MAPS BETWEEN NON-COMMUTATIVE SPACES

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Abstract. Let \( J \) be a graded ideal in a not necessarily commutative graded \( k \)-algebra \( A = A_0 \oplus A_1 \oplus \cdots \) in which \( \dim_k A_i < \infty \) for all \( i \). We show that the map \( A \to A/J \) induces a closed immersion \( i : \text{Proj}nc A/J \to \text{Proj}nc A \) between the non-commutative projective spaces with homogeneous coordinate rings \( A \) and \( A/J \). We also examine two other kinds of maps between non-commutative spaces. First, a homomorphism \( A \to B \) between not necessarily commutative \( N \)-graded rings induces an affine map \( \text{Proj}nc B \supset U \to \text{Proj}nc A \) from a non-empty open subspace \( U \subset \text{Proj}nc B \). Second, if \( A \) is a right noetherian connected graded algebra (not necessarily generated in degree one), and \( A^{(n)} \) is a Veronese subalgebra of \( A \), there is a map \( \text{Proj}nc A \to \text{Proj}nc A^{(n)} \); we identify open subspaces on which this map is an isomorphism. Applying these general results when \( A \) is (a quotient of) a weighted polynomial ring produces a non-commutative resolution of (a closed subscheme of) a weighted projective space.

1. Introduction

This paper concerns maps between non-commutative projective spaces of the form \( \text{Proj}nc A \). Before summarizing our main results we define the relevant terms.

Following Rosenberg [8, p. 278] and Van den Bergh [13], a non-commutative space \( X \) is a Grothendieck category \( \text{Mod}X \). A map \( g : Y \to X \) between two spaces is an adjoint pair of functors \((g^*, g_*)\) with \( g_* : \text{Mod}Y \to \text{Mod}X \) and \( g^* \) left adjoint to \( g_* \). The map \( g \) is affine [8, page 278] if \( g_* \) is faithful and has a right adjoint. For example, a ring homomorphism \( \varphi : R \to S \) induces an affine map \( g : Y \to X \) between the affine spaces defined by \( \text{Mod}Y := \text{Mod}S \) and \( \text{Mod}X := \text{Mod}R \).

Let \( k \) be a field. An \( N \)-graded \( k \)-algebra \( A \) is locally finite if \( \dim_k A_i < \infty \) for all \( i \). The non-commutative projective space \( X \) with homogeneous coordinate ring \( A \) is defined by

\[
\text{Mod}X := \text{GrMod}A/F\dim A
\]

(see Section 2), and

\[
\text{Proj}nc A := (\text{Mod}X, \mathcal{O}_X),
\]

where \( \mathcal{O}_X \) is the image of \( A \) in \( \text{Mod}X \). Thus \( \text{Proj}nc A \) is an enriched quasi-scheme in the language of [13]. Let \( Y \) be another non-commutative projective space with homogeneous coordinate ring \( B \). A map \( f : \text{Proj}nc B \to \text{Proj}nc A \) is a map \( f : Y \to X \) such that \( f^* \mathcal{O}_X \cong \mathcal{O}_Y \).

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When $A$ is a commutative $\mathbb{N}$-graded $k$-algebra we write $\text{Proj} A$ for the usual projective scheme. We will always view a quasi-separated, quasi-compact scheme $X$ as a non-commutative space by associating to it the enriched space $(\text{Qcoh} X, \mathcal{O}_X)$. The rule $X \mapsto (\text{Qcoh} X, \mathcal{O}_X)$ is a faithful functor.

**Summary of results.** The main results in this paper are Theorems 3.2, 3.3, 4.1, and Proposition 4.8.

A map $g : Y \to X$ is a closed immersion if it is affine and the essential image of $\text{Mod} Y$ in $\text{Mod} X$ under $g_*$ is closed under submodules and quotients. Theorem 3.2 shows that a surjective homomorphism $A \to A/J$ of graded rings induces a closed immersion $i : \text{Proj} nc A/J \to \text{Proj} nc A$. The functors $i^*$ and $i_*$ are the obvious ones (see the proof of 3.2). It seems to be a folklore result that $i^*$ is left adjoint to $i_*$, but we could not find a proof in the literature so we provide one here. Several people have been aware for some time that this is the appropriate intuitive picture, but, as far as I know, no formal definition of a closed immersion has been given and so no explicit proof has been given.

