TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 5, Pages 2081–2091
S 0002-9947(03)03384-1
Article electronically published on November 12, 2003

POSITIVE LAWS IN FIXED POINTS

PAVEL SHUMYATSKY

Abstract. Let A be an elementary abelian group of order at least q^3 acting on a finite q'-group G in such a manner that $C_G(a)$ satisfies a positive law of degree n for any $a \in A^{\#}$. It is proved that the entire group G satisfies a positive law of degree bounded by a function of q and n only.

1. Introduction

Let A be a finite group acting coprimely on a finite group G. It is well known that the structure of the centralizer $C_G(A)$ (the fixed-point subgroup) of A has strong influence over the structure of G. The best illustration for this phenomenon is the fact that if G admits a fixed-point-free automorphism of prime order, then G is nilpotent (Thompson, [20]) and the nilpotency class of G is bounded by a function depending only on the order of the automorphism (Higman, [8]). Thus we see that in certain situations restrictions on centralizers of coprime automorphisms result in very specific identities that hold in G. An interesting problem is to describe as many such situations as possible. Powerful Lie-theoretic results of Zel'manov and of Bahturin, Linchenko and Zaicev (see Section 3 of this paper) provide us with very effective tools for dealing with the problem. Those tools were recently used with some success in the papers [9] and [6]. Our goal in the present paper is to give a proof of the following theorem that belongs to the same category as the main results in [9] and [6].

Theorem A. *Let q be a prime. Let A be an elementary abelian group of order q^3 acting on a finite q'-group G in such a manner that $C_G(a)$ satisfies a positive law of degree n for any $a \in A^{\#}$. Then the entire group G satisfies a positive law of degree bounded by a function of n and q only.*

As usual, the symbol $A^{\#}$ stands for the set of all non-trivial elements of the group A. A positive law in a group can be defined as follows.

Let F denote the free group on $X = \{x_1, x_2, \dots\}$. A positive word in X is any non-trivial element of F not involving the inverses of the x_i. A positive (or semigroup) law of a group G is a non-trivial identity of the form $u \equiv v$, where u, v are positive words in F, holding under every substitution $X \to G$. The maximum of lengths of u and v is called the degree of the law $u \equiv v$. By a result of Mal'cev [13] (see also [14]) a group that is an extension of a nilpotent group by a group of finite exponent satisfies a positive law. More precisely, Mal'cev has discovered

Received by the editors January 2, 2003 and, in revised form, April 15, 2003.
2000 *Mathematics Subject Classification.* Primary 20D45.
Key words and phrases. Automorphisms, centralizers, associated Lie rings.
The author was supported by CNPq-Brazil.

a positive law $M_c(x,y)$ of 2 variables and of degree 2^c that holds in any nilpotent group of class c. Therefore, if G is an extension of a nilpotent group of class c by a group of exponent e, then G satisfies the positive law $M_c(x^e, y^e)$. The explicit form of the Mal'cev law will not be required in this paper.

The question of whether any group satisfying a positive law is necessarily an extension of a nilpotent group by a group of finite exponent was settled negatively by Ol'shanskiĭ and Storozhev in [15]. In contrast with this negative result, Burns, Macedońska and Medvedev answered the question in the affirmative for a large class of groups including all solvable and residually finite groups [2]. In particular they showed that there exist functions $c(n)$ and $e(n)$ of n only, such that any finite group satisfying a positive law of degree n is an extension of a nilpotent group of class at most $c(n)$ by a group of exponents dividing $e(n)$.

Theorem A fails if the group A has order q^2. This can be shown using an example from [10]. Let p be an odd prime and let t denote the largest odd divisor of $p-1$. Let G_k be the group formed by the matrices

$$M = \begin{pmatrix} u+pa & pb \\ pc & v+pd \end{pmatrix}$$

of determinant 1, where a, b, c, d, u, v lie in the ring of residue classes modulo p^{k+1} and $uv = u^t = 1$ modulo p. Then G_k is of derived length m or $m+1$, where m is the least integer such that $2^m \geq k+1$. Let α_k and β_k be the automorphisms of G_k such that for any $M = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) \in G_k$ we have $M^{\alpha_k} = (M^{-1})^T$ and $\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)^{\beta_k} = \left(\begin{smallmatrix} a_1 & -a_2 \\ -a_3 & a_4 \end{smallmatrix}\right)$. It is easy to check that $V_k = \langle \alpha_k, \beta_k \rangle$ is a four group acting fixed-point-freely on G_k. Further analysis will show that the centralizer in G_k of any $\alpha \in V_k^{\#}$ is cyclic (so it satisfies the positive law $xy \equiv yx$) and G_k can be generated by 3 elements. Thus there is no positive law that holds in all groups G_k.

The reduction of Theorem A to the case that G is nilpotent uses the classification of finite simple groups (see Lemmas 2.5, 2.6 and 2.8 in the next section). The author expresses his thanks to Robert M. Guralnick for providing the important Lemma 2.5 and suggesting the idea to use it in the proof of Lemma 2.6. Thus, the proof of Lemma 2.6 presented here has a number of advantages as compared to the proof in the original version of the paper.

