DIALGEBRA COHOMOLOGY AS A G-ALGEBRA

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Abstract. It is well known that the Hochschild cohomology $H^*(A,A)$ of an associative algebra $A$ admits a G-algebra structure. In this paper we show that the dialgebra cohomology $HY^*(D,D)$ of an associative dialgebra $D$ has a similar structure, which is induced from a homotopy G-algebra structure on the dialgebra cochain complex $CY^*(D,D)$.

1. Introduction

It is well known, since the pioneering work of M. Gerstenhaber [2], that the Hochschild cochain complex $C^*(A,A)$ of an associative algebra $A$ admits a brace algebra structure. Moreover, in [3], M. Gerstenhaber and A. A. Voronov have shown that $C^*(A,A)$ admits a homotopy G-algebra structure which induces the G-algebra structure on the Hochschild cohomology as introduced in [2]. These structures on $C^*(A,A)$ are in fact induced from a natural operad structure on $C^*(A,A)$, where only the non-$\Sigma$ part of the operad is responsible for inducing the above structures.

The notions of Leibniz algebras and associative dialgebras were introduced in [6], by J.-L. Loday. Leibniz algebras are a non-commutative variation of Lie algebras, and associative dialgebras are a variation of associative algebras. Recall that an associative algebra gives rise to a Lie algebra by $[x,y] = xy - yx$. The notion of associative dialgebra is introduced in order to build an analogue of the couple

$$\text{Lie algebras} \leftrightarrow \text{associative algebras},$$

when Lie algebras are replaced by Leibniz algebras. A cohomology theory associated with dialgebras has been developed by Loday, called dialgebra cohomology, where in the construction of the dialgebra complex which defines the dialgebra cohomology, planar binary trees play a crucial role. Dialgebra cohomology with coefficients has been studied by A. Frabetti in [1]. In [7], it has been shown that the dialgebra complex $CY^*(D,D)$ admits the structure of an associative algebra, and also of a pre-Lie algebra. The aim of this paper is to show that, as in the case of a Hochschild complex, $CY^*(D,D)$ admits a homotopy G-algebra structure which comes from a non-$\Sigma$ operad structure on $CY^*(D,D)$. As a consequence, the dialgebra cohomology $HY^*(D,D)$ becomes a G-algebra.
2. Dialgebra complex

In this section, we recall the construction of a dialgebra complex. Throughout this paper, by dialgebra we mean associative dialgebra.

Definition 2.1. Let \( K \) be a field. A dialgebra \( D \) over \( K \) is a vector space over \( K \) along with two \( K \)-linear maps, \( \div: D \otimes D \rightarrow D \) (called left) and \( \vdash: D \otimes D \rightarrow D \) (called right), satisfying the following axioms:

\[
\begin{cases}
  x \div (y \vdash z) = (x \div y) \div z = x \div (y \vdash z), \\
  (x \div y) \vdash z = x \div (y \vdash z), \\
  (x \vdash y) \vdash z = x \vdash (y \vdash z).
\end{cases}
\]

(2.1)

for all \( x, y, z \in D \).

A planar binary tree with \( n \) vertices (in short, an \( n \)-tree) is a planar tree with \( (n+1) \) leaves, one root and each vertex trivalent. Let \( Y_n \) denote the set of all \( n \)-trees. Let \( Y_0 \) be the singleton set consisting of a root only. The \( n \)-trees for \( 0 \leq n \leq 3 \) are given by the following diagrams:

\[
Y_0 = \{ \ |
, \quad Y_1 = \{ \ _\}
, \quad Y_2 = \{ \ _\ , \_\, \}
, \quad Y_3 = \{ \ _\ , \_\ , \_\ , \_\ , \_\}
\]

For any \( y \in Y_n \), the \((n+1)\) leaves are labelled by \( \{0, 1, \ldots, n\} \) from left to right, and the vertices are labelled \( \{1, 2, \ldots, n\} \), so that the \( i \)th vertex is between the leaves \((i-1)\) and \( i \). Recall from \( 8 \) that the only element \(| \) of \( Y_0 \) is denoted by \( [0] \). The only element of \( Y_1 \) is denoted by \( [1] \). The grafting of a \( p \)-tree \( y_1 \) and a \( q \)-tree \( y_2 \) is a \((p+q+1)\)-tree denoted by \( y_1 \vee y_2 \) which is obtained by joining the roots of \( y_1 \) and \( y_2 \) and creating a new root from that vertex. This is denoted by \([y_1 \ p + q + 1 \ y_2] \) with the convention that all zeros are deleted except for the element in \( Y_0 \). With this notation, the trees pictured above from left to right are \([0], [1], [12], [21], [123], [213], [131], [312], [321] \).

For any \( i, 0 \leq i \leq n \), there is a map, called the face map, \( d_i: Y_n \rightarrow Y_{n-1} \), \( y \mapsto d_i y \), where \( d_i y \) is obtained from \( y \) by deleting the \( i \)th leaf. The face maps satisfy the relation \( d_i d_j = d_{j-1} d_i \), for all \( i < j \).

Let \( D \) be a dialgebra over a field \( K \). The cochain complex \( CY^*(D, D) \) which defines the dialgebra cohomology \( HY^*(D, D) \) is defined as follows. For any \( n \geq 0 \), let \( K[Y_n] \) denote the \( K \)-vector space spanned by \( Y_n \), and let

\[
CY^n(D, D) := \text{Hom}_K(K[Y_n] \otimes D^\otimes n, D)
\]

be the module of \( n \)-cochains of \( D \) with coefficients in \( D \). The coboundary operator \( \delta: CY^n(D, D) \rightarrow CY^{n+1}(D, D) \) is defined as the \( K \)-linear map \( \delta = \sum_{i=0}^{n+1} (-1)^i \delta^i \), where

\[
(\delta^i f)(y; a_1, a_2, \ldots, a_{n+1}) := \begin{cases}
  a_1 \circ_0 f(d_0 y; a_2, \ldots, a_{n+1}), & \text{if } i = 0, \\
  f(d_i y; a_1, \ldots, a_i \circ_i a_{i+1}, \ldots, a_{n+1}), & \text{if } 1 \leq i \leq n, \\
  f(d_{n+1} y; a_1, \ldots, a_n) \circ_{n+1} a_{n+1}, & \text{if } i = n + 1,
\end{cases}
\]

for all \( x, y, z \in D \).
for any \( y \in Y_{n+1} \); \( a_1, \ldots, a_{n+1} \in D \) and \( f : K[Y_n] \otimes D^\otimes n \to D \). Here, for any \( i \), \( 0 \leq i \leq n + 1 \), the maps \( \partial_i : Y_{n+1} \to \{+, |\} \) are defined by

\[
\partial_0(y) = \partial^y_0 := \begin{cases} + & \text{if } y \text{ is of the form } | \vee y_1, \text{ for some } n\text{-tree } y_1, \\ | & \text{otherwise}, \end{cases}
\]

\[
\partial_i(y) = \partial^y_i := \begin{cases} + & \text{if the } i\text{th leaf of } y \text{ is oriented like } \backslash, \\ | & \text{if the } i\text{th leaf of } y \text{ is oriented like } /', \end{cases}
\]

for \( 1 \leq i \leq n \), and

\[
\partial_{n+1}(y) = \partial^y_{n+1} := \begin{cases} | & \text{if } y \text{ is of the form } y_1 \vee |, \text{ for some } n\text{-tree } y_1, \\ + & \text{otherwise}, \end{cases}
\]

where the symbol ‘\( \vee \)’ stands for grafting of trees [6].

