

UNITS IN SOME FAMILIES OF ALGEBRAIC NUMBER FIELDS

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ABSTRACT. Multi-dimensional continued fractions associated with $GL_n(\mathbf{Z})$ are introduced and applied to find systems of fundamental units in some families of totally real fields and fields with signature $(2,1)$.

1. INTRODUCTION

Let F be an algebraic number field of degree n . There exist exactly n field embeddings of F in \mathbf{C} . Let s be the number of embeddings of F whose images lie in \mathbf{R} , and let $2t$ be the number of non-real complex embeddings so that $n = s + 2t$. The pair (s, t) is said to be the *signature* of F . Let \mathbf{Z}_F be the ring of integers of the field F . A *unit* in F is an invertible element of \mathbf{Z}_F . The set of units in F forms a multiplicative group which will be denoted by \mathbf{Z}_F^\times . In 1840 P. G. Lejeune-Dirichlet determined the structure of the group \mathbf{Z}_F^\times . He showed that \mathbf{Z}_F^\times is a finitely generated Abelian group of rank $r = s + t - 1$, i.e. \mathbf{Z}_F^\times is isomorphic to $\mu_F \times \mathbf{Z}^r$, where μ_F is a finite cyclic group. μ_F is called the *torsion* subgroup of \mathbf{Z}_F^\times . Thus, there exist units $\epsilon_1, \dots, \epsilon_r$ such that every element of \mathbf{Z}_F^\times can be written in a unique way as $\zeta \epsilon_1^{n_1} \dots \epsilon_r^{n_r}$, where $n_i \in \mathbf{Z}$ and ζ is a root of unity in F . Such a set $\{\epsilon_1, \dots, \epsilon_r\}$ is called a *system of fundamental units* of F . Finding a system of fundamental units of F is one of the main computational problems of algebraic number theory (see e.g. [6], p. 217). Much work has been done to solve this problem for certain classes of algebraic number fields (see e.g. [22]). In the case of the real quadratic fields, the continued fraction algorithm provides a very efficient method for solving this problem (see e.g. [22], p. 119). This approach goes back to L. Euler, who applied continued fractions to solve Pell's equation $x^2 - dy^2 = \pm 1$. (If a square-free positive integer $d \equiv 2$ or $3 \pmod{4}$ and x, y is an integral solution of this equation, then $x + \sqrt{d}y$ is a unit in the real quadratic field $\mathbf{Q}(\sqrt{d})$. Moreover, any unit in $\mathbf{Q}(\sqrt{d})$ can be obtained this way.) Many attempts have been made to develop a similar algorithm that would find a system of fundamental units in other algebraic number fields. In the case of a cubic field, one of the most successful such algorithms was introduced by G. F. Voronoi [28]. A review of the multi-dimensional continued fraction algorithms and their properties that were known by 1980 can be found in [2].

In [30] and [31], a continued fraction algorithm associated with a discrete group acting in a hyperbolic space was defined. The purpose of the present paper is to extend this definition to the case of the group $\Gamma = GL_n(\mathbf{Z})/\{\pm 1\}$ acting on

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$\mathcal{P} = SL_n(\mathbf{R})/SO_n(\mathbf{R})$ and apply it to the problem of finding a system of fundamental units in an algebraic number field F . The symmetric space \mathcal{P} can be identified with the set of definite quadratic forms in n real variables with the leading coefficient one. \mathcal{P} with the metric $ds^2 = \text{trace}((X^{-1}dX)^2)$, where $X = (x_{ij}) \in \mathcal{P}$ and $dX = (dx_{ij})$, is a Riemannian manifold (see e.g. [13] or [27]).

Assume that $g \in GL_n(\mathbf{R})$. Let $ga_i = \lambda_i a_i$, $i = 1, \dots, n$, so that a_i is an eigenvector of g corresponding to its eigenvalue λ_i . For simplicity, assume that all the eigenvalues of g are distinct. Let $P = (a_1, \dots, a_n)$ be the matrix with columns a_1, \dots, a_n . The set of points in \mathcal{P} fixed by g will be called the *axis* L_P of g . The axis L_P of g depends only on eigenvectors of g , i.e. on P , but not on its eigenvalues (see Section 3). L_P is a totally geodesic submanifold of \mathcal{P} of dimension $s + t - 1$, where s is the number of real and $2t$ the number of non-real complex eigenvalues of g , so that $n = s + 2t$. If $t = 0$, then L_P is an $(n - 1)$ -flat in \mathcal{P} (see e.g. [13]).

Let $(1, \omega_2, \dots, \omega_n)$ be a \mathbf{Z} -basis of the ring of integers \mathbf{Z}_F of a number field F of degree n . Let $a_1 = (1, \omega_2, \dots, \omega_n)^T$. Let $\gamma \in \mathbf{Z}_F$. Then $\gamma\omega_i = \sum m_{ij}\omega_j$ or $\gamma a_1 = M_\gamma a_1$, where $\omega_1 = 1$, $m_{ij} \in \mathbf{Z}$ and $M_\gamma = (m_{ij})$ is a square matrix of order n . Let σ_i be the n distinct embeddings of F in \mathbf{C} . Let $a_i = \sigma_i(a_1)$ and $\gamma_i = \sigma_i(\gamma)$, where $\gamma_1 = \gamma$. Then $\gamma_i a_i = M_\gamma a_i$ for $i = 1, \dots, n$. Thus, a_i is an eigenvector of M_γ corresponding to its eigenvalue γ_i . It is clear that the map $\gamma \mapsto M_\gamma$ is an isomorphism of the ring of integers \mathbf{Z}_F and the commutative ring of \mathbf{Z} -integral square matrices of order n with the common axis L_P . The norm of γ equals $\det(M_\gamma)$, so that γ is a unit in \mathbf{Z}_F if and only if $M_\gamma \in GL_n(\mathbf{Z})$. The torsion-free subgroup Γ_L of the stabilizer of L_P is isomorphic to $\mathbf{Z}_F^\times / \mu_F$. Thus, the problem of finding a system of fundamental units of F is equivalent to the problem of finding a set of generators of Γ_L . The [multi-dimensional continued fraction] Algorithm II introduced in this paper can be used to solve the latter problem. Here, a set of generators of Γ_L and therefore a system of fundamental units is found in some families of fields F of degree $n \leq 4$.

In Section 2, the notion of the height of a point in \mathcal{P} is introduced. Let $w = (1, 0, \dots, 0)^T$ and $W = ww^T$. In what follows, the point W , which belongs to the boundary of \mathcal{P} , is analogous to the point ∞ in the upper half-space model $H^{n+1} = \{(z, t) : z \in \mathbf{R}^n, t > 0\}$ of the $(n + 1)$ -dimensional hyperbolic space (see [30] and [31]). The set $K = K(w)$ in \mathcal{P} is defined so that, for every point $X \in \mathcal{P}$, the points in the Γ -orbit of X with the largest height belong to $K(w)$. The images $K[g]$ of K , $g \in \Gamma$, under the action of Γ form the *K-tessellation* of \mathcal{P} . The *K-tessellation* of \mathcal{P} is Γ -invariant.

If $L_P \cap K[g] \neq \emptyset$, $g \in \Gamma$, then the vector $u = g^{-1}w \in \mathbf{Z}^n$ is called a *convergent* of L_P . In Section 3, it is shown that if u is a convergent of L_P , then $|\langle a_1, u \rangle \dots \langle a_n, u \rangle / \det P|$, where $\langle \cdot, \cdot \rangle$ denotes the dot product in \mathbf{R}^n , is small (Theorem 7). Algorithm II, which is introduced in Section 3, can be applied to find the sets $R(g^{-1}w) = L_P \cap K[g] \neq \emptyset$, which form a tessellation of L_P , and the set of convergents of L_P .

If $g \in \Gamma$, then there are only finitely many sets $R(u)$ which are not congruent modulo the action of Γ . The union of non-congruent sets $R(u)$ forms a fundamental domain of Γ_L in L_P . Assume that the characteristic polynomial $p(x)$ of g is irreducible. Let $p(\epsilon) = 0$. In Section 4, the problem of finding a system of fundamental units in F is solved for some families of totally real fields $F = \mathbf{Q}(\epsilon)$ by reducing it, as explained above, to the problem of finding a set of generators of Γ_L . In Examples

1-4, new proofs of certain known results are given. The new results obtained in Examples 5 and 6 can be presented as follows.

Theorem 1. *Assume that $t > 3$ is a positive integer. Let $f(x) = x^4 + tx^3 - x^2 - tx + \alpha = x(x^2 - 1)(x + t) + \alpha$, $\alpha = \pm 1$. Let $f(\epsilon) = 0$. Assume that $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a \mathbf{Z} -basis of the maximal order \mathbf{Z}_F of the totally real quadric field $F = \mathbf{Q}(\epsilon)$. Then $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon - 1, \epsilon, \epsilon + 1 \rangle$.*

Note that the Galois group of F is D_4 if $\alpha = 1$, and it is S_4 if $\alpha = -1$.

In Section 5, the problem of finding a system of fundamental units is solved for some families of fields with signature $(2, 1)$. The following theorems are proved in Examples 7, 8, and 9 respectively.

Theorem 2. *Let $f(x) = x^4 + tx^3 + x^2 + tx + 1 = x(x^2 + 1)(x + t) + 1$, where $t \in \mathbf{Z}$. Let $f(\epsilon) = 0$. Assume that $\eta = \epsilon + \epsilon^{-1} = (-t \pm \sqrt{t^2 + 4})/2$ is a fundamental unit of the quadratic subfield $K = \mathbf{Q}(\sqrt{d})$, $d = t^2 + 4$, of the dihedral quartic field $F = \mathbf{Q}(\epsilon)$ with signature $(2, 1)$. Assume that $4t^2 - 9$ is square-free. Then $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a \mathbf{Z} -basis of the maximal order \mathbf{Z}_F of the field F , and $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \eta \rangle$.*

Theorem 3 (cf. [20]). *Let $t \geq 4$ be an integer. Let $f(x) = x^4 + tx^3 + x^2 + tx - 1 = x(x^2 + 1)(x + t) - 1$. Let ϵ be a real root of $f(x)$. Assume that the discriminant of the quartic field $F = \mathbf{Q}(\epsilon)$ with signature $(2, 1)$ is square-free. Then $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a \mathbf{Z} -basis of the maximal order \mathbf{Z}_F of the field F , and $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon + t \rangle$.*

Theorem 4. *Let $\alpha = \pm 1$. Let $t \geq s$ be positive integers and let $\eta = \frac{1}{2}(t + \sqrt{d})$, where $d = t^2 + 4\alpha$. Let $f(x) = f_1(x)f_2(x) = x^4 + stx^3 + (t - \alpha s^2)x^2 + s(t^2 + 2\alpha)x - \alpha$, where $f_1(x) = (x^2 + s\eta x - 1/\eta)$, $f_2(x) = (x^2 - sx/\eta + \eta)$. Let $f_1(\epsilon) = 0$. Then $\eta \in F = \mathbf{Q}(\epsilon)$. Assume that η is a fundamental unit of the quadratic subfield $K = \mathbf{Q}(\sqrt{d})$ of F and that $\Delta = 4s^2t^3 + 12\alpha s^2t - s^4 + 16\alpha$ is square-free. Denote $p(x) = (s + (\alpha t - s^2)x + \alpha x^3)/(ts^2 + 1)$. Then $\{1, \epsilon, \epsilon^2, p(\epsilon)\}$ is a \mathbf{Z} -basis of the maximal order \mathbf{Z}_F of the field F , the discriminant of F is $d^2\Delta$, and $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \eta \rangle$.*

Families of cyclic quartic fields are considered in [15] and [35]. In [33], a one-dimensional version of Algorithm II is applied to find fundamental units in a two-parameter family of complex cubic fields.

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2. FUNDAMENTAL DOMAINS AND K -TESSELLATION

Let $n \geq 2$ be a positive integer. Let V_n be the space of symmetric $n \times n$ real matrices. The dimension of V_n is $N = n(n + 1)/2$. The action of $g \in G = GL(n, \mathbf{R})$ on $X \in V_n$ is given by

$$X \longmapsto X[g] = g^T X g.$$

For a subset S of V_n , denote $S[g] = \{X[g] \in V_n : X \in S\}$.

The one-dimensional subspaces of V_n form the real projective space V of dimension $N - 1$, so that for any fixed nonzero $X \in V_n$, all the vectors $kX \in V_n$, $0 \neq k \in \mathbf{R}$, represent one point in V . Denote by $\mathcal{P} \subset V$ the set of (positive) definite elements of V and by \mathcal{B} the boundary of \mathcal{P} (\mathcal{B} can be identified with the non-negative elements of V of rank less than n). The group G preserves both \mathcal{P} and \mathcal{B} , as does its arithmetic subgroup $GL(n, \mathbf{Z})$.

The space V_n (and V) can be also identified with the set of quadratic forms $A[x] = x^T Ax$, $A \in V_n$, $x \in \mathbf{R}^n$. With each point $a = (a_1, \dots, a_n)^T \in \mathbf{R}^n$, we associate the matrix $A = aa^T \in \mathcal{B}$ and the quadratic form

$$(1) \quad A[x] = \langle a, x \rangle^2 = (a_1x_1 + \dots + a_nx_n)^2$$

of rank one. For $g \in G$, we have $\langle ga, x \rangle = a^T g^T x = \langle a, g^T x \rangle$.

Let $w = (1, 0, \dots, 0)^T$ and $W = ww^T$. Then $\langle w, x \rangle^2 = x_1^2$ and $W[g] = U = uu^T$, where $u = g^T w$.