If $A$ is a graded subalgebra of $B$, commutative results suggest there should be a closed subspace $Z$ of $Y = \text{Proj} nc B$ and an affine map $g : Y \setminus Z \to \text{Proj} nc A$. Theorem 3.3 establishes such a result under reasonable hypotheses on $A$ and $B$. In fact, that result is set in a more general context, namely a homomorphism $\phi : A \to B$ of graded rings. Corollary 3.4 then says that if $\phi : A \to B$ and $B$ is a finitely presented left $A$-module, then there is an affine map $g : \text{Proj} nc B \to \text{Proj} nc A$. This is a (special case of a) non-commutative analogue of the commutative result that a finite morphism is affine.

If $A$ is a quotient of a commutative polynomial ring, and $A^{(n)}$ is the graded subring with components $(A^{(n)})_i = A_{ni}$, then there is an isomorphism of schemes $\text{Proj} A \cong \text{Proj} A^{(n)}$. Verevkin [12] proved that $\text{Proj} nc A \cong \text{Proj} nc A^{(n)}$ when $A$ is no longer commutative, but is connected and generated in degree one. Theorem 4.1 shows that when $A$ is not required to be generated in degree one, there is still a map $\text{Proj} nc A \to \text{Proj} nc A^{(n)}$, and Proposition 4.8 describes open subspaces on which this map is an isomorphism.

The results here are modelled on the commutative case, and none is a surprise. In large part the point of this paper is to make the appropriate definitions so that results from commutative algebraic geometry carry over verbatim to the non-commutative setting. Thus we formalize and make precise some of the terminology and intuition in papers like [2] and [7].

In Example 4.9 we show how our results apply to a quotient of a weighted polynomial ring to obtain a birational isomorphism $g : \text{Proj} nc A \to X = \text{Proj} A$, where $X$ is a commutative subscheme of a weighted projective space. It can happen that $X$ is singular whereas $\text{Proj} nc A$ is smooth. Thus we can view $\text{Proj} nc A \to \text{Proj} A$ as something like a non-commutative resolution of singularities. Furthermore, in this situation $g_* g^* \cong \text{id}$.

We freely use basic notions and terminology for non-commutative spaces from the papers [4], [10], and [13].

## 2. Definitions and preliminaries

Throughout this paper we assume that $A$ is a locally finite $\mathbb{N}$-graded algebra over a field $k$. Thus $A = A_0 \oplus A_1 \oplus \cdots$, and $\dim_k A_i < \infty$ for all $i$. The **augmentation ideal** $\mathfrak{m}$ of $A$ is $A_1 \oplus A_2 \oplus \cdots$. If $A_0$ is finite dimensional and $A$ is right noetherian, then it...
follows that \( \dim_k A_i < \infty \) for all \( i \) because \( A_{\geq i}/A_{\geq i+1} \) is a noetherian \( A/\mathfrak{m} \)-module. We write \( \text{GrMod} A \) for the category of \( \mathbb{Z} \)-graded right \( A \)-modules, and define

\[
\text{Tails} A := \text{GrMod} A / \text{Fdim} A,
\]

where \( \text{Fdim} A \) is the full subcategory consisting of direct limits of finite dimensional \( A \)-modules. Equivalently, \( \text{Fdim} A \) consists of those modules in which every element is annihilated by a suitably large power of \( \mathfrak{m} \). We write \( \pi \) for the quotient functor \( \text{GrMod} A \to \text{Tails} A \) and \( \omega \) for its right adjoint.

The projective space with homogeneous coordinate ring \( A \) is the space \( X \) defined by \( \text{Mod} X := \text{Tails} A \). We write \( \text{Proj}_n A = (\text{Mod} X, \mathcal{O}_X) \), where \( \mathcal{O}_X \) denotes the image of \( A \) in \( \text{Tails} A \).