### 3. Braces for a Dialgebra Complex

In this section, we introduce braces or multilinear operations in \( CY^*(D, D) \) of a dialgebra \( D \), generalizing the \( \partial_i \) products as introduced in [7], which endow \( CY^*(D, D) \) with a brace algebra structure.

**Definition 3.1.** A brace algebra is a graded vector space with a collection of braces (or multilinear operations) \( x\{x_1, x_2, \ldots, x_n\} \) of degree \(-n\) satisfying the identity (brace identity)

\[
x\{x_1, x_2, \ldots, x_m\}\{y_1, y_2, \ldots, y_n\} = \sum_{0 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq n} \begin{pmatrix} \varepsilon \end{pmatrix} \cdot x\{y_{i_1+1}, \ldots, y_{i_1}\}, y_{j_1+1}, \ldots, y_{i_2}\}, y_{j_2+1}, \ldots, y_{i_3}\}, \ldots, y_{j_m+1}, \ldots, y_{i_m}\}, y_{j_m+1}, \ldots, y_{j_m}\}
\]

\[
x\{x_1, x_2, \ldots, x_n\} = \sum_{i=1}^n \deg x\{x_1, \ldots, x_n\} = \deg x + \sum_{i=1}^n \deg x_i - n, \quad |x| = \deg x - 1, \quad \text{and } \varepsilon = \sum_{p=1}^m |x_p| \sum_{q=1}^{|y_p|} |y_q|.
\]

**Definition 3.2.** Let \( n, i_1, i_2, \ldots, i_r, m_1, m_2, \ldots, m_r \) be non-negative integers with \( n, m_1, \ldots, m_r \geq 1 \) such that

\[
0 \leq i_1, i_1 + m_1 \leq i_2, \ldots, i_{r-1} + m_{r-1} \leq i_r, i_r + m_r \leq N = n + \sum_{i=1}^r m_i - r.
\]

For each \( j, 0 \leq j \leq r \), we define maps

\[
R_{j+1}^{i_1, \ldots, i_r}(N; n, m_1, \ldots, m_r) : Y_N \to Y_{m_j},
\]

with \( m_0 = n \), in the following way. For \( j = 0 \),

\[
R_1^{i_1, \ldots, i_r}(N; n, m_1, \ldots, m_r) = \prod_{1 \leq i \leq r} (d_{i_1+1} \cdots d_{i_1+m_{i-1}}) \text{ if } 2 \leq m_0 < N,
\]

where \( \Pi \) stands for composition of terms and \( R_1^{i_1, \ldots, i_r}(N; n, m_1, \ldots, m_r) \) is the identity or the obvious constant map according to whether \( m_0 \) is \( N \) or \( 1 \).
For $1 \leq j \leq r$, if $2 \leq m_j < N$ we have

$$R^{i_1, \ldots, i_r}_{j+1}(N; n, m_1, \ldots, m_r) = \begin{cases} (d_0 \cdots d_{i_j-1})(d_{i_j+m_j+1} \cdots d_N), & i_j \geq 1 \text{ and } i_j + m_j + 1 \leq N, \\ (d_{m_j+1} \cdots d_N), & i_j = 0, \\ (d_0 \cdots d_{i_j-1}), & i_j + m_j + 1 > N, \end{cases}$$

and $R^{i_1, \ldots, i_r}_{j+1}(N; n, m_1, \ldots, m_r)$ is the identity or the obvious constant map according to whether $m_j = N$ or $m_j = 1$.

**Definition 3.3.** Let $D$ be a dialgebra over a field $K$. For non-negative integers $n, i_1, \ldots, i_r, m_1, \ldots, m_r$ with $0 \leq i_1, i_1 + m_1 \leq i_2, \ldots, i_r-1 + m_{r-1} \leq i_r, i_r + m_r \leq N = n + \sum_{i=1}^{r} m_i - r$, the multilinear maps

$\circ_{i_1, \ldots, i_r} : CY^n(D, D) \otimes \bigotimes_{j=1}^{r} CY^{m_j}(D, D) \rightarrow CY^N(D, D)$

are defined as follows. Let $f \in CY^n(D, D), g_j \in CY^{m_j}(D, D), 1 \leq j \leq r$. For $y \in Y_N$ and $x_1, \ldots, x_N \in D$ we have

$$f \circ_{i_1, \ldots, i_r} (g_1, \ldots, g_r)(y; x_1, \ldots, x_N) = f(R^{i_1, \ldots, i_r}_{j+1}(N; n, m_1, \ldots, m_r)y; x_1, \ldots, x_{i_1},
\quad g_1(R^{i_1, \ldots, i_r}_{j+1}(N; n, m_1, \ldots, m_r)y; x_{i_1+1}, \ldots, x_{i_1+m_1}), \ldots,
\quad g_r(R^{i_1, \ldots, i_r}_{j+1}(N; n, m_1, \ldots, m_r)y; x_{i_r+1}, \ldots, x_{i_r+m_r}), \ldots, x_N).$$

In the above definition, if $m_j = 0$ for some $j$, then $g_j \in CY^0(D, D) \cong \text{Hom}_K(K, D) = D$ and the corresponding input is simply $g_j$.

Next we use these generalized $\circ_i$ products to define braces as follows.

**Definition 3.4.** For $f \in CY^n(D, D), g_\nu \in CY^{m_\nu}(D, D), \nu = 1, \ldots, r,$

$$f\{g_1, \ldots, g_r\} = \sum_{i_1, \ldots, i_r} (-1)^{\eta} f \circ_{i_1, \ldots, i_r} (g_1, \ldots, g_r),$$

where $\eta = \sum_{\nu=1}^{r} |g_\nu|_{i_\nu}$, and $|g_\nu|_{i_\nu} = \deg g_\nu - 1 = m_\nu - 1$.

**Remark 3.5.** It may be noted that by the above definition of braces on $CY^*(D, D)$, $f\{g\}$ coincides with the pre-Lie product $f \circ g$ as introduced in [7].

Henceforth, we shall use the symbol $f \circ g$ in order to denote $f\{g\}$. The following proposition will follow from Lemma 3.1.

**Proposition 3.6.** The braces as defined above make the dialgebra cochain complex $CY^*(D, D)$ into a brace algebra.

4. Operad structure

In this section we show that the dialgebra complex $CY^*(D, D)$ of a dialgebra $D$ admits the structure of a non-$\Sigma$ operad.