Denote by G_∞ and Γ_∞ the stabilizers of w in G and $\Gamma = GL(n, \mathbf{Z})/\{\pm 1\}$ respectively. Then

$$G_\infty = \{g \in G : gw = w\} = \{g \in G : g_1 = w\},$$

where g_1 is the first column of g . Thus, $g \in G_\infty$ if and only if $W[g^T] = W$.

We shall say that $A \in V$ is *extremal* if $|A[x]| \geq |A[w]| = a_{11}^2$ for any $x \in \mathbf{Z}^n$, $x \neq (0, \dots, 0)$. Let $\mathcal{A}_n = \{X \in V : X[w] \neq 0\}$. It is clear that $\mathcal{P} \subset \mathcal{A}_n$. For $X \in \mathcal{A}_n$, we shall say that

$$\text{ht}(X) = |\det(X)|^{1/n} / |X[w]|$$

is the *height* of X and, for a subset S of V , we define the *height* of S as

$$\text{ht}(S) = \max \text{ht}(X), \quad X \in S.$$

The elements of \mathcal{A}_n will be normalized so that $X[w] = 1$. For a fixed $g \in \Gamma$, the set $\{X \in \mathcal{A}_n : |X[gw]| < 1\}$ is called the *g-strip* (cf. [32], [29], where this definition is introduced for $n = 2$). It is clear that the *gh-strip* coincides with the *g-strip* for any $h \in \Gamma_\infty$. The plane

$$L^+(gw) = L^+(g) = \{X \in \mathcal{A}_n : X[gw] = 1\}$$

is the boundary of the *g-strip* which cuts \mathcal{P} . Let \mathcal{R}_w be the set of all extremal points of V . Denote

$$K = K(w) = \mathcal{P} \cap \mathcal{R}_w.$$

(In the notation of [1], p.148, K is the dual core of K_{perf} .) Note that $K \subset \mathcal{A}_n$ is bounded by the planes $L^+(g)$. If $h \in \Gamma_\infty$, then $X[hw] = X[w]$ and, therefore, $\text{ht}(X[h]) = \text{ht}(X)$. Thus,

$$(2) \quad K[h] = K, \quad h \in \Gamma_\infty.$$

Let $q(x)$ be an indefinite quadratic form in $n > 2$ variables. By Margulis' theorem [19], if $q(x)$ is not a multiple of a quadratic form with integer coefficients, then the infimum of $|q(x)|$ taken over all $x \in \mathbf{Z}^n$, $x \neq (0, \dots, 0)$, is equal to zero. Hence, all the points of $\mathcal{R}_w - \mathcal{P}$ are rational if $n > 2$. By Meyer's theorem (see e.g. [5]), if the coefficients of $q(x)$ are rational and $n = 5$, then there is $x_0 \in \mathbf{Z}^5$, $x_0 \neq (0, \dots, 0)$, such that $q(x_0) = 0$. Thus, $\mathcal{R}_w - \mathcal{P}$ is empty and $K = \mathcal{R}_w$ if $n > 4$.

Let D be any of the fundamental domains of Γ obtained by Minkowski, Korkine and Zolotarev (see e.g. [24], p.13), or Grenier [16]. For $X \in D$ we have $X[w] = \inf X[gw]$, $g \in \Gamma$, in any of these cases. Hence, $\bigcup D[g] = K$, the union being taken over all $g \in \Gamma_\infty$. Note that the fundamental domain described in [16] coincides with the domain found by Korkine and Zolotarev in 1873 (see [18] or [24]). Unless a point $X \in \mathcal{P}$ is integral, like point I in Example 1 below, in order to prove that X is extremal, we shall show that $X[h]$ is Minkowski reduced for some $h \in \Gamma_\infty$.

The main features of our approach to the problem of finding a system of fundamental units in an algebraic number field can be seen in the following simple example.

Example 1. Let $\Gamma = GL_2(\mathbf{Z})$. Let $X = (x_{ij}) \in V$. A point X lies in \mathcal{B} if and only if $\det(X) = x_{11}x_{22} - x_{12}^2 = 0$. Thus, when $n = 2$, \mathcal{B} is a conic in the projective plane V , and \mathcal{P} , which consists of the points X with $\det(X) > 0$, is the Klein model of the hyperbolic plane. Let $f(x) = x^2 - tx - 1$, where $t \in \mathbf{Z}$. Let $f(\epsilon) = 0$. Assume that either $t^2 + 4$ or $t^2/4 + 1$ is a square-free integer. Then $\{1, \epsilon\}$ is a \mathbf{Z} -basis of the maximal order \mathbf{Z}_F of the field $F = \mathbf{Q}(\epsilon)$. Let

$$E = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}.$$

Then $f(x)$ is a characteristic polynomial of E ; and $a_1 = (1, \epsilon)^T$ and $a_2 = (1, -1/\epsilon)^T$ are eigenvectors of E corresponding to its eigenvalues ϵ and $-1/\epsilon$ respectively. Let $A_i = a_i a_i^T$. Let L_P be the axis of E . Then L_P is the interval $\mu A_1 + (1 - \mu)A_2$, $0 < \mu < 1$, which is a geodesic in \mathcal{P} . The identity matrix I is the intersection of L_P with $L^+(E)$, and the interval $(I, I[E])$ is a fundamental domain of Γ_L on L_P . Thus, $\Gamma_L = \langle E \rangle$ and, therefore, $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon \rangle$.

The period length of the corresponding continued fraction is one (see Remark 2 at the end of Section 3). More examples of this type can be found in [30] and [31].

Similar families of algebraic number fields of degree three and four are considered in Sections 4 and 5.

By (2), $K[hg] = K[g]$ for any $g \in \Gamma$ and $h \in \Gamma_\infty$. Thus, the sets $K[g]$ are parameterized by the classes $\Gamma_\infty \backslash \Gamma$ or by primitive vectors $u = g^{-1}h^{-1}w = g^{-1}w$, so that $\pm u$ represent the same $K[g]$. The sets $K[g]$, $g \in \Gamma_\infty \backslash \Gamma$, form a tessellation of \mathcal{P} which will be called the K -tessellation. It is clear that the K -tessellation of \mathcal{P} is Γ -invariant. The vertices of K are called the *perfect forms* (see e.g. [1] or [24]). The perfect forms are known for $n \leq 7$.

3. AXES OF ELEMENTS OF G

Let $g \in G$. Let $ga_i = \lambda_i a_i$, $i = 1, \dots, n$, where, for simplicity, we assume that $\lambda_i \neq \lambda_j$ if $i \neq j$. Here a_i is an eigenvector of g corresponding to its eigenvalue λ_i . Assume that $\langle a_i, w \rangle \neq 0$, $i = 1, \dots, n$. Then we can choose a_i so that

$$(3) \quad \langle a_i, w \rangle = 1, \quad i = 1, \dots, n.$$

$g \in \Gamma$ is said to be *irreducible* if its characteristic polynomial is irreducible over \mathbf{Q} . If $g \in \Gamma$ is irreducible, then all its eigenvalues are distinct. Assume that $\lambda_1, \lambda_2, \dots, \lambda_s$ are real and $\lambda_{s+1}, \lambda_{s+2}, \dots, \lambda_{s+t}, \bar{\lambda}_{s+1}, \bar{\lambda}_{s+2}, \dots, \bar{\lambda}_{s+t}$ are non-real complex eigenvalues of g , so that $n = s + 2t$. Let $\lambda_k \neq \pm 1$. Let

$$P = (a_1, \dots, a_s, a_{s+1}, \bar{a}_{s+1}, \dots, a_{s+t}, \bar{a}_{s+t})$$

be the matrix with columns $a_1, \dots, a_s, a_{s+1}, \bar{a}_{s+1}, \dots, a_{s+t}, \bar{a}_{s+t}$ and let

$$H = \text{diag}(\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \bar{\lambda}_{s+1}, \dots, \lambda_{s+t}, \bar{\lambda}_{s+t}).$$

Then $g = PHP^{-1}$. For $k > s$, let $a_k = \alpha_k + i\beta_k$, where $\alpha_k, \beta_k \in \mathbf{R}^n$. Then

$$\det P = (2i)^t \det(a_1, \dots, a_s, \beta_{s+1}, \alpha_{s+1}, \dots, \beta_{s+t}, \alpha_{s+t}).$$

The totally geodesic submanifold L_P of \mathcal{P} fixed by $g = PHP^{-1}$ will be called the *axis* of g . The dimension of L_P is $s + t - 1$. It can be identified with the set of quadratic forms in \mathcal{A}_n

$$(4) \quad q[x] = \sum_{i=1}^s \mu_i \langle x, a_i \rangle^2 + \sum_{i=1}^t \mu_{s+i} |\langle x, a_{s+i} \rangle|^2, \quad \mu_i \geq 0, \quad \sum_{i=1}^{s+t} \mu_i = 1.$$

or

$$q[x] = \sum_{i=1}^s \mu_i \langle x, a_i \rangle^2 + \sum_{i=1}^t \mu_{s+i} (\langle x, \alpha_{s+i} \rangle^2 + \langle x, \beta_{s+i} \rangle^2).$$

Hence

$$\det q = (-4)^{-t} \mu_1 \dots \mu_s \mu_{s+1}^2 \dots \mu_{s+t}^2 (\det P)^2.$$

It follows from (4) that L_P is the axis of $h \in G$ if and only if $a_i, i = 1, \dots, n$, are eigenvectors of h . Hence, the axis of g depends only on its set of eigenvectors, i.e. on P , but not on the eigenvalues of g . A point $q \in L_P$ can be also represented as

$$q = \sum_{i=1}^{s+t} \mu_i A_i, \quad A_i = a_i a_i^T, \quad i \leq s, \quad A_{s+i} = \alpha_{s+i} \alpha_{s+i}^T + \beta_{s+i} \beta_{s+i}^T.$$

Thus, L_P is the simplex with vertices $A_i, i = 1, \dots, s + t$. All the faces of L_P belong to \mathcal{B} . Note that $L_P[g^T] = L_P$. The curve $q[g^v] = (\sum \mu_i |\lambda_i|^{2v} A_i) / \sum \mu_i |\lambda_i|^{2v}$ is a geodesic in L_P through q . If $|\lambda_1| < |\lambda_2| < \dots < |\lambda_{s+t}|$, then $q[g^v] \rightarrow A_1$ as the real parameter $v \rightarrow -\infty$, and $q[g^v] \rightarrow A_{s+t}$ as $v \rightarrow \infty$. Note that if $t = 0$, then L_P is an $(n - 1)$ -flat in \mathcal{P} and the stabilizer of L_P in G is a maximal commutative subgroup of G (see e.g. [13]).

Denote $K(g^{-1}w) = K[g]$ and

$$(5) \quad R(g^{-1}w) = K[g] \cap L_P \neq \emptyset, \quad g \in \Gamma_\infty \setminus \Gamma.$$

The sets $R(u), u = g^{-1}w$, form a tessellation of L_P , which is invariant modulo the action of Γ since the K -tessellation of \mathcal{P} is Γ -invariant. We say that this tessellation is *periodic* if there are only a finite number of non-congruent sets $R(u)$ modulo the action of $\text{Stab}(L_P, \Gamma)$. In that case, the union of all non-congruent sets $R(u)$ is a fundamental domain of $\text{Stab}(L_P, \Gamma)$. The number of non-congruent sets $R(u)$ in the tessellation of L_P will be called the *period length*.

Denote

$$\gamma_n = 1/h_n, \quad h_n = \inf \text{ht}(A), \quad A \in K(w).$$

Hermite's constant γ_n is known for $n \leq 8$ (see e.g. [4]). Let $C_n = \gamma_n/n$. Then

$$C_2 = 1/\sqrt{3}, \quad C_3 = 2^{1/3}/3, \quad C_4 = 1/\sqrt{8}, \quad C_5 = 8^{1/5}/5, \\ C_6 = 1/3^{7/6}, \quad C_7 = 64^{1/7}/7, \quad C_8 = 1/4,$$

and for large n (see [8]),

$$\frac{1}{2\pi e} \leq C_n \leq \frac{1.744}{2\pi e}.$$

Let

$$N_P(x) = \langle x, a_1 \rangle \dots \langle x, a_n \rangle = \langle x, a_1 \rangle \dots \langle x, a_s \rangle |\langle x, a_{s+1} \rangle \dots \langle x, a_{s+t} \rangle|^2,$$

where $\langle x, a_i \rangle = x^T a_i$. Define

$$\nu(L_P) = \inf \left| \frac{N_P(gw)}{\det P} \right|,$$

where the infimum is taken over all $g \in \Gamma$. It is clear that $\nu(L_P) = \nu(L_{MP}[h])$ for any $h \in \Gamma$, and $M = \text{diag}(\mu_1, \dots, \mu_s, \mu_{s+1}, \bar{\mu}_{s+1}, \dots, \mu_{s+t}, \bar{\mu}_{s+t})$, where $\mu_1, \dots, \mu_s \in \mathbf{R}$, $\mu_{s+1}, \dots, \mu_{s+t} \in \mathbf{C}$ and $\mu_1 \dots \mu_{s+t} \neq 0$. The projective invariant $\nu(L_P)$ is well known in the geometry of numbers (see e.g. [4] or [17]). In particular, for $n = s = 3$, it was shown by Davenport [9], [10], [11] that $\nu(L_P) \leq 1/7$, where the equality holds only if $ga_i = (1, \alpha_i, \alpha_i^2)$, $i = 1, 2, 3$, the α_i being the roots of $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$, for some $g \in \Gamma$. For other matrices P , $\nu(L_P) \leq 1/9$. Swinnerton-Dyer [26] found the consequent seventeen values of $\nu(L_P)$.