2. Preliminaries

Throughout the article we use the term "$\{a, b, c \dots\}$-bounded" to mean "bounded from above by some function depending only on the parameters $a, b, c \dots$".

The first two lemmas are well known (see for example [5, 5.3.16, 6.2.2, 6.2.4]).

Lemma 2.1. *Let A be a group of automorphisms of the finite group G with $(|A|, |G|) = 1$.*

(1) *If N is any A-invariant normal subgroup of G, we have*

$$C_{G/N}(A) = C_G(A)N/N.$$

(2) *If H is an A-invariant p-subgroup of G, then H is contained in an A-invariant Sylow p-subgroup of G.*

(3) *If P is an A-invariant Sylow p-subgroup of G, then $C_P(A)$ is a Sylow p-subgroup of $C_G(A)$.*

Lemma 2.2. *Let q be a prime, and G a finite q'-group acted on by an elementary abelian q-group A of rank at least 2. Let $A_1, \dots, A_s$ be the maximal subgroups of A. If H is an A-invariant subgroup of G we have $H = \langle C_H(A_1), \dots, C_H(A_s)\rangle$. Furthermore, if H is nilpotent, then $H = \prod_i C_H(A_i)$.*

Lemma 2.3. *Let q be a prime, and G a finite q'-group acted on by an elementary abelian q-group A of rank r at least 2. Let $A_1, \dots, A_s$ be the maximal subgroups of A. Let m be the maximum of orders of $C_G(A_i)$, where i ranges from 1 to s. Then the order of G is $\{m, q, r\}$-bounded.*

Proof. Let p be a prime divisor of $|G|$. By Lemma 2.1 A normalizes some Sylow p-subgroup P of G. Lemma 2.2 tells us that $P = \prod C_P(A_i)$. It is immediate now that $|P| \le m^s$. We notice that this in particular shows that any prime divisor of $|G|$ is $\{m, s\}$-bounded. Hence, $|G|$ is $\{m, s\}$-bounded. Since s depends only on q and r, the lemma follows. □

The following lemma will be helpful later on.

Lemma 2.4. *Let A be an elementary abelian group of order q^3 acting on a finite q'-group G in such a manner that $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. Suppose there exist a subgroup $B \le A$ of order q^2 and an A-invariant normal abelian subgroup $M \le G$ such that $G = MC_G(B)$. Then G is nilpotent of class at most c.*

Proof. Let $A_1, A_2, \dots, A_s$ be the subgroups of order q^2 of A and for any $i = 1, \dots, s$ set $M_i = C_M(A_i)$. By Lemma 2.2 $M = \prod M_j$. Therefore to show that G is nilpotent of class at most c it is sufficient to prove that so is $\langle M_j, C_G(B)\rangle$ for any $j = 1, \dots, s$. Since $|A| = q^3$, we can choose a non-trivial element $b \in B \cap A_j$. It remains to note that $C_G(B)$ and M_j are both contained in $C_G(b)$, which is nilpotent of class at most c, as required. □

In what follows $F(G)$ denotes the Fitting subgroup of a group G and $E(G)$ the product of quasisimple subnormal subgroups of G. In the case where $F(G) = 1$, the subgroup $E(G)$ is just the socle of G, that is, the direct product of minimal normal subgroups of G. An important property is that $[F(G), E(G)] = 1$ in any finite group G [3, 10.1]. The next lemma is due to Robert M. Guralnick.

Lemma 2.5. *Let A be a non-cyclic group of order q^2 acting on a finite q'-group N which is a direct product of simple groups. Then $N = \langle E(C_N(a)); a \in A^{\#}\rangle$.*

Proof. There is no loss in assuming that A acts transitively on the simple factors of N. Let t be the number of simple factors, and let L denote one of them. Suppose first that $t = 1$. We will use the fact that if B is any coprime group of automorphisms of a finite simple group, then B is cyclic—this is where we use the classification of finite simple groups (see [6]). In the case that $t = 1$, some non-trivial $a \in A$ centralizes N. Then $N = E(C_N(a))$ and we have nothing to prove.

If $t = q$, then some non-trivial element of A normalizes each factor and $q^2 - q$ elements of A permute the factors. Choose generators a and b for A among the elements not normalizing any of the factors. Then $C_N(a)$ and $C_N(b)$ are distinct diagonal subgroups of N. These are simple and so $C_N(a) = E(C_N(a))$ and $C_N(b) = E(C_N(b))$. The subgroup $\langle E(C_N(a)), E(C_N(b))\rangle$ is A-invariant and, since q is prime, this can only be N.

If $t = q^2$, then A permutes the factors regularly and $E(C_N(a))$ is a product of q copies of L, where each factor is a diagonal subgroup of an orbit of a. It is easy to see that, since these orbits are distinct for generators a, b of A, two of these generate an A-invariant subgroup of N containing L. This subgroup is necessarily all of N since A permutes the factors regularly. □

The next lemma is well known in the case that G is solvable (see for example [4, Lemma 2.5]). The importance of the lemma in the context of the present paper is due to the fact that it will enable us to reduce Theorem A to the case that G is a p-group.