**Definition 4.1.** A non-$\Sigma$ operad $\mathcal{C}$ of $K$-vector spaces consists of vector spaces $\mathcal{C}(j), j \geq 0,$ together with a unit map $K \rightarrow \mathcal{C}(1)$ and multilinear maps

$$\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \rightarrow \mathcal{C}(j)$$

for $k \geq 1; j_\nu \geq 0$ and $j = \sum_{\nu=1}^{k} j_\nu$. The maps $\gamma$ are required to be associative and unital as in [8].
The following maps on trees will be used to define a non-$\Sigma$ operad structure on $\text{CY}^*(D,D)$.

**Definition 4.2.** Given an integer $j$, with $j = \sum_{r=1}^{k} j_r$, $k \geq 1$ and $j_r \geq 1$, define maps

$$
\Gamma^0(k; j_1, \ldots, j_k) : Y_j \longrightarrow Y_k,
\Gamma^r(k; j_1, \ldots, j_k) : Y_j \longrightarrow Y_{j_r}, \quad 1 \leq r \leq k,
$$

by

$$
\Gamma^0(k; j_1, \ldots, j_k) = d_1 \cdots d_{j_1-1} d_{j_1+1} \cdots d_{j_1+j_2-1} d_{j_1+j_2+1} \cdots d_{\sum_{s=1}^{r} j_s-1} d_{\sum_{s=1}^{r} j_s+1} \cdots d_{j-1},
$$

and

$$
\Gamma^r(k; j_1, \ldots, j_k) = d_1 \cdots d_{p_1-1} \cdots d_{p_r-1} d_{p_r+1} \cdots d_{j-1},
$$

where $p_r = j_1 + j_2 + \cdots + j_r, 1 \leq r \leq k$, and the symbol $\hat{d}_i$ appearing in any expression means that the map $d_i$ has been omitted.

**Remark 4.3.** Given integers $j$, $k \geq 1$, $j_r \geq 1$ with $j = \sum_{r=1}^{k} j_r$, we shall often write the map $\Gamma^r(k; j_1, \ldots, j_k)$ simply as $\Gamma^r$, for all $r = 0, 1, \ldots, k$. However, to avoid confusion we shall write the maps $\Gamma^r$ explicitly, along with the values of $k$, $j_1, \ldots, j_k$, whenever necessary.

**Theorem 4.4.** For a dialgebra $D$ over a field $K$, the dialgebra complex $\text{CY}^*(D,D)$ is a non-$\Sigma$ operad of $K$-vector spaces.

To prove the above theorem we need the following lemma.

**Lemma 4.5.** Let $j_s \geq 1$, $1 \leq r \leq k$ be integers with $j = \sum_{r=1}^{k} j_r$. Let $i = \sum_{t=1}^{s} i_t$, with integers $i_t \geq 1$. Set $p_s = j_1 + j_2 + \cdots + j_s$ and $q_s = i_{p_s-1+1} + \cdots + i_{p_s}$. Then for $1 \leq s \leq j_r$, $1 \leq r \leq k$ the corresponding maps

$$
\Gamma^0(k; j_1, \ldots, j_k) : Y_j \longrightarrow Y_k,
\Gamma^0(j; i_1, \ldots, i_j) : Y_i \longrightarrow Y_j,
\Gamma^0(k; q_1, \ldots, q_k) : Y_i \longrightarrow Y_k,
$$

satisfy

(a) $\Gamma^0(k; j_1, \ldots, j_k) \Gamma^0(j; i_1, \ldots, i_j) = \Gamma^0(k; q_1, \ldots, q_k),$

(b) $\Gamma^r(k; j_1, \ldots, j_k) \Gamma^0(j; i_1, \ldots, i_j) = \Gamma^0(j; i_{p_r-1+1}, \ldots, i_{p_r-1+j_r}) \Gamma^r(k; q_1, \ldots, q_k),$

(c) $\Gamma^{p_{r-1}+s}(j; i_1, \ldots, i_j) = \Gamma^s(j; i_{p_{r-1}+1}, \ldots, i_{p_{r-1}+j_r}) \Gamma^r(k; q_1, \ldots, q_k).$
Proof. The above lemma is a repeated application of the simplicial identity \(d_i d_j = d_{j-1} d_i, \ i < j\). We sketch below the proof of (a); the proofs of the other cases are similar. The operator \(\Gamma^0 \Gamma^0\) on the left hand side of (a) is given by two strings of operators as

\[
\Gamma^0 \Gamma^0 = (d_1 \cdots d_{p_1} \cdots d_{p_2} \cdots d_{p_{k-1}} \cdots d_{p_{k-1}}) \\
(d_1 \cdots d_{i_1} \cdots d_{i_1+i_2} \cdots d_{\sum_{i=1}^{j-1} i} \cdots d_{i-1}).
\]

Now that the operator \(d_1\) at the extreme left in

\[
d_1 \cdots d_{p_1} \cdots d_{p_{k-1}} \cdots d_{p_k-1}
\]

can be brought to the extreme right by successive application of \(d_i d_j = d_{j-1} d_i, \ i < j\), yielding

\[
d_1 \cdots d_{p_1-1} \cdots d_{p_2-1} \cdots d_{p_{k-1}} \cdots d_{p_k-2} d_1.
\]

Now, by applying \(d_{j-1} d_i = d_i d_j, \ i < j\), the operator \(d_1\) at the right of the above string can be pushed into the string

\[
d_1 \cdots d_{i_1} \cdots d_{i_1+i_2} \cdots d_{\sum_{i=1}^{j-1} i} \cdots d_{i-1},
\]

to recover the operator \(d_{i_1}\), thus yielding

\[
\Gamma^0 \Gamma^0 = (d_1 \cdots d_{p_1-1} \cdots d_{p_2-1} \cdots d_{p_{k-1}} \cdots d_{p_k-2}) \\
(d_1 \cdots d_{i_1} \cdots d_{i_1+i_2} \cdots d_{\sum_{i=1}^{j-1} i} \cdots d_{i-1}).
\]

We repeat the above method, each time starting with the operator \(d_1\) at the left of the first string to recover an omitted operator in the second string. After \((p_1 - 1)\) steps, we get

\[
\Gamma^0 \Gamma^0 = (d_2 \cdots d_{p_2-p_1} d_{p_2-(p_1-2)} \cdots d_{p_{r-p_1}} d_{p_{r-(p_1-2)}} \cdots \\
(d_{p_{k-1}-p_1} d_{p_{k-1}-(p_1-2)} \cdots d_{p_{k-p_1}})(d_1 \cdots d_{i_1} \cdots d_{i_1+i_2} \cdots \\
(d_{q_1} \cdots d_{\sum_{i=1}^{j-1} i} \cdots d_{i-1}),
\]

since \(q_1 = i_1 + \cdots + i_{p_1}\). Again we apply the above method starting with the operators \(d_2, \ldots, d_{p_2-p_1}\) at the left end of the first string to replace all the omitted operators between \(d_{q_1+1}\) and \(d_{q_1+q_2-1}\), of the second string. Proceeding this way, all the operators of the first string can be exhausted to yield

\[
\Gamma^0 \Gamma^0 = d_1 \cdots d_{q_1} d_{q_1+1} \cdots d_{q_1+q_2-1} d_{q_1+q_2+1} \cdots \\
(d_{q_1} \cdots d_{\sum_{i=1}^{j-1} i} q_{s+1}+1 \cdots d_{\sum_{i=1}^{j-1} i} q_{s+1} \cdots d_{\sum_{i=1}^{j-1} i} q_{s+1} \cdots d_{i-1}.
\]

Observe that \(\sum_{s=1}^{k} q_s = i\).

