a G-regular point in the normal slice at q. Then $\Sigma \cap T_q(Gq) = r_*\mathcal{D}_{2p}$, and this is the orthogonal complement of $r_*\mathcal{E}_{2p} = \mathcal{E}_{2p}$ in $T_q(Gq)$.*

Proof. Write $p = q + tv$, where $v \in \nu_q(Gq)$. Let $u \in \mathcal{D}_{2p} \subset T_p(Gp)$. A computation similar to the one done in Lemma 5.4 shows that

$$r_*(u) = \underbrace{(\mathrm{id} + tA_v)u}_{\in \mathcal{D}_2 p} - \underbrace{t\nabla_u^\perp \hat{v}}_{\in \nu_p(Gp)}$$

is a vector in Σ and therefore orthogonal to $\mathcal{E}_{2p}$. Now

$$T_q(Gq) = r_*T_p(Gp) = r_*\mathcal{D}_p + r_*\mathcal{E}_p = \underbrace{r_*\mathcal{D}_{2p}}_{\subset \Sigma} + \underbrace{r_*\mathcal{E}_{2p}}_{=\mathcal{E}_{2p}}.$$

Since Σ is orthogonal to $\mathcal{E}_{2p}$, we conclude that the last sum is an orthogonal direct sum. □

Corollary 5.12. *We have $r_*\mathcal{D}_{2p} \oplus N_v = \mathcal{D}_{2p} \oplus \nu_p(Gp)$.*

Proof. Consider the orthogonal decomposition $\Sigma = \Sigma \cap T_q(Gq) \oplus \Sigma \cap \nu_q(Gq)$ from Lemma 5.10. On the one hand, we know that $\Sigma = \nu_p(Gp) \oplus \mathcal{D}_{2p} \oplus \mathcal{D}_{1p}$. On the other hand, we have that $\Sigma \cap T_q(Gq) = r_*\mathcal{D}_{2p}$ by Lemma 5.11 and $\Sigma \cap \nu_q(Gq) = N_v \oplus \mathcal{D}_{1p}$ by the proof of Lemma 5.5. This gives the result. □

5.3. **Reducible representations.** In this section we prove that the copolarity of a direct sum of representations is not smaller than the copolarity of its summand representations. The result is

Theorem 5.13. *Let (G, V) be an orthogonal representation and suppose that $V = V_1 \oplus V_2$ is an invariant decomposition. If* $\mathrm{copol}(G, V) \leq k$*, then* $\mathrm{copol}(G, V_i) \leq k$ *for $i = 1, 2$.*

Lemma 5.14. *Given a (G, V_1)-regular point $p_1 \in V_1$, there is $p_2 \in V_2$ such that $p = (p_1, p_2)$ is (G, V)-regular.*

Proof. Consider the representation (G_{p_1}, V_2) and take a (G_{p_1}, V_2)-regular point $p_2 \in V_2$. We claim that $p = (p_1, p_2)$ is (G, V)-regular. In fact, let $q = (q_1, q_2) \in V_1 \oplus V_2$. Since p_1 is (G, V_1)-regular, there is $h \in G$ such that $G_{p_1} \subset hG_{q_1}h^{-1} = G_{hq_1}$, and therefore $(G_{p_1})_{hq_2} \subset (G_{hq_1})_{hq_2} = G_{hq}$. Now p_2 is (G_{p_1}, V_2)-regular, so there is $g \in G$ such that $G_p = (G_{p_1})_{p_2} \subset g(G_{p_1})_{hq_2}g^{-1} \subset gG_{hq}g^{-1} = (gh)G_q(gh)^{-1}$. Hence, the result. □

Lemma 5.15. *Let Σ be a k-section of (G, V). Given a (G, V_1)-regular point $p_1 \in \Sigma \cap V_1$, there is $p_2 \in V_2$ such that $p = (p_1, p_2)$ is (G, V)-regular and $p \in \Sigma$.*

Proof. We already know from Lemma 5.14 that there is $p_2' \in V_2$ such that $p' = (p_1, p_2')$ is (G, V)-regular. Let $g \in G$ be such that $gp' = (gp_1, gp_2') \in \Sigma$. Note that $gp_1 \in \nu_{gp'}(Gp')$. Since gp' is (G, V)-regular, it follows that $\nu_{gp'}(Gp') \subset \Sigma$, so that $gp_1 \in \Sigma$. Now $p_1 \in \Sigma \cap g^{-1}\Sigma$. Therefore, by Lemma 5.7, there is $k \in G_{p_1}$ such that $kg^{-1}\Sigma = \Sigma$. Hence $p = kp' = (p_1, kp_2') \in \Sigma$. □