A point $q_m \in L_P$ is said to be the *summit* of L_P if $|\det(q_m)| = \max |\det(q)|$, the maximum being taken over all $q \in L_P$. It is clear that if $R = L_P \cap K(w) \neq \emptyset$, then $q_m \in R$.

Lemma 5. *Let L_P be the totally geodesic manifold fixed by $g \in G$ and defined by (4), where $ga_i = \lambda_i a_i$. Let $P = (a_1, \dots, a_n)$ be the matrix with columns a_1, \dots, a_n . Then*

$$(6) \quad q_m[x] = \frac{1}{n} \sum_{i=1}^s \langle x, a_i \rangle^2 + \frac{2}{n} \sum_{i=1}^t |\langle x, a_{s+i} \rangle|^2$$

is the summit of L_P ,

$$\text{ht}(L_P) = \frac{1}{n} \left| \frac{\det P}{N_P(w)} \right|^{2/n},$$

and

$$\nu(L_P) = \inf(n \text{ht}(L_P[g]))^{-n/2}, \quad g \in \Gamma.$$

Proof. The maximum of $m = \mu_1 \dots \mu_s \mu_{s+1}^2 \dots \mu_{s+t}^2$ subject to $\mu_i \geq 0$, $\sum_{i=1}^{s+t} \mu_i = 1$, is attained when $\mu_k = 1/n$, $k \leq s$, and $\mu_k = 2/n$, $k > s$. Thus, $\max m = 4^t n^{-n}$, (6) holds, and $\max \det(q) = \det(q_m) = |\det P|^2 n^{-n}$. By (3), $N_P(w) = q[w] = 1$. Hence

$$\text{ht}(L_P) = |\det(q_m)|^{1/n} = \frac{1}{n} \left| \frac{\det P}{N_P(w)} \right|^{2/n},$$

and $\nu(L_P) = \inf(n \text{ht}(g L_P))^{-n/2}$, $g \in \Gamma$, as required. □

It follows that, for any L_P , $\text{ht}(X) = |\det(X)|^{1/n} \rightarrow 0$ as $X \in L_P$ approaches the boundary of L_P . Note that $nq_m[x]$ is the form $\text{size}(M_x)$ from [6], p. 169.

In Example 1 above, by Lemma 5, $q_m = (A_1 + A_2)/2 = (I + I[E])/2$ and $\det(q_m) = t^2/4 + 1$ and $\text{ht}(L_P) = \sqrt{t^2/4 + 1}$.

Assume that $L_P \cap K(w) = \emptyset$. Let q_m be the summit of L_P . Since $q_m \notin K(w)$, there is $g \in \Gamma$ such that $\text{ht}(L_P[g]) \geq \text{ht}(q_m[g]) > \text{ht}(q_m) = \text{ht}(L_P)$. We have obtained the following.

Lemma 6. *Let L_P be the totally geodesic manifold fixed by $g \in G$ and defined by (4), where $ga_i = \lambda_i a_i$. Then*

$$\nu(L_P) = \inf(n \text{ht}(L_P[g_j]))^{-n/2}, \quad L_P \cap K(g_j w) \neq \emptyset, \quad g_j \in \Gamma.$$

By Lemma 6,

$$\nu(L_P) < (nh_n)^{-n/2} = C_n^{n/2}.$$

For $n = 2$, $\nu(L_P) < C_2 = 1/\sqrt{3}$. $\sup \nu(L_P) = 1/\sqrt{5}$ is a well-known approximation constant. It is attained only if $ga_i = (1, (1 \pm \sqrt{5})/2)$ for some $g \in \Gamma$. For other matrices P , $\nu(L_P) \leq 1/\sqrt{8}$ (cf. Example 1 above, where, for $t = 1$ and 2, the

discriminants of the fields F are 5 and 8). The set of values of $\nu(L_P) > 1/3$ is discrete and consists of numbers $(9 - 4/m^2)^{-1/2}$, where m runs through the set of all positive integers such that (m, m_1, m_2) is a solution of the Diophantine equation $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$. This result was obtained by Markov in 1879 (see e.g. [17], Section. 43).

When $n = 3$, we have $\nu(L_P) < \sqrt{2/27} = 0.2722$. This estimate does not depend on the dimension of L_P . It was shown by Davenport [12] that $\nu(L_P) \leq 1/\sqrt{23} = 0.2085$ when $s = t = 1$, where the equality holds only if $ga_i = (1, \alpha_i, \alpha_i^2)$, $i = 1, 2, 3$, the α_i being the roots of $\alpha^3 - \alpha - 1 = 0$, for some $g \in \Gamma$. As mentioned above, $\nu(L_P) \leq 1/7 = 0.1429$ when $\dim L_P = 2$.

Assume that $L_P \cap K(gw) \neq \emptyset$, where $g \in \Gamma$. Since $L_P[g] \cap K(w) \neq \emptyset$, by Lemma 5,

$$\text{ht}(L_P[g]) = \text{ht}(L_{g^T P}) = \frac{1}{n} \left| \frac{\det P}{N_{g^T P}(w)} \right|^{2/n} > h_n = 1/\gamma_n.$$

But $N_{g^T P}(x) = \langle x, g^T a_1 \rangle \dots \langle x, g^T a_n \rangle = \langle gx, a_1 \rangle \dots \langle gx, a_n \rangle$. Hence $N_{g^T P}(w) = N_P(gw)$.

A vector $gw \in \mathbf{Z}^n$ such that $L_P \cap K(gw) \neq \emptyset$ will be called a *convergent* of L_P . We have proved the following.

Theorem 7. *If a vector u is a convergent of L_P (that is, if $L_P \cap K(u) \neq \emptyset$), then*

$$|N_P(u)| < C_n^{n/2} |\det P|,$$

where $C_n = \gamma_n/n$ and γ_n is Hermite's constant. Hence if L_P cuts infinitely many sets $K(u)$, then this inequality has infinitely many solutions in $u \in \mathbf{Z}^n$.

A component of the boundary of a set $R(u)$ of codimension one will be called a *face* of $R(u)$. We shall say that sets $R(u)$ and $R(u')$ are *neighbors* if they have a common face, and if the sets $R(u)$ and $R(u')$ are neighbors then the convergents u and u' are *neighbors*. The following lemma can be used to find the faces of $R = L_P \cap K(w) \neq \emptyset$ (see e.g. Example 4 below).

Lemma 8. *Let L_P be the axis of $g \in G$. Assume that $R = L_P \cap K(w) \neq \emptyset$. Let $R_i = L_P \cap K(u_i)$, $i = 1, 2, \dots$, be the neighbors of R , so that R and R_i have a common face ϕ_i . Then $\phi_i \subset L^+(u_i)$.*

Proof. Assume that $K(w)$ and $K(gw)$ have a common face and that $X \in \overline{K}(w) \cap \overline{K}(gw)$. Then $X[g] \in \overline{K}(w)$. Hence $X[w] = X[gw] = 1$ and $X \in L^+(gw)$. Thus, the common face of $K(w)$ and $K(gw)$ lies in $L^+(gw)$. \square

Algorithm II below can be used to enumerate all the sets $R(u)$ which form the tessellation of L_P . Our ability to apply this algorithm is based on the assumption that one can find all the faces of $R = L_P \cap K(w)$ for any L_P . Then all the faces of any $R_i = L_P \cap K(u_i) \neq \emptyset$, where $u_i = g_i w$ and $g_i \in \Gamma$, can be found since $R_i[g_i] = L_P[g_i] \cap K(w)$. It is clear that g_i is not unique. Algorithm II will make the choice of g_i more specific.

Let D be a fundamental domain of Γ . The *floor* of D consists of the faces of D which do not contain the point W . Let $\mathcal{F}_D = \{\phi_1, \dots, \phi_m\}$ be the set of faces in the floor of D . Let $\mathcal{S}_D = \{S_1, \dots, S_m\}$, $S_i \in \Gamma$, where the face $\phi_i \subset L^+(S_i^{-1})$. When $n = 2$ or 3 , the floor of D consists of only one face, and one can choose S_1 to be the reflection with respect to this face (see [33], p. 1310, and Example 1).

As mentioned above, the tessellation of L_P is invariant with respect to the action of Γ . Hence, we can assume that $L_P \cap K(w) \neq \emptyset$. Indeed, if $L_P \cap K(w) = \emptyset$, take a point $X \in L_P$ and find $h \in \Gamma$ such that $X[h] \in K(w)$. (Any of the reduction algorithms (see e.g. [8] for references) can be used to find such an h .) Then $L_P[h] \cap K(w) \neq \emptyset$.

Denote by V_L the set $\{R(u)\}$, where the sets $R(u)$ form the tessellation of $L = L_P$. There is a unique graph $\overline{G}_L = (V_L, \overline{E}_L)$ associated with L whose set of vertices is V_L , and there is an edge $(R, R') \in \overline{E}_L$ if and only if $R, R' \in V_L$ are neighbors.

Algorithm II. This algorithm finds a spanning tree $G_L = (V_L, E_L)$ of the graph \overline{G}_L . An edge $(R, R') \in E_L$ is labeled by $T \in \Gamma$ if $R = L \cap K[g]$ and $R' = L \cap K[Tg]$. If the dimension of L is one, then $\overline{G}_L = G_L$.

Input. A simplex $L \subset \mathcal{P}$ with vertices at $A_i \in \mathcal{B}$, where $A_i = a_i a_i^T$ for $i = 1, \dots, s$, $A_{s+i} = \alpha_{s+i} \alpha_{s+i}^T + \beta_{s+i} \beta_{s+i}^T$ for $i = 1, \dots, t$, $a_i, \alpha_{s+i}, \beta_{s+i} \in \mathbf{R}^n$, and $\det(a_1, \dots, a_s, \beta_{s+1}, \alpha_{s+1}, \dots, \beta_{s+t}, \alpha_{s+t}) \neq 0$.

$R_0 = L \cap K(w) \neq \emptyset$.

Output. A tree $G_L = (V_L, E_L)$ where $V_L = \{R(u)\}$, where sets $R(u)$ form the tessellation of L .

Denote by $(\mathcal{V}, \mathcal{E})$ a subtree of G_L where $\mathcal{V} \subset V_L$ and $\mathcal{E} \subset E_L$ are current sets of vertices and edges of G_L found.

Let \mathcal{L} be an ordered list of leaves of the subtree $(\mathcal{V}, \mathcal{E})$ of G_L .

Let $R \in \mathcal{L}$. Let $\{R_k, k = 1, 2, \dots\}$ be the set of neighbors of R such that $R_k \notin \mathcal{V}$. Denote by $\mathcal{N}(R)$ an ordered set of faces $\psi_k = \overline{R} \cap \overline{R}_k$.

The root of G_L is $R_0 = R(w)$.

$\mathcal{V} = \mathcal{L} = \{R_0\}$ and $\mathcal{E} = \{\emptyset\}$.

Step 1. Let $\psi_k \in \mathcal{N}(R_0)$. Find $U_k \in \Gamma_\infty$ such that $\psi_k \subset \phi_k[U_k]$ for some $\phi_k \in \mathcal{F}_D$. Let $L'_k = L[U_k^{-1}]$.

Then the face $\psi'_k = \psi_k[U_k^{-1}]$ of $R' = L'_k \cap K(w)$ lies in ϕ_k .

Step 2. Let $\psi'_k \subset \phi_k \subset L^+(S_k^{-1})$, where $S_k \in \mathcal{S}_D$, be as in step 1. Denote $T_k = S_k U_k$.

(Then $L_k = L'_k[S_k^{-1}] = L[T_k^{-1}]$ cuts $K(w)$, and $L = L_k[T_k]$.)

Add $R_k = L \cap K[T_k]$ to \mathcal{V} and \mathcal{L} , and add the edge (R_0, R_k) labeled by T_k to \mathcal{E} .

Step 3. If $\mathcal{N}(R_0) \neq \emptyset$, then go to step 1. Otherwise, remove R_0 from \mathcal{L} .

(When R_0 is removed from \mathcal{L} , \mathcal{L} consists of all the neighbors of R_0 and $\mathcal{V} = L \cup R_0$.)

Step 4. Let $R \in \mathcal{L}$. Let $(R_0, R_1, \dots, R_{i-1}, R)$ be the path in G_L from R_0 to R , whose edges (R_{s-1}, R_s) are labeled by $T_s \in \Gamma$, $s = 1, \dots, i$, so that

$$R_s = L \cap K[T_s \dots T_1]$$

and $R = L \cap K[g]$, where $g = T_i \dots T_1$.

(Then $R[g^{-1}] = L[g^{-1}] \cap K(w) \neq \emptyset$. If $L[g^{-1}] = L$, then $R[g^{-1}] = R_0$ and $g \in \Gamma_L$.)

Step 5. Let $\psi_k \in \mathcal{N}(R)$. Find $U_k \in \Gamma_\infty$ such that $\psi_k[g^{-1}] \subset \phi_k[U_k]$ for some $\phi_k \in \mathcal{F}_D$. Let $L'_k = L[g^{-1}U_k^{-1}]$.

Then the face $\psi'_k = \psi_k[g^{-1}U_k^{-1}]$ of $R' = L'_k \cap K(w)$ lies in ϕ_k .

Step 6. Let $\psi'_k \subset \phi_k \subset L^+(S_k^{-1})$, where $S_k \in \mathcal{S}_D$, be as in step 6. Denote $T_k = S_k U_k$.

