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DIALGEBRA COHOMOLOGY AS A G-ALGEBRA

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Abstract. It is well known that the Hochschild cohomology $H^*(A,A)$ of an associative algebra A admits a G-algebra structure. In this paper we show that the dialgebra cohomology $HY^*(D,D)$ of an associative dialgebra D has a similar structure, which is induced from a homotopy G-algebra structure on the dialgebra cochain complex $CY^*(D,D)$.

1. Introduction

It is well known, since the pioneering work of M. Gerstenhaber [2], that the Hochschild cochain complex $C^*(A,A)$ of an associative algebra A admits a brace algebra structure. Moreover, in [3], M. Gerstenhaber and A. A. Voronov have shown that $C^*(A,A)$ admits a homotopy G-algebra structure which induces the G-algebra structure on the Hochschild cohomology as introduced in [2]. These structures on $C^*(A,A)$ are in fact induced from a natural operad structure on $C^*(A,A)$, where only the non-Σ part of the operad is responsible for inducing the above structures.

The notions of Leibniz algebras and associative dialgebras were introduced in [6], by J.-L. Loday. Leibniz algebras are a non-commutative variation of Lie algebras, and associative dialgebras are a variation of associative algebras. Recall that an associative algebra gives rise to a Lie algebra by $[x,y] = xy - yx$. The notion of associative dialgebra is introduced in order to build an analogue of the couple

$$\text{Lie algebras} \leftrightarrow \text{associative algebras},$$

when Lie algebras are replaced by Leibniz algebras. A cohomology theory associated with dialgebras has been developed by Loday, called dialgebra cohomology, where in the construction of the dialgebra complex which defines the dialgebra cohomology, planar binary trees play a crucial role. Dialgebra cohomology with coefficients has been studied by A. Frabetti in [1]. In [7], it has been shown that the dialgebra complex $CY^*(D,D)$ admits the structure of an associative algebra, and also of a pre-Lie algebra. The aim of this paper is to show that, as in the case of a Hochschild complex, $CY^*(D,D)$ admits a homotopy G-algebra structure which comes from a non-Σ operad structure on $CY^*(D,D)$. As a consequence, the dialgebra cohomology $HY^*(D,D)$ becomes a G-algebra.

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2. Dialgebra complex

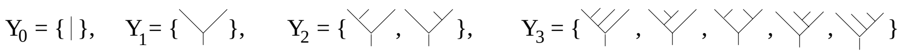
In this section, we recall the construction of a dialgebra complex. Throughout this paper, by dialgebra we mean associative dialgebra.

Definition 2.1. Let K be a field. A dialgebra D over K is a vector space over K along with two K-linear maps, $\dashv\colon D \otimes D \longrightarrow D$ (called left) and $\vdash\colon D \otimes D \longrightarrow D$ (called right), satisfying the following axioms:

$$\begin{cases} x \dashv (y \dashv z) \overset{1}{=} (x \dashv y) \dashv z \overset{2}{=} x \dashv (y \vdash z), \\ (x \vdash y) \dashv z \overset{3}{=} x \vdash (y \dashv z), \\ (x \dashv y) \vdash z \overset{4}{=} x \vdash (y \vdash z) \overset{5}{=} (x \vdash y) \vdash z, \end{cases} \tag{2.1}$$

for all $x, y, z \in D$.

A planar binary tree with n vertices (in short, an n-tree) is a planar tree with $(n+1)$ leaves, one root and each vertex trivalent. Let Y_n denote the set of all n-trees. Let Y_0 be the singleton set consisting of a root only. The n-trees for $0 \leq n \leq 3$ are given by the following diagrams:



For any $y \in Y_n$, the $(n+1)$ leaves are labelled by $\{0, 1, \ldots, n\}$ from left to right, and the vertices are labelled $\{1, 2, \ldots, n\}$, so that the ith vertex is between the leaves $(i-1)$ and i. Recall from [6] that the only element $|$ of Y_0 is denoted by [0]. The only element of Y_1 is denoted by [1]. The grafting of a p-tree y_1 and a q-tree y_2 is a $(p+q+1)$-tree denoted by $y_1 \vee y_2$ which is obtained by joining the roots of y_1 and y_2 and creating a new root from that vertex. This is denoted by $[y_1\ p+q+1\ y_2]$ with the convention that all zeros are deleted except for the element in Y_0. With this notation, the trees pictured above from left to right are [0], [1], [12], [21], [123], [213], [131], [312], [321].

For any i, $0 \leq i \leq n$, there is a map, called the face map, $d_i : Y_n \longrightarrow Y_{n-1}$, $y \mapsto d_i y$, where $d_i y$ is obtained from y by deleting the ith leaf. The face maps satisfy the relation $d_i d_j = d_{j-1} d_i$, for all $i < j$.

Let D be a dialgebra over a field K. The cochain complex $CY^*(D, D)$ which defines the dialgebra cohomology $HY^*(D, D)$ is defined as follows. For any $n \geq 0$, let $K[Y_n]$ denote the K-vector space spanned by Y_n, and let

$$CY^n(D, D) := \mathrm{Hom}_K(K[Y_n] \otimes D^{\otimes n}, D)$$

be the module of n-cochains of D with coefficients in D. The coboundary operator $\delta : CY^n(D, D) \longrightarrow CY^{n+1}(D, D)$ is defined as the K-linear map $\delta = \sum_{i=0}^{n+1} (-1)^i \delta^i$, where

$$(\delta^i f)(y; a_1, a_2, \ldots, a_{n+1}) := \begin{cases} a_1 \circ_0^y f(d_0 y; a_2, \ldots, a_{n+1}), & \text{if } i = 0, \\ f(d_i y; a_1, \ldots, a_i \circ_i^y a_{i+1}, \ldots, a_{n+1}), & \text{if } 1 \leq i \leq n, \\ f(d_{n+1} y; a_1, \ldots, a_n) \circ_{n+1}^y a_{n+1}, & \text{if } i = n+1, \end{cases}$$

for any $y \in Y_{n+1}$; $a_1, \ldots, a_{n+1} \in D$ and $f : K[Y_n] \otimes D^{\otimes n} \longrightarrow D$. Here, for any i, $0 \leq i \leq n+1$, the maps $\circ_i : Y_{n+1} \longrightarrow \{\dashv, \vdash\}$ are defined by

$$\circ_0(y) = \circ_0^y := \begin{cases} \dashv & \text{if } y \text{ is of the form } | \vee y_1, \text{ for some } n\text{-tree } y_1, \\ \vdash & \text{otherwise,} \end{cases}$$

$$\circ_i(y) = \circ_i^y := \begin{cases} \dashv & \text{if the } i^{th} \text{ leaf of } y \text{ is oriented like `}\backslash\text{'}, \\ \vdash & \text{if the } i^{th} \text{ leaf of } y \text{ is oriented like `/'}, \end{cases}$$

for $1 \leq i \leq n$, and

$$\circ_{n+1}(y) = \circ_{n+1}^y := \begin{cases} \vdash & \text{if } y \text{ is of the form } y_1 \vee |, \text{ for some } n\text{-tree } y_1, \\ \dashv & \text{otherwise,} \end{cases}$$

where the symbol '$\vee$' stands for grafting of trees [6].

3. Braces for a dialgebra complex

In this section, we introduce braces or multilinear operations in $CY^*(D,D)$ of a dialgebra D, generalizing the $\circ_i$ products as introduced in [7], which endow $CY^*(D,D)$ with a brace algebra structure.