A closed subspace \( Z \) of a space \( X \) is a full subcategory \( \text{Mod} Z \) of \( \text{Mod} X \) that is closed under submodules and quotient modules in \( \text{Mod} X \) and such that the inclusion functor \( i_* : \text{Mod} Z \to \text{Mod} X \) has both a left adjoint \( i^* \) and a right adjoint \( i^! \).

Two spaces are isomorphic if their module categories are equivalent. Hence a map \( Y \to X \) is a closed immersion if and only if it is an isomorphism from \( Y \) to a closed subspace of \( X \).

The complement \( X \setminus Z \) to a closed subspace \( Z \) is defined by

\[
\text{Mod} X \setminus Z := \text{Mod} X / T,
\]

the quotient category of \( \text{Mod} X \) by the localizing subcategory \( T \) consisting of those \( X \)-modules \( M \) that are the direct limit of modules \( N \) with the property that \( N \) has a finite filtration \( N = N_n \supseteq N_{n-1} \supseteq \cdots \supseteq N_1 \supseteq N_0 = 0 \) such that each \( N_i/N_{i-1} \) is in \( \text{Mod} Z \). Because \( T \) is a localizing category, there is an exact quotient functor \( j^* : \text{Mod} X \to \text{Mod} X \setminus Z \), and its right adjoint \( j_* : \text{Mod} X \setminus Z \to \text{Mod} X \). The pair \( (j^*, j_*) \) defines a map \( j : X \setminus Z \to X \). We call it an open immersion.

We sometimes write \( \text{Mod}_Z X \) for the category \( T \) and call it the category of \( X \)-modules supported on \( Z \).

Let \( f : Y \to X \) be a map. If \( f_* \) is faithful, then the counit \( \text{id}_Y \to f^* f_* \) is monic and the unit \( f^* f_* \to \text{id}_Y \) is epic.

**Watt’s Theorem for graded modules.** Let \( A \) and \( B \) be \( \mathbb{Z} \)-graded \( k \)-algebras. We recall the analogue of Watt’s Theorem proved by Del Rio [23 Proposition 3] that describes the \( k \)-linear functors \( \text{GrMod} A \to \text{GrMod} B \) that have a right adjoint.

A bigraded \( A \)-\( B \)-bimodule is an \( A \)-\( B \)-bimodule

\[
M = \bigoplus_{(p,q) \in \mathbb{Z}^2} pM_q
\]

such that \( A_i \cdot pM_q \cdot B_j \subseteq i + pM_{q+j} \) for all \( i, j, p, q \in \mathbb{Z} \). Write \( \otimes \) for \( \otimes_k \). If \( L \) is a graded right \( A \)-module, we define

\[
L \overset{A}{\otimes} M := \bigoplus_{q \in \mathbb{Z}} (L \overset{A}{\otimes} M)_q,
\]

where \( (L \overset{A}{\otimes} M)_q \) is the image of \( \bigoplus_p (L_{-p} \otimes pM_q) \) under the canonical map \( L \otimes M \to L \overset{A}{\otimes} M \). This gives \( L \overset{A}{\otimes} M \) the structure of a graded right \( B \)-module; it is a \( B \)-module direct summand of the usual tensor product \( L \overset{A}{\otimes} M \).

If \( N \) is a graded right \( B \)-module, we define

\[
\text{Hom}_B(M, N) := \{ f \in \text{Hom}_{\text{Gr}}(M, N) \mid f(pM_n) = 0 \text{ for almost all } p \}.
\]
This is made into a graded right $A$-module by declaring that $\deg f = p$ if $f(i, M_i) = 0$ for all $i \neq -p$. Hence $\underline{\text{Hom}}_{B}(M, N)_p$ is naturally isomorphic to $\text{Hom}_{GrB}(-pM_\ast, N)$, and there is a natural isomorphism

$$\underline{\text{Hom}}_{B}(M, N) = \bigoplus_p \text{Hom}_{GrB}(-pM_\ast, N).$$

The usual adjoint isomorphism between $\text{Hom}$ and $\otimes$ then induces an isomorphism

$$\text{Hom}_{GrB}(L \otimes_A M, N) \cong \text{Hom}_{GrA}(L, \underline{\text{Hom}}_{B}(M, N)),$$

showing that $- \otimes_A M : \text{GrMod} A \to \text{GrMod} B$ is left adjoint to $\underline{\text{Hom}}_{B}(M, -)$.