(Then $L_k = L'_k[S_k^{-1}] = L[g^{-1}T_k^{-1}]$ cuts $K(w)$, and $L = L_k[T_k g]$.)

Add $R_k = L \cap K[T_k g]$ to \mathcal{V} and \mathcal{L} , and add the edge (R, R_k) labeled by T_k to \mathcal{E} .

Step 7. If $\mathcal{N}(R) \neq \emptyset$, then go to step 5. Otherwise, remove R from \mathcal{L} and go to step 4.

(When R is removed from \mathcal{L} , all the neighbors of R have been added to \mathcal{V} .)

Remarks. 1. By (5), $R = L \cap K[g] = R(g^{-1}w)$, and the convergent associated with the set R is $u = g^{-1}w$, where $g = T_i \dots T_1$. Thus, Algorithm II can be used to find all the convergents of L .

2. Assume that the sets R_i form the tessellation of the axis L_P of an irreducible $g \in \Gamma$. Then the number of non-congruent sets R_i is finite and the tessellation of L_P is periodic. Let $R_1 \cup \dots \cup R_p = D_L$ be a fundamental domain of $\Gamma_L = \langle E_1, \dots, E_{s+t-1} \rangle$ in L_P . Let $V'_L = \{u_1, \dots, u_p\}$ be the set of all convergents of L_P associated with the vertices $R_1, \dots, R_p \in V_L$. Then the set of all convergents of L_P is $\{gu : g \in \Gamma_L, u \in V'_L\}$. In Example 1 above, the set of convergents of L_P is $\{E^n w, n \in \mathbf{Z}\}$. In Example 4 below, $p = 4$. In all other examples, $p = 1$, i.e., it will be shown in each of these cases that $R(w)$ is a fundamental domain of Γ_L .

3. If vertices R_k on level i of the tree G_L have been added to \mathcal{V} in step 6, then all the vertices of G_L on levels $< i$ belong to \mathcal{V} . If L is the axis of an irreducible $g \in \Gamma$, then in a finite number of steps \mathcal{V} will contain a complete set of non-congruent sets $R(u)$. Thus, Algorithm II finds a fundamental domain and a set of generators of the group Γ_L in a finite number of steps.

4. For the axis L_P defined by (4), it can be shown that $\langle u, a_1 \rangle \rightarrow 0$ as $R(u) \rightarrow A_1 = a_1 a_1^T$ (see [33], Lemma 9).

4. UNITS IN TOTALLY REAL FIELDS

Let $(1, \omega_2, \dots, \omega_n)$ be a \mathbf{Z} -basis of the ring of integers \mathbf{Z}_F of a number field F of degree n . Let $a_1 = (1, \omega_2, \dots, \omega_n)^T$. Let $\gamma \in \mathbf{Z}_F$. Then $\gamma \omega_i = \sum m_{ij} \omega_j$ or $\gamma a_1 = M_\gamma a_1$, where $\omega_1 = 1$, $m_{ij} \in \mathbf{Z}$ and $M_\gamma = (m_{ij})$ is a square matrix of order n . As explained in Section 1, the map $\gamma \mapsto M_\gamma$ is an isomorphism of the ring of integers \mathbf{Z}_F and the commutative ring of \mathbf{Z} -integral square matrices of order n with the common axis L_P defined by (4), where (s, t) is the signature of F . The norm of γ equals $\det(M_\gamma)$, so that γ is a unit in \mathbf{Z}_F if and only if $M_\gamma \in \text{GL}_n(\mathbf{Z})$. The group of units in \mathbf{Z}_F is not isomorphic to $\text{Stab}(L_P, \Gamma)$. On the one hand, $\Gamma = \text{GL}_n(\mathbf{Z})/\{\pm 1\}$. On the other hand, the group $\text{Stab}(L_P, \Gamma)$ is not necessarily commutative, since there may exist a non-trivial homomorphism $\text{Gal}(F) \rightarrow \text{Stab}(L_P, \Gamma)$. However, the torsion-free subgroup Γ_L of $\text{Stab}(L_P, \Gamma)$ is isomorphic to $\mathbf{Z}_F^\times/\mu_F$. Thus, the problem of finding a system of fundamental units of F is equivalent to the problem of finding a set of generators of Γ_L . If $s > 0$, then the torsion group $\mu_F = \{\pm 1\}$, and Γ_L is isomorphic to $\mathbf{Z}_F^\times/\{\pm 1\}$. Note that, by Lemma 5,

$$\text{ht}(L_P) = \frac{1}{n} |d(F)|^{1/n},$$

where $d(F) = \det^2(P)$ is the *discriminant* of F .

A point $X = (x_{ij}) \in \mathcal{P}$ is said to be rational over the field K if all $x_{ij} \in K$. A subset S of \mathcal{P} is rational over K if the set of rational points of S is dense in S . It is clear that the summit q_m (see (6)) of L_P is rational over some real subfield F_L of the Galois closure of F . Let $\Gamma_L(\mathbf{Q})$ be the stabilizer of L_P in $\text{GL}_n(\mathbf{Q})$. The $\Gamma_L(\mathbf{Q})$ -orbit of q_m is dense in L_P . Hence L_P is rational over F_L . Assume that $\dim(L_P) = r$. Then any vertex of $R = L_P \cap K(w)$ is the intersection of L_P with r rational (over \mathbf{Q}) planes $L^+(g_i)$. Hence all the vertices of R are rational over F_L . In this section,

F is a totally real field, $\dim(L_P) = n - 1$, and $F_L = \mathbf{Q}$. In Section 5, the signature of F is $(2,1)$.

Let $f(x) = x^n - c_{n-1}x^{n-1} - \dots - c_1x - c_0$ be an irreducible polynomial with integral coefficients. Let $f(\delta) = 0$. Let $F = \mathbf{Q}(\delta)$. Assume that \mathbf{Z}_F has the power basis $\{1, \delta, \dots, \delta^{n-1}\}$. Let $a_1 = (1, \delta, \dots, \delta^{n-1})^T$. The integral matrix C such that $Ca_1 = \delta a_1$ is said to be the companion matrix of $f(x)$. Let $a_i = (1, \delta_i, \dots, \delta_i^{n-1})^T$, $i = 1, 2, \dots, n$, where $\delta_1 = \delta$. An equation of the axis L_P of C^T is

$$q = \sum \mu_i A_i, \quad A_i = (a_{km}) = a_i a_i^T, \quad \mu_i > 0, \quad \mu_1 + \dots + \mu_n = 1,$$

where $a_{km} = \delta_i^{k+m-2}$. Thus a point $q = (q_{km})$ of L_P can be identified with the vector $[p_0, p_1, p_2, \dots, p_{2n-2}]$, where $q_{km} = p_{k+m-2}$ and

$$p_i = \mu_1 \delta_1^i + \dots + \mu_n \delta_n^i, \quad i = 0, 1, \dots, n - 1.$$

Since $\delta_j^n = c_{n-1} \delta_j^{n-1} + \dots + c_1 \delta_j + c_0$,

(7)

$$p_i = c_{n-1} p_{i-1} + c_{n-2} p_{i-2} + \dots + c_1 p_{i-n+1} + c_0 p_{i-n}, \quad i = n, n + 1, \dots, 2n - 2.$$

Using p_0, \dots, p_{n-1} as parameters on L_P instead of $\{\mu_i, i = 1, \dots, n\}$, we obtain the following.

Lemma 9. *Assume that the ring of integers \mathbf{Z}_F of a totally real field F has a power basis. A point $q = [p_0, p_1, p_2, \dots, p_{2n-2}]$, defined as above, belongs to the plane spanned by L_P if and only if (7) holds, in which case the point $q \in L_P$ is uniquely determined by its first row:*

$$q = [p_0, p_1, \dots, p_{n-1}].$$

In Example 1 above, $I = [1, 0]$ and $I[E] = [1, t]$ in these coordinates. In the examples below, we shall use these coordinates for a point $q \in L_P$ whenever the ring of integers \mathbf{Z}_F of a totally real field F has a power basis.

Example 2. Here we consider the case of the simplest cubic fields (see [25]). These are the cyclic fields of discriminant $(t^2 + 3t + 9)^2$. The field $F = \mathbf{Q}(\epsilon_1)$ is generated by a root ϵ_1 of $f(x) = x^3 - tx^2 - (t + 3)x - 1$. Assume that $t^2 + 3t + 9$ is square-free. Then $\{1, \epsilon_1, \epsilon_1^2\}$ is a basis of \mathbf{Z}_F , units ϵ_1 and $\epsilon_2 = -1/(1 + \epsilon_1)$ both are the roots of this polynomial, and $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon_1, \epsilon_2 \rangle$ (see [34]). For $t = -1, 0$, and 1 , the discriminants of F are $7^2, 9^2$, and 13^2 respectively. These fields were considered by Davenport [9], [10], [11] (see also [3]), and Swinnerton-Dyer [26].

Let E^T be the companion matrix of $f(x)$ and let $E_1 = E + I$. Let L_P be the axis of E . Then the torsion-free subgroup Γ_L of the stabilizer of L_P in Γ is generated by E and E_1 . Let $E^T a_i = \epsilon_i a_i$ and $A_i = a_i a_i^T$, where $a_i = (1, \epsilon_i, \epsilon_i^2)$, $i = 1, 2, 3$. Then $q(\mu_1, \mu_2, \mu_3) = \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3$, $\mu_i > 0$, $\mu_1 + \mu_2 + \mu_3 = 1$, is an equation of L_P .

Let $R = L_P \cap K(w)$. An edge of R is the intersection of L_P with some $L^+(g)$, $g \in \Gamma$, which contains a face of $K(w)$, and the vertices of R are the points of intersection of L_P with some faces of $K(w)$ of codimension two. Denote $E_2 = EE_1^{-1}$. Let F_1 be the intersection of L_P , $L^+(E)$, and $L^+(E_2)$, and let G_1 be the intersection of L_P , $L^+(E)$, and $L^+(E_1)$. Since \mathbf{Z}_F has a power basis, Lemma 9 is applicable. In our case, R is the hexagon with vertices at $F_1 = [1, 0, 1]$, $F_2 = F_1[E]$, $F_3 = F_1[E_2]$, $G_1 = [1, -1/2, 1]$, $G_2 = G_1[E_1]$, $G_3 = G_1[E]$ with $\det(F_i) = 1$, $\det(G_i) = (t^2 + 3t + 9)/8$. The sides of R are identified as follows: $E : F_1 G_1 \rightarrow F_2 G_3$;

$E_1 : F_3G_1 \rightarrow F_2G_2$; $E_2 : F_1G_2 \rightarrow F_3G_3$. Since the forms F_i are integral, they are extremal.

Let $t = 3u + k$, where $|k| \leq 1$. Denote

$$h = \begin{bmatrix} 1 & 0 & h_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & k/2 \\ 0 & k/2 & g_{33} \end{bmatrix},$$

where $h_{13} = -1 - u$, $h_{23} = -2u$, and $g_{33} = (4 \det(G_1) + k^2)/3$. Then $G_0 = G_1[h]$ is Minkowski reduced ([8], pp. 396–397). Hence the points G_i are extremal.

Hence $R = L_P \cap K(w)$ is a fundamental domain of $\langle E, E_1 \rangle$, and therefore we have given a new proof that $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon + 1 \rangle$.

Note that if $F_1 = q(\mu_1, \mu_2, \mu_3)$, then $F_2 = q(\mu_2, \mu_3, \mu_1)$ and $F_3 = q(\mu_3, \mu_1, \mu_2)$. The same relations hold for G_1, G_2 , and G_3 . Also, for the summit q_m of L_P , we have

$$q_m = \frac{1}{3} \sum A_i = \frac{1}{3} \sum F_i = \frac{1}{3} \sum G_i.$$

Example 3. Let $t \geq 3$ be a positive integer. Let $f(x) = x^3 + (t-1)x^2 - tx - 1$. Let $f(\epsilon) = 0$. Assume that the discriminant of $f(x)$ is square-free. Then $\{1, \epsilon, \epsilon^2\}$ is a basis of \mathbf{Z}_F , where $F = \mathbf{Q}(\epsilon)$ is a totally real field. The field F is non-Galois, and it is exceptional; that is, $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon - 1 \rangle$ (see e.g. [14], [21]). Since \mathbf{Z}_F has a power basis, Lemma 9 is applicable.

Let E^T be the companion matrix of $f(x)$, $E_1 = E - I$, and $E_2 = EE_1^{-1}$. Let L_P be the axis of E . Let F_1 be the intersection of L_P , $L^+(E)$, and $L^+(E_1)$, and let G_1 be the intersection of L_P , $L^+(E)$, and $L^+(E_2)$. Let $c = t^2 + 3t + 1$. The region $R = K(w) \cap L_P$ is the hexagon with vertices at $F_1 = [1, 1/2, 1]$, $F_2 = F_1[E]$, $F_3 = F_1[E_1]$, $G_1 = [1, -(2t+4)/c, 1]$, $G_2 = G_1[E]$, and $G_3 = G_1[E_2]$ with $\det(F_i) = (t^2 + 3t - 9)/8$ and $\det(G_i) \sim 1$ as $t \rightarrow \infty$. Let h be as in Example 2 with $h_{13} = -u$, $h_{23} = -2 + 2u$. Then $F_1[h]$ is Minkowski reduced. Hence the points F_i , $i = 1, 2, 3$, are extremal. Denote

$$h = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1 & -(2t+4)/c & (2t+2)/c \\ -(2t+4)/c & 1 & -2/c \\ (2t+2)/c & -2/c & 1 \end{bmatrix}.$$

The point $G_0 = G_1[h]$ is Minkowski reduced ([8], pp. 396–397); hence the points G_i , $i = 0, 1, 2, 3$, are extremal. The sides of R are identified as follows: $E : F_1G_1 \rightarrow G_2F_2$; $E_1 : F_1G_3 \rightarrow F_3G_2$; $E_2 : F_3G_1 \rightarrow F_2G_3$. Hence R is a fundamental domain of $\langle E, E_1 \rangle$, and therefore we have given a new proof that $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon - 1 \rangle$.