Definition 3.1. A brace algebra is a graded vector space with a collection of braces (or multilinear operations) $x\{x_1, x_2, \ldots, x_n\}$ of degree $-n$ satisfying the identity (brace identity)

$$\begin{aligned} x\{x_1, x_2, \ldots, x_m\}\{y_1, y_2, \ldots, y_n\} = \sum_{0 \leq i_1 \leq j_1 \leq i_2 \leq \cdots \leq i_m \leq j_m \leq n} & (-1)^{\epsilon} x\{y_1, \ldots, y_{i_1}, \\ & x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_2}, \\ & x_2\{y_{i_2+1}, \ldots, y_{j_2}\}, y_{j_2+1}, \ldots, y_{i_m}, \\ & x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_n\} \end{aligned}$$

where $x\{\}$ is understood as just x, $\deg\ x\{x_1, \ldots, x_n\} = \deg\ x + \sum_{i=1}^n \deg\ x_i - n$, $|x| = \deg\ x - 1$, and $\epsilon = \sum_{p=1}^m |x_p| \sum_{q=1}^{i_p} |y_q|$.

Definition 3.2. Let $n, i_1, i_2, \ldots, i_r$, $m_1, m_2, \ldots, m_r$ be non-negative integers with $n, m_1, \ldots, m_r \geq 1$ such that

$$0 \leq i_1, i_1 + m_1 \leq i_2, \ldots, i_{r-1} + m_{r-1} \leq i_r, i_r + m_r \leq N = n + \sum_1^r m_i - r.$$

For each j, $0 \leq j \leq r$, we define maps

$$R_{j+1}^{i_1, \ldots, i_r}(N; n, m_1, \ldots, m_r) : Y_N \longrightarrow Y_{m_j},$$

with $m_0 = n$, in the following way. For $j = 0$,

$$R_1^{i_1, \ldots, i_r}(N; n, m_1, \ldots, m_r) = \prod_{\substack{m_\ell \geq 2 \\ 1 \leq \ell \leq r}} (d_{i_\ell+1} \cdots d_{i_\ell+m_\ell-1}) \text{ if } 2 \leq m_0 < N,$$

where Π stands for composition of terms and $R_1^{i_1, \ldots, i_r}(N; n, m_1, \ldots, m_r)$ is the identity or the obvious constant map according to whether m_0 is N or 1.

For $1 \leq j \leq r$, if $2 \leq m_j < N$ we have

$$R_{j+1}^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r) = \begin{cases} (d_0 \cdots d_{i_j-1})(d_{i_j+m_j+1} \cdots d_N), & i_j \geq 1 \text{ and} \\ & i_j + m_j + 1 \leq N, \\ (d_{m_j+1} \cdots d_N), & i_j = 0, \\ (d_0 \cdots d_{i_j-1}), & i_j + m_j + 1 > N, \end{cases}$$

and $R_{j+1}^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r)$ is the identity or the obvious constant map according to whether $m_j = N$ or $m_j = 1$.

Definition 3.3. Let D be a dialgebra over a field K. For non-negative integers $n, i_1, \dots i_r, m_1, \dots, m_r$ with $0 \leq i_1$, $i_1 + m_1 \leq i_2, \dots, i_{r-1} + m_{r-1} \leq i_r$, $i_r + m_r \leq N = n + \sum_1^r m_i - r$, the multilinear maps

$$\circ_{i_1,\dots,i_r} : CY^n(D, D) \otimes \bigotimes_{j=1}^{r} CY^{m_j}(D, D) \longrightarrow CY^N(D, D)$$

are defined as follows. Let $f \in CY^n(D, D), g_j \in CY^{m_j}(D, D), 1 \leq j \leq r$. For $y \in Y_N$ and $x_1, \dots, x_N \in D$ we have

$$\begin{aligned} & f \circ_{i_1,\dots,i_r} (g_1, \dots, g_r)(y; x_1, \dots, x_N) \\ = \; & f(R_1^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r)y; x_1, \dots, x_{i_1}, \\ & g_1(R_2^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r)y; x_{i_1+1}, \dots, x_{i_1+m_1}), \dots, \\ & g_r(R_{r+1}^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r)y; x_{i_r+1}, \dots, x_{i_r+m_r}), \dots, x_N). \end{aligned}$$

In the above definition, if $m_j = 0$ for some j, then

$$g_j \in CY^0(D, D) \cong \operatorname{Hom}_K(K, D) = D$$

and the corresponding input is simply g_j.

Next we use these generalized $\circ_i$ products to define braces as follows.

Definition 3.4. For $f \in CY^n(D, D)$, $g_\nu \in CY^{m_\nu}(D, D)$, $\nu = 1, \dots, r$,

$$f\{g_1, \dots, g_r\} = \sum_{i_1,\dots,i_r} (-1)^\eta f \circ_{i_1,\dots,i_r} (g_1, \dots, g_r),$$

where $\eta = \sum_{\nu=1}^r |g_\nu| i_\nu$, and $|g_\nu| = \deg g_\nu - 1 = m_\nu - 1$.

Remark 3.5. It may be noted that by the above definition of braces on $CY^*(D, D)$, $f\{g\}$ coincides with the pre-Lie product $f \circ g$ as introduced in [7].

Henceforth, we shall use the symbol $f \circ g$ in order to denote $f\{g\}$. The following proposition will follow from Lemma 5.1.

Proposition 3.6. *The braces as defined above make the dialgebra cochain complex $CY^*(D, D)$ into a brace algebra.*

4. OPERAD STRUCTURE

In this section we show that the dialgebra complex $CY^*(D, D)$ of a dialgebra D admits the structure of a non-Σ operad.

Definition 4.1. A non-Σ operad $\mathcal{C}$ of K-vector spaces consists of vector spaces $\mathcal{C}(j)$, $j \geq 0$, together with a unit map $K \longrightarrow \mathcal{C}(1)$ and multilinear maps

$$\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \longrightarrow \mathcal{C}(j)$$

for $k \geq 1; j_s \geq 0$ and $j = \sum_{s=1}^k j_s$. The maps γ are required to be associative and unital as in [8].

The following maps on trees will be used to define a non-Σ operad structure on $CY^*(D,D)$.

Definition 4.2. Given an integer j, with $j=\sum_{r=1}^{k} j_r,\ k\geq 1$ and $j_r\geq 1$, define maps

$$\begin{aligned}\Gamma^0(k;j_1,\dots,j_k) &\ :\ Y_j\longrightarrow Y_k,\\ \Gamma^r(k;j_1,\dots,j_k) &\ :\ Y_j\longrightarrow Y_{j_r},\ 1\leq r\leq k,\end{aligned}$$

by

$$\begin{aligned}&\Gamma^0(k;j_1,\dots,j_k)\\ &\quad = d_1\cdots d_{j_1-1}d_{j_1+1}\cdots d_{j_1+j_2-1}d_{j_1+j_2+1}\cdots d_{\sum_{s=1}^{r} j_s-1}d_{\sum_{s=1}^{r} j_s+1}\cdots\\ &\qquad d_{\sum_{s=1}^{k-1} j_s-1}d_{\sum_{s=1}^{k-1} j_s+1}\cdots d_{j-1}\\ &\quad = d_1\cdots \check{d}_{p_1}\cdots\check{d}_{p_2}\cdots\check{d}_{p_r}\cdots\check{d}_{p_{k-1}}\cdots d_{p_k-1} \text{ for all } 1\leq r\leq k-1,\end{aligned}$$

and

$$\begin{aligned}\Gamma^r(k;j_1,\dots,j_k) &= d_0\cdots d_{\sum_{s=1}^{r-1} j_s-1}d_{\sum_{s=1}^{r} j_s+1}\cdots d_{\sum_{s=1}^{k} j_s}\\ &= d_0\cdots d_{p_{r-1}-1}d_{p_r+1}\cdots d_j,\end{aligned}$$

where $p_r=j_1+j_2+\cdots+j_r, 1\leq r\leq k$, and the symbol $\check{d}_i$ appearing in any expression means that the map d_i has been omitted.