Lemma 2.6. *Let A be a non-cyclic group of order q^2 acting on a finite q'-group G. Let $C = \bigcap_{a\in A^\#} F(C_G(a))$. Then $C \leq F(G)$.*

Proof. Let G be a counterexample of minimal possible order. Let N be a minimal A-invariant normal subgroup of G. By induction we assume that $NC/N \leq F(G/N)$. If $F(G) \neq 1$, we can assume that $N \leq F(G)$. Then NC is a solvable group satisfying the hypothesis of the lemma and, since the lemma is known to be true for solvable groups, we conclude that $C \leq F(NC)$, which quickly gives us a contradiction. So we can assume that $F(G) = 1$ and N is a direct product of simple groups.

Taking any element $c \in C$ of prime order p and considering the subgroup $N\langle c\rangle$ we notice that the subgroup satisfies the hypothesis of the lemma, whence it follows by the induction hypothesis that $G = N\langle c\rangle$. Since $c \in F(C_G(a))$, it follows that c actually centralizes $E(C_G(a))$ for any $a \in A^\#$. Combining this with Lemma 2.5, we derive that $c \in Z(G)$, a contradiction. □

We now require the following theorem obtained in [9].

Theorem 2.7. *Let q be a prime and e a positive integer. Let A be an elementary abelian group of order q^2 acting on finite q'-group G. Assume that the exponent of $C_G(a)$ divides e for any $a \in A^\#$. Then the exponent of G is $\{e, q\}$-bounded.*

The next lemma will be deduced by combining Lemma 2.6 and Theorem 2.7.

Lemma 2.8. *Let q be a prime and e a positive integer. Let A be an elementary abelian group of order q^3 acting on a finite q'-group G. Assume that the exponent of the quotient $C_G(a)/F(C_G(a))$ divides e for any $a \in A^\#$. Then the exponent of the quotient $G/F(G)$ is $\{e, q\}$-bounded.*

Proof. Let $A_1, A_2, \ldots, A_s$ be the subgroups of order q^2 of A and for any $i = 1, \ldots, s$ set $F_i = \bigcap_{a\in A_i^\#} F(C_G(a))$ and $G_i = C_G(A_i)$. Evidently F_i is a normal subgroup of G_i and the exponent of G_i/F_i is $\{e, q\}$-bounded. Lemma 2.6 tells us that $F_i \leq F(G)$ for any i. It becomes clear that $\bar{G}_i$, the image of G_i in $\bar{G} = G/F(G)$, has $\{e, q\}$-bounded exponent. If $a \in A^\#$, the group A induces a group of automorphisms, say B, of $C_{\bar{G}}(a)$ that acts in such a manner that the exponent of the centralizer of any $b \in B^\#$ coincides with some $\bar{G}_i$. If B has order q^2, it follows by Theorem 2.7 that $C_{\bar{G}}(a)$ has an $\{e, q\}$-bounded exponent. If the order of B is less than q^2, we conclude that $C_{\bar{G}}(a) \leq \bar{G}_i$ for some i so the exponent of $C_{\bar{G}}(a)$ is $\{e, q\}$-bounded anyway.

Thus, for any $a \in A^\#$ we have shown that $C_{\bar{G}}(a)$ has an $\{e, q\}$-bounded exponent. Theorem 2.7 now shows that $\bar{G}$ has an $\{e, q\}$-bounded exponent, as required. □

Lemma 2.9. *Let A be an elementary abelian group of order q^3 acting on a finite q'-group G that is solvable with derived length at most d. Assume that $C_G(a)$ has a nilpotent subgroup of class at most c and index at most k for any $a \in A^{\#}$. Then G has a characteristic nilpotent subgroup whose class and index are both bounded in terms of q, c, d and k alone.*

Proof. Clearly, without any loss of generality we can assume that $C_G(a)$ has a characteristic nilpotent subgroup of class at most c and index at most k for any $a \in A^{\#}$. Let $A_1, A_2, \ldots, A_s$ be the subgroups of order q^2 of A and set $G_i = C_G(A_i)$ for $i = 1, \ldots, s$. For any $a \in A^{\#}$ we let D_a denote some characteristic nilpotent subgroup of class at most c and index at most k in $C_G(a)$. Next, we set $D_i = \bigcap_{a \in A_i^{\#}} D_a$. Then the index of D_i in G_i is $\{q, k\}$-bounded. We have an important property of these subgroups: if M is any A-invariant normal abelian subgroup of G, then MD_i is nilpotent of class at most $c+1$.

This can be proved in the same way as Lemma 2.4. Indeed, by Lemma 2.2 $M = \prod M_j$, where $M_j = G_j \cap M$. Therefore to show that MD_i is nilpotent of class at most $c+1$ it is sufficient to prove that so is $\langle M_j, D_i \rangle$ for any $j = 1, \ldots, s$. Since $|A| = q^3$, we can choose a non-trivial element $b \in A_i \cap A_j$. It remains to note that D_i is contained in a characteristic nilpotent subgroup of class at most c of $C_G(b)$ while M_j is contained in a normal abelian subgroup of $C_G(b)$. It follows that $\langle M_j, D_i \rangle$ is nilpotent of class at most $c+1$, as required.