Note that now neither $(\sum F_i)/3$ nor $(\sum G_i)/3$ is the summit of L_P .

Example 4. Let $f(x) = x^3 + (t+2)x^2 + (2t-1)x - 1$, where $t \in \mathbf{Z}$. Let $f(\epsilon) = 0$. Assume that the discriminant of $f(x)$ is square-free. Then $\{1, \epsilon, \epsilon^2\}$ is a basis of \mathbf{Z}_F , where $F = \mathbf{Q}(\epsilon)$ is a totally real field. The field F is non-Galois, and $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon + 2 \rangle$ (see [21]). When $t = 3$, the discriminant of the field is 148. This case was considered by Swinnerton-Dyer [26] (see also [23]).

Let E^T be the companion matrix of $f(x)$, $E_1 = E + 2I$, and $E_2 = EE_1$. Let L_P be the axis of E . Since \mathbf{Z}_F has a power basis, Lemma 9 is applicable.

Let $d = 2t^2 - 5t + 1$. Denote

$$\begin{aligned} F_1 &= [1, -1/2 + 1/(4t - 6), 1], & F_2 &= F_1[E], & F_3 &= F_1[E_2], \\ G_1 &= [1, -1, 2], & G_2 &= G_1[E_2], \\ H_1 &= [1, -3/2, 3], & H_2 &= H_1[E_1], \\ K_1 &= [1, (2 - t)/d, 1], & K_2 &= K_1[E], \\ M_1 &= [1, -1/(2t - 1), 1], & M_2 &= M_1[E_1^{-1}], & M_3 &= M_2[E_2] \end{aligned}$$

with $\det(G_1) = 1$, $\det(H_1) = (3t + 1)/8$, $\det(M_1) = (6t + 1)(t - 1)^2/(2t - 1)^3$, and $\det(F_1), \det(K_1) \sim 3/4$ as $t \rightarrow \infty$.

Denote $u_1 = (0 \ t \ 1)^T$, $u_2 = (1 \ 1 \ 0)^T$, $u_3 = (t \ 1 \ 0)^T$ and $h_1 = [u_1, e_1, e_2]$, $h_i = [u_i, e_1, e_3]$, $i = 2, 3$, where $[e_1, e_2, e_3]$ is the identity matrix. Then $F_1 = L_P \cap L^+(E) \cap L^+(E_2)$, $G_1 = L_P \cap L^+(E_2) \cap L^+(u_2)$, $H_1 = L_P \cap L^+(E_1) \cap L^+(u_2)$, $K_1 = L_P \cap L^+(E) \cap L^+(u_1)$ and $M_1 = L_P \cap L^+(E) \cap L^+(E_1^{-1})$.

The points G_1 and H_1 are extremal since they are integral. Let

$$h(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $F_1[h(2)]$ and $K_1[h(t)]$ are Minkowski reduced ([8], pp. 396-397). Hence F_i and K_i are also extremal. The points M_i , $i = 1, 2, 3$, are not extremal. But the points $M'_i = (2t - 1)/(2t - 2)M_i[h_i]$ are extremal with $\det(M'_i) = (6t + 1)/(8t - 8)$. In particular, $M_1[h_1]$ is Minkowski reduced.

The fundamental domain of $\Gamma_L = \langle E, E_1 \rangle$ is the hexagon $D_L = R \cup R_1 \cup R_2 \cup R_3$ with vertices at F_1, M_1, F_3, M_3, F_2 and M_2 . The region $R = L_P \cap K(w)$ is the 9-gon with vertices at $F_1, K_1, H_2, F_3, G_2, K_2, F_2, H_1$ and G_1 . Regions R_1, R_2 and R_3 are triangles. The vertices of R_1 are M_1, K_1 and H_2 , of R_2 are M_2, H_1 and G_1 , and of R_3 are M_3, G_2 and K_2 . The intersection of the plane $L^+(u_1)$ with D_L is the interval K_1H_2 which is the common boundary of R and R_1 . Similarly, $L^+(u_2) \cap D_L = G_1H_1 = R \cap R_2$ and $L^+(u_3) \cap D_L = G_2K_2 = R \cap R_3$. Note that $R_i[h_i] = L_P[h_i] \cap K(w)$, $i = 1, 2, 3$.

The sides of D_L are identified as follows: $E : F_1M_1 \rightarrow F_2M_3$; $E_1 : F_2M_2 \rightarrow F_3M_1$; $E_2 : F_1M_2 \rightarrow F_3M_3$. Hence D_L is a fundamental domain of $\langle E, E_1 \rangle$, and therefore we have given a new proof that $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon + 2 \rangle$.

The set of convergents of L_P is $\{gu, g \in \Gamma_L, u \in V'_L\}$, where $V'_L = \{w, u_1, u_2, u_3\}$.

Remark. Six sides of R are identified as follows: $E : F_1K_1 \rightarrow F_2K_2$; $E_1 : F_2H_1 \rightarrow F_3H_2$; $E_2 : F_1G_1 \rightarrow F_3G_2$.

Example 5. Assume that $t > 3$ is a positive integer which is not divisible by 3. Then $4t^2 + 9$ is a square-free integer. Let $f(x) = x^4 + tx^3 - x^2 - tx + 1 = x(x^2 - 1)(x + t) + 1$. Let $f(\epsilon) = 0$. The discriminant of $f(x)$ is $(4t^2 + 9)(t^2 - 4)^2$. Let $\eta = \epsilon - \epsilon^{-1}$. Since

$$x^{-2}f(x) = (x - x^{-1})^2 + t(x - x^{-1}) + 1,$$

it follows that $\eta^2 + t\eta + 1 = 0$, $\eta = (-t \pm \sqrt{t^2 - 4})/2$, and $K = \mathbf{Q}(\eta)$ is a quadratic subfield of the totally real quartic field $F = \mathbf{Q}(\epsilon)$. We have $\epsilon^2 - \eta\epsilon - 1 = 0$, and the roots of $f(x)$ are

$$\epsilon_{i,i+2} = \frac{\eta_i}{2} \pm \sqrt{\frac{\eta_i^2}{4} + 1}, \quad \eta_i = -\frac{t}{2} \pm \sqrt{\frac{t^2}{4} - 1}, \quad i = 1, 2.$$

Clearly, $\epsilon_1\epsilon_3 = \epsilon_2\epsilon_4 = -1$. Hence, $F_i = K(\sqrt{\eta_i^2 + 4})$ and the discriminant of F_i is

$$D_F = D_K^2 N(D_{F/K}) = (t^2 - 4)^2 N(\eta_i^2 + 4) = (t^2 - 4)^2 (4t^2 + 9)$$

provided $t^2 - 4$ is the discriminant of K , in which case $\mathbf{Z}_K^\times / \{\pm 1\} = \langle \eta_i \rangle$, where \mathbf{Z}_K^\times is the unit group in the maximal order \mathbf{Z}_K of K . Since $N(D_{F/K}) = 4t^2 + 9$ is square-free, $\{1, \epsilon\}$ is a basis of $\mathbf{Z}_{F/K}$ (see e.g. [7], p. 79). It follows that $\{1, \epsilon, \epsilon\eta, \eta\}$ or $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a basis of \mathbf{Z}_F . The regulator obtained from the units ϵ_1, ϵ_2 , and η is of order $\log^3 t$ as $t \rightarrow \infty$. To show that the Galois group of $\mathbf{Q}(\epsilon)$ is D_4 , it is enough to show that F_i is not a normal field. But, if it is, then $\sqrt{\eta_1^2 + 4}\sqrt{\eta_2^2 + 4} = \sqrt{N(\eta_i^2 + 4)} = \sqrt{4t^2 + 9}$ belongs F_i and therefore D_F is divisible by $(4t^2 + 9)^2$, which is not the case since $t > 3$.

Let E^T be the companion matrix of $f(x)$, $E_1 = E + I$, $E_2 = E - E^{-1}$, and $E_3 = E - I = EE_2E_1^{-1}$. Let L_P be the axis of E . Since \mathbf{Z}_F has a power basis, Lemma 9 is applicable. Now we shall show that the following 28 points are the vertices of $R = L_P \cap K(w)$.

The points $A = [1, 1/2, 1, -2/t]$ and $B = [1, -1/2, 1, -1/2 - 2/t]$ with $\det(B) = \det(A) \sim \frac{3}{8}t$ as $t \rightarrow \infty$ are the intersections of L_P with $L^+(E)$, $L^+(E_2)$, $L^+(E_3)$ and with $L^+(E_1)$, $L^+(E)$, $L^+(EE_3)$ respectively. Denote $A_1 = A[E]$, $A_2 = A[E_2]$, $A_3 = A[E_3]$, and $B_1 = B[E_1]$, $B_2 = B[E]$, $B_3 = B[EE_2]$.

The points $C = [1, -1/2, 1, -t/2 + 1/t - 1/4]$ and $D = [1, 1/2, 1, -t/2 + 1/t + 1/4]$ with $\det(D) = \det(C) \sim \frac{7}{64}t^2$ as $t \rightarrow \infty$ are the intersections of L_P with $L^+(E_1)$, $L^+(E_1E_2^{-1})$, $L^+(E)$ and with $L^+(E)$, $L^+(EE_1^{-1})$, $L^+(E_3)$ respectively. Denote $C_1 = C[E_1]$, $C_2 = C[E_1E_2^{-1}]$, $C_3 = C[E]$ and $D_1 = D[E]$, $D_2 = D[EE_1^{-1}]$, $D_3 = D[E_3]$.

Let $s = 1/(2t^2 - t - 2)$. The points $K = [1, -s, 1, 2/t - t]$ and $L = [1, s, 1, s + 2/t - t]$ with $\det(L) = \det(K) \sim 3/4$ as $t \rightarrow \infty$ are the intersections of L_P with $L^+(E)$, $L^+(E_1E_2^{-1})$, $L^+(E_2^{-1})$ and $L^+(E_1)$, $L^+(E_1E_2^{-1})$, $L^+(E_1E^{-1})$ respectively. Denote $K_1 = K[E]$, $K_2 = K[E_1E_2^{-1}]$, $K_3 = K[E_2^{-1}]$ and $L_1 = L[E_1]$, $L_2 = L[E_1E_2^{-1}]$, $L_3 = L[E_1E^{-1}]$.

The point $M = [1, 0, 1, -2/t]$ with $\det(M) = (1 - 4/t^2)^2$ is the intersection of L_P with $L^+(E)$, $L^+(E_2)$, and $L^+(EE_2)$. Denote $M_1 = M[E]$, $M_2 = M[E_2]$, $M_3 = M[EE_2]$. Note that the summit q_m of the 3-flat L_P is $q_m = \frac{1}{4}(M + M_1 + M_2 + M_3)$ with $\det(q_m) = D_F/4^4$.

To show that all the points enumerated above are extremal we shall use Minkowski reduction. (If for some $h \in \Gamma_\infty$ the point $X[h]$ is Minkowski reduced ([8], pp. 396-397), then X is extremal). Below we shall indicate such an h for one of the vertices in each of the Γ_L -orbits of vertices of R . The polytope R is bounded by four octagons lying in $L^+(g^{\pm 1})$, $g = E, E_2$, eight pentagons lying in $L^+(g^{\pm 1})$, $g = E_1, E_3, EE_1^{-1}, E_1E_2^{-1}$, and four triangular faces lying in $L^+(g^{\pm 1})$, $g = EE_2, EE_2^{-1}$. It has 28 vertices, 42 edges and 16 faces. The projections of the boundary of R into a plane which is ‘perpendicular’ to its octagonal faces are shown in Figure 1.