Remark 4.3. Given integers $j,\ k\geq 1,\ j_r\geq 1$ with $j=\sum_{r=1}^{k} j_r$, we shall often write the map $\Gamma^r(k;j_1,\dots,j_k)$ simply as Γ^r, for all $r=0,1,\dots,k$. However, to avoid confusion we shall write the maps Γ^r explicitly, along with the values of $k,\ j_1,\dots,j_k$, whenever necessary.

Theorem 4.4. *For a dialgebra D over a field K, the dialgebra complex $CY^*(D,D)$ is a non-Σ operad of K-vector spaces.*

To prove the above theorem we need the following lemma.

Lemma 4.5. *Let $j_r\geq 1$, $1\leq r\leq k$ be integers with $j=\sum_{r=1}^{k} j_r$. Let $i=\sum_{t=1}^{j} i_t$, with integers $i_t\geq 1$. Set $p_s=j_1+j_2+\cdots+j_s$ and $q_s=i_{p_{s-1}+1}+\cdots+i_{p_s}$. Then for $1\leq s\leq j_r$, $1\leq r\leq k$ the corresponding maps*

$$\begin{aligned}&\Gamma^0(k;j_1,\dots,j_k):Y_j\longrightarrow Y_k,\\ &\Gamma^0(j;i_1,\dots,i_j):Y_i\longrightarrow Y_j,\\ &\Gamma^0(k;q_1,\dots,q_k):Y_i\longrightarrow Y_k,\\ &\Gamma^0(j_r;i_{p_{r-1}+1},\dots,i_{p_{r-1}+j_r}):Y_{q_r}\longrightarrow Y_{j_r},\\ &\Gamma^s(j_r;i_{p_{r-1}+1},\dots,i_{p_{r-1}+j_r}):Y_{q_r}\longrightarrow Y_{i_{p_{r-1}+s}},\\ &\Gamma^r(k;j_1,\dots,j_k):Y_j\longrightarrow Y_{j_r},\\ &\Gamma^{p_{r-1}+s}(j;i_1,\dots,i_j):Y_i\longrightarrow Y_{i_{p_{r-1}+s}},\\ &\Gamma^r(k;q_1,\dots,q_k):Y_i\longrightarrow Y_{q_r}\end{aligned}$$

satisfy

(a) $\Gamma^0(k;j_1,\dots,j_k)\Gamma^0(j;i_1,\dots,i_j)=\Gamma^0(k;q_1,\dots,q_k)$,
(b) $\Gamma^r(k;j_1,\dots,j_k)\Gamma^0(j;i_1,\dots,i_j)=\Gamma^0(j_r;i_{p_{r-1}+1},\dots,i_{p_{r-1}+j_r})\Gamma^r(k;q_1,\dots,q_k)$,
(c) $\Gamma^{p_{r-1}+s}(j;i_1,\dots,i_j)=\Gamma^s(j_r;i_{p_{r-1}+1},\dots,i_{p_{r-1}+j_r})\Gamma^r(k;q_1,\dots,q_k)$.

Proof. The above lemma is a repeated application of the simplicial identity $d_i d_j = d_{j-1} d_i$, $i < j$. We sketch below the proof of (a); the proofs of the other cases are similar. The operator $\Gamma^0\Gamma^0$ on the left hand side of (a) is given by two strings of operators as

$$\begin{aligned}\Gamma^0\Gamma^0 \quad = \quad & (d_1 \cdots \check{d}_{p_1} \cdots \check{d}_{p_2} \cdots \check{d}_{p_{k-1}} \cdots d_{p_k - 1}) \\ & (d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1}).\end{aligned}$$

Now that the operator d_1 at the extreme left in

$$d_1 \cdots \check{d}_{p_1} \cdots \check{d}_{p_2} \cdots \check{d}_{p_{k-1}} \cdots d_{p_k - 1}$$

can be brought to the extreme right by successive application of $d_i d_j = d_{j-1} d_i$, $i < j$, yielding

$$d_1 \cdots \check{d}_{p_1 - 1} \cdots \check{d}_{p_2 - 1} \cdots \check{d}_{p_{k-1} - 1} \cdots d_{p_k - 2} d_1.$$

Now, by applying $d_{j-1} d_i = d_i d_j, i < j$, the operator d_1 at the right of the above string can be pushed into the string

$$d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1},$$

to recover the operator d_{i_1}, thus yielding

$$\begin{aligned}\Gamma^0\Gamma^0 = {} & (d_1 \cdots \check{d}_{p_1 - 1} \cdots \check{d}_{p_2 - 1} \cdots \check{d}_{p_{k-1} - 1} \cdots d_{p_k - 2}) \\ & \qquad\qquad (d_1 \cdots d_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1}).\end{aligned}$$

We repeat the above method, each time starting with the operator d_1 at the left of the first string to recover an omitted operator in the second string. After $(p_1 - 1)$ steps, we get

$$\begin{aligned}\Gamma^0\Gamma^0 \quad = \quad & (d_2 \cdots d_{p_2 - p_1} d_{p_2 - (p_1 - 2)} \cdots d_{p_r - p_1} d_{p_r - (p_1 - 2)} \cdots \\ & d_{p_{k-1} - p_1} d_{p_{k-1} - (p_1 - 2)} \cdots d_{p_k - p_1})(d_1 \cdots d_{i_1} \cdots d_{i_1 + i_2} \cdots \\ & \check{d}_{q_1} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1}),\end{aligned}$$

since $q_1 = i_1 + \cdots + i_{p_1}$. Again we apply the above method starting with the operators $d_2, \ldots, d_{p_2 - p_1}$ at the left end of the first string to replace all the omitted operators between d_{q_1+1} and $d_{q_1+q_2-1}$, of the second string. Proceeding this way, all the operators of the first string can be exhausted to yield

$$\begin{aligned}\Gamma^0\Gamma^0 \quad = \quad & d_1 \cdots d_{q_1 - 1} d_{q_1 + 1} \cdots d_{q_1 + q_2 - 1} d_{q_1 + q_2 + 1} \cdots \\ & d_{\sum_{s=1}^{r} q_s - 1} d_{\sum_{s=1}^{r} q_s + 1} \cdots d_{\sum_{s=1}^{k-1} q_s - 1} d_{\sum_{s=1}^{k-1} q_s + 1} \cdots d_{i-1}.\end{aligned}$$

Observe that $\sum_{s=1}^{k} q_s = i$.