But this is the operator \(\Gamma^0\) of the right hand side of the equality (a). This proves part (a).

Proof of the theorem. For each \(j \geq 0\), set

\[
C(j) = CY^j(D, D) = \text{Hom}_K(K[Y_j] \otimes D^\otimes j, D).
\]

Note that

\[
C(1) = \text{Hom}_K(K[Y_1] \otimes D, D) \cong \text{Hom}_KD(D, D).
\]
Define the unit map \( \eta : K \rightarrow C(1) \) by \( \eta(1) = id_D \). Now, for \( k \geq 1, j_r \geq 0 \) and \( j = \sum j_r \) we define multilinear maps

\[
(4.1) \quad \gamma : CY^k(D, D) \otimes^{\bigotimes} \bigotimes_{r=1}^{k} CY^{j_r}(D, D) \rightarrow CY^j(D, D)
\]
as follows: For \( f \in CY^k(D, D), g_r \in CY^{j_r}(D, D) \)

\[
\begin{align*}
\gamma(f; g_1, \ldots, g_k)(y; x_1, \ldots, x_j) &= f(\Gamma^0(y); g_1(\Gamma^1(y); x_1, \ldots, x_{j_1}), g_2(\Gamma^2(y); x_{j_1+1}, \ldots, x_{j_1+j_2}), \ldots, \\
g_k(\Gamma^k(y); x_{\sum_{l=1}^{k-1} j_l + 1}, \ldots, x_{\sum_{l=1}^{k-1} j_{l-1} + j_l})) = f(\Gamma^0(y); g_1(\Gamma^1(y); x_1, \ldots, x_{p_1}), g_2(\Gamma^2(y); x_{p_1+1}, \ldots, x_{p_2}), \ldots, \\
g_k(\Gamma^k(y); x_{p_{k-1}+1}, \ldots, x_{p_k})),
\end{align*}
\]
where \( \Gamma^0 = \Gamma^0(k; j_1, \ldots, j_k) : Y_j \rightarrow Y_k \) and \( \Gamma^r = \Gamma^r(k; j_1, \ldots, j_k) : Y_j \rightarrow Y_r \) are the maps as defined in Definition 4.2 \( x_1, \ldots, x_j \in D \) and \( y \in Y_j \).

Note that if \( j_r = 0 \) for some \( r \), then \( g_r \in CY^0(D, D) \cong Hom_K(K, D) = D \), and the corresponding input in \( f \) is simply \( g_r \).

To check associativity, let \( f \in CY^k(D, D), g_r \in CY^{j_r}(D, D), r = 1, \ldots, k \), and \( h_t \in CY^{t}(D, D), t = 1, \ldots, j = \sum_{r=1}^{k} j_r \). As in the above lemma, let \( i = \sum_{t=1}^{j} i_t, p_s = j_1 + j_2 + \cdots + j_s, q_s = i_{p_{s-1}+1} + \cdots + i_{p_s} \). Also set \( q_{i,s} = i_{p_{s-1}+1} + i_{p_{s-1}+2} + \cdots + i_{p_s} \). Then \( 1 \leq s \leq j_r \).

\[
(4.2) \quad \gamma \circ (\gamma \otimes id)((f; g_1, \ldots, g_k); h_1, h_2, \ldots, h_j) = \gamma(\gamma(f; g_1, \ldots, g_k); h_1, \ldots, h_j).
\]

On the other hand, shuffle yields

\[
((f, g_1, \ldots, g_k), h_1, \ldots, h_j) \overset{\text{shuffle}}{\rightarrow} (f, (g_1, h_1, \ldots, h_{j_1}), (g_2, h_{j_1+1}, \ldots, h_{p_2}), \ldots, (g_k, h_{p_{k-1}+1}, \ldots, h_{p_k})).
\]

Now, composing with \( \gamma \circ (id \otimes (\otimes \gamma)) \), we get

\[
(4.3) \quad \gamma \circ (id \otimes (\otimes \gamma)) \circ (\text{shuffle})((f, g_1, \ldots, g_k), h_1, \ldots, h_j)
\]

To show that (4.2) and (4.3) are the same cochain in \( CY^j(D, D) \), let \( y \in Y_j \) and \( x_1, x_2, \ldots, x_i \in D \). Then,

\[
(4.4) \quad \gamma(\gamma(f; g_1, \ldots, g_k); h_1, \ldots, h_j)(y; x_1, \ldots, x_i)
\]

where

\[
\begin{align*}
\Gamma^y &= \Gamma^0(j; i_1, \ldots, i_j)y = d_1 \cdot \cdots \cdot d_i \cdot \cdots \cdot d_{\sum_{i=1}^{j-1} i} \cdot \cdots \cdot d_i \cdot 1 - y, \\
\Gamma^u &= \Gamma^0(j; i_1, \ldots, i_j)y = d_0 \cdot \cdots \cdot d_{\sum_{i=1}^{j-1} i} \cdot d_{\sum_{i=1}^{j-1} i+1} \cdot \cdots \cdot d_{i} y, 1 \leq u \leq j.
\end{align*}
\]

Now by definition of \( \gamma \), as given in (4.1), the equation (4.4) is

\[
(4.5) \quad f(\Gamma^0; g_1(\Gamma^1; y, x_1, \ldots, x_{i_1}), \ldots, \\
g_k(\Gamma^k; y, x_{\sum_{i=1}^{k-1} i + 1}, \ldots, x_{\sum_{i=1}^{k-1} i + i_k})), \ldots)
\]
where

\[ \Gamma^0 \Gamma^0 y = \Gamma^0(k; j_1, \ldots, j_k) \Gamma^0(j; i_1, \ldots, i_j)y \]
\[ = d_1 \cdots d_{p_1} \cdots \hat{d}_{p_2} \cdots \hat{d}_{p_k} \cdots d_{p_{k-1}} \hat{d}_1 \]
\[ \cdots \hat{d}_1 \cdots \hat{d}_{i_1+i_2} \cdots \hat{d}_{i_1+i_2+i_3} \cdots d_{i-1}y \]

and for \( 1 \leq r \leq k \)

\[ \Gamma^r \Gamma^0 y = \Gamma^r(k; j_1, \ldots, j_k) \Gamma^0(j; i_1, \ldots, i_j)y \]
\[ = d_0 \cdots d_{p_{r-1}} \cdots d_{p_r}d_1 \cdots d_{i_1+i_2} \cdots d_{i-1}y. \]