Let

$$U_A = \begin{bmatrix} 1 & 1 & -t & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad U_B = \begin{bmatrix} 1 & 0 & -1 & 1-t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

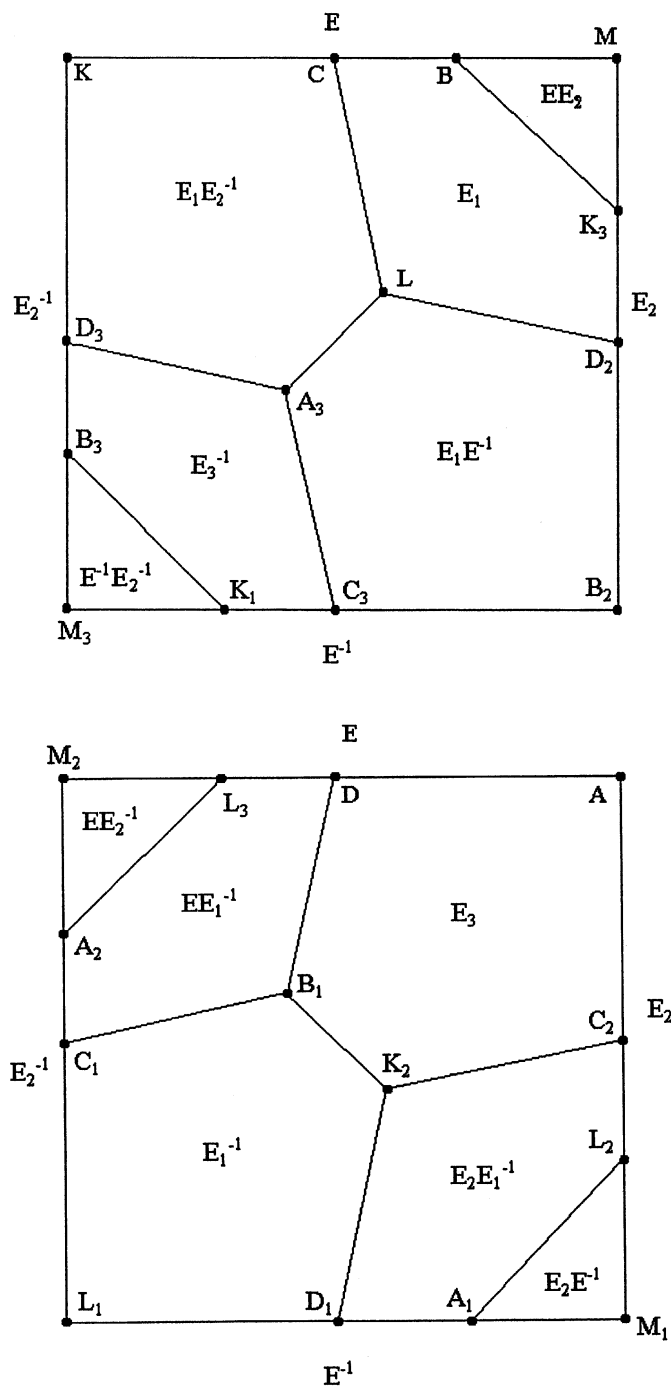


FIGURE 1.

Let $b = 1/2 - 2/t$, $c = t/2 + 1 - 4/t$, $d = -1/2 - 2/t$. Then

$$A^U = \begin{bmatrix} 1 & 1/2 & -2/t & -1/2 \\ 1/2 & 1 & 0 & -2/t \\ -2/t & 0 & 1 & -b \\ -1/2 & -2/t & -b & c \end{bmatrix}, \quad B^U = \begin{bmatrix} 1 & -1/2 & 0 & b \\ -1/2 & 1 & -2/t & -1/2 \\ 0 & -2/t & 1 & 1/2 \\ b & -1/2 & 1/2 & c \end{bmatrix},$$

where $A^U = A[U_A]$ and $B^U = B[U_B]$, are Minkowski reduced. Hence A , A_i and B , B_i , $i = 1, 2, 3$, are extremal. Let

$$U_K = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & t+1 & -t \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad U_L = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & t & -t \\ 0 & 0 & 1 & t-1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $a = 2t^3 - t^2 - 2t$, $b = 3t^2 - t - 4$, $c = -t^3 + t^2 + t$, $d = t - b$. Then

$$aK^U = \begin{bmatrix} a & -t & -d & -2a/t \\ -t & a & b & t^2 \\ -d & b & a & c \\ -2a/t & t^2 & c & a \end{bmatrix}, \quad aL^U = \begin{bmatrix} a & t & t^2 & b \\ t & a & 2a/t & d \\ t^2 & 2a/t & a & c \\ b & d & c & a \end{bmatrix},$$

where $K^U = K[U_K]$ and $L^U = L[U_L]$, are Minkowski reduced. Hence K , K_i and L , L_i , $i = 1, 2, 3$, are extremal. For

$$U = \begin{bmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the matrix

$$tM[U] = \begin{bmatrix} t & 0 & 0 & -2 \\ 0 & t & -2 & 0 \\ 0 & -2 & t & 0 \\ -2 & 0 & 0 & t \end{bmatrix}$$

is Minkowski reduced. Thus, M , M_i , $i = 1, 2, 3$, are extremal.

Reduction of C and D is more difficult. Let $t = 22k + l$. Let

$$U_D = \begin{bmatrix} 1 & 0 & 1 & u_{14} \\ 0 & 1 & -t & u_{24} \\ 0 & 0 & t-1 & u_{34} \\ 0 & 0 & 1 & u_{44} \end{bmatrix},$$

where $u_{44} = -[t/11]$, $u_{34} = u_{44}(t-1) + 1$, $u_{14} = -9k - a$, while

$$u_{24} = 44k^2 + (2l + 15)k + b,$$

where $a = [l/3] + 1$, $b = [(2l + 1)/3]$ if $1 \leq l \leq 10$, and

$$u_{24} = 44k^2 + (2l - 7)k + b,$$

where $a = -[-l/3]$, $b = [(1-l)/3]$ if $-11 \leq l \leq 0$. Then $D[U_D]$ is Minkowski reduced. For example, when $l = 0$,

$$D[U_D] = \begin{bmatrix} 1 & 1/2 & 1/4 + 1/t & 7/44 + 1/t \\ 1/2 & 1 & 1/4 - 1/t & 1/11 \\ 1/4 + 1/t & 1/4 - 1/t & 1 & 1/2 \\ 7/44 + 1/t & 1/11 & 1/2 & 7/44(t^2 - 4) \end{bmatrix}.$$

Let

$$U_C = \begin{bmatrix} 1 & 0 & -1 & u_{14} \\ 0 & 1 & t & u_{24} \\ 0 & 0 & t+1 & u_{34} \\ 0 & 0 & 1 & u_{44} \end{bmatrix},$$

where $u_{44} = -[t/11]$, $u_{34} = u_{44}(t+1) + 1$, $u_{14} = 9k + a$, while

$$u_{24} = -44k^2 - (2l + 7)k - b,$$

where $a = [l/3]$, $b = [(l+1)/3]$ if $0 \leq l \leq 11$, and

$$u_{24} = -44k^2 - (2l - 15)k - b,$$

where $a = [l/3] - 1$, if $-10 \leq l \leq -1$, $l \neq -5$, and $a = [l/3] - 2$ if $l = -5$, and $b = [(1-2l)/3]$ if $-10 \leq l \leq -1$. Then $C[U_C]$ is Minkowski reduced. For example, when $l = 0$,

$$C[U_C] = \begin{bmatrix} 1 & -1/2 & 1/t - 1/4 & -1/11 \\ -1/2 & 1 & 1/t + 1/4 & 7/44 + 1/t \\ 1/t - 1/4 & 1/t + 1/4 & 1 & 1/2 \\ -1/11 & 7/44 + 1/t & 1/2 & 7/44(t^2 - 4) \end{bmatrix}.$$

Hence, C , C_i and D , D_i , $i = 1, 2, 3$, are extremal.

Thus, $R = L_P \cap K(w)$ is a fundamental domain of $\Gamma_L = \langle E, E_1, E_2 \rangle$, and therefore $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon + 1, \epsilon - 1/\epsilon \rangle = \langle \epsilon, \epsilon + 1, \epsilon - 1 \rangle$.

Example 6. Let $f(x) = x^4 + tx^3 - x^2 - tx - 1 = x(x^2 - 1)(x + t) - 1$. Let $f(\epsilon) = 0$. The field $F = \mathbf{Q}(\epsilon)$ is totally real and $\text{Gal}(F) = S_4$. Assume that the discriminant of $f(x)$ is square-free. Then \mathbf{Z}_F has a power basis, and Lemma 9 is applicable.

Let E^T be the companion matrix of $f(x)$, $E_1 = E - I$, $E_2 = E - E^{-1}$, and $E_3 = E + I = EE_2E_1^{-1}$. Let L_P be the axis of E . We shall show that the region $R = L_P \cap K(w)$ is the same as in Example 5.

The point $A = [1, -1/2, 1, -1-2/t]$ with $\det(A) \sim \frac{3}{8}t$ as $t \rightarrow \infty$ is the intersection of L_P with $L^+(E)$, $L^+(E_2)$ and $L^+(E_3)$. Denote $A_1 = A[E]$, $A_2 = A[E_2]$, $A_3 = A[E_3]$. Let $c = -1/2 + 2/t$,

$$U_A = \begin{bmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^U = \begin{bmatrix} 1 & -1/2 & -2/t & c \\ -1/2 & 1 & 0 & -c \\ -2/t & 0 & 1 & c + 2/t \\ c & -c & c + 2/t & t/2 - 6c \end{bmatrix}.$$

Then $A[U_A] = A^U$ is Minkowski reduced. Hence A , A_1 , A_2 and A_3 are extremal.

The point $B_3 = [1, 1/2, 1, 1/2]$ with $\det(B) = (6t - 19)/16$ is the intersection of L_P with $L^+(E^{-1}E_2^{-1})$, $L^+(E_2^{-1})$ and $L^+(E_3^{-1})$. Denote $B = B_3[E^{-1}E_2^{-1}]$, $B_1 = B_3[E_3^{-1}]$, $B_2 = B_3[E_2^{-1}]$. Since B_3 is an integral matrix, it is extremal.

The points $C = [1, 1/2, 1, -t/2 + 7/4 + 1/(2t-4)]$ and $D = [1, -1/2, 1, -t/2 - 7/4 + 1/(2t+4)]$ with $\det(C), \det(D) \sim \frac{7}{64}t^2$ as $t \rightarrow \infty$ are the intersections of L_P with $L^+(E)$, $L^+(E_1)$, $L^+(E_1E_2^{-1})$ and with $L^+(E)$, $L^+(E_3)$, $L^+(EE_1^{-1})$ respectively. Denote $C_1 = C[E_1]$, $C_2 = C[E_1E_2^{-1}]$, $C_3 = C[E]$ and $D_1 = D[E]$, $D_2 = D[EE_1^{-1}]$, $D_3 = D[E_3]$. As in Example 5, reduction of C and D is divided into particular cases modulo 22. We consider only one case of such reduction. Let $t = 22k + 13$.

Let

$$U_C = \begin{bmatrix} 1 & 0 & -1 & -u_{14} \\ 0 & 1 & -t & -u_{24} \\ 0 & 0 & t-1 & u_{34} \\ 0 & 0 & 1 & u_{44} \end{bmatrix},$$

where $u_{44} = 2 + 2k$, $u_{34} = u_{44}(t - 1) + 1$, $u_{14} = 9k + 6$, $u_{24} = 44k^2 + 55k + 19$. Then

$$C[U_C] = \begin{bmatrix} 1 & 1/2 & -1/4 + c & 1/22 + c \\ 1/2 & 1 & -1/4 - c & -1/4 - 1/22 \\ 1/4 + c & 1/4 - c & 1 & c \\ 1/22 + c & -1/4 - 1/22 & c & 7/44t^2 + 3/11t - 31/22 - c \end{bmatrix},$$

where $c = 1/(22 + 44s)$, is Minkowski reduced. Hence C is extremal. Similarly, the other cases of reduction of C and D can be considered.

Denote by K_3 the intersection of L_P with $L^+(E_3)$, $L^+(E_2)$ and $L^+(EE_2)$, and by L_3 the intersection of L_P with $L^+(E)$, $L^+(EE_2^{-1})$ and $L^+(EE_1^{-1})$. Let $K = K_3[E_2]$, $K_1 = K[E]$, $K_2 = K[E_1E_2^{-1}]$ and $L = L_3[EE_1^{-1}]$, $L_1 = L[E_1]$, $L_2 = L[E_1E_2^{-1}]$. Let

$$U_K = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & t-1 & t \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad U_L = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & t & t \\ 0 & 0 & 1 & t+1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $a_1 = t(t - 2)(2t + 3)$, $b_1 = (t - 4)/a_1$, $c_1 = (3t^2 - 4)/a_1$, $d_1 = (t^2 - 3t - 4)/a_2$, $a_2 = t(t + 2)(2t - 5)$, $b_2 = (t + 4)/a_2$, $c_2 = (3t^2 - 3t - 8)/a_2$, $d_2 = -(t^2 + 2t - 8)/a_2$. Then

$$K[U_K] = \begin{bmatrix} 1 & -2/t - b_1 & c_1 & 2/t - 2b_1 \\ -2/t - b_1 & 1 & d_1 & tb_1 \\ c_1 & d_1 & 1 & 1/2 - tb_1/2 \\ 2/t - 2b_1 & tb_1 & 1/2 - tb_1/2 & 1 \end{bmatrix},$$

$$L_3[U_L] = \begin{bmatrix} 1 & -2/t - b_2 & -tb_2 & c_2 \\ -2/t - b_2 & 1 & 2b_2 & d_2 \\ -tb_2 & 2b_2 & 1 & 1/2 - tb_2/2 \\ c_2 & d_2 & 1/2 - tb_2/2 & 1 \end{bmatrix}$$

are Minkowski reduced. Hence K and L are extremal. Note that $\det(K) \sim 3/4$, $\det(L) \sim 3/4$ as $t \rightarrow \infty$.

The point $M = [1, 2/t, 1, 2/t]$ with $\det(M) = (1 - 4/t^2)^2 - 4/t^2$ is the intersection of L_P with $L^+(E)$, $L^+(E_2)$ and $L^+(EE_2)$. Denote $M_1 = M[E]$, $M_2 = M[E_2]$, $M_3 = M[EE_2]$. Let

$$U_M = \begin{bmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_U = \begin{bmatrix} 1 & 2/t & 0 & -2/t \\ 2/t & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/t \\ -2/t & 0 & 2/t & 1 \end{bmatrix}.$$

Since $M[U_M] = M_U$ is Minkowski reduced, the points M , M_1 , M_2 , M_3 are extremal.

Thus, the polytope $R = L_P \cap K(w)$ is the same as in Example 5 (see Figure 1). It is a fundamental domain of $\Gamma_L = \langle E, E_1, E_2 \rangle$. Hence $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon + 1, \epsilon - 1/\epsilon \rangle = \langle \epsilon, \epsilon + 1, \epsilon - 1 \rangle$.