But this is the operator Γ^0 of the right hand side of the equality (a). This proves part (a). □

Proof of the theorem. For each $j \geq 0$, set

$$\mathcal{C}(j) = CY^j(D, D) = \operatorname{Hom}_K(K[Y_j] \otimes D^{\otimes j}, D).$$

Note that

$$\begin{aligned}\mathcal{C}(1) \quad & = \quad \operatorname{Hom}_K(K[Y_1] \otimes D, D) \\ & \cong \quad \operatorname{Hom}_K(D, D).\end{aligned}$$

Define the unit map $\eta : K \longrightarrow \mathcal{C}(1)$ by $\eta(1) = id_D$. Now, for $k \geq 1, j_r \geq 0$ and $j = \sum j_r$ we define multilinear maps

$$\gamma : CY^k(D,D) \otimes \bigotimes_{r=1}^{k} CY^{j_r}(D,D) \longrightarrow CY^j(D,D) \tag{4.1}$$

as follows: For $f \in CY^k(D,D), g_r \in CY^{j_r}(D,D)$

$$\begin{aligned} & \gamma(f; g_1, \ldots, g_k)(y; x_1 \ldots, x_j) \\ = \ & f(\Gamma^0(y); g_1(\Gamma^1(y); x_1, \ldots, x_{j_1}), g_2(\Gamma^2(y); x_{j_1+1}, \ldots, x_{j_1+j_2}), \ldots, \\ & g_k(\Gamma^k(y); x_{\sum_{s=1}^{k-1} j_s+1}, \ldots, x_{\sum_{s=1}^{k} j_s})) \\ = \ & f(\Gamma^0(y); g_1(\Gamma^1(y); x_1, \ldots, x_{p_1}), g_2(\Gamma^2(y); x_{p_1+1}, \ldots, x_{p_2}), \ldots, \\ & g_k(\Gamma^k(y); x_{p_{k-1}+1}, \ldots, x_{p_k})), \end{aligned}$$

where $\Gamma^0 = \Gamma^0(k; j_1, \ldots, j_k) : Y_j \longrightarrow Y_k$, and $\Gamma^r = \Gamma^r(k; j_1, \ldots, j_k) : Y_j \longrightarrow Y_{j_r}$ are the maps as defined in Definition 4.2, $x_1, \ldots, x_j \in D$ and $y \in Y_j$.

Note that if $j_r = 0$ for some r, then $g_r \in CY^0(D,D) \cong \mathrm{Hom}_K(K,D) = D$, and the corresponding input in f is simply g_r.

To check associativity, let $f \in CY^k(D,D), g_r \in CY^{j_r}(D,D), r = 1, \ldots, k$, and $h_t \in CY^{i_t}(D,D), t = 1, \ldots, j = \sum_{r=1}^{k} j_r$. As in the above lemma, let $i = \sum_{t=1}^{j} i_t, p_s = j_1 + j_2 + \cdots + j_s, q_s = i_{p_{s-1}+1} + \cdots + i_{p_s}$. Also set $q_{(r,s)} = i_{p_{r-1}+1} + i_{p_{r-1}+2} + \cdots + i_{p_{r-1}+s}$, $1 \leq s \leq j_r$. Then

$$\gamma \circ (\gamma \otimes id)((f; g_1, \ldots, g_k), h_1, h_2, \ldots, h_j) = \gamma(\gamma(f; g_1, \ldots, g_k); h_1, \ldots, h_j). \tag{4.2}$$

On the other hand, shuffle yields

$$\begin{aligned} ((f, g_1, \ldots, g_k), h_1, \ldots, h_j) \ \overset{\text{shuffle}}{\longrightarrow} \ & (f, (g_1, h_1, \ldots, h_{j_1}), (g_2, h_{j_1+1}, \ldots, h_{p_2}), \ldots, \\ & (g_k, h_{p_{k-1}+1}, \ldots, h_{p_k=j})). \end{aligned}$$

Now, composing with $\gamma \circ (id \otimes (\otimes_r \gamma))$, we get

$$\begin{aligned} & \gamma \circ (id \otimes (\otimes_r \gamma)) \circ (\text{shuffle})((f, g_1, \ldots, g_k), h_1, \ldots, h_j) \\ = \ & \gamma(f; \gamma(g_1; h_1, \ldots, h_{p_1}), \gamma(g_2; h_{p_1+1}, \ldots, h_{p_2}), \ldots, \\ & \gamma(g_k; h_{p_{k-1}+1}, \ldots, h_{p_k=j})). \end{aligned} \tag{4.3}$$

To show that (4.2) and (4.3) are the same cochain in $CY^i(D,D)$, let $y \in Y_i$ and $x_1, x_2, \ldots, x_i \in D$. Then,

$$\begin{aligned} & \gamma(\gamma(f; g_1, \ldots, g_k); h_1, \ldots, h_j)(y; x_1, \ldots, x_i) \\ = \ & \gamma(f; g_1, \ldots, g_k)(\Gamma^0 y; h_1(\Gamma^1 y; x_1, \ldots, x_{i_1}), h_2(\Gamma^2 y; x_{i_1+1}, \ldots, x_{i_1+i_2}), \\ & \ldots, h_j(\Gamma^j y; x_{\sum_{t=1}^{j-1} i_t+1}, \ldots, x_i)), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \Gamma^0 y &= \Gamma^0(j; i_1, \ldots, i_j) y = d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1} y, \\ \Gamma^u y &= \Gamma^u(j; i_1, \ldots, i_j) y = d_0 \cdots d_{\sum_{t=1}^{u-1} i_t} d_{\sum_{t=1}^{u} i_t+1} \cdots d_i y, \ 1 \leq u \leq j. \end{aligned}$$

Now by definition of γ, as given in (4.1), the equation (4.4) is

$$\begin{aligned} = \ & f(\Gamma^0 \Gamma^0 y; g_1(\Gamma^1 \Gamma^0 y; h_1(\Gamma^1 y; x_1, \ldots, x_{i_1}), \ldots, \\ & h_{j_1}(\Gamma^{j_1} y; x_{\sum_{t=1}^{j_1-1} i_t+1}, \ldots, x_{\sum_{t=1}^{j_1} i_t=q_1})), \ldots, \\ & g_k(\Gamma^k \Gamma^0 y; h_{p_{k-1}+1}(\Gamma^{p_{k-1}+1} y; x_{\sum_{t=1}^{p_{k-1}} i_t+1}, \ldots, x_{\sum_{t=1}^{p_{k-1}+1} i_t}), \ldots, \\ & h_j(\Gamma^j y; x_{\sum_{t=1}^{j-1} i_t+1}, \ldots, x_i))) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned}\Gamma^0\Gamma^0 y &= \Gamma^0(k; j_1, \ldots, j_k)\Gamma^0(j; i_1, \ldots, i_j)y\\ &= d_1 \cdots \check{d}_{p_1} \cdots \check{d}_{p_2} \cdots \check{d}_{p_{k-1}} \cdots d_{p_k - 1} d_1\\ &\quad \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1} y\end{aligned}$$