**Theorem 2.1** (Del Rio [3]). Let $A$ and $B$ be graded $k$-algebras, and $F : \text{GrMod} A \to \text{GrMod} B$ a $k$-linear functor having a right adjoint. Then $F \cong - \otimes_A M$, where $M$ is the bigraded $A$-$B$-bimodule

$$M = \bigoplus_{p \in \mathbb{Z}} F(A(p))$$

with homogeneous components $pM_q = F(A(p))_q$.

If $F$ also commutes with the twists by degree, then $F$ is given by tensoring with a graded $A$-$B$-bimodule, say $V = \bigoplus_n V_n$. The corresponding $M$ in this case is $M = \bigoplus V(p)$ with $pM_q = V(p)q$.

The left $A$-action on $M$ is given by declaring that $x \in A$ acts on $pM_\ast$ as $F(\lambda_x)$, where $\lambda_x : A(p) \to A(p + i)$ denotes left multiplication by $x$.

### 3. Maps induced by graded ring homomorphisms

Throughout this section we assume that $A$ and $B$ are locally finite $\mathbb{N}$-graded algebras over a field $k$.

We consider the problem of when a homomorphism $\phi : A \to B$ of graded rings induces a map $g : \text{Proj}_{nc}B \to \text{Proj}_{nc} A$ and, if it does, how the properties of $g$ are determined by the properties of $\phi$.

Associated to $\phi$ is an adjoint triple $(f^*, f_*, f')$ of functors between the categories of graded modules. Explicitly, $f^* = - \otimes_A B$, $f_* = - \otimes_B A$ is the restriction map, and $f' = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{GrB}(B(-p), -)$. We wish to establish conditions on $\phi$ which imply that these functors factor through the quotient categories in the following diagrams:

\[
\begin{array}{ccc}
\text{GrMod}B & \xrightarrow{f_*} & \text{GrMod}A \\
\pi' \downarrow & & \pi \downarrow \\
\text{Tails}B & \xrightarrow{f'} & \text{Tails}A
\end{array}
\quad
\begin{array}{ccc}
\text{GrMod}B & \xleftarrow{f^*} & \text{GrMod}A \\
\pi' \downarrow & & \pi \downarrow \\
\text{Tails}B & \xleftarrow{f'} & \text{Tails}A
\end{array}
\]

**Lemma 3.1.** Let $A$ and $B$ be Grothendieck categories with localizing subcategories $S \subset A$ and $T \subset B$. Let $\pi : A \to A/S$ and $\pi' : B \to B/T$ be the quotient functors, and let $\omega$ and $\omega'$ be their right adjoints. Consider the following diagram of functors:

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\pi \downarrow & & \pi' \downarrow \\
A/S & \xrightarrow{} & B/T
\end{array}
\]
Suppose that $F(S) \subset T$.

1. There is a unique functor $G : A/S \rightarrow B/T$ such that $\pi'F = G\pi$.
2. If $H : B \rightarrow A$ is a right adjoint to $F$, then $\pi H\omega'$ is a right adjoint to $G$.
3. If $H$ is a right adjoint to $F$ and $G'$ is a right adjoint to $G$, then $H(T) \subset S$ if and only if $G'\pi' \cong \pi H$.

Proof. (1) The existence and uniqueness of $G$ is due to Gabriel [6, Corollaire 2, p. 368].

(2) To show that $G$ has a right adjoint it suffices to show that it is right exact and commutes with direct sums. If $\mathcal{M}_\lambda$ is a collection of objects in $A/S$, then each is of the form $\mathcal{M}_\lambda = \pi M_\lambda$ for some object $M_\lambda$ in $A$. Both $\pi'$ and $F$ commute with direct sums because they have right adjoints, so $G\pi$ commutes with direct sums; $\pi$ also commutes with direct sums. Therefore

$$G \left( \bigoplus \mathcal{M}_\lambda \right) = G \left( \bigoplus \pi M_\lambda \right) \cong G\pi \left( \bigoplus \mathcal{M}_\lambda \right) \cong \bigoplus G\pi M_\lambda = \bigoplus GM_\lambda.$$

Thus $G$ commutes with direct sums.