Lemma 5.16. *Let Σ be a k-section of (G, V). Given a (G, V_1)-regular point $p_1 \in \Sigma \cap V_1$, we have that $\Sigma \cap V_1$ contains $\nu_{p_1}^{V_1}(Gp_1)$ as a subspace of codimension at most k, where $\nu_{p_1}^{V_1}(Gp_1)$ denotes the normal space to Gp_1 at p_1 in V_1.*

Proof. Use Lemma 5.15 to find $p_2 \in V_2$ such that $p = (p_1, p_2)$ is (G, V)-regular and $p \in \Sigma$. Now Σ contains $\nu_p(Gp)$ with codimension k, and $\nu_{p_1}^{V_1}(Gp_1) = \nu_p(Gp) \cap V_1$. □

Proof of Theorem 5.13. Fix a (G, V_1)-regular $p_1 \in V_1$ and fix a k-section Σ for (G, V) with $p_1 \in \Sigma$. Define $\Sigma_1 = \bigcap_{h \in G_{p_1}} h\Sigma \cap V_1$ and let $q_1 \in \Sigma_1$ be (G, V_1)-regular (for instance, q_1 could be equal to p_1). Then, for all $h \in G_{p_1}$, we have $q_1 \in h\Sigma \cap V_1$ and $h\Sigma$ is a k-section for (G, V). It follows by Lemma 5.16 that $h\Sigma \cap V_1$ contains $\nu_{q_1}^{V_1}(Gq_1)$ as a subspace of codimension at most k for all $h \in G_{p_1}$. Therefore Σ_1 contains $\nu_{q_1}^{V_1}(Gq_1)$ as a subspace of codimension at most k, and hence Σ_1 satisfies conditions (C1), (C2) and (C3). This already implies that $\mathrm{copol}(G, V_1) \leq k$, but we go on to show that Σ_1 itself satisfies condition (C4).

Suppose that $p_1 \in \Sigma_1 \cap g^{-1}\Sigma_1$ for some $g \in G$. Then $p_1 \in h\Sigma \cap g^{-1}h\Sigma$ for all $h \in G_{p_1}$. We use Lemma 5.7 to find $l = l(h) \in G_{p_1}$ such that $lh\Sigma = g^{-1}h\Sigma$. This gives $g^{-1}\Sigma_1 = \bigcap_{h \in G_{p_1}} lh\Sigma \cap V_1 \supset \Sigma_1$. Therefore $g\Sigma_1 = \Sigma_1$. This shows that the distribution $\mathcal{D}_1$ defined by $\mathcal{D}_{1q_1} = \Sigma_1 \cap T_{q_1}(Gq_1)$ for $q_1 \in Gp_1$ satisfies assertion (c) in Proposition 2.1 with respect to the principal (G, V_1)-orbit Gp_1. It follows that Σ_1 satisfies condition (C4), and hence it is a k_1-section with $k_1 = \dim \mathcal{D}_1 \leq \dim \mathcal{D} = k$. □

5.4. **Minimal k-sections, the osculating spaces of orbits and the integrability of $\mathcal{E}$.** In this section we study properties of minimal k-sections of orthogonal representations with nontrivial copolarity. (Recall that an isometric action (G, M) has nontrivial copolarity if a minimal k-section is properly contained in M.) In particular, we show that in the irreducible case there can be no minimal k-sections of codimension one or two in the ambient space (Theorem 5.22). We also characterize the case for which the distribution $\mathcal{E}$ is integrable (Theorem 5.23).

Let (G, V) be an orthogonal representation. Define an equivalence relation in the set of G-regular points by declaring two points to be equivalent if they can be joined by a polygonal path which is (at smooth points) tangent to the distribution of normal spaces of the G-orbits. It is clear that the group action permutes the equivalence classes. For a G-regular $p \in V$, denote the equivalence class of p by $\mathfrak{S}_p$. Note that $0 \in \mathfrak{S}_p$, as p is a vector in the normal space of Gp at p. Therefore the affine hull $\langle \mathfrak{S}_p \rangle$ is a vector subspace of V.

Next suppose that (G, V) admits a k-section Σ. Let $p \in \Sigma$ be G-regular. It is clear that $\langle \mathfrak{S}_p \rangle \subset \Sigma$. Let $\mathfrak{g}$ be the Lie algebra of G and consider its induced action by linear skew-symmetric endomorphisms on V. For each $X \in \mathfrak{g}$, let $f_X : \Sigma \to \Sigma$ be defined by $f_X(q) = \Pi \circ X_q$, where $\Pi : V \to \Sigma$ is orthogonal projection. Then f_X is a linear map and $W_X := \ker f_X$ is a subspace of Σ. Let $p \in \Sigma$ be G-regular. Define the following subspace of Σ:

$$\bar{\Sigma}_p := \bigcap_{W_X \ni p} W_X.$$

We have that $\bar{\Sigma}_p$ is the subset of Σ that comprises all common zeros of the vector fields that are orthogonal projections onto Σ of the G-Killing fields that are orthogonal to Σ at p. It is clear from this definition that if $q \in \bar{\Sigma}_p$ is a G-regular point, then $\bar{\Sigma}_q \subset \bar{\Sigma}_p$.