On the other hand,

\[ \gamma(f; \gamma(g_1; h_1, \ldots, h_{p_1}), \ldots, \gamma(g_k; h_{p_k-1}+1, \ldots, h_{p_k}+j))(y; x_1, \ldots, x_i) \]
\[ = f(\Gamma^0 y; \gamma(g_1; h_1, \ldots, h_{p_1})(\Gamma^1 y; x_1, \ldots, x_{q_1}), \ldots, \gamma(g_k; h_{p_k-1}+1, \ldots, h_{p_k}+j)(\Gamma^k y; x_{\sum_{s=1}^{k} q_s+1}, \ldots, x_{\sum_{s=1}^{k} q_s-i})), \]

where

\[ \Gamma^0 y = \Gamma^0(k; q_1, \ldots, q_k)y \]
\[ = d_1 \cdots d_{q_1} \cdots d_{q_1+q_2} \cdots \hat{d}_{\sum_{s=1}^{k} q_s} \cdots d_{\sum_{s=1}^{k} q_s-1}y \]

and, for \( 1 \leq r \leq k \),

\[ \Gamma^r y = \Gamma^r(k; q_1, \ldots, q_k)y \]
\[ = d_0 \cdots d_{\sum_{s=1}^{r} q_s-1} d_{\sum_{s=1}^{r} q_s+1} \cdots d_{\sum_{s=1}^{k} q_s-1}y. \]

By definition of \( \gamma \), (4.6) can further be written as

\[ = f(\Gamma^0 y; g_1(\Gamma^0 \Gamma^1 y; h_1(\Gamma^1 \Gamma^1 y; x_1, \ldots, x_{i_1}), \ldots, h_{j_1}(\Gamma^1 y; x_{\sum_{s=1}^{i_1} i_s+1}, \ldots, x_{q_1})), \ldots, \]
\[ g_k(\Gamma^0 \Gamma^k y; h_{p_k-1+1}(\Gamma^1 \Gamma^k y; x_{\sum_{s=1}^{k} q_s+1}, \ldots, x_{\sum_{s=1}^{k} q_s-i})), \ldots, \]
\[ h_{j_2}(\Gamma^j y; x_{\sum_{s=1}^{i_2} i_s+1}, \ldots, x_{i_2})), \]

where

\[ \Gamma^0 \Gamma^r y = \Gamma^0(j_r; i_{p_{r-1}+1}, \ldots, i_{p_{r-1}+j_r}) \Gamma^r(k; q_1, \ldots, q_k)y \]
\[ = (d_1 \cdots d_{q(r,1)} \cdots d_{q(r,2)} \cdots d_{q(r,j_r-1)} \cdots d_{q(r,j_r-1)}) \]
\[ d_{\sum_{s=1}^{r} q_s} d_{\sum_{s=1}^{r} q_s+1} \cdots d_{\sum_{s=1}^{k} q_s-1}y \]

and

\[ \Gamma^s \Gamma^r y = \Gamma^s(j_r; i_{p_{r-1}+1}, \ldots, i_{p_{r-1}+j_r}) \Gamma^r(k; q_1, \ldots, q_k)y \]
\[ = (d_0 \cdots d_{q(r,s-1)} \cdots d_{q(r,s)+1} \cdots d_{q(r,s)}) \]
\[ d_{\sum_{s=1}^{r} q_s} d_{\sum_{s=1}^{r} q_s+1} \cdots d_{\sum_{s=1}^{k} q_s-1}y \]

for \( 1 \leq s \leq r \) and \( 1 \leq r \leq k \).

Comparing (4.5) and (4.7), and using Lemma 4.5, it follows that the cochains in (4.2) and (4.3) are the same.

To check commutativity of unit diagrams, let \( f \in \mathcal{C}(k) = CY^k(D, D) \) and \( \alpha_1, \ldots, \alpha_k \in K \).

Then,

\[ \gamma \circ (\text{id} \otimes \eta^k)(f \otimes (\alpha_1, \ldots, \alpha_k)) = \gamma(f; \alpha_1, \ldots, \alpha_k), \]

where we identify \( \alpha_i \in K \) with the map

\[ \alpha_i : K[Y_1] \otimes D \rightarrow D, \]
\[ (y; a) \mapsto \alpha_i a, \]

for all \( i = 1, 2, \ldots, k \). If \( \phi \) denotes the isomorphism

\[ \mathcal{C}(k) \otimes K^k \cong \mathcal{C}(k), \]
then

\[ \phi(f \otimes (\alpha_1, \ldots, \alpha_k))(y; x_1, \ldots, x_k) = f(y; \alpha_1 x_1, \ldots, \alpha_k x_k). \]

Now,

\[ \gamma(f; \alpha_1, \ldots, \alpha_k)(y; x_1, \ldots, x_k) = f(\Gamma^0 y; \alpha_1 (\Gamma^1 y; x_1), \ldots, \alpha_k (\Gamma^k y; x_k)), \]

where \( \Gamma^0 y = y \), as \( \Gamma^0 = \Gamma^0(k; 1, \ldots, 1) \) and \( \Gamma^r y = d_0 \cdots d_{r-2} d_{r+1} \cdots d_k y, 1 \leq r \leq k \).

Therefore,

\[ \gamma(f; \alpha_1, \ldots, \alpha_k)(y; x_1, \ldots, x_k) = f(y; \alpha_1 x_1, \ldots, \alpha_k x_k). \]

Hence,

\[ \gamma \circ (\text{id} \otimes \eta^k)(f \otimes (\alpha_1, \ldots, \alpha_k)) = \phi(f \otimes (\alpha_1, \ldots, \alpha_k)). \]

Also, for \( f \in C(j) \) and \( \alpha \in K \),

\[ \gamma(\eta \otimes \text{id})(\alpha \otimes f) = \gamma(\alpha; f), \]

where \( \alpha \) is regarded as an element of \( C(1) \) as above.

Now,

\[ \gamma(\alpha; f)(y; x_1, \ldots, x_j) = \alpha(\Gamma^0 y; f(\Gamma^1 y; x_1, \ldots, x_j)), \]

where \( \Gamma^0 y = \Gamma^0(1; j)y = d_1 \cdots d_{j-1} y \) and \( \Gamma^1 y = \Gamma^1(1; j)y = y \). Thus

\[ \gamma(\alpha; f)(y; x_1, \ldots, x_j) = \alpha(y'; f(y; x_1, \ldots, x_j)) = \alpha f(y; x_1, \ldots, x_j), \]

where \( y' \) is the only tree in \( Y_1 \).