5. UNITS IN FIELDS WITH SIGNATURE (2,1)

Let $n = 4$. Let $g \in \Gamma$. Assume that the characteristic polynomial of g is irreducible with signature (2, 1). Let ϵ_1, ϵ_2 be real and $\epsilon_3, \epsilon_4 = \bar{\epsilon}_3$ non-real complex eigenvalues of g . Let $ga_i = \epsilon_i a_i$. Assume that the field $F = \mathbf{Q}(\epsilon_1)$ is dihedral and K is its real quadratic subfield. Let σ be the non-trivial automorphism of F/K . Then $\sigma(\epsilon_i) = \epsilon_{i+1}$, $i = 1, 3$. Hence $\sigma(A_1) = A_2$ and $\sigma(A_3) = A_4$, where A_1, A_2 , and A_3 are vertices of L_P , the axis of g . Thus, the entries of $A_1 + A_2$ and of A_3 lie in K , and $q_m = (A_1 + A_2 + 2A_3)/3$ is rational over K . Hence L_P and the vertices of $R = L_P \cap K(w)$ are rational over K . It follows that $F_L = K$ in this case. In Examples 7 and 9 below, the signature of the field F is (2,1), F has a real quadratic subfield K , and $F_L = K$. In Example 8, the degree of F_L is six.

Example 7. Let $f(x) = x^4 + tx^3 + x^2 + tx + 1 = x(x^2 + 1)(x + t) + 1$. Let $f(\epsilon) = 0$, where $\epsilon \in \mathbf{R}$. The discriminant of $f(x)$ is $-(4t^2 - 9)(t^2 + 4)^2$. Let $\eta = \epsilon + \epsilon^{-1}$. Since

$$x^{-2}f(x) = (x + x^{-1})^2 + t(x + x^{-1}) - 1,$$

it follows that $\eta^2 + t\eta - 1 = 0$, $\eta = (-t \pm \sqrt{t^2 + 4})/2$, and $K = \mathbf{Q}(\sqrt{d})$, $d = t^2 + 4$, is a quadratic subfield of the dihedral quartic field $F = \mathbf{Q}(\epsilon)$ with signature (2, 1). We have $\psi(\epsilon) = \epsilon^2 - \eta\epsilon + 1 = 0$, and the roots of $f(x)$ are

$$\epsilon_{i,i+2} = \frac{1}{2}(\eta_i \pm \sqrt{\eta_i^2 - 4}), \quad \eta_i = \frac{1}{2}(-t \mp \sqrt{t^2 + 4}), \quad i = 1, 2.$$

Hence $\epsilon_1\epsilon_3 = \epsilon_2\epsilon_4 = -1$. If $N(\eta^2 - 4) = 4t^2 - 9$, the norm of the discriminant of $\psi(x)$, is square-free, then $\{1, \epsilon\}$ is a basis of $\mathbf{Z}_{F/K}$. Hence $\{1, \epsilon, \eta\}$ or $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a basis of \mathbf{Z}_F provided η is a fundamental unit of K .

Let E^T be the companion matrix of $f(x)$. Then $E^T a_i = \epsilon_i a_i$, where $a_i = (1, \epsilon_i, \epsilon_i^2, \epsilon_i^3)$. Denote $E_1 = E + E^{-1}$. Let M be the intersection of L_P , $L^+(E)$, and $L^+(E_1)$. Then

$$M[x] = \mu_1(x, a_1)^2 + \mu_3(x, a_3)^2 + \mu_2|(x, a_2)|^2,$$

where $\mu_{1,3} = (1 + (t \pm (2t\sqrt{d} + 2t^2 - 12)^{1/2})/\sqrt{d})/4$, $\mu_2 = (1 - t/\sqrt{d})/2$, and $\det(M) = (4t^2 - 9)(t^2 + 2 - t\sqrt{d})/2$.

Let $b = -2t(\eta\sqrt{d})^{-1} - 3(2\sqrt{d})^{-1}$, $c = -2t(\eta\sqrt{d})^{-1} + 3(2\sqrt{d})^{-1}$, $e = 5(4\eta\sqrt{d})^{-1}$ and

$$h = \begin{bmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 1 & b & e & 0 \\ b & 1 & 0 & -e \\ e & 0 & 1 & c \\ 0 & -e & c & 1 \end{bmatrix}.$$

Then $M_0 = M[h]$ is Minkowski reduced (see e.g. [8], p. 397). Hence the points M , $M[E]$, $M[E_1]$, and $M[EE_1]$ are extremal, and they are the vertices of the ‘square’ $R = L_P \cap K(w)$, which is a fundamental domain of $\langle E, E_1 \rangle$. Thus $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \eta \rangle$.

Note that $q_m = (M + M[E] + M[E_1] + M[EE_1])/4$ is the summit of L_P .

Example 8 (cf. [20]). Let $t \geq 4$ and $f_t(x) = f(x) = x^4 + tx^3 + x^2 + tx - 1 = x(x^2 + 1)(x + t) - 1$. Note that $f_{-t}(-x) = f_t(x)$. Let ϵ and ϵ_1 be the real and $\epsilon_2, \bar{\epsilon}_2$ non-real complex roots of $f(x) = 0$. The signature of $F = \mathbf{Q}(\epsilon)$ is (2, 1), and $\text{Gal}(F) = S_4$. Assume that the discriminant of F is square-free. Then $\{1, \epsilon, \epsilon^2, \epsilon^3\}$ is a basis of the maximal order \mathbf{Z}_F of F .

Let E^T be the companion matrix of $f(x)$. Let $E_1 = E^2(E + tI)$. Let L_P be the axis of E . It can be identified with the set

$$F(\nu_1, \nu_2, \nu_3) = \nu_1 a^T a + \nu_2 a_1^T a_1 + \nu_3 (a_{2R}^T a_{2R} + a_{2I}^T a_{2I}),$$

where $\nu_k > 0$, $\nu_1 + \nu_2 + \nu_3 = 1$. Here $a = (1, \epsilon, \epsilon^2, \epsilon^3)$, $a_i = (1, \epsilon_i, \epsilon_i^2, \epsilon_i^3)$, $i = 1, 2$, are the eigenvectors of E corresponding to its eigenvalues $\epsilon, \epsilon_1, \epsilon_2$ respectively. Let A and B be the intersections of L_P with $L^+(E)$, $L^+(E_1)$ and with $L^+(E^{-1})$, $L^+(E_1^{-1})$ respectively.

Let $\alpha = |\epsilon_2|^2 = -1/(\epsilon\epsilon_1)$. Then $\alpha \rightarrow 1$ as $t \rightarrow \infty$. Over $F_L = \mathbf{Q}(\alpha)$ we have $f(x) = f_1(x)f_2(x)$, where

$$f_1(x) = x^2 + t \frac{\alpha + 1}{\alpha^2 + 1} x - \frac{1}{\alpha},$$

$$f_2(x) = x^2 + t \frac{\alpha^2 - \alpha}{\alpha^2 + 1} x + \alpha,$$

so that $f_i(\epsilon_i) = 0$, $i = 1, 2$. The degree of F_L is 6, since $g(\alpha) = 0$, where $g(x) = x^6 - x^5 + (t^2 + 1)x^4 - 2x^3 - (t^2 + 1)x - x - 1$.

Let d_1 be the discriminant of $f_1(x)$. Let $\nu_{1,2} = \mu_1 \pm \mu_2 \sqrt{d_1}$, $\mu_3 = \nu_3$, $a^T a = M_1 + M_2 \sqrt{d_1}$, $a_1^T a_1 = M_1 - M_2 \sqrt{d_1}$. Then L_P can be identified with the set $\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3$, where the entries of M_k , $k = 1, 2, 3$, belong to F_L . Thus, the summit q_m of L_P , L_P itself and all the vertices of $R = L_P \cap K(w)$ are rational over F_L .

Let Δ_A be the determinant of the system of three linear equations

$$\begin{aligned} 2\mu_1 + \mu_3 &= 1, \\ (\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3)[wE] &= 1, \\ (\mu_1 M_1 + \mu_2 d_1 M_2 + \mu_3 M_3)[wE_1] &= 1, \end{aligned}$$

with respect to μ_1, μ_2, μ_3 . Then

$$\begin{aligned} D_A &= \Delta_A(\alpha^2 + 1)\alpha/(4t) \\ &= \alpha^5 + 2\alpha^4 + (2t^2 - 1)\alpha^3 + (t^2 - 2)(\alpha^2 - \alpha) - 2. \end{aligned}$$

Let Δ_B be the determinant of the system of three similar linear equations where E and E_1 are replaced by E and EE_1^{-1} . Then

$$\begin{aligned} D_B &= \Delta_B(\alpha^2 + 1)/(4t) \\ &= -(\alpha^3 + (t^2 + 5)\alpha + t^2 + 2). \end{aligned}$$

Let

$$U_A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & t \\ 0 & 0 & t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad U_B = \begin{bmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and let $A_U = A[U_A] = (a_{ij})$ and $B_U = B[U_B] = (b_{ij})$. Then $a_{11} = a_{22} = a_{33} = 1$,

$$\begin{aligned} 2tD_Aa_{12} &= (1 - \alpha)(1 + \alpha^2 - \alpha^3)t^2 - 4\alpha(2 - \alpha + 2\alpha^2 - \alpha^3), \\ 2tD_Aa_{13} &= (\alpha - \alpha^3)t^4 + (1 + 5\alpha - 7\alpha^2 + 4\alpha^3 + 2\alpha^4 - \alpha^5)t^2 - 6 - 14\alpha^2 \\ &\quad + 6\alpha^3 - 8\alpha^4 + 6\alpha^5, \\ 2D_Aa_{14} &= (1 - \alpha)(1 + \alpha - 4\alpha^2)t^2 - 6 - 3\alpha - 13\alpha^2 + 2\alpha^3 - 7\alpha^4 + 5\alpha^5, \\ 2D_Aa_{23} &= (1 - \alpha)(\alpha + 6\alpha^2 + \alpha^3)t^2 + 11 - 3\alpha + 18\alpha^2 - 9\alpha^3 + 7\alpha^4 - 6\alpha^5, \\ 2tD_Aa_{24} &= (1 + 2\alpha - 2\alpha^2 + 7\alpha^3 + \alpha^4 - \alpha^5)t^2 - 6 + 8\alpha - 18\alpha^2 + 14\alpha^3 \\ &\quad - 12\alpha^4 + 6\alpha^5, \\ 2tD_Aa_{34} &= 4(\alpha^2 - \alpha^3)t^4 + (9 - 9\alpha + 23\alpha^2 - 14\alpha^3 + 5\alpha^4 - 4\alpha^5)t^2 \\ &\quad + 12 - 8\alpha + 32\alpha^2 - 20\alpha^3 + 20\alpha^4 - 12\alpha^5, \\ D_Aa_{44} &= (2\alpha + 3\alpha^2 - 3\alpha^3)t^2 + 8 + 12\alpha^2 + \alpha^3 + 2\alpha^4 - 3\alpha^5, \end{aligned}$$

and $b_{11} = b_{22} = b_{33} = 1$,

$$\begin{aligned} 2tD_Bb_{12} &= 4(\alpha^3 - \alpha)t^4 + (-3 - 9\alpha - 12\alpha^2 + 10\alpha^3 - 4\alpha^4 + 4\alpha^5)t^2 \\ &\quad - 4 + 2\alpha - 14\alpha^2 + 8\alpha^3 - 10\alpha^4 + 6\alpha^5, \\ 2D_Bb_{13} &= (\alpha^2 - 1)(2 - 3\alpha)t^2 - 4 - 11\alpha + 11\alpha^2 - 4\alpha^3 + 5\alpha^4 - 3\alpha^5, \\ 2tD_Bb_{14} &= (\alpha^2 - 1)(\alpha + 2)t^4 + (-7 - 12\alpha - 2\alpha^2 - \alpha^3 + \alpha^4 + \alpha^5)t^2 \\ &\quad + 4\alpha(1 - \alpha + \alpha^2 - \alpha^3), \\ 2tD_Bb_{23} &= 2(\alpha^3 - \alpha)t^4 + (5 + 4\alpha - 7\alpha^2 - 2\alpha^3 - 2\alpha^4 + 2\alpha^5)t^2 \\ &\quad + 8 + 4\alpha + 8\alpha^2 - 4\alpha^5, \\ 2D_Bb_{24} &= 2(\alpha^3 - \alpha)t^4 + (1 - 8\alpha - 7\alpha^2 + 6\alpha^3 - 2\alpha^4 + 2\alpha^5)t^2 \\ &\quad - 3 - 13\alpha - 8\alpha^2 + 5\alpha^3 - 7\alpha^4 + 4\alpha^5, \\ 2tD_Bb_{34} &= 2(\alpha - \alpha^3)t^4 + (7 + 7\alpha - 4\alpha^2 - 10\alpha^3 + 2\alpha^4 - 2\alpha^5)t^2 \\ &\quad + 10 + 14\alpha + 8\alpha^2 + 6\alpha^3 - 2\alpha^4 - 8\alpha^5, \\ D_Bb_{44} &= (2 - 3\alpha - 5\alpha^2)t^2 - 4 - 5\alpha - 4\alpha^2 + 7\alpha^3 - 6\alpha^4. \end{aligned}$$

Let $c = (\alpha^2 + 1)^2$. The identity

$$t^2(\alpha^3 - \alpha) = c - \left(\frac{c}{\alpha t}\right)^2 + \frac{c^3}{\alpha^2 t^2(\alpha^2 t^2 + c)}$$

implies $a_{ij} \rightarrow 0$ and $b_{ij} \rightarrow 0$ as $t \rightarrow \infty$ for $i \neq j$. It follows that $\det(A) \sim 1$ and $\det(B) \sim 3$ as $t \rightarrow \infty$. Thus, A_U and B_U are Minkowski reduced. Hence the points $A, A[E], A[E_1], B, B[E^{-1}],$ and $B[E_1^{-1}]$ are extremal, and they are the vertices of the hexagon $R = L_P \cap K(w)$. Thus, R is a fundamental domain of $\langle E, E_1 \rangle$, and $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \epsilon + t \rangle$.