Now suppose N is an A-invariant normal subgroup of G, that is, nilpotent of class u. Then ND_i is nilpotent of class at most $c\frac{u(u+1)}{2} + u$. This is because we already know from the previous paragraph that ND_i/N' is of class at most $c+1$ so the bound $c\frac{u(u+1)}{2} + u$ follows from well-known Hall's theorem [7].

We are now in a position to complete the proof of the lemma using induction on d. Assume by induction that G' has a characteristic nilpotent subgroup with the required properties. Now set $L_0 = G'$ and $L_j = L_{j-1}G_j$ for $j = 1, \ldots, s$. It is clear from Lemma 2.2 that $L_s = G$. We will now use induction on j to show that L_j has a characteristic nilpotent subgroup with the required properties for all j. Assume that this is true for L_{j-1} and let Q be a characteristic nilpotent subgroup of L_{j-1} whose class u and index t are both bounded in terms of q, c, d and k alone. Consider the subgroup QD_j of L_j. One can easily check that the index of QD_j in L_j is at most kt and we have seen above that QD_j is nilpotent of $\{c, u\}$-bounded class. Thus the characteristic closure of QD_j in L_j is a characteristic subgroup with the required properties in L_j. The proof is complete. □

In [19] we proved the following theorem.

Theorem 2.10. *Let A be an elementary abelian group of order q^3 acting on a finite q'-group G. Assume that $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. Then G is nilpotent and the class of G is bounded by a function depending only on q and c.*

We will now extend the above result in the following way.

Theorem 2.11. *Let A be an elementary abelian group of order q^3 acting on a finite q'-group G. Assume that $C_G(a)$ has a nilpotent subgroup of class at most c and index at most k for any $a \in A^{\#}$. Then G has a normal nilpotent subgroup whose class and index are both bounded in terms of q, c and k alone.*

Proof. Let $A_1, A_2, \dots, A_s$ be the subgroups of order q^2 of A. Then, combining Lemma 2.6 with Lemma 2.1(1), we deduce that in the quotient $G/F(G)$ the centralizer of any A_i has $\{k,q\}$-bounded order. It follows from Lemma 2.3 that $G/F(G)$ has $\{k,q\}$-bounded order. Thus it is sufficient to prove the lemma under the hypothesis that G is nilpotent. For any A-invariant subgroup H of G and any $a \in A^{\#}$ we define the parameter $j_a(H)$ to be the least positive integer j such that $C_H(a)$ has a normal subgroup of class at most c and index at most j. We set $k(H) = \sum_{a \in A^{\#}} j_a(H)$. It is clear that $k(G)$ is $\{k,q\}$-bounded. The lemma will be proved by induction on $k(G)$, the case $k(G) = q^3 - 1$ (the smallest possible value for $k(G)$—it occurs if and only if $C_G(a)$ is nilpotent of class c for any $a \in A^{\#}$) being immediate from Theorem 2.10. Assume by induction that if H is any A-invariant subgroup of G with the property that $k(H) \le k(G)$, then H has a normal nilpotent subgroup whose class and index are both bounded in terms of q, c and k alone. Let $f = f(c,q)$ be the function depending only on q and c that bounds the class of the group in Theorem 2.10. Denote by R the $(f+2)$th term of the lower central series of G. Suppose $k(R) = k(G)$. It is fairly easy to check that in this case $C_{G/R}(a)$ is nilpotent of class c for any $a \in A^{\#}$. Now Theorem 2.10 tells us that G/R has class at most f. Since $R = \gamma_{f+2}(G)$, we are forced to conclude that $R = 1$ and we are done.

Assume now that $k(R) \le k(G)$. Then, by the induction hypothesis, R has a normal nilpotent subgroup whose class and index are both bounded in terms of q, c and k alone. It follows that the derived length of G is $\{c,k,q\}$-bounded. Now the theorem is straightforward from Lemma 2.9. □

The remaining part of the proof of Theorem A is based on so-called Lie methods, which we will proceed to describe in the next section.

3. Some Lie-theoretic machinery

Let L be a Lie algebra over a field $\mathfrak{k}$. Let k, n be positive integers and let $x_1, x_2, \dots, x_k, x, y$ be elements of L. We define inductively

$$[x_1] = x_1;\ [x_1, x_2, \dots, x_k] = [[x_1, x_2, \dots, x_{k-1}], x_k]$$

and

$$[x, {}_0y] = x;\ [x, {}_ny] = [[x, {}_{n-1}y], y].$$

An element $a \in L$ is called ad-nilpotent if there exists a positive integer n such that $[x, {}_na] = 0$ for all $x \in L$. If n is the least integer with the above property, then we say that a is ad-nilpotent of index n. Let $X \subseteq L$ be any subset of L. By a commutator in elements of X we mean any element of L that can be obtained as a Lie product of elements of X with some system of brackets. Denote by F the free Lie algebra over $\mathfrak{k}$ on countably many free generators $x_1, x_2, \dots$. Let $f = f(x_1, x_2, \dots, x_n)$ be a non-zero element of F. The algebra L is said to satisfy the identity $f \equiv 0$ if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_1, a_2, \dots, a_n \in L$. In this case we say that L is PI. A deep result of Zel'manov says that if a Lie algebra L is PI and is generated by finitely many elements all commutators in which are ad-nilpotent, then L is nilpotent [22, III(0.4)]. Using this and some routine universal arguments, the next theorem can be deduced (see [9]).