Proposition 5.17. *$\langle \mathfrak{S}_p \rangle \subset \bar{\Sigma}_p \subset \Sigma$, and $\bar{\Sigma}_p$ is an l-section of (G, V) with $l \leq k$.*

Proof. Let $X \in \mathfrak{g}$ be such that $p \in W_X$. Then $X_p \perp \Sigma$. If $q \in \mathfrak{S}_p$, then q can be joined to p by a polygonal path which is normal to the orbits. It follows by an iterated application of Lemmas 4.3 and 4.4 that $X_q \perp \Sigma$, that is, $q \in W_X$. This shows that $\mathfrak{S}_p \subset W_X$. Therefore $\langle \mathfrak{S}_p \rangle \subset W_X$. Since X can be any element in $\mathfrak{g}$ satisfying $W_X \ni p$, we get that $\langle \mathfrak{S}_p \rangle \subset \bar{\Sigma}_p$.

We next prove that $\bar{\Sigma}_p$ is an l-section. Since $\bar{\Sigma}_p \subset \Sigma$, it will follow that $l \leq k$. In fact, condition (C1) for $\bar{\Sigma}_p$ is obvious, and condition (C2) follows from the facts that $\bar{\Sigma}_p \supset \mathfrak{S}_p$ and $\mathfrak{S}_p$ intersects all orbits. Let us verify condition (C4). Let $q \in \bar{\Sigma}_p$ be a G-regular point and $g \in G$ with $gq \in \bar{\Sigma}_p$. We need to show that $g\bar{\Sigma}_p = \bar{\Sigma}_p$. First we note that it is clear from the definition of $\bar{\Sigma}_p$ that $\bar{\Sigma}_{gq} \subset \bar{\Sigma}_p$. Next, since $\mathfrak{S}_p$ intersects all orbits and $\mathfrak{S}_p \subset \bar{\Sigma}_p$, we may assume that $q \in \mathfrak{S}_p$, and this implies, as above, via Lemmas 4.3 and 4.4, that $\bar{\Sigma}_q = \bar{\Sigma}_p$. Finally, since q, $gq \in \bar{\Sigma}_p \subset \Sigma$, condition (C4) for Σ gives that $g\Sigma = \Sigma$, and then we have $gW_X = W_{gXg^{-1}}$, which shows that $g\bar{\Sigma}_q = \bar{\Sigma}_{gq}$. Putting all this together, we have $g\bar{\Sigma}_p = g\bar{\Sigma}_q = \bar{\Sigma}_{gq} \subset \bar{\Sigma}_p$, and hence $g\bar{\Sigma}_p = \bar{\Sigma}_p$.

In order to check (C3), let $q \in \bar{\Sigma}_p$ be a G-regular point. Since $\mathfrak{S}_p$ intersects all orbits, there is $g \in G$ such that $gq \in \mathfrak{S}_p$. It is clear that $\nu_{gq}(Gq) \subset \langle \mathfrak{S}_p \rangle \subset \bar{\Sigma}_p$. Then $\nu_q(Gq) = g^{-1}\nu_{gq}(Gq) \subset g^{-1}\bar{\Sigma}_p = \bar{\Sigma}_p$, where the last equality follows from (C4) for $\bar{\Sigma}_p$. □

Corollary 5.18. *If Σ is a* minimal *k-section for (G,V) and $p \in \Sigma$ is G-regular, then for all $X \in \mathfrak{g}$ we have that $X_p \perp \Sigma$ if and only if $X_q \perp \Sigma$ for all $q \in \Sigma$.*

Proof. This is clear, because $\Sigma = \bar{\Sigma}_p$. □

Lemma 5.19. *If Σ is a* minimal *k-section for (G,V), then, given a Killing field X of V induced by G, we can write $X = X_1 + X_2$, where X_1, X_2 are Killing fields induced by G such that $X_1|_\Sigma$ is always tangent to Σ and $X_2|_\Sigma$ is always perpendicular to Σ.*

Proof. Fix a G-regular $p \in \Sigma$. Let $\bar{X}$ be the intrinsic Killing field of Σ which is obtained by projecting $X|_\Sigma$ to the tangent space of Σ. Since X_p is perpendicular to the normal space $\nu_p(Gp) \subset \Sigma$, we must have that $\bar{X}_p$ is tangent to $Gp \cap \Sigma$ at p. By Lemma 5.8, G_Σ acts transitively on $Gp \cap \Sigma$, so there exists a Killing field X_1 induced by $G_\Sigma \subset G$ such that $X_{1p} = \bar{X}_p$ (observe that the restriction of X_1 to Σ is always tangent to Σ). Then $X_2 = X - X_1$ is perpendicular to Σ at p, and, by Corollary 5.18, we have that $X_2|_\Sigma$ is always perpendicular to Σ. □