Note that \( \psi: K \otimes C(j) \xrightarrow{\cong} C(j) \) is given by

\[ \psi(\alpha \otimes f)(y; x_1, \ldots, x_j) = \alpha f(y; x_1, \ldots, x_j). \]

This completes the proof of the theorem. \( \square \)

5. Braces induced by the operad structure

We recall from [3] that if \( C(j), j \geq 0 \), is a (non-\( \Sigma \)) operad with multiplication map \( \gamma \), then the graded vector space \( C = \bigoplus C(j) \) admits a brace algebra structure. For \( C(j) = CY^j(D, D) \), the brace algebra structure is given by

\[ f\{g_1, \ldots, g_n\} = \sum (-1)^\epsilon \gamma(f; \text{id}_D, \ldots, \text{id}_D, g_1, \text{id}_D, \ldots, \text{id}_D, g_n, \text{id}_D, \ldots, \text{id}_D) \]

where the summation runs over all possible substitutions of \( g_1, \ldots, g_n \) into \( f \) in the prescribed order, and \( \epsilon = \sum_{p=1}^{n} |g_p| i_p \), \( i_p \) being the total number of variables one has to input in front of \( g_p \). Here \( \text{id}_D \) represents \( \eta(1) \). The brace identity is a consequence of the commutativity of associative and unit diagrams. Therefore, in view of Theorem 4.4, we see that \( CY^*(D, D) \) admits a brace algebra structure. The following lemma now shows that the braces as introduced in Definition 4.4 make the dialgebra cochain complex into a brace algebra.

**Lemma 5.1.** The braces on \( CY^*(D, D) \) induced by the operad structure coincide with the braces as introduced in Definition 4.4.

**Proof.** Let \( f \in C(k) = CY^k(D, D) \) and \( g_i \in C(m_i) = CY^{m_i}(D, D), 1 \leq i \leq n \).

Then, according to M. Gerstenhaber and A. A. Voronov [3], the brace induced by the multilinear maps \( \gamma \) is given by

\[ f\{g_1, \ldots, g_n\} = \sum (-1)^\epsilon \gamma(f; \text{id}, \ldots, \text{id}, g_1, \text{id}, \ldots, \text{id}, g_n, \text{id}, \ldots, \text{id}), \]
where id = id₂ = η(1) and the summation is over all possible substitutions of 
g₁, . . . , g₀ into f, in the given order, and ε = \sum_{p=1}^{n} |g_p| i_p, i_p being the total number of inputs in front of g_p.

Observe that in the term

\((-1)^{r} \gamma(f; \text{id}, \ldots, \text{id}, g_1, \text{id}, \ldots, \text{id}, g_n, \text{id}, \ldots, \text{id})\)

of the above summation, the total number of identity entries in γ is k − n, the total number of identity entries in front of g₁ is i₁ and the total number of identity entries in front of g_r is i_r − \sum_{t=1}^{r-1} m_t. 2 ≤ r ≤ n. Moreover, the following inequalities hold:

0 ≤ i₁, i₁ + m₁ ≤ i₂, . . . , i_{r-1} + m_{r-1} ≤ i_r, i_n + m_n ≤ k + \sum_{t=1}^{n} m_t − n = N (say).

By definition of γ as given in (4.1), we have, for y ∈ Y_N,

\[\gamma(f; \text{id}, \ldots, \text{id}, g_1, \text{id}, \ldots, \text{id}, g_n, \text{id}, \ldots, \text{id})(y; x_1, \ldots, x_N) ∈ \mathbb{F}_{N}^{2452 ANITA MAJUMDAR AND GOUTAM MUKHERJEE}
\]

\[\text{(5.1)} \]

\[\Gamma^p = \Gamma^p(k; 1, \ldots, 1, m_1, 1, \ldots, 1, m_2, \ldots, m_{r-1}, 1, \ldots, 1, m_r, 1, \ldots, 1, n_{m_n-i_n}) \]

for 0 ≤ p ≤ k. Note that in the definition of γ as given in (4.1), the map Γ^r yields the only tree in Y_1 when operated on y if j_r = 1 by Definition 4.2. In other words, Γ^r is the obvious constant map. For instance, by Definition 4.2 the map Γ^{i+2} appearing in (5.1) is given by

\[\Gamma^{i+2} = d_0 \cdots d_{(i_1+m_1+1)-1}d_{(i_1+m_1+2)} \cdots d_N \]

and consists of N − 1 face maps d_i; hence Γ^{i+2}y = y', where y' is the only tree in Y_1. Hence the corresponding input id(y'y; x_i) in γ is simply x_i.

Now according to Definition 4.2 we have

\[\Gamma^{0} = d_1 \cdots d_{i_1}d_{i_1+1} \cdots d_{i_1+m_1-1}d_{i_1+m_1} \cdots d_{i_2}d_{i_2+1} \cdots d_{i_2+m_2-1}d_{i_2+m_2} \cdots d_{i_3} \cdots d_{i_{r-1}+1} \cdots d_{i_{r-1}+m_{r-1}} \cdots d_N \]

\[= d_{i_1+1} \cdots d_{i_1+m_1-1}d_{i_2+1} \cdots d_{i_2+m_2-1}d_{i_2+m_2} \cdots d_{i_{r-1}+m_{r-1}} \cdots d_{i_{r-1}+m_{r-1}+1} \cdots d_N \]

\[= R^{i_1 \cdots i_n}_{1 \cdots m_n}, \text{as introduced in Definition 4.2} \]

Also the operator Γ^{i_r−\sum_{t=1}^{r-1} m_t+r}, corresponding to g_r, is given by

\[\Gamma^{i_r−\sum_{t=1}^{r-1} m_t+r} = d_0 \cdots d_{(i_r−\sum_{t=1}^{r-1} m_t)-1}d_{(i_r−\sum_{t=1}^{r-1} m_t)+1} \cdots d_N \]

Recall that the number of identity entries in front of g_r is i_r − \sum_{t=1}^{r-1} m_t and their degrees sum up to i_r − \sum_{t=1}^{r-1} m_t, while the sum of the degrees of g_1, . . . , g_{r-1}
is $\sum_{i=1}^{r-1} m_i$. Thus,
\[
\Gamma^i_r - \sum_{k=1}^{r-1} m_k + r = d_0 \cdots d_{i-1} d_{i+r+1} \cdots d_N = R^{i+1}_{i+1}, \text{ as introduced in Definition 5.2.}
\]

It follows that the $N$-cochain
\[
\gamma(f; \text{id}, \ldots, \text{id}, g_1, \text{id}, \ldots, \text{id}, g_n, \text{id}, \ldots, \text{id})
\]
is the same as $f \circ_{i_1, \ldots, i_n} (g_1, \ldots, g_n)$. This sets up a sign-preserving bijective correspondence between the terms of the summation
\[
\sum (-1)^{\epsilon} \gamma(f; \text{id}, \ldots, \text{id}, g_1, \text{id}, \ldots, \text{id}, g_n, \text{id}, \ldots, \text{id}),
\]
where the summation is over all possible substitutions of $g_1, \ldots, g_n$ into $f$, in the given order, $\epsilon = \sum_{p=1}^{n} |g_p| i_p, i_p$ being the total number of inputs in front of $g_p$, and the terms of the summation
\[
\sum (-1)^{\eta} f \circ_{i_1, \ldots, i_n} (g_1, \ldots, g_n),
\]
where the summation is over all $i_1, \ldots, i_n$ such that $0 \leq i_1, i_1 + m_1 \leq i_2, \ldots, i_{n-1} + m_{n-1} \leq i_n, i_n + m_n \leq k + \sum_{i=1}^{r-1} m_i - n$ and $\eta = \sum_{p=1}^{n} |g_p| i_p$.