Theorem 3.1. *Let L be a Lie algebra over a field $\mathfrak{k}$ generated by $a_1, a_2, \ldots, a_m$. Assume that L satisfies an identity $f \equiv 0$ and that each commutator in the generators $a_1, a_2, \ldots, a_m$ is ad-nilpotent of index at most n. Then L is nilpotent of $\{f, n, m, \mathfrak{k}\}$-bounded class.*

An important criterion for a Lie algebra to be PI is the following

Theorem 3.2 (Bahturin-Linchenko-Zaicev)**.** *Let L be a Lie algebra over a field $\mathfrak{k}$. Assume that a finite group A acts on L by automorphisms in such a manner that $C_L(A)$, the subalgebra formed by fixed elements, is PI. Assume further that the characteristic of $\mathfrak{k}$ is either 0 or prime to the order of A. Then L is PI.*

This theorem was proved by Bahturin and Zaicev for solvable groups A [1] and extended by Linchenko to the general case [12].

Corollary 3.3 ([18])**.** *Let F be the free Lie algebra of countable rank over $\mathfrak{k}$. Denote by F^* the set of non-zero elements of F. For any finite group A there exists a mapping*

$$\phi : F^* \to F^*$$

such that if L and A are as in Theorem 3.2, and if $C_L(A)$ satisfies an identity $f \equiv 0$, then L satisfies the identity $\phi(f) \equiv 0$.

To be able to use Theorem 3.1 we need a tool allowing us to deduce that certain elements of L are ad-nilpotent. In this context the following lemma is quite helpful.

Lemma 3.4 ([9])**.** *Suppose that L is a Lie algebra, and K a subalgebra of L generated by r elements $h_1, \ldots, h_r$ such that all commutators in the h_i are ad-nilpotent in L of index t. If K is nilpotent of class c, then for some $\{r, t, c\}$-bounded number u we have $[L, \underbrace{K, \ldots, K}_{u}] = 0$.*

We now turn to groups. Throughout the rest of the section p will denote an arbitrary but fixed prime. Let G be any group. A series of subgroups

$$(*) \qquad G = G_1 \geq G_2 \geq \ldots$$

is called an N_p-series if $[G_i, G_j] \leq G_{i+j}$ and $G_i^p \leq G_{pi}$ for all i, j. To any N_p-series $(*)$ of a group G one can associate a Lie algebra $L^*(G)$ over $\mathbb{F}_p$, the field with p elements. Let us briefly describe the construction.

Given an N_p-series $(*)$, let us view the quotients $L_i^* = G_i/G_{i+1}$ as linear spaces over $\mathbb{F}_p$, and let $L^*(G)$ be the direct sum of these spaces. Commutation in G induces a binary operation $[,]$ in L. For homogeneous elements $xG_{i+1} \in L_i^*, yG_{j+1} \in L_j^*$ the operation is defined by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L_{i+j}^*$$

and extended to arbitrary elements of $L^*(G)$ by linearity. It is easy to check that the operation is well-defined and that $L^*(G)$ with the operations $+$ and $[,]$ is a Lie algebra over $\mathbb{F}_p$.

We are now concerned with the relationship between G and $L^*(G)$. For any $x \in G_i \setminus G_{i+1}$ let x^* denote the element xG_{i+1} of $L^*(G)$.

Proposition 3.5 (Lazard, [11])**.** *For any $x \in G$ we have $(ad\, x^*)^p = ad\, (x^p)^*$.*

The following proposition can be extracted from the proof of Theorem 1 in the paper of Wilson and Zel'manov [21].

Proposition 3.6. *Let G be a group satisfying a group identity $w \equiv 1$. Then there exists a non-zero multilinear Lie polynomial f over $\mathbb{F}_p$ depending only on p and w such that for any N_p-series $(*)$ of G the algebra $L^*(G)$ satisfies the identity $f \equiv 0$.*

In fact Wilson and Zel'manov describe in [21] an effective algorithm allowing one to write f explicitely for any p and w, but we do not require this.

In general a group G has many N_p-series. The series described below is particularly important. To simplify the notation we write γ_i for $\gamma_i(G)$, the ith term of the lower central series of G. Set $D_i = D_i(G) = \prod_{jp^k \geq i} \gamma_j^{p^k}$. The subgroups D_i form an N_p-series $G = D_1 \geq D_2 \geq \dots$ in the group G. This is known as the Jennings-Lazard-Zassenhaus series.