Fix a minimal k-section for (G,V) and a G-regular $p \in \Sigma$. Let $\mathfrak{k}$ be the Lie subalgebra of $\mathfrak{g}$ generated by all the $X \in \mathfrak{g}$ such that $X_p \perp \Sigma$ and let K be the associated connected subgroup of G. Let G^0_Σ be the connected component of the identity in G_Σ. It follows from Lemma 5.19 that G is generated by G^0_Σ and K. Next we show that K is a normal subgroup of G. It is enough to show that if $X \in \mathfrak{g}$ satisfies $X_p \perp \Sigma$ and $g \in G_\Sigma$, then $Y = gXg^{-1} \in \mathfrak{k}$. In fact, $Y_{gp} = gX_p \perp g\Sigma = \Sigma$, and hence $Y \in \mathfrak{k}$ by Corollary 5.18. Since K is normal in G, we get that G is a quotient of the semidirect product of G^0_Σ and K.

Proposition 5.20. *Let (G,V) be irreducible with nontrivial copolarity k and assume that Σ is a k-section. Then, for every G-regular $p \in \Sigma$, there does not exist a nonzero $\xi \in \nu_p(Gp)$ such that the Weingarten operator $A_\xi|_{\mathcal{E}_p} = 0$.*

Proof. Suppose there is a nonzero $\xi \in \nu_p(Gp)$ such that the Weingarten operator $A_\xi|_{\mathcal{E}_p} = 0$. Let $\hat{\xi}$ be the equivariant normal vector field along Gp which extends ξ. Then $\nabla^\perp_u \hat{\xi} = 0$ for all $u \in \mathcal{E}_p$ by Corollary 4.6. This implies that $\hat{\xi}$ is constant on Kp as a vector in V, so that ξ is in the fix-point set of K. Since K is normal in G, the fix-point set of K is G-invariant. Since G is irreducible on V, K must be trivial on V and then $\Sigma = V$, but this is impossible as (G,V) has nontrivial copolarity. □

Theorem 5.21. *Let (G,V) be irreducible with nontrivial copolarity k. Then the cohomogeneity of (G,V) is at most $\frac{l(l+1)}{2}$, where l is the codimension of a k-section in V.*

Proof. Let Gp be a principal orbit and choose a minimal k-section $\Sigma \ni p$. It follows from Proposition 5.20 that the map

$$\xi \in \nu_p(Gp) \mapsto A_\xi|_{\mathcal{E}_p} \in \mathrm{Sym}^2(\mathcal{E}_p^*)$$

is injective, where $\mathrm{Sym}^2(\mathcal{E}_p^*)$ denotes the symmetric square of the dual space of $\mathcal{E}_p$. Now we need just note that $\dim \mathcal{E}_p = l$. □

Theorem 5.22. *Let (G, V) be irreducible with nontrivial copolarity $k > 0$ and let n be the dimension of a principal orbit. Then $k \leq n - 3$.*

Proof. If $k = n - 1$, then $l = n - k = 1$, and by Theorem 5.21 the codimension of a principal orbit is 1. In this case G is transitive on the unit sphere and therefore polar, so this case is impossible.

If $k = n - 2$, then $l = 2$ and the codimension of a principal orbit is at most 3. Since (G, V) is not polar, it must have cohomogeneity 3 and therefore be one of the three irreducible representations of copolarity $k = 1$ listed in the table of Theorem 1.1 (see (i), Section 3.3). Then $n = 3$, but no such representation has principal orbits of dimension 3, so this case is impossible as well. □

We next characterize the orthogonal representations of nontrivial copolarity whose distribution $\mathcal{E}$ is integrable.

Theorem 5.23. *Let (G, V) be an orthogonal representation with nontrivial copolarity. Suppose that the distribution $\mathcal{E}$ is integrable. Then there are an orthogonal decomposition $V = V_1 \oplus V_2$, a polar representation (K, V_1) and another orthogonal representation (H, V_2) such that (G, V) is orbit equivalent to the direct product representation $(K \times H, V_1 \oplus V_2)$. Here the leaves of $\mathcal{E}$ correspond to the K-orbits.*