and for $1 \le r \le k$

$$\begin{aligned}\Gamma^r\Gamma^0 y &= \Gamma^r(k; j_1, \ldots, j_k)\Gamma^0(j; i_1, \ldots, i_j)y\\ &= d_0 \cdots d_{p_{r-1}-1} d_{p_r+1} \cdots d_{p_k} d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1} y.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\gamma(f; \gamma(g_1; h_1, \ldots, h_{p_1}), \ldots, \gamma(g_k; h_{p_{k-1}+1}, \ldots, h_{p_k=j}))(y; x_1, \ldots, x_i)\\ = \;& f(\Gamma^0 y; \gamma(g_1; h_1, \ldots, h_{p_1})(\Gamma^1 y; x_1, \ldots, x_{q_1}), \ldots,\\ & \gamma(g_k; h_{p_{k-1}+1}, \ldots, h_{p_k=j})(\Gamma^k y; x_{\sum_{s=1}^{k-1} q_s+1}, \ldots, x_{\sum_{s=1}^{k} q_s=i})),\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}\Gamma^0 y &= \Gamma^0(k; q_1, \ldots, q_k)y\\ &= d_1 \cdots \check{d}_{q_1} \cdots \check{d}_{q_1+q_2} \cdots \check{d}_{\sum_{s=1}^{k-1} q_s} \cdots d_{\sum_{s=1}^{k} q_s - 1} y\end{aligned}$$

and, for $1 \le r \le k$,

$$\begin{aligned}\Gamma^r y &= \Gamma^r(k; q_1, \ldots, q_k)y\\ &= d_0 \cdots d_{\sum_{s=1}^{r-1} q_s - 1} d_{\sum_{s=1}^{r} q_s + 1} \cdots d_{\sum_{s=1}^{k} q_s = i} y.\end{aligned}$$

By definition of γ, (4.6) can further be written as

$$\begin{aligned} = \;& f(\Gamma^0 y; g_1(\Gamma^0\Gamma^1 y; h_1(\Gamma^1\Gamma^1 y; x_1, \ldots, x_{i_1}), \ldots,\\ & h_{j_1}(\Gamma^{j_1}\Gamma^1 y; x_{\sum_{t=1}^{j_1-1} i_t+1}, \ldots, x_{q_1})), \ldots,\\ & g_k(\Gamma^0\Gamma^k y; h_{p_{k-1}+1}(\Gamma^1\Gamma^k y; x_{\sum_{s=1}^{k-1} q_s+1}, \ldots, x_{\sum_{t=1}^{p_{k-1}+1} i_t}), \ldots,\\ & h_j(\Gamma^{j_k}\Gamma^k y; x_{\sum_{t=1}^{j-1} i_t+1}, \ldots, x_i))),\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}\Gamma^0\Gamma^r y &= \Gamma^0(j_r; i_{p_{r-1}+1}, \ldots, i_{p_{r-1}+j_r})\Gamma^r(k; q_1, \ldots, q_k)y\\ &= (d_1 \cdots \check{d}_{q(r,1)} \cdots \check{d}_{q(r,2)} \cdots \check{d}_{q(r,j_r-1)} \cdots d_{q_r-1})\\ &\quad (d_0 \cdots d_{\sum_{s=1}^{r-1} q_s - 1} d_{\sum_{s=1}^{r} q_s+1} \cdots d_{\sum_{s=1}^{k} q_s=i})y\end{aligned}$$

and

$$\begin{aligned}\Gamma^s\Gamma^r y &= \Gamma^s(j_r; i_{p_{r-1}+1}, \ldots, i_{p_{r-1}+j_r})\Gamma^r(k; q_1, \ldots, q_k)y\\ &= (d_0 \cdots d_{q(r,s-1)-1} d_{q(r,s)+1} \cdots d_{q_r})\\ &\quad (d_0 \cdots d_{\sum_{s=1}^{r-1} q_s - 1} d_{\sum_{s=1}^{r} q_s+1} \cdots d_{\sum_{s=1}^{k} q_s=i})y\end{aligned}$$

for $1 \le s \le j_r$ and $1 \le r \le k$.

Comparing (4.5) and (4.7), and using Lemma 4.5, it follows that the cochains in (4.2) and (4.3) are the same.

To check commutativity of unit diagrams, let $f \in \mathcal{C}(k) = CY^k(D, D)$ and $\alpha_1, \ldots, \alpha_k \in K$. Then,

$$\gamma \circ (\mathrm{id} \otimes \eta^k)(f \otimes (\alpha_1, \ldots, \alpha_k)) = \gamma(f; \alpha_1, \ldots, \alpha_k),$$

where we identify $\alpha_i \in K$ with the map

$$\begin{aligned}\alpha_i \quad : \quad & K[Y_1] \otimes D \longrightarrow D,\\ & (y; a) \mapsto \alpha_i a,\end{aligned}$$

for all $i = 1, 2, \ldots, k$. If ϕ denotes the isomorphism

$$\mathcal{C}(k) \otimes K^k \cong \mathcal{C}(k),$$

then

$$\phi(f \otimes (\alpha_1, \dots, \alpha_k))(y; x_1, \dots, x_k) = f(y; \alpha_1 x_1, \dots, \alpha_k x_k).$$

Now,

$$\gamma(f; \alpha_1, \dots, \alpha_k)(y; x_1, \dots, x_k) = f(\Gamma^0 y; \alpha_1(\Gamma^1 y; x_1), \dots, \alpha_k(\Gamma^k y; x_k)),$$

where $\Gamma^0 y = y$, as $\Gamma^0 = \Gamma^0(k; 1, \dots, 1)$ and $\Gamma^r y = d_0 \cdots d_{r-2} d_{r+1} \cdots d_k y, 1 \leq r \leq k$.

Therefore,

$$\gamma(f; \alpha_1, \dots, \alpha_k)(y; x_1, \dots, x_k) = f(y; \alpha_1 x_1, \dots, \alpha_k x_k).$$

Hence,

$$\gamma \circ (\mathrm{id} \otimes \eta^k)(f \otimes (\alpha_1, \dots, \alpha_k)) = \phi(f \otimes (\alpha_1, \dots, \alpha_k)).$$

Also, for $f \in \mathcal{C}(j)$ and $\alpha \in K$,

$$\gamma(\eta \otimes \mathrm{id})(\alpha \otimes f) = \gamma(\alpha; f),$$

where α is regarded as an element of $\mathcal{C}(1)$ as above.

Now,

$$\gamma(\alpha; f)(y; x_1, \dots, x_j) = \alpha(\Gamma^0 y; f(\Gamma^1 y; x_1, \dots, x_j)),$$

where $\Gamma^0 y = \Gamma^0(1; j) y = d_1 \dots d_{j-1} y$ and $\Gamma^1 y = \Gamma^1(1; j) y = y$. Thus

$$\begin{aligned} \gamma(\alpha; f)(y; x_1, \dots, x_j) &= \alpha(y'; f(y; x_1, \dots, x_j)) \\ &= \alpha f(y; x_1, \dots, x_j), \end{aligned}$$

where y' is the only tree in Y_1.

Note that $\psi : K \otimes \mathcal{C}(j) \xrightarrow{\cong} \mathcal{C}(j)$ is given by

$$\psi(\alpha \otimes f)(y; x_1, \dots, x_j) = \alpha f(y; x_1, \dots, x_j).$$

This completes the proof of the theorem. □

5. Braces induced by the operad structure

We recall from [3] that if $\mathcal{C}(j), j \geq 0$, is a (non-Σ) operad with multiplication map γ, then the graded vector space $\mathcal{C} = \bigoplus \mathcal{C}(j)$ admits a brace algebra structure. For $\mathcal{C}(j) = CY^j(D, D)$, the brace algebra structure is given by

$$f\{g_1, \dots, g_n\} = \sum (-1)^\epsilon \gamma(f; \mathrm{id}_D, \dots, \mathrm{id}_D, g_1, \mathrm{id}_D, \dots, \mathrm{id}_D, g_n, \mathrm{id}_D, \dots, \mathrm{id}_D)$$

where the summation runs over all possible substitutions of $g_1, \dots, g_n$ into f in the prescribed order, and $\epsilon = \sum_{p=1}^n |g_p| i_p$, i_p being the total number of variables one has to input in front of g_p. Here id_D represents $\eta(1)$. The brace identity is a consequence of the commutativity of associative and unit diagrams. Therefore, in view of Theorem 4.4, we see that $CY^*(D, D)$ admits a brace algebra structure. The following lemma now shows that the braces as introduced in Definition 3.4 make the dialgebra cochain complex into a brace algebra.