To see that $G$ is right exact, consider an exact sequence

$$(3-1) \quad 0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$$

in $A/S$. By Gabriel [6, Corollaire 1, p. 368], $(3-1)$ is obtained by applying $\pi$ to an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ in $A$. Both $\pi'$ and $F$ are right exact because they have right adjoints, so $\pi'FL \rightarrow \pi'FM \rightarrow \pi'FN \rightarrow 0$ is exact. In other words, $G\mathcal{L} \rightarrow GM \rightarrow GN \rightarrow 0$ is exact.

Hence $G$ has a right adjoint, say $G'$. It follows that $\omega G'$ is a right adjoint to $G\pi$. But $G\pi = \pi'F$ has $H\omega'$ as a right adjoint, so $\omega G' \cong H\omega'$. Since $\pi\omega \cong \text{id}_{A/S}$, $G' \cong \pi H\omega'$. Since a right adjoint is only determined up to natural equivalence, $\pi H\omega'$ is a right adjoint to $G$.

(3) If $H(T) \subset S$, then $\pi H$ vanishes on $T$ so, by Gabriel [6, Corollaire 2, page 368], there is a functor $V : B/T \rightarrow A/S$ such that $V\pi' = \pi H$. Thus $V \cong \pi H\omega'$, and this is isomorphic to $G'$ by (2). Hence $G'\pi' \cong \pi H$. Conversely, if $G'\pi' \cong \pi H$, then $\pi H(T) = 0$, so $H(T) \subset S$. \qed

Warning. The functor $H$ in part (2) of Lemma 3.1 need not have the property that $H(T)$ is contained in $S$. An explicit example of this is provided by taking $B = A$, $F = H = \text{id}_A$, $S = 0$, and $T = B$.

Theorem 3.2. Let $J$ be a graded ideal in an $\mathbb{N}$-graded $k$-algebra $A$. Then the homomorphism $A \rightarrow A/J$ induces a closed immersion $i : \text{Proj}_{nc} A/J \rightarrow \text{Proj}_{nc} A$.

Proof. Write $X = \text{Proj}_{nc} A$ and $Z = \text{Proj}_{nc} A/J$. Write $m = A_1 \oplus A_2 \oplus \cdots$. Thus $\text{Mod}X = \text{GrMod}A/F\text{dim}A$. We write $\pi : \text{GrMod}A \rightarrow \text{Mod}X$ for the quotient functor and $\omega$ for a right adjoint to it. Similarly, $\pi' : \text{GrMod}A/J \rightarrow \text{Mod}Z$ is the quotient functor, and $\omega'$ is a right adjoint to it. See [12] and [11, Section 2] for more information about this.

Let $f_* : \text{GrMod}A/J \rightarrow \text{GrMod}A$ be the inclusion functor. It has a left adjoint $f^* = - \otimes_A A/J$, and a right adjoint $f^!$ that sends a graded $A$-module to the largest submodule of it that is annihilated by $J$. 

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By [6] Corollaire 2, p. 368 (= part (1) of Lemma 3.1), there is a unique functor $i_*$ such that

$$\begin{array}{ccc}
\text{GrMod} A/J & \xrightarrow{f_*} & \text{GrMod} A \\
\pi' \downarrow & & \downarrow \\
\text{Mod} Z & \xrightarrow{i_*} & \text{Mod} X
\end{array}$$

commutes, and $i_*$ is exact because $f_*$ is [6] Corollaire 3, p. 369]. Thus $i_*\pi' = \pi f_*$. By Lemma 3.1(2), $i_*$ has a right adjoint, namely $i^! := \pi' f^! \omega$. It is clear that $f^!$ sends $\text{Fdim} A$ to $\text{Fdim} A/J$, so $\pi' f^! \cong i^! \pi$ by Lemma 3.1(3).