Proof. Let K be the normal subgroup of G as above. We first show that the leaves of $\mathcal{E}$ coincide with the K-orbits. For that purpose, note that $\mathcal{E}_q \subset T_q(Kq)$ for every G-regular $q \in \Sigma$, and then $\mathcal{E}_q \subset T_q(Kq)$ for every G-regular $q \in V$, as every K-orbit intersects Σ and $\mathcal{E}$ is K-invariant. It follows that, if $p \in \Sigma$ is a fixed G-regular point and β is the leaf of $\mathcal{E}$ through p, then $\beta \subset Kp$. Let K^0_β denote the connected component of the stabilizer of β in K. It is clear that $K^0_\beta p = \beta$. Let $X \in \mathfrak{g}$ be such that $X_p = u \in \mathcal{E}_p$. Since $\mathcal{E}_p = T_p\beta = T_p(K^0_\beta p)$, there is a Y in the Lie algebra of K^0_β such that $Y_p = u$. Now $Z = X - Y \in \mathfrak{k}$ and $Z_p = 0$. Therefore the one-parameter subgroup of K generated by Z is in the connected component of the isotropy subgroup K^0_p. But $K^0_p \subset K^0_\beta$, since K maps $\mathcal{E}$-leaves onto $\mathcal{E}$-leaves. It follows that Z is in the Lie algebra of K^0_β, and so is X. Since X is an arbitrary generator of $\mathfrak{k}$, it follows that $K^0_\beta = K$, and hence $\beta = Kp$. Note that the G-regular points in V are also K-regular.

Now Σ intersects all the principal K-orbits orthogonally, so these are complete isoparametric submanifolds and therefore closed, which implies that they are invariant under the closure $\bar{K}$ of K in G. Let $h \in \bar{K}$. Since $\bar{K}$ is connected, there is a continuous curve h_t in $\bar{K}$ joining the identity to h; namely, $h_0 = 1$ and $h_1 = h$. Since $\bar{K}p = Kp$, there is a continuous curve k_t in K such that $k_0 = 1$ and $k_t p = h_t p$. This implies that $k_t^{-1} h_t$ belongs to the connected component of the isotropy subgroup G^0_p. Since $G^0_p \subset K$, we deduce that $h \in K$. It follows that K is a closed subgroup of G, Σ is a section of (K, V) and (K, V) is polar. Let N denote the normalizer of K in the orthogonal group $\mathbf{O}(V)$. Then N maps K-orbits onto K-orbits. Let $n \in N$. Since Σ intersects all K-orbits, there is $k \in K$ such that $knp \in \Sigma$. Now kn maps Σ onto Σ. The principal K-orbits are isoparametric in V, and kn preserves their common focal set. Therefore kn preserves the focal hyperplanes in Σ. Decompose Σ into an orthogonal sum $\Sigma_1 \oplus V_2$, where Σ_1 is the span of the curvature normals of the principal K-orbits. If $q \in \Sigma$ is K-regular, then Kq is full in the affine subspace $q + V_1$, where V_1 is the orthogonal complement of V_2 in V, and K acts trivially on V_2. The decomposition $V = V_1 \oplus V_2$ is N-invariant. Now kn maps Σ_1 onto Σ_1 and maps the K-orbits in V_1 onto K-orbits in V_1. Therefore kn is in the Weyl group of (K, V_1), which is a finite group generated by the reflections on the focal hyperplanes in Σ_1. It follows that kn maps a K-orbit in V_1 onto the *same* K-orbit in V_1. In particular, if n is in the connected component N^0 of N, then kn is the identity on V_1. In any case, N and K have the same orbits in V_1.

Since G normalizes K, we have $Kp_1 = Gp_1$ for $p_1 \in V_1$, and then $G_\Sigma p_1 = Gp_1 \cap \Sigma = Kp_1 \cap \Sigma_1$ is finite. Now $G^0_\Sigma p_1 = \{p_1\}$. This shows that G^0_Σ is trivial on V_1. Since K is trivial on V_2, the intersection $G^0_\Sigma \cap K$ is in the kernel of the G-action on V. Therefore we may assume that $G^0_\Sigma \cap K = \{1\}$. Now the action of G on V is orbit-equivalent to the action of the direct product of G^0_Σ and K, and this completes the proof of the theorem. □

6. Representations of copolarity one

In this section we describe the structure of a principal orbit of a representation of copolarity one (Theorem 6.3) and classify the irreducible representations of copolarity one (Theorem 6.4). We start with two lemmas which are interesting on their own.

Lemma 6.1. *Let (G, V) be an orthogonal representation and let $p \in V$. If Gp is a full, homogeneous curve in V, then Gp is a principal orbit. In particular, if (G, V) is effective, then $G = S^1$.*

Proof. We have that $G/G_p = Gp$ is a group, since Gp is a circle. Therefore G_p is a normal subgroup of G. Now $G_{gp} = gG_pg^{-1} = G_p$ for any $g \in G$. This implies that G_p fixes every point in Gp. Since Gp is full in V, it follows that G_p acts trivially on V. □

Lemma 6.2. *Let (G, V) be an orthogonal representation, N any orbit, N_0 an extrinsically irreducible factor of N, and V_0 the linear span of N_0. Then V_0 is G-invariant.*

Proof. Let N' be the product of the other extrinsically irreducible factors of N and let V' be the linear span of N'. Since $V_0 \oplus V'$ is the linear span of the orbit N, it is G-invariant, and we may assume that $V_0 \oplus V' = V$.