Lemma 5.1. *The braces on $CY^*(D, D)$ induced by the operad structure coincide with the braces as introduced in Definition 3.4.*

Proof. Let $f \in \mathcal{C}(k) = CY^k(D, D)$ and $g_i \in \mathcal{C}(m_i) = CY^{m_i}(D, D), 1 \leq i \leq n$.

Then, according to M. Gerstenhaber and A. A. Voronov [3], the brace induced by the multilinear maps γ is given by

$$f\{g_1, \dots, g_n\} = \sum (-1)^\epsilon \gamma(f; \mathrm{id}, \dots, \mathrm{id}, g_1, \mathrm{id}, \dots, \mathrm{id}, g_n, \mathrm{id}, \dots, \mathrm{id}),$$

where $\mathrm{id} = \mathrm{id}_D = \eta(1)$ and the summation is over all possible substitutions of $g_1, \ldots, g_n$ into f, in the given order, and $\epsilon = \sum_{p=1}^{n} |g_p| i_p$, i_p being the total number of inputs in front of g_p.

Observe that in the term

$$(-1)^{\epsilon}\gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})$$

of the above summation, the total number of identity entries in γ is $k-n$, the total number of identity entries in front of g_1 is i_1 and the total number of identity entries in front of g_r is $i_r - \sum_{t=1}^{r-1} m_t$, $2 \le r \le n$. Moreover, the following inequalities hold:

$$0 \le i_1, i_1 + m_1 \le i_2, \ldots, i_{r-1} + m_{r-1} \le i_r, i_n + m_n \le k + \sum_{t=1}^{n} m_t - n = N \text{ (say)}.$$

By definition of γ as given in (4.1), we have, for $y \in Y_N$,

$$\begin{aligned} & \gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})(y; x_1, \ldots, x_N) \\ = \; & f(\Gamma^0 y; x_1, \ldots, x_{i_1}, g_1(\Gamma^{i_1+1} y; x_{i_1+1}, \ldots, x_{i_1+m_1}), x_{i_1+m_1+1}, \ldots, x_{i_2}, \\ & g_2(\Gamma^{i_2-m_1+2} y; x_{i_2+1}, \ldots, x_{i_2+m_2}), x_{i_2+m_2+1}, \ldots, x_{i_n}, \\ & g_n(\Gamma^{i_n - \sum_{t=1}^{n-1} m_t + n} y; x_{i_n+1}, \ldots, x_{i_n+m_n}), \ldots, x_N), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \Gamma^p = \; & \Gamma^p(k; \underbrace{1, \ldots, 1}_{i_1}, m_1, \underbrace{1, \ldots, 1}_{i_2-m_1-i_1}, m_2, \ldots, m_{r-1}, \underbrace{1, \ldots, 1}_{i_r-m_{r-1}-i_{r-1}}, \\ & m_r, 1, \ldots, m_n, \underbrace{1, \ldots, 1}_{N-m_n-i_n}) \end{aligned}$$

for $0 \le p \le k$. Note that in the definition of γ as given in (4.1), the map Γ^r yields the only tree in Y_1 when operated on y if $j_r = 1$ by Definition 4.2. In other words, Γ^r is the obvious constant map. For instance, by Definition 4.2, the map Γ^{i_1+2} appearing in (5.1) is given by

$$\begin{aligned} \Gamma^{i_1+2} & = d_0 \cdots d_{(i_1+m_1+1)-1} d_{(i_1+m_1+2)+1} \cdots d_N \\ & = d_0 \cdots \check{d}_{i_1+m_1+1} \check{d}_{i_1+m_1+2} \cdots d_N \end{aligned}$$

and consists of $N-1$ face maps d_i; hence $\Gamma^{i_1+2} y = y'$, where y' is the only tree in Y_1. Hence the corresponding input $\mathrm{id}(y'; x_i)$ in γ is simply x_i.

Now according to Definition 4.2, we have

$$\begin{aligned} \Gamma^0 & = \check{d}_1 \cdots \check{d}_{i_1} d_{i_1+1} \cdots d_{i_1+m_1-1} \check{d}_{i_1+m_1} \cdots \check{d}_{i_2} d_{i_2+1} \cdots d_{i_2+m_2-1} \check{d}_{i_2+m_2} \\ & \quad \cdots \check{d}_{i_3} \cdots \check{d}_{i_r+m_r} \cdots \check{d}_{i_r+1} \cdots \check{d}_{i_n+m_n} \cdots \check{d}_N \\ & = d_{i_1+1} \cdots d_{i_1+m_1-1} d_{i_2+1} \cdots d_{i_2+m_2-1} \cdots d_{i_r+1} \\ & \quad \cdots d_{i_r+m_r-1} \cdots d_{i_n+1} \cdots d_{i_n+m_n-1} \\ & = R_1^{i_1, \ldots, i_n}, \text{ as introduced in Definition 3.2.} \end{aligned}$$

Also the operator $\Gamma^{i_r - \sum_{t=1}^{r-1} m_t + r}$, corresponding to g_r, is given by

$$\begin{aligned} & \Gamma^{i_r - \sum_{t=1}^{r-1} m_t + r} \\ & \quad = d_0 \cdots d_{(i_r - \sum_{t=1}^{r-1} m_t) + \sum_{t=1}^{r-1} m_t - 1} d_{(i_r - \sum_{t=1}^{r-1} m_t) + \sum_{t=1}^{r-1} m_t + m_r + 1} \cdots d_N \end{aligned}$$

Recall that the number of identity entries in front of g_r is $i_r - \sum_{t=1}^{r-1} m_t$ and their degrees sum up to $i_r - \sum_{t=1}^{r-1} m_t$, while the sum of the degrees of $g_1, \ldots, g_{r-1}$

is $\sum_{t=1}^{r-1} m_t$. Thus,

$$\begin{aligned}\Gamma^{i_r-\sum_{t=1}^{r-1} m_t+r} &= d_0 \cdots d_{i_r-1} d_{i_r+m_r+1} \cdots d_N \\ &= R_{r+1}^{i_1,\ldots,i_n}, \text{ as introduced in Definition 3.2.}\end{aligned}$$

It follows that the N-cochain

$$\gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})$$

is the same as $f \circ_{i_1,\ldots,i_n} (g_1, \ldots, g_n)$. This sets up a sign-preserving bijective correspondence between the terms of the summation

$$\sum (-1)^{\epsilon} \gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id}),$$

where the summation is over all possible substitutions of $g_1, \ldots, g_n$ into f, in the given order, $\epsilon = \sum_{p=1}^{n} |g_p| i_p$, i_p being the total number of inputs in front of g_p, and the terms of the summation

$$\sum (-1)^{\eta} f \circ_{i_1,\ldots i_n} (g_1, \ldots, g_n),$$

where the summation is over all $i_1, \ldots, i_n$ such that $0 \leq i_1, i_1 + m_1 \leq i_2, \ldots, i_{n-1} + m_{n-1} \leq i_n, i_n + m_n \leq k + \sum_{i=1}^{n} m_i - n$ and $\eta = \sum_{p=1}^{n} |g_p| i_p$.