Thus the braces as defined in section 3 are precisely the braces induced by the (non-$\Sigma$) operad structure. \hfill \Box

6. $G$-algebra structure on cohomology

In this final section we show that the dialgebra cohomology $HY^*(D, D)$ of a dialgebra $D$ has a $G$-algebra structure which is induced from a homotopy $G$-algebra structure on the dialgebra cochain complex $CY^*(D, D)$ with the differential altered by a sign.

Let us first recall the following definitions from [3].

**Definition 6.1.** A homotopy $G$-algebra is a brace algebra $V = \bigoplus_{n} V^n$ provided with a differential $d$ of degree one and a dot product $x \cdot y$ of degree zero making $V$ into a differentially graded associative algebra. The dot product must satisfy the following compatibility identities:

\[
(x_1 \cdot x_2)\{y_1, \ldots, y_n\} = \sum_{k=0}^{n} (-1)^{\epsilon} x_1 \{y_1, \ldots, y_k\} \cdot x_2 \{y_{k+1}, \ldots, y_n\},
\]

where $\epsilon = |x_2| + 1 \sum_{p=1}^{k} |y_p|$, and

\[
d(x\{x_1, \ldots, x_{n+1}\}) = (dx)\{x_1, \ldots, x_{n+1}\} - \epsilon x_1 \{dx, x_2, \ldots, x_{n+1}\} + (|x_2| + 1 \sum_{i=1}^{n+1} (-1)^{|x_1|+\cdots+|x_{i-1}|} x_1 \cdot x_2, \ldots, x_{n+1}) \cdot x_1 \{x_2, \ldots, x_{n+1}\} \cdot x_1 \{x_2, \ldots, x_{n+1}\} \cdot x_1 \{x_2, \ldots, x_{n+1}\}.
\]

**Remark 6.2.** It should be mentioned here that the notion of homotopy $G$-algebras as defined above is different from the notion of strong homotopy $G$-algebras ($G_\infty$-algebras, for short) as considered in [4]. A $G_\infty$-algebra is an algebra over the minimal model of the Koszul operad describing $G$-algebras. However, the notion of homotopy $G$-algebras that we are considering do not really fit the general scheme of quadratic operad theory [5].
Definition 6.3. A multiplication on an operad $C$ of vector spaces is an element $m \in C(2)$ such that $m \circ m = 0$, where $m \circ m := m\{m\}$ and $\{ \}$ denote the associated braces.

The following lemma shows that the operad $CY^*(D, D)$ is equipped with a multiplication.

Lemma 6.4. The 2-cochain $\pi \in CY^2(D, D)$ defined by

\[
\begin{align*}
\pi([21]; a, b) &= a \cdot b, \\
\pi([12]; a, b) &= a \cdot b
\end{align*}
\]

for all $a, b \in D$ is a multiplication on the operad $CY^*(D, D)$.

Proof. By Remark 3.5, we only need to verify that $\pi \circ \pi = 0$. Now, by definition of the pre-Lie product as introduced in [7], we have, for $y \in Y_3$ and $a, b, c \in D$,

\[
\pi \circ \pi(y; a, b, c) = (\pi \circ \pi - \pi \circ \pi)(y; a, b, c).
\]

The proof now follows from the dialgebra axioms.\qed

In order to show that the dialgebra cochain complex $CY^*(D, D)$ admits a homotopy $G$-algebra structure, we shall make use of Proposition 2(3) from [3], which we describe below. Let $C$ denote an operad, $m$ a multiplication on $C$, and let $m \circ x$ denote $m\{x\}$.

Proposition 6.5. The product

\[
x \cdot y := (-1)^{|x|+1}m\{x, y\}
\]

of degree 0 and the differential

\[
dx = m \circ x - (-1)^{|x|}x \circ m, \quad d^2 = 0, \quad \deg d = 1,
\]

define the structure of a differential graded (DG) associative algebra on $C$.

First, we observe the following two facts.

Remark 6.6. Note that by Lemma 6.12 of [7], the coboundary operator

\[
\delta : CY^n(D, D) \longrightarrow CY^{n+1}(D, D)
\]

can be expressed in the form

\[
\delta f = (-1)^{|f|}(\pi \circ f - (-1)^{|f|}f \circ \pi) = (-1)^{|f|}df.
\]

Remark 6.7. The $*$ product, as introduced in Definition 6.8 of [7], can be expressed in terms of braces as

\[
f * g = (-1)^{|f|+1}|g|\pi\{f, g\}.
\]
This is because, by the definition of braces on \( CY^*(D, D) \),
\[
\pi(f, g)(y; x_1, \ldots, x_{p+q}) = \begin{cases} 
(-1)^{p(q-1)} \pi_{0, p}(f, g)(y; x_1, \ldots, x_{p+q}) & \text{if } p < q \\
(-1)^{p(q-1)} \pi(R_1^p(p+q; 2, p, q)y; f(R_1^p(p+q; 2, p, q)y; x_1, \ldots, x_p), \\
g(R_1^p(p+q; 2, p, q)y; x_{p+1}, \ldots, x_{p+q})) & \text{if } q < p \\
(-1)^{p(q-1)} \pi(d_1 \cdots d_{p-1}d_{p+1} \cdots d_{p+q-1}(y); \\
f(d_{p+1} \cdots d_{p+q}(y); x_1, \ldots, x_p), \\
g(d_0 \cdots d_{p-1}(y); x_{p+1}, \ldots, x_{p+q})) & \text{if } p = q \\
(-1)^{p(q-1)} \pi(R_2^p(p+1; 2, p)R_2^p(p+q; p+1, q)(y); \\
f(R_2^p(p+1; 2, p)R_2^p(p+q; p+1, q)(y); x_1, \ldots, x_p), \\
g(R_2^p(p+q; p+1, q)((y); x_{p+1}, \ldots, x_{p+q})) & \text{if } q = p \\
(-1)^{p(q-1)} f * g(y; x_1, \ldots, x_{p+q}). 
\end{cases}
\]

Here we make use of the fact that the operator \( d_{p+q} \) in the string of operators \( d_{p+1} \cdots d_{p+q} \) can be moved to the extreme left of the same string using \( d_i d_j = d_j d_i, i < j \), to yield \( d_{p+1}d_{p+1} \cdots d_{p+q-1} \).

Therefore by equation (6.5), the dot product \( f \cdot g \) determined by the multiplication \( m \) as in Proposition 6.5 is in this case nothing but the \( \ast \) product, up to the sign \( (-1)^{(|f|+1)(|g|+1)} \). Moreover, the differential \( d \) determined by \( m \) as in Proposition 6.5 is merely the coboundary \( \delta \), up to the sign \( (-1)^{|f|} \); that is, \( df = (-1)^{|f|}\delta f \).