Let $DL(G) = \bigoplus L_i$ be the Lie algebra over $\mathbb{F}_p$ corresponding to the Jennings-Lazard-Zassenhaus series of G. Here $L_i = D_i/D_{i+1}$. Let $L_p(G) = \langle L_1 \rangle$ be the subalgebra of $DL(G)$ generated by L_1. The following result was obtained in Riley [16].

Lemma 3.7. *Suppose that G is a d-generator finite p-group such that the Lie algebra $L_p(G)$ is nilpotent of class c. Then the rank of G is $\{p, c, d\}$-bounded.*

Recall that the rank of a finite group G is the least integer r such that any subgroup of G can be generated by at most r elements.

Given a subgroup H of the group G, we denote by $L(G, H)$ the linear span in $DL(G)$ of all homogeneous elements of the form hD_{j+1}, where $h \in D_j \cap H$. Clearly, $L(G, H)$ is always a subalgebra of $DL(G)$. Moreover, it is isomorphic with the Lie algebra associated with H using the N_p-series of H formed by $H_j = D_j \cap H$. We also set $L_p(G, H) = L_p(G) \cap L(G, H)$. Let A be any group of automorphisms of the group G. Then A acts naturally on every quotient of the Jennings-Lazard-Zassenhaus series of G. This action induces an automorphism group of the Lie algebra $DL(G)$. So when convenient we will consider A as a group acting on $DL(G)$ (or on $L_p(G)$). Lemma 2.1 implies that if G is finite and $(|G|, |A|) = 1$, then $L_p(G, C_G(A)) = C_{L_p(G)}(A)$.

The following lemma is taken from [6].

Lemma 3.8. *Suppose that any Lie commutator in homogeneous elements $x_1, \dots, x_r$ of $DL(G)$ is ad-nilpotent of index at most t. Let $K = \langle x_1, \dots, x_r \rangle$ and assume that $K \leq L(G, H)$ for some subgroup H of G satisfying a group identity $w \equiv 1$. Then for some $\{r, t, w, p\}$-bounded number u we have $[DL(G), \underbrace{K, \dots, K}_{u}] = 0$.*

Proof. In view of Lemma 3.4 it is sufficient to show that K has a $\{r, t, w, p\}$-bounded nilpotency class. We know from Proposition 3.6 that K satisfies a certain multilinear polynomial identity depending only on w. Thus Theorem 3.1 shows that K has $\{r, t, w, p\}$-bounded nilpotency class. □

4. Proof of Theorem A

Proposition 4.1. *Let p and q be distinct primes. Let A be an elementary abelian group of order q^3 acting on a finite m-generated p-group G. Assume that $C_G(a)$ is an extension of a nilpotent subgroup of class at most c by a group of exponents e for any $a \in A^{\#}$. Then the rank of G is $\{c, e, m, q\}$-bounded.*

Proof. First of all we note that if $p > e$, then $C_G(a)$ is necessarily nilpotent of class at most c for any $a \in A^{\#}$, in which case the result follows from Theorem 2.10. So we assume that $p \leq e$ and e is a p-power. Set $D_j = D_j(G)$, $L = L_p(G)$, $L_j = L \cap (D_j/D_{j+1})$, so that $L = \bigoplus L_j$. We can view A as a group acting on L. Let $A_1, A_2, \ldots, A_s$ be the distinct subgroups of order q^2 of A and for any i, j set $L_{ij} = C_{L_j}(A_i)$. Then, by Lemma 2.2, for any j we have

$$L_j = \sum_{1 \leq i \leq q+1} L_{ij}.$$

By Lemma 2.1 (1) for any $l \in L_{ij}$ there exists $x \in D_j \cap C_G(A_i)$ such that $l = xD_{j+1}$. If M is any A-invariant abelian section of G, then $\langle M, x^e \rangle$ is nilpotent of class at most c. This can be shown exactly as Lemma 2.4. One only needs to notice that $\langle C_M(A_k), x^e \rangle$ is nilpotent of class c for any $k \leq s$. Since L is a direct sum of abelian sections of G, it follows that the linear transformation of L induced by commutation with x^e is nilpotent of index at most c. Combining this with Lazard's Lemma 3.5 yields

$(*)$ any element in L_{ij} is ad-nilpotent of index at most ce.

Let ω be a primitive qth root of unity, and let $\overline{L} = L \otimes \mathbb{F}_p[\omega]$. If we show that $\overline{L}$ is nilpotent of $\{e, c, q\}$-bounded class, this will, of course, imply the same nilpotency result for L.

It is natural to identify L with the $\mathbb{F}_p$-subalgebra $L \otimes 1$ of $\overline{L}$. We note that if an element $x \in L$ is ad-nilpotent of index n, say, then the "same" element $x \otimes 1$ is ad-nilpotent in $\overline{L}$ of the same index n.

Put $\overline{L_j} = L_j \otimes \mathbb{F}_p[\omega]$; then $\overline{L} = \langle \overline{L_1} \rangle$, since $L = \langle L_1 \rangle$, and $\overline{L}$ is the direct sum of the homogeneous components $\overline{L_j}$. Since the $\mathbb{F}_p$-space L_1 is m-dimensional, so is the $\mathbb{F}_p[\omega]$-space $\overline{L_1}$.