Fix $p = (p_0, p') \in N_0 \times N' = N$ and let $g \in G$. Then $gp = (q_0, q')$ for some $q_0 \in N_0$ and $q' \in N'$. Now

$$gp = gp_0 + gp' = q_0 + q'$$

as elements of V, which implies

$$gp' - q' = q_0 - gp_0.$$

Note that since $g : N \to N$ is the restriction of an ambient isometry of V, it must preserve the decomposition of N into extrinsically irreducible factors, up to permutation of extrinsically isometric factors. Since G is connected, g can be deformed continuously to the identity and therefore there can be no such permutations. Hence

$$g : N_0 \times \{p'\} \to N_0 \times \{q'\}.$$

For any $x \in N_0$, we now have $g(x, p') \in N_0 \times \{q'\}$. Therefore, as elements of V, $gx + gp' - q' \in N_0$, which implies that $gx - gp_0 + q_0 \in N_0 \subset V_0$. Since $q_0 \in V_0$, we get

$$g(x - p_0) \in V_0$$

for all $x \in N_0$. Since N_0 is full in V_0 (because N is full in V), $\{x - p_0 : x \in N_0\}$ spans V_0. Therefore $gV_0 \subset V_0$. Since $g \in G$ is arbitrary, V_0 is G-invariant. (It is now easy to show that $N_0 = Gp_0 = G_{p'}p_0$ and $G = G_{p_0} \cdot G_{p'}$ is a factorization, but we will not need that.) □

Theorem 6.3. *Let (G, V) be an orthogonal representation with copolarity $k = 1$ and let $N = Gp$ be a principal orbit. Then the submanifold N of V splits extrinsically as $N = N_0 \times N_1$, where N_0 is either a homogeneous isoparametric submanifold or a point, and N_1 is one of the following:*

(i) *a nonisoparametric homogeneous curve;*
(ii) *a focal manifold of a homogeneous irreducible isoparametric submanifold which is obtained by focalizing a one-dimensional distribution;*
(iii) *a codimension 3 nonisoparametric homogeneous submanifold.*

Proof. It follows from the proof of Theorem B in [OS95] that the Lie algebra $\mathfrak{k}$, which is algebraically generated by the projection to $\nu_p(N)$ of Killing fields induced by G restricted to $\nu_p(N)$, contains the normal holonomy algebra, and it is contained in its normalizer. By Lemma 5.19, $\mathfrak{k}$ is generated by the projection of the Killing fields induced by G_Σ, where Σ is a 1-section through p. But if X, $Y \neq 0$ are such Killing fields, then they must be proportional at p, since $\dim(Gp \cap \Sigma) = 1$. By multiplying one of them by a nonzero scalar, we may assume that $X_p = Y_p$. Now, by Corollary 5.18, $(X - Y)|_\Sigma$ must always be perpendicular to Σ and is thus zero. Therefore $\mathfrak{k}$ is generated by $X|_\Sigma$, and so it has dimension 1. In particular, the restricted normal holonomy group of N has dimension 0 or 1.

Let N_1 be a nonisoparametric extrinsically irreducible factor of N (there exists such a factor because the copolarity of (G, V) is 1). Let V_1 be the linear span of N_1. If N_1 has flat normal bundle, then, by Theorem A in [Olm94], N_1 is a homogeneous curve (which is not an extrinsic circle). Assume that N_1 has nonflat normal bundle. Orthogonally decompose the normal bundle in V_1

$$\nu(N_1) = \nu_0(N_1) \oplus \nu_s(N_1),$$

where $\nu_0(N_1)$ is the maximal parallel and flat subbundle of $\nu(N_1)$. By the Normal Holonomy Theorem [Olm90], since the restricted normal holonomy group of N_1 has dimension 1, $\nu_s(N_1)$ has dimension 2 over N_1 (and the restricted normal holonomy group acts as the circle action in a two-dimensional Euclidean space). If the codimension of N_1 in V_1 is greater than 3, then $\mathrm{rank}(N_1)$ (i.e., the dimension of $\nu_0(N_1)$ over N_1) is at least 2. Then, by Theorem A in [Olm94], G can be enlarged to a group admitting a representation which is the isotropy representation of an irreducible symmetric space and has N_1 as an orbit. The normal holonomy tube, which has one dimension more and coincides with a principal orbit of the isotropy representation, is a homogeneous irreducible isoparametric submanifold which has N_1 as a focal manifold. This completes the argument to show that a nonisoparametric extrinsically irreducible factor of N must be of one of the three types listed in the theorem.