Consequently, by Proposition 6.5 and Theorem 4.4 we deduce the following corollary.

**Corollary 6.8.** The graded cochain module \( CY^*(D, D) \) equipped with the \( \ast \) product \( f \ast g \), as introduced in [7], altered by the sign \( (-1)^{|f|+1)|g|+1} \) and the coboundary \( df = (-1)^{|f|}\delta f \), is a differential graded associative algebra.

Next we recall Theorem 3 of [3].

**Theorem 6.9.** A multiplication on an operad \( C \) defines the structure of a homotopy G-algebra on \( \bigoplus_k C(k) \). A multiplication on a brace algebra is equivalent to the structure of a homotopy G-algebra on it.

Thus in view of Theorem 4.4 Theorem 6.9 and Lemma 4.5 we have the following corollary.

**Corollary 6.10.** The cochain complex \( (CY^*(D, D), d) \), where \( df = (-1)^{|f|}\delta f \), is a homotopy G-algebra with the dot product \( f \cdot g = (-1)^{|f|+1)|g|+1} f \ast g \).

As a consequence, we have the following corollary.

**Corollary 6.11.** The cochain complex \( (CY^*(D, D), d) \) is a differential graded Lie algebra with respect to the commutator \( [x, y] = x \circ y - (-1)^{|x||y|}y \circ x \).

**Proof.** The brace identity, for \( m = n = 1 \), implies that
\[
x\{x_1\} \{y_1\} = x\{x_1, y_1\} + x\{x_1\} \{y_1\} + (-1)^{|x_1||y_1|}x\{y_1, x_1\},
\]
as \( 0 \leq i_1 \leq j_1 \leq 1 \).
Using Remark 3.3, we deduce from above that

\[(x \circ x_1) \circ y_1 - x \circ (x_1 \circ y_1) = x \{x_1, y_1\} + (-1)^{|x_1||y_1|} x \{y_1, x_1\}.\]

A straightforward computation using equation (6.6) and the fact that \(|x \circ y| = |x| + |y|\) shows that the commutator satisfies the graded Jacobi identity.

Moreover, the dot product is always homotopy graded commutative; that is,

\[(x, y) = x \cdot y - (-1)^{|x|(|y|+1)} y \cdot x = (-1)^{|x|} (d(x \circ y) - dx \circ y - (-1)^{|x|} x \circ dy).\]

This follows directly from equation (6.2), as

\[(-1)^{|x|} (d(x \circ y) - dx \circ y - (-1)^{|x|} x \circ dy) = (-1)^{|x|} (-(-1)^{|x|+1}|y| y \cdot x + (-1)^{|x|} x \cdot y) = x \cdot y - (-1)^{|x|+1} |y| y \cdot x.\]

Also, the differential is a derivation of the bracket. In other words,

\[d[x, y] - [dx, y] - (-1)^{|x|}[x, dy] = 0,\]

which is a direct consequence of the homotopy graded commutativity of the dot product. This shows that every homotopy G-algebra is a differential graded Lie algebra with respect to the commutator \([x, y] = x \circ y - (-1)^{|x||y|} y \circ x\).

Next we recall the following definition from \[3\].

**Definition 6.12.** A G-algebra is a graded vector space \(H\) with a dot product \(x \cdot y\) defining the structure of a graded commutative algebra with a bracket \([x, y]\) of degree \(-1\) defining the structure of a graded Lie algebra such that the bracket with an element is a derivation of the dot product:

\[[x, y \cdot z] = [x, y] \cdot z + (-1)^{|x||y|+1} y \cdot [x, z].\]

**Corollary 6.13.** The * product \(x * y\), altered by the sign \((-1)^{|x|(|y|+1)}\) and the bracket \([x, y] = x \circ y - (-1)^{|x||y|} y \circ x\), defines the structure of a G-algebra on the dialgebra cohomology \(HY^*(D, D)\) of a dialgebra \(D\).

**Proof.** First observe that

\[HY^n(D, D) = H^n((CY^*(D, D), \delta)) = H^n((CY^*(D, D), d)).\]

The fact that the dot product \(x \cdot y = (-1)^{|x|+1} \pi \{x, y\}\) lifts to the cohomology follows from Proposition 6.5. Equation (6.7) implies that this dot product is graded commutative. Moreover, by Corollary 6.11, the bracket \([x, y] = x \circ y - (-1)^{|x||y|} y \circ x\) of degree \(-1\) defines the structure of a graded Lie algebra on \(HY^*(D, D)\). It remains to show that the bracket with an element is a derivation of the dot product.

First we show that the commutator \([x, y] = x \circ y - (-1)^{|x||y|} y \circ x\) for all \(x, y \in CY^*(D, D)\) is a graded derivation of the dot product up to null homotopy; that is,

\[[x, y \cdot z - [x, y] \cdot z - (-1)^{|x||y|+1} y \cdot [x, z] = (-1)^{|x||y|+1}(d(x \{y, z\}) - (dx)\{y, z\} - (-1)^{|x|} x \{dy, z\} - (-1)^{|x||y|} x \{y, dz\}).\]
By definition of the commutator, we have

\[ [x, y \cdot z] - [x, y] \cdot z - (-1)^{[x][y]+1} y \cdot [x, z] = x \circ (y \cdot z) - (-1)^{[x][y]} (y \cdot z) \circ x - (x \circ y) - (-1)^{[x][y]} (y \circ x) \cdot z - (-1)^{[x][y]+1} y \cdot (x \circ z) - (-1)^{[x][y]} x \circ y \cdot z \]

\[ = (x \circ y \cdot z) - (-1)^{[x][y]} y \cdot (x \circ z) - (x \circ y) \cdot z \]

\[ - (-1)^{[x][y]} (y \cdot z) \circ x - (-1)^{[x][y]} (y \circ x) \cdot z + (-1)^{[x][y]+1} x \circ y \cdot z \]

\[ = (x \circ y \cdot z - (-1)^{[x][y]} y \cdot (x \circ z) - (x \circ y) \cdot z \]

\[ - (-1)^{[x][y]+1} (y \cdot z) \circ x - (-1)^{[x][y]} (y \circ x) \cdot z + (-1)^{[x][y]+1} x \circ y \cdot z \]

\[ = x \circ y \cdot z - (-1)^{[x][y]+1} y \cdot (x \circ z) - (x \circ y) \cdot z \]

(as \((y \cdot z) \circ x - y \cdot (z \circ x) + (-1)^{[x][y]+1} y \circ x \cdot z\) = 0, by equation (6.1))

\[ = (-1)^{[x][y]} d(x \{y, z\}) - (dx) \{y, z\} - (-1)^{[x]} x \{dy, z\} - (-1)^{[x][y]} x \{y, dz\} \]

by equation (6.2). This implies that \([x, y \cdot z] = [x, y] \cdot z + (-1)^{[x][y]+1} y \cdot [x, z]\) for all \(x, y, z \in HY^*(D, D)\). Thus \(HY^*(D, D)\) admits a G-algebra structure. \(\square\)

References


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