The group A acts naturally on $\overline{L}$, and we have $\overline{L_{ij}} = C_{\overline{L_j}}(A_i)$, where $\overline{L_{ij}} = L_{ij} \otimes \mathbb{F}_p[\omega]$. Let us show that

$(**)$ any element $y \in \overline{L_{ij}}$ is ad-nilpotent of $\{q, n\}$-bounded index.

Since $\overline{L_{ij}} = L_{ij} \otimes \mathbb{F}_p[\omega]$, we can write

$$y = x_0 + \omega x_1 + \omega^2 x_2 + \cdots + \omega^{q-2} x_{q-2}$$

for some $x_0, x_1, x_2, \ldots, x_{q-2} \in L_{ij}$, which are all ad-nilpotent of index ce by $(*)$. Set $H = \langle x_0, \omega x_1, \ldots, \omega^{q-2} x_{q-2} \rangle$ and notice that $H \subseteq C_{\overline{L}}(A_i)$, since $x_0, x_1, x_2, \ldots, x_{q-2} \in C_{\overline{L}}(A_i)$. A commutator of weight k in the $\omega^n x_n$ has the form $\omega^t x$ for some $x \in L_{iu}$, where $u = kj$. By $(*)$ such an x is ad-nilpotent of index ce and hence so is $\omega^t x$.

Combining Proposition 3.6 with the fact that $L_p(G, C_G(A_i)) = C_{L_p(G)}(A_i)$, we conclude that $C_L(A_i)$ satisfies a multilinear polynomial identity of a $\{c, e\}$-bounded degree. This identity, being multilinear, is also satisfied by $C_L(A_i) \otimes \mathbb{F}_p[\omega] = C_{\overline{L}}(A_i)$. We have already observed that $H \subseteq C_{\overline{L}}(A_i)$, whence the identity is satisfied in H. Theorem 3.1 now tells us that H is nilpotent of a $\{c, e, q\}$-bounded class. Because of Lemma 3.8 it follows that $[L, \underbrace{H, \ldots, H}_{v}] = 0$ for some $\{c, e, q\}$-bounded number v. This establishes $(**)$.

Since A is abelian, and the ground field is now a splitting field for A, every $\overline{L_j}$ decomposes in the direct sum of common eigenspaces for A. In particular, $\overline{L_1}$ is

spanned by common eigenvectors for A, and it requires at most m of them to span $\overline{L_1}$. Hence $\overline{L}$ is generated by m common eigenvectors for A from $\overline{L_1}$. Every common eigenspace is contained in the centralizer $C_{\overline{L}}(A_i)$ for some $1 \leq i \leq q+1$, since A acts on it as a cyclic group. Note that any commutator in common eigenvectors is again a common eigenvector. Therefore if $l_1, \ldots, l_m \in \overline{L_1}$ are common eigenvectors for A generating $\overline{L}$, then any commutator in these generators belongs to some $\overline{L_{ij}}$ and hence, by $(**)$, is ad-nilpotent of a $\{c, e, q\}$-bounded index.

We already know that some polynomial identity $f \equiv 0$ of $\{c, e\}$-bounded degree is satisfied in $C_L(A_i) \otimes \mathbb{F}_p[\omega] = C_{\overline{L}}(A_i)$. So by Corollary 3.3, $\overline{L}$ satisfies some identity $\phi(f) \equiv 0$ which depends only on c, e and q. Theorem 3.1 now shows that $\overline{L}$ (hence L) is nilpotent of a $\{c, e, q\}$-bounded class. In view of Riley's Lemma 3.7 the theorem follows. □

Proof of Theorem A. In accordance with the theorem by Burns, Macedońska and Medvedev [2] there exist n-bounded numbers c and e such that for any $a \in A^{\#}$ the centralizer $C_G(a)$ is an extension of a nilpotent group of class at most c by a group of exponent dividing e. By Lemma 2.8 we conclude that the exponent ϵ of the quotient $G/F(G)$ is $\{e, q\}$-bounded. Let $\{p_1, \ldots, p_t\}$ be the set of prime divisors of $|F(G)|$, and let us assume that $p_1, \ldots, p_r$ divide e while $p_{r+1}, \ldots, p_t$ do not. Let S_i denote the Sylow p_i-subgroup of $F(G)$ and write $S = S_{r+1} \times \cdots \times S_t$. It is clear that $C_S(a)$ is nilpotent of class at most c for any $a \in A^{\#}$, so Theorem 2.10 tells us that S is nilpotent of $\{c, q\}$-bounded class. Let g, h be arbitrary elements either in S_i, for some $i \leq r$, or in S. Let H be the minimal A-invariant subgroup of G containing g and h. Obviously H has at most $2q^3$ generators. Using Proposition 4.1 and the fact that S is of $\{c, q\}$-bounded nilpotency class, yields that the rank of H is $\{n, q\}$-bounded. Now, for any $a \in A^{\#}$ the centralizer $C_H(a)$ is an extension of a nilpotent group of class at most c by a group of exponent dividing e, and, at the same time, $C_H(a)$ has an $\{n, q\}$-bounded rank. It follows (see for example the proof of Lemma 2.2 in [17]) that $C_H(a)$ has a subgroup of class at most c and of index bounded in terms of n and q alone. Applying Theorem 2.11 shows that H has a subgroup of $\{n, q\}$-bounded nilpotency class and of $\{n, q\}$-bounded index. The upper bound for the latter will be denoted by k.