Thus the braces as defined in section 3 are precisely the braces induced by the (non-Σ) operad structure. □

6. G-ALGEBRA STRUCTURE ON COHOMOLOGY

In this final section we show that the dialgebra cohomology $HY^*(D, D)$ of a dialgebra D has a G-algebra structure which is induced from a homotopy G-algebra structure on the dialgebra cochain complex $CY^*(D, D)$ with the differential altered by a sign.

Let us first recall the following definitions from [3].

Definition 6.1. A homotopy G-algebra is a brace algebra $V = \bigoplus_n V^n$ provided with a differential d of degree one and a dot product $x \cdot y$ of degree zero making V into a differentially graded associative algebra. The dot product must satisfy the following compatibility identities:

$$(x_1 \cdot x_2)\{y_1, \ldots, y_n\} = \sum_{k=0}^{n} (-1)^{\epsilon} x_1\{y_1, \ldots, y_k\} \cdot x_2\{y_{k+1}, \ldots, y_n\}, \tag{6.1}$$

where $\epsilon = (|x_2| + 1) \sum_{p=1}^{k} |y_p|$, and

$$\begin{aligned} & d(x\{x_1, \ldots, x_{n+1}\}) - (dx)\{x_1, \ldots, x_{n+1}\} \\ & -(-1)^{|x|} \textstyle\sum_{i=1}^{n+1} (-1)^{|x_1|+\cdots+|x_{i-1}|} x\{x_1, \ldots, dx_i, \ldots x_{n+1}\} \\ = \; & (-1)^{(|x|+1)|x_1|} x_1 \cdot x\{x_2, \ldots, x_{n+1}\} \\ & -(-1)^{|x|} \textstyle\sum_{i=1}^{n} (-1)^{|x_1|+\cdots+|x_i|} x\{x_1, \ldots, x_i \cdot x_{i+1}, \ldots x_{n+1}\} \\ & +(-1)^{|x|+|x_1|+\cdots|x_n|} x\{x_1, \ldots, x_n\} \cdot x_{n+1}. \end{aligned} \tag{6.2}$$

Remark 6.2. It should be mentioned here that the notion of homotopy G-algebras as defined above is different from the notion of strong homotopy G-algebras ($\mathcal{G}_\infty$-algebras, for short) as considered in [4]. A $\mathcal{G}_\infty$-algebra is an algebra over the minimal model of the Koszul operad describing G-algebras. However, the notion of homotopy G-algebras that we are considering do not really fit the general scheme of quadratic operad theory [5].

Definition 6.3. A multiplication on an operad $\mathcal{C}$ of vector spaces is an element $m \in \mathcal{C}(2)$ such that $m \circ m = 0$, where $m \circ m := m\{m\}$ and $\{\ \}$ denote the associated braces.

The following lemma shows that the operad $CY^*(D,D)$ is equipped with a multiplication.

Lemma 6.4. *The* 2*-cochain* $\pi \in CY^2(D,D)$ *defined by*

$$\left\{ \begin{array}{rcl} \pi([21];a,b) & = & a \dashv b, \\ \pi([12];a,b) & = & a \vdash b \end{array} \right. \tag{6.3}$$

for all $a, b \in D$ *is a multiplication on the operad* $CY^*(D,D)$.

Proof. By Remark 3.5, we only need to verify that $\pi \circ \pi = 0$. Now, by definition of pre-Lie product as introduced in [7], we have, for $y \in Y_3$ and $a, b, c \in D$,

$$\pi \circ \pi(y;a,b,c) = (\pi \circ_0 \pi - \pi \circ_1 \pi)(y;a,b,c).$$

The proof now follows from the dialgebra axioms. □

In order to show that the dialgebra cochain complex $CY^*(D,D)$ admits a homotopy G-algebra structure, we shall make use of Proposition 2(3) from [3], which we describe below. Let $\mathcal{C}$ denote an operad, m a multiplication on $\mathcal{C}$, and let $m \circ x$ denote $m\{x\}$.

Proposition 6.5. *The product*

$$x \cdot y := (-1)^{|x|+1} m\{x,y\}$$

of degree 0 *and the differential*

$$dx = m \circ x - (-1)^{|x|} x \circ m, \ d^2 = 0, \ \deg d = 1,$$

define the structure of a differential graded (DG) associative algebra on $\mathcal{C}$.

First, we observe the following two facts.

Remark 6.6. Note that by Lemma 6.12 of [7], the coboundary operator

$$\delta : CY^n(D,D) \longrightarrow CY^{n+1}(D,D)$$

can be expressed in the form

$$\delta f = (-1)^{|f|}(\pi \circ f - (-1)^{|f|} f \circ \pi) = (-1)^{|f|} df. \tag{6.4}$$

Remark 6.7. The $*$ product, as introduced in Definition 6.8 of [7], can be expressed in terms of braces as

$$f * g = (-1)^{(|f|+1)(|g|)} \pi\{f,g\}. \tag{6.5}$$

This is because, by the definition of braces on $CY^*(D,D)$,

$$\begin{array}{rcl} \pi\{f,g\}(y;x_1,\dots,x_{p+q}) & = & (-1)^{p(q-1)}\pi\circ_{0,p}(f,g)(y;x_1,\dots,x_{p+q}) \\ & = & (-1)^{p(q-1)}\pi(R_1^{0,p}(p+q;2,p,q)y; \\ & & f(R_2^{0,p}(p+q;2,p,q)y;x_1,\dots,x_p), \\ & & g(R_3^{0,p}(p+q;2,p,q)y;x_{p+1},\dots,x_{p+q})) \\ & = & (-1)^{p(q-1)}\pi(d_1\cdots d_{p-1}d_{p+1}\cdots d_{p+q-1}(y); \\ & & f(d_{p+1}\cdots d_{p+q}(y);x_1,\dots x_p), \\ & & g(d_0\cdots d_{p-1}(y);x_{p+1},\dots,x_{p+q})) \\ & = & (-1)^{p(q-1)}\pi(d_1\cdots d_{p-1}d_{p+1}\cdots d_{p+q-1}(y); \\ & & f(d_{p+1}d_{p+1}\cdots d_{p+q-1}(y);x_1,\dots x_p), \\ & & g(d_0\cdots d_{p-1}(y);x_{p+1},\dots,x_{p+q})) \\ & = & (-1)^{p(q-1)}\pi(R_1^0(p+1;2,p)R_1^p(p+q;p+1,q)(y); \\ & & f(R_2^0(p+1;2,p)R_1^p(p+q;p+1,q)(y);x_1,\dots,x_p), \\ & & g(R_2^p(p+q;p+1,q)((y);x_{p+1},\dots,x_{p+q})) \\ & = & (-1)^{p(q-1)}f*g(y;x_1,\dots,x_{p+q}). \end{array}$$

Here we make use of the fact that the operator d_{p+q} in the string of operators $d_{p+1}\cdots d_{p+q}$ can be moved to the extreme left of the same string using $d_id_j = d_{j-1}d_i, i<j$, to yield $d_{p+1}d_{p+1}\cdots d_{p+q-1}$.