Example 9. Let

$$f(x) = x^4 + stx^3 + (t - \alpha s^2)x^2 + s(t^2 + 2\alpha)x - \alpha = x(x^2 + t)(x + st) - \alpha(sx - 1)^2,$$

where $t \geq s$ are positive integers and $\alpha = \pm 1$. Let $d = t^2 + 4\alpha$. It can be easily verified that

$$f(x) = (x^2 + s\eta x - \alpha/\eta)(x^2 - \alpha s x/\eta + \eta),$$

where $\eta = \frac{1}{2}(t + \sqrt{d})$ satisfies the equation $\eta^2 - t\eta - \alpha = 0$. The roots of $f(x) = 0$ are

$$\epsilon_{i,i+2} = \frac{1}{2}(-s\eta_i \pm \sqrt{s^2\eta_i^2 + 4\alpha/\eta_i}), \quad \eta_i = \frac{1}{2}(t \pm \sqrt{d}), \quad i = 1, 2,$$

where $\eta_1 = \eta$ and real ϵ_1 and ϵ_3 are the roots of $x^2 + s\eta x - \alpha/\eta = 0$. The discriminant of $f(x)$ is $\Delta(f) = -d^2(4s^2t^3 + 12\alpha s^2t - s^4 + 16\alpha)(1 + s^2t)^2$. Let $\epsilon = \epsilon_1$. Since $\eta = (\epsilon^2 + t)/(1 - s\epsilon)$, $K = \mathbf{Q}(\eta)$ is a quadratic subfield of the dihedral quartic field $F = \mathbf{Q}(\epsilon)$ with signature (2,1). Assume that η is a fundamental unit of K . Then $\{1, \eta\}$ is a basis of \mathbf{Z}_K . Let $c = ts^2 + 1$. Denote $p(x) = (s + (\alpha t - s^2)x + \alpha x^3)/c$. Then $\epsilon\eta = p(\epsilon)$. Assume that $4s^2t^3 + 12\alpha s^2t - s^4 + 16\alpha$ is square-free. Then $\{1, \epsilon, \epsilon^2, p(\epsilon)\}$ is a basis of \mathbf{Z}_F , and the discriminant of F is $D_F = \Delta(f)/(ts^2 + 1)^2$.

Let $a_i = (1, \epsilon_i, \epsilon_i^2, p(\epsilon_i))$. Let E_0 be the companion matrix of f . Denote

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s/c & (\alpha t - s^2)/c & 0 & \alpha/c \end{bmatrix}.$$

Let $E^T = \tau E_0 \tau^{-1}$, $E_1 = p(E)$, $E_2 = E_1 E^{-1}$. Then $E^T a_i = \epsilon_i a_i$, and the axis L_P of E has equation $q[x] = \mu_1(x, a_1)^2 + \mu_3(x, a_3)^2 + \mu_2|(x, a_2)|^2$, $\mu_i > 0$, $\mu_1 + \mu_3 + \mu_2 = 1$.

First let $\alpha = 1$. Let A and B be the intersections of L_P with $L^+(E)$, $L^+(E_1)$ and with $L^+(E)$, $L^+(E_2^{-1})$ respectively. Then $\det(A) = \det(B)$. Let $a_{ii} = b_{ii} = 1$, $i = 1, 2, 3$, $a_{44} = b_{44} = \sqrt{d} - 1 + \eta^{-2}$, $a_{23} = b_{13} = 0$, $a_{14} = b_{24} = \eta^{-1}$,

$$\begin{aligned} a_{24} &= -b_{14} = \left(-\frac{s}{2\sqrt{d}} + \frac{t}{s\eta^2\sqrt{d}}\right)(1 - \eta^{-1}), \\ a_{34} &= -b_{34} = \left(\frac{s}{2\eta\sqrt{d}} + \frac{t}{s\eta\sqrt{d}}\right)(1 - \eta^{-1}), \\ a_{13} &= -b_{23} = \frac{1}{s\eta} - \frac{2}{s\eta\sqrt{d}} - \frac{s}{2\eta^2\sqrt{d}}, \\ a_{12} &= -b_{12} = \frac{1}{s\eta^2} - \frac{2}{s\eta^2\sqrt{d}} + \frac{s}{2\eta\sqrt{d}}. \end{aligned}$$

Let

$$U_A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad U_B = \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & s & 1 \end{bmatrix}.$$

Then $A_U = (a_{ij}) = A[U_A]$ and $B_U = (b_{ij}) = B[E_2^{-1}][U_B]$ are Minkowski reduced. Hence the points $A, A[E], A[E_1], B, B[E], B[E_2^{-1}]$ are extremal, and they are the vertices of the hexagon $R = L_P \cap K(w)$.

Now let $\alpha = -1$. Let A and B be the intersections of L_P with $L^+(E^{-1})$, $L^+(E_2)$ and with $L^+(E_1)$, $L^+(E_2)$ respectively. Then $\det(A) = \det(B)$. Note that if $A = q(\gamma_1, \gamma_2, \gamma_3)$, then $B = q(\gamma_3, \gamma_2, \gamma_1)$. Here $\gamma_2 = \eta$ and $\sigma(\gamma_1) = \gamma_3$, where $\sigma \in \text{Gal}(F/K)$.

Let $a_{ii} = b_{ii} = 1$, $i = 1, 2, 3$, $a_{44} = b_{44} = \sqrt{d} - 1 - 2\eta^{-1} + \eta^{-2} - 4/(t\eta)$,
 $a_{14} = -b_{24} = -(1 - 2/t)/\eta$, $a_{23} = b_{13} = -2/t$,

$$\begin{aligned} a_{12} &= -b_{23} = -\frac{s}{2t\eta} - \frac{1}{s\eta^2}, \\ a_{13} &= -b_{12} = \frac{s}{2\eta^2 t} + \frac{1}{s\eta}, \\ a_{24} &= b_{34} = \left(\frac{s}{2\sqrt{d}} \left(1 - \frac{2}{t} \right) - \frac{t-2}{s\eta^2\sqrt{d}} \right) (1 + \eta^{-1}), \\ a_{34} &= b_{14} = \frac{2t - s^2}{2s\eta\sqrt{d}} (1 + \eta^{-1})(1 - 2/t). \end{aligned}$$

Let

$$U_A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -s \end{bmatrix}, \quad U_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t-1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -s & 1 \end{bmatrix}.$$

Then $A_U = (a_{ij}) = A[E^{-1}][U_1^T]$ and $B_U = (b_{ij}) = B[U]$ are Minkowski reduced. Hence the points A , $A[E^{-1}]$, $A[E_2]$, B , $B[E_1]$, $B[E_2]$ are extremal, and they are the vertices of the hexagon $R = L_P \cap K(w)$.

Thus, for any α , R is a fundamental domain of $\langle E, E_2 \rangle$, and $\mathbf{Z}_F^\times / \{\pm 1\} = \langle \epsilon, \eta \rangle$.

REFERENCES

- [1] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth Compactification of Locally Symmetric Varieties*, Math. Sci. Press, Brookline, Mass., 1975. MR **56**:15642
- [2] A. J. Brentjes, *Multi-dimensional continued fraction algorithms*, *Mathematical Centre Tracts* 145, Mathematisch Centrum, Amsterdam, 1981. MR **83b**:10038
- [3] Bryuno, A.D. and Parusnikov, V.I., Klein polyhedra for two Davenport cubic forms, (Russian) *Mat. Zametki*, **56** (1994) 9-27; *translation in Math. Notes* **56** (1994) 994-1007 (1995). MR **96a**:11061
- [4] J.W.S. Cassels, *An Introduction to the Geometry of Numbers*, Springer-Verlag, 1959. MR **28**:1175
- [5] J.W.S. Cassels, *Rational Quadratic Forms*, Ac. Press, NY, 1978. MR **80m**:10019
- [6] H. Cohen, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Math. **138**, Springer-Verlag, 1993. MR **94i**:11105
- [7] H. Cohen, *Advanced Topics in Computational Algebraic Number Theory*, Graduate Texts in Math. **193**, Springer-Verlag, 1999. MR **2000k**:11144
- [8] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, Springer-Verlag, New York and Berlin, 1988. MR **89a**:11067
- [9] H. Davenport, On the product of three homogeneous linear forms. I, *Proc. London Math. Soc.*, **44** (1938), 412-431.
- [10] H. Davenport, Note on the product of three homogeneous linear forms, *J. London Math. Soc.*, **16** (1941), 98-101. MR **3**:70f
- [11] H. Davenport, On the product of three homogeneous linear forms. IV, *Math. Proc. Camb. Phil. Soc.*, **39** (1943), 1-21. MR **4**:212a
- [12] H. Davenport, C. A. Rogers, Diophantine inequalities with an infinity of solutions, *Philos. Trans. Roy. Soc. London*, **242** (1950), 311-344. MR **12**:394b
- [13] P. B. Eberlein, *Geometry of Nonpositively Curved Manifolds*, The University of Chicago Press, 1996. MR **98h**:53002
- [14] Ennola, V., Cubic number fields with exceptional units, *Computational number theory (Drebrecken, 1989)*, 103-128, de Gruyter, Berlin, 1991. MR **93e**:11131
- [15] M.-N. Gras, Table numérique du nombre de classes et des unités des extensions cycliques de degré de \mathbf{Q} , *Publ. Math. Fac. Sci. Besançon*, fasc. 2 (1977/78). MR **52**:10678

- [16] D. Grenier, Fundamental domains for the general linear group, *Pacific J. Math.* **132** (1988), 293-317. MR **89d**:11055
- [17] P. M. Gruber and C. G. Lekkerkerker, *Geometry of Numbers*, 2nd ed. North-Holland Publishing Co., Amsterdam, 1987. MR **88j**:11034
- [18] A. Korkine and G. Zolotarev, Sur les formes quadratiques, *Math. Ann.* **6** (1873), 366-389.
- [19] G. A. Margulis, Indefinite quadratic forms and unipotent flows on homogeneous spaces, *C. R. Acad. Sci. Paris Sér 1 Math.* **304** (1987), 251-257. MR **88f**:11027
- [20] G. Niklasch and N.P. Smart, Exceptional units in a family of quartic number fields, *Math. Comp.* **67** (1998), 759-772. MR **98g**:11125
- [21] Minemura, Kenji, On totally real cubic fields whose unit groups are of type $\{\theta + r, \theta + s\}$, *Proc. Japan Acad. Ser. A Math. Sci.* **74** (1998), 156-159. MR **99m**:11126
- [22] Wladislaw Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, Springer-Verlag, Berlin and New York, 1990. MR **91h**:11107
- [23] Parusnikov, V.I., Klein polyhedra for complete decomposable forms, *Number theory (Eger, 1996)*, 453-463, de Gruyter, Berlin, 1998. MR **99f**:11083
- [24] S. S. Ryshkov and E. P. Baranovskii, Classical methods in the theory of lattice packings, *Russian Math. Surveys*, **34** (1979), 1-68. MR **81a**:10045
- [25] D. Shanks, The simplest cubic fields, *Math. Comp.* **28** (1974), 1137-1152. MR **50**:4537
- [26] Swinnerton-Dyer, H.P.F., On the product of three homogeneous linear forms, *Acta Arith.* **18** (1971), 371-385. MR **45**:1844
- [27] A. Terras, *Harmonic analysis on symmetric spaces and applications*. II, Springer-Verlag, Berlin and New York, 1988. MR **89k**:22017
- [28] G. Voronoi, On a generalization of the Algorithm of Continued Fractions, Doctoral dissertation, Warsaw, 1896. (= *Collected works, vol. 1, Izdat. Akad. Nauk Ukrain. SSR, Kiev 1952*, 197-391) (Russian). MR **16**:2d
- [29] L.Ya. Vulakh, The Markov spectrum for triangle groups, *J. Number Theory*, **67** (1997), 11-28. MR **99e**:11093
- [30] L.Ya. Vulakh, Farey polytopes and continued fractions associated with discrete hyperbolic groups, *Trans. Amer. Math. Soc.*, **351** (1999), 2295-2323. MR **99i**:11054
- [31] L.Ya. Vulakh, Diophantine approximation in Euclidean spaces, *CRM Proceedings and Lecture Notes*, **19** (1999), 341-351. MR **2000j**:11099
- [32] L.Ya. Vulakh, The Markov spectrum for Fuchsian groups, *Trans. Amer. Math. Soc.*, **352** (2000), 4067-4094. MR **2000m**:11056
- [33] L.Ya. Vulakh, Continued fractions associated with $SL_3(\mathbf{Z})$ and units in complex cubic fields, *Canad. J. Math.* **54** (2002), 1305-1318.
- [34] L. C. Washington, Class numbers of the simplest cubic fields, *Math. Comp.* **48** (1987), 371-384. MR **88a**:11107
- [35] L. C. Washington, A family of cyclic quartic fields arising from modular curves, *Math. Comp.* **57** (1991), 763-775. MR **92a**:11120

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