Now let x, y be arbitrary elements of G. We have just shown that any Sylow subgroup of $\langle x^{k\epsilon}, y^{k\epsilon} \rangle$ has $\{n, q\}$-bounded nilpotency class, so this also holds for $\langle x^{k\epsilon}, y^{k\epsilon} \rangle$ (recall that ϵ is the exponent of the quotient $G/F(G)$ and this is $\{n, q\}$-bounded). Let v be the maximum of the classes of subgroups $\langle x^{k\epsilon}, y^{k\epsilon} \rangle$, where x, y range through G. Then G satisfies the Mal'cev law on two variables $M_v(x^{k\epsilon}, y^{k\epsilon})$ whose degree is $\{n, q\}$-bounded. The proof is complete. □

References

[1] Yu. A. Bahturin, M. V. Zaicev, Identities of graded algebras, J. Algebra, **205** (1998), 1–12. MR **99f:**17034

[2] R. G. Burns, O. Macedońska, Yu. Medvedev, Groups satisfying semigroup laws, and nilpotent-by-Burnside varieties, J. Algebra, **195** (1997), 510-525. MR **98h:**20039

[3] T. M. Gagen, "Topics in finite groups", Cambridge University Press, London Math. Soc. Lecture Note Series, **16**, 1976. MR **53:**10910

[4] D. M. Goldschmidt, Solvable signalizer functors on finite groups, J. Algebra, **21** (1972), 137–148. MR **45:**6913

[5] D. Gorenstein, "Finite Groups", Harper and Row, New York, 1968. MR **38:**229

[6] R. Guralnick and P. Shumyatsky, Derived subgroups of fixed points, Israel J. Math., **126** (2001), 345–362. MR **2002k:**20036

[7] P. Hall, Some sufficient conditions for a group to be nilpotent, Illinois J. Math., **2** (1958), 787–801. MR **21:**4183

[8] G. Higman, Groups and Lie rings having automorphisms without non-trivial fixed points, J. London Math. Soc., **32** (1957), 321–334. MR **19:**633c

[9] E. I. Khukhro and P. V. Shumyatsky, Bounding the exponent of a finite group with automorphisms, J. Algebra, **212** (1999), 363–374. MR **2000c:**20039

[10] L. G. Kovacs and G. E. Wall, Involutory automorphisms of groups of odd order and their fixed point groups, Nagoya Math. J., **27** (1966), 113–120. MR **33:**2714

[11] M. Lazard, Sur les groupes nilpotents et les anneaux de Lie, Ann. Sci. École Norm. Supr., **71** (1954), 101-190. MR **19:**529b

[12] V. Linchenko, Identities of Lie algebras with actions of Hopf algebras, Comm. Algebra, **25** (1997), 3179–3187. MR **99c:**16040

[13] A. I. Mal'cev, Nilpotent semigroups, Uchen. Zap. Ivanovsk. Ped. Inst., **4** (1953), 107–111. MR **17:**825d

[14] B. H. Neumann and T. Taylor, Subsemigroups of nilpotent groups, Proc. Roy. Soc. London Ser. A, **274** (1963), 1–4. MR **28:**3100

[15] A. Yu. Ol'shanskiĭ and A. Storozhev, A group variety defined by a semigroup law, J. Austral Math. Soc. Ser. A, **60** (1996), 255-259. MR **97b:**20033

[16] D. M. Riley, Analytic pro-p groups and their graded group rings, J. Pure Appl. Algebra, **90** (1993), 69–76. MR **94k:**20051

[17] P. Shumyatsky, Involutory automorphisms of finite groups and their centralizers, Arch. Math. (Basel), **71** (1998), 425–432. MR **99j:**20031

[18] P. Shumyatsky, Exponent of finite groups admitting an involutory automorphism, J. Group Theory, **2** (1999), 367–372. MR **2000j:**20033

[19] P. Shumyatsky, Finite groups and the fixed points of coprime automorphisms, Proc. Amer. Math. Soc., **129** (2001), 3479–3484. MR **2002h:**20029

[20] J. G. Thompson, Finite groups with fixed-point-free automorphisms of prime order, Proc. Nat. Acad. Sci. U.S.A., **45** (1959), 578–581. MR **21:**3484

[21] J. S. Wilson and E. Zelmanov, Identities for Lie algebras of pro-p groups, J. Pure Appl. Algebra, **81** (1992), 103-109. MR **93m:**17004

[22] E. Zelmanov, "Nil Rings and Periodic Groups", The Korean Math. Soc. Lecture Notes in Math., Seoul, 1992. MR **94c:**16027

Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil
E-mail address: pavel@ipe.mat.unb.br