Suppose now that N is not extrinsically irreducible, and let N_0 be the product of the extrinsically irreducible factors of N other than N_1. We need to show that N_0 is isoparametric. Suppose, on the contrary, that this is not the case. Notice that there can be at most one extrinsically irreducible factor of N with nonflat normal bundle, for otherwise the restricted normal holonomy group of N would have dimension bigger than one. So, since we are supposing that N_0 contains a nonisoparametric extrinsically irreducible factor N_2 of N, after renaming these factors if necessary, we may assume that N_1 is a homogeneous curve.

Suppose that N_2 is also a homogeneous curve. Let V_2 be the linear span of N_2. Then V_i is G-invariant and N_i is a principal orbit of (G, V_i) for $i = 1, 2$, by Lemmas 6.1 and 6.2. This implies that $(G, V_1 \oplus V_2)$ has copolarity 2, and so contradicts Theorem 5.13. It follows that N_2 cannot be a homogeneous curve. Therefore N_2 has nonflat normal bundle, and this implies that the normal holonomy algebra of N has dimension 1.

Let $p = (p_0, p_1) \in N_0 \times N_1 = N$, and let Z be a Killing field induced by G on V such that Z_p is tangent to $\{p_0\} \times N_1$. Since N_1 is not an extrinsic circle, the projection of Z to the normal space of N_1 in V_1 is a nontrivial intrinsic Killing field. This implies that the projection $\bar{Z}$ of Z to the normal space of N in V is also a nontrivial intrinsic Killing field. Therefore $\bar{Z}$ generates the Lie algebra $\mathfrak{k}$ referred to above. Since the normal holonomy algebra of N is not trivial and is contained in $\mathfrak{k}$, it follows that it coincides with $\mathfrak{k}$. But the normal holonomy algebra of N acts trivially on the normal space of N_1 in V_1 (since curves have flat normal bundle), and this is a contradiction to the fact that the projection of Z to the normal space of N_1 in V_1 is nontrivial. Hence N_0 cannot contain any nonisoparametric extrinsically irreducible factor of N, and thus N_0 is isoparametric. □

Theorem 6.4. *Let (G, V) be an irreducible representation of nontrivial copolarity 1. Then (G, V) is one of the three orthogonal representations listed in the table of Theorem 1.1.*

Proof. Since (G, V) is irreducible, any orbit is extrinsically irreducible by Lemma 6.2. Now we know from Theorem 6.3 that any principal orbit is either a codimension 3 homogeneous submanifold or a focal manifold of an irreducible homogeneous isoparametric submanifold. (It cannot be a homogeneous curve because the copolarity is not trivial, nor a homogeneous isoparametric submanifold since the action is not polar.) If there is a principal orbit which falls into the first case, then the cohomogeneity is 3 and the result follows from the classification of cohomogeneity 3 irreducible representations (see (i), Section 3.3). Suppose, on the contrary, that no principal orbit falls into the first case. Then any principal orbit is a focal manifold of an isoparametric submanifold; thus it is taut (see [HPT88]). If we can show that the nonprincipal orbits are also taut, then it will follow from the classification of taut irreducible representations [GT03] that the cohomogeneity is 3, and this will be a contradiction.

Let Gq be a nonprincipal orbit. We need to prove that it is tautly embedded in V. We can find a vector $v \in \nu_q(Gq)$ and a decreasing sequence $\{t_n\}$ such that $t_n \to 0$ and $p_n = q + t_n v$ are G-regular points. For each n there are a group $K_n \supset G$ and an isotropy representation of a symmetric space (K_n, V) such that $Gp_n = K_n p_n$. Since the number of isotropy representations of symmetric spaces of a given dimension is finite, by passing to a subsequence we can assume that there is a sequence $h_n \in \mathbf{SO}(V)$ such that $K_n = h_n K_1 h_n^{-1}$ for $n \geq 2$. By compactness of $\mathbf{SO}(V)$, again by passing to a subsequence we can write $h_n \to h \in \mathbf{SO}(V)$. Let $K_\infty = hK_1h^{-1}$. We finally prove that $K_\infty q = Gq$. Since (K_∞, V) is conjugate to the isotropy representation of a symmetric space, its orbits are tautly embedded [BS58], and this shows that Gq is taut.