Therefore by equation (6.5), the dot product $f\cdot g$ determined by the multiplication m as in Proposition 6.5 is in this case nothing but the $*$ product, up to the sign $(-1)^{(|f|+1)(|g|+1)}$. Moreover, the differential d determined by m as in Proposition 6.5 is merely the coboundary operator δ, up to the sign $(-1)^{(|f|)}$; that is, $df = (-1)^{(|f|)}\delta(f)$.

Consequently, by Proposition 6.5 and Theorem 4.4, we deduce the following corollary.

Corollary 6.8. *The graded cochain module $CY^*(D,D)$ equipped with the $*$ product $f*g$, as introduced in* [7]*, altered by the sign $(-1)^{(|f|+1)(|g|+1)}$ and the coboundary $df=(-1)^{|f|}\delta f$, is a differential graded associative algebra.*

Next we recall Theorem 3 of [3].

Theorem 6.9. *A multiplication on an operad $\mathcal{C}$ defines the structure of a homotopy G-algebra on $\bigoplus_k \mathcal{C}(k)$. A multiplication on a brace algebra is equivalent to the structure of a homotopy G-algebra on it.*

Thus in view of Theorem 4.4, Theorem 6.9 and Lemma 4.5 we have the following corollary.

Corollary 6.10. *The cochain complex $(CY^*(D,D),d)$, where $df=(-1)^{|f|}\delta f$, is a homotopy G-algebra with the dot product $f\cdot g = (-1)^{(|f|+1)(|g|+1)}f*g$.*

As a consequence, we have the following corollary.

Corollary 6.11. *The cochain complex $(CY^*(D,D),d)$ is a differential graded Lie algebra with respect to the commutator $[x,y]=x\circ y-(-1)^{|x||y|}y\circ x$.*

Proof. The brace identity, for $m=n=1$, implies that

$$x\{x_1\}\{y_1\} = x\{x_1,y_1\} + x\{x_1\{y_1\}\} + (-1)^{|x_1||y_1|}x\{y_1,x_1\},$$

as $0\le i_1\le j_1\le 1$.

Using Remark 3.5, we deduce from above that

$$(x \circ x_1) \circ y_1 - x \circ (x_1 \circ y_1) = x\{x_1, y_1\} + (-1)^{|x_1||y_1|} x\{y_1, x_1\}. \tag{6.6}$$

A straightforward computation using equation (6.6) and the fact that $|x \circ y| = |x| + |y|$ shows that the commutator satisfies the graded Jacobi identity.

Moreover, the dot product is always *homotopy* graded commutative; that is,

$$x \cdot y - (-1)^{(|x|+1)(|y|+1)} y \cdot x = (-1)^{|x|}(d(x \circ y) - dx \circ y - (-1)^{|x|} x \circ dy). \tag{6.7}$$

This follows directly from equation (6.2), as

$$\begin{aligned} &(-1)^{|x|}(d(x \circ y) - dx \circ y - (-1)^{|x|} x \circ dy) \\ &= (-1)^{|x|}((-1)^{(|x|+1)|y|} y \cdot x + (-1)^{|x|} x \cdot y) \\ &= x \cdot y - (-1)^{(|x|+1)(|y|+1)} y \cdot x. \end{aligned}$$

Also, the differential is a derivation of the bracket. In other words,

$$d[x, y] - [dx, y] - (-1)^{|x|}[x, dy] = 0,$$

which is a direct consequence of the *homotopy* graded commutativity of the dot product. This shows that every homotopy G-algebra is a differential graded Lie algebra with respect to the commutator $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$. □

Next we recall the following definition from [3].

Definition 6.12. A G-algebra is a graded vector space H with a dot prduct $x \cdot y$ defining the structure of a graded commutative algebra with a bracket $[x, y]$ of degree -1 defining the structure of a graded Lie algebra such that the bracket with an element is a derivation of the dot product:

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{|x|(|y|+1)} y \cdot [x, z].$$

Corollary 6.13. *The $*$ product $x * y$, altered by the sign $(-1)^{(|x|+1)(|y|+1)}$ and the bracket $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$, defines the structure of a G-algebra on the dialgebra cohomology $HY^*(D, D)$ of a dialgebra D.*

Proof. First observe that

$$HY^n(D, D) = H^n((CY^*(D, D), \delta)) = H^n((CY^*(D, D), d)).$$

The fact that the dot product $x \cdot y = (-1)^{|x|+1} \pi\{x, y\}$ lifts to the cohomology follows from Proposition 6.5. Equation (6.7) implies that this dot product is graded commutative. Moreover, by Corollary 6.11, the bracket $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$ of degree -1 defines the structure of a graded Lie algebra on $HY^*(D, D)$. It remains to show that the bracket with an element is a derivation of the dot product.

First we show that the commutator $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$ for all $x, y \in CY^*(D, D)$ is a graded derivation of the dot product up to null homotopy; that is,

$$\begin{aligned} &[x, y \cdot z] - [x, y] \cdot z - (-1)^{|x|(|y|+1)} y \cdot [x, z] \\ =&(-1)^{|x|+|y|+1}(d(x\{y, z\}) - (dx)\{y, z\} - (-1)^{|x|} x\{dy, z\} - (-1)^{|x|+|y|} x\{y, dz\}). \end{aligned}$$

By definition of the commutator, we have

$$\begin{aligned}
&[x, y\cdot z] - [x,y]\cdot z - (-1)^{|x|(|y|+1)} y\cdot [x,z]\\
&= x\circ (y\cdot z) - (-1)^{|x||y\cdot z|}(y\cdot z)\circ x - (x\circ y - (-1)^{|x||y|} y\circ x)\cdot z\\
&\quad - (-1)^{|x|(|y|+1)} y\cdot (x\circ z - (-1)^{|x||z|} z\circ x)\\
&= (x\circ (y\cdot z) - (-1)^{|x|(|y|+1)} y\cdot (x\circ z) - (x\circ y)\cdot z)\\
&\quad - (-1)^{|x||y\cdot z|}(y\cdot z)\circ x - (-1)^{|x||y|}(y\circ x)\cdot z + (-1)^{|x|(|y|+|z|+1)} y\cdot (z\circ x)\\
&= (x\circ y.z - (-1)^{|x|(|y|+1)} y\cdot (x\circ z) - (x\circ y)\cdot z)\\
&\quad - (-1)^{|x|(|y|+|z|+1)}((y\cdot z)\circ x - y\cdot (z\circ x) + (-1)^{|x|(|z|+1)} y\circ x.z)\\
&= x\circ y.z - (-1)^{|x|(|y|+1)} y\cdot (x\circ z) - (x\circ y)\cdot z
\end{aligned}$$

(as $((y\cdot z)\circ x - y\cdot (z\circ x) + (-1)^{|x|(|z|+1)} y\circ x.z) = 0$, by equation (6.1))

$$\begin{aligned}
&= (-1)^{|x|+|y|+1}(d(x\{y,z\}) - (dx)\{y,z\} - (-1)^{|x|} x\{dy,z\}\\
&\qquad\qquad - (-1)^{|x|+|y|} x\{y,dz\})
\end{aligned}$$

by equation (6.2). This implies that $[x, y\cdot z] = [x,y]\cdot z + (-1)^{|x|(|y|+1)} y\cdot [x,z]$ for all $x,y,z \in HY^*(D,D)$. Thus $HY^*(D,D)$ admits a G-algebra structure. □

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