Let $k \in K_\infty$. We have that $k = hk_1h^{-1}$ for some $k_1 \in K_1$, and then $k = \lim k_n$, where $k_n = h_nk_1h_n^{-1} \in K_n$ for $n \geq 2$. Since $K_np_n = Gp_n$, there is $g_n \in G$ such that $g_np_n = k_np_n$. By passing to a subsequence we may assume that $g_n \to g \in G$. Now $kq = \lim k_np_n = \lim g_np_n = gq \in Gq$, and this proves that $K_\infty q \subset Gq$. Since the reverse inclusion is clear, this completes the proof of the claim and the proof of the theorem. □

Remark 6.5. We do not know of any examples of reducible representations of copolarity 1 showing that (ii) in Theorem 6.3 indeed can occur.

Final questions. (1) Is it true that a minimal k-section Σ of an orthogonal representation (G, V), where G is the maximal (not necessarily connected) subgroup of $\mathbf{O}(V)$ with its orbits, always coincides with a connected component of the fixed point set of a principal isotropy group at a point $p \in \Sigma$?

(2) Is there an example of a focal manifold of a homogeneous irreducible isoparametric submanifold, obtained by focalizing a one-dimensional distribution, which is an extrinsic factor of a principal orbit of a *reducible* representation of nontrivial copolarity 1 and cohomogeneity bigger than 3?

(3) Classify representations of nontrivial copolarity (in the irreducible case we believe that there should be not too many examples).

References

[Bot56] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251–281. MR **19:**291a

[BS58] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029, correction in Amer. J. Math. **83** (1961), 207–208. MR **30:**589

[Con71] L. Conlon, *Variational completeness and K-transversal domains*, J. Differential Geom. **5** (1971), 135–147. MR **45:**4320

[Dad85] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), 125–137. MR **86k:**22019

[DO01] A. J. Di Scala and C. Olmos, *Variationally complete representations are polar*, Proc. Amer. Math. Soc. **129** (2001), 3445–3446. MR **2002d:**53065

[GS00] K. Grove and C. Searle, *Global G-manifold reductions and resolutions*, Ann. Global Anal. and Geom. **18** (2000), 437–446, Special issue in memory of Alfred Gray (1939–1998). MR **2001g:**57068

[GTa] C. Gorodski and G. Thorbergsson, *Representations of compact Lie groups and the osculating spaces of their orbits*, Preprint, University of Cologne, 2000 (also E-print math. DG/0203196).

[GTb] C. Gorodski and G. Thorbergsson, *Variationally complete actions on compact symmetric spaces*, J. Differential Geom. **62** (2002), 39–48.

[GT02] C. Gorodski and G. Thorbergsson, *Cycles of Bott-Samelson type for taut representations*, Ann. Global Anal. Geom. **21** (2002), 287–302. MR **2003c:**53067

[GT03] C. Gorodski and G. Thorbergsson, *The classification of taut irreducible representations*, J. Reine Angew. Math. **555** (2003), 187–235.

[HPT88] W.-Y. Hsiang, R.S. Palais, and C.-L. Terng, *The topology of isoparametric submanifolds*, J. Differential Geom. **27** (1988), 423–460. MR **89m:**53104

[Olm90] C. Olmos, *The normal holonomy group*, Proc. Amer. Math. Soc. **110** (1990), 813–818. MR **91b:**53069

[Olm94] C. Olmos, *Homogeneous submanifolds of higher rank and parallel mean curvature*, J. Differential Geom. **39** (1994), 605–627. MR **95h:**53079

[OS95] C. Olmos and M. Salvai, *Holonomy of homogeneous vector bundles and polar representations*, Indiana Univ. Math. J. **44** (1995), 1007–1015. MR **97a:**53078

[PT87] R. S. Palais and C.-L. Terng, *A general theory of canonical forms*, Trans. Amer. Math. Soc. **300** (1987), 771–789. MR **88f:**57069

[SS95] T. Skjelbred and E. Straume, *A note on the reduction principle for compact transformation groups*, preprint, 1995.

[Str94] E. Straume, *On the invariant theory and geometry of compact linear groups of cohomogeneity* ≤ 3, Diff. Geom. and its Appl. **4** (1994), 1–23. MR **95c:**20061

[Uch80] F. Uchida, *An orthogonal transformation group of* $(8k-1)$*-sphere*, J. Differential Geom. **15** (1980), 569–574. MR **83a:**57056

[Yas86] O. Yasukura, *A classification of orthogonal transformation groups of low cohomogeneity*, Tsukuba J. Math. **10** (1986), 299–326. MR **88b:**57036

Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, São Paulo, SP 05508-090, Brazil
E-mail address: gorodski@ime.usp.br

Facultad de Matemática, Astronomía y Física, Universidad Nacional Córdoba, Medina Allende y Haya de la Torre, Ciudad Universitaria, 5000 Córdoba, Argentina
E-mail address: olmos@mate.uncor.edu

Departamento de Matemática, Universidade Federal de São Carlos, Rodovia Washington Luiz, Km 235, São Carlos, SP 13565-905, Brazil
E-mail address: tojeiro@dm.ufscar.br