It is clear that $f^*$ sends $\text{Fdim} A$ to $\text{Fdim} A/J$, so by [6] Corollaire 2, p. 368], there is a functor $i^* : \text{Mod} X \to \text{Mod} Z$ such that $\pi' f^* = i^* \pi$. Since $f_*$ is right adjoint to $f^*$, it follows from Lemma 3.1(2) that $\pi f_* \omega'$ is a right adjoint to $i^*$. But $\pi f_* \omega' = i_* \pi \omega' = i_*$. Hence $i^*$ is left adjoint to $i_*$.

We now show that $i_*$ is faithful. Since $i_*$ has a left and a right adjoint, it is exact, so it suffices to show that if $i_* M = 0$, then $M = 0$. Suppose that $i_* \pi' M = 0$ for some $M \in \text{GrMod} A/J$. Then $\pi f_* M = 0$, and we conclude that $M$ is in $\text{Fdim} A$, and hence in $\text{Fdim} A/J$; therefore $\pi' M = 0$. Hence $i_*$ is faithful.

We will show that $i_*$ is full after establishing the following fact.

Claim. $\omega \pi f_* \cong f_* \omega' $.

Proof. Let $M \in \text{GrMod} A/J$, let $\tau M$ denote the largest submodule of $M$ that is in $\text{Fdim} A/J$ (equivalently, in $\text{Fdim} A$), and set $M = M/\tau M$. Then $\pi' M = \pi' M$ and $\pi f_* M = \pi f_* M$, so the two functors take the same value on $M$ if and only if they take the same value on $\tau M$. Hence we can, and will, assume that $M = \tau M$; i.e., $\tau M = 0$.

We must show that $\omega \pi M = \omega' \pi' M$. By definition $\omega \pi M$ is the largest essential extension $0 \to M \to \omega \pi M \to T \to 0$ such that $T \in \text{Fdim} A$. The definition of $\omega' \pi' M$ is analogous, although $T$ is now required to belong to $\text{Fdim} A/J$. It therefore suffices to prove that $\omega \pi M$ is in $\text{GrMod} A/J$. The top row in the diagram

$$
\begin{array}{ccc}
M \otimes_A J & \xrightarrow{\omega \pi M} & \otimes_A J \\
\downarrow & & \downarrow \\
M J & \xrightarrow{(\omega \pi M) J}
\end{array}
$$

is exact, and the first vertical map is zero because $M$ is an $A/J$-module, so there is an induced map $T \otimes_A J \to (\omega \pi M) J$. This map is surjective. However, $T \otimes_A J$ belongs to $\text{Fdim} A$ because $T$ does, so $(\omega \pi M) J \in \text{Fdim} A$. This implies that $M \cap (\omega \pi M) J \in \text{Fdim} A$. But $\tau M = 0$, so $M \cap (\omega \pi M) J = 0$, and it follows that $(\omega \pi M) J = 0$ because $M$ is essential in $\omega \pi M$. In other words, $\omega \pi M \in \text{GrMod} A/J$. This completes the proof of the claim.

We have $f^* f_* \cong \text{id}_{\text{GrMod} A/J}$ and $\pi' \omega' \cong \text{id}_{\text{Mod} Z}$, so

$$i^* i_* \cong (\pi' f^! \omega)(\pi f_* \omega') \cong \pi' f^! \omega' \pi' \omega' \cong \text{id}_{\text{Mod} Z} \, .$$

It follows from this that $i_*$ is full.

To see that $i_*$ is a closed immersion, it remains to check that $i_*(\text{Mod} Z)$ is closed under submodules and quotients in $\text{Mod} X$. Let $M \in \text{Mod} Z$ and suppose that $0 \to L \to i_* M \to N \to 0$ is an exact sequence in $\text{Mod} X$. There is an exact sequence
Theorem 3.3. Suppose that $\phi : A \to B$ is a map of locally finite $\mathbb{N}$-graded $k$-algebras. Write $X = \text{Proj}_{nc} A$ and $Y = \text{Proj}_{nc} B$. Let $m$ be the augmentation ideal of $A$, and let $I$ be the largest two-sided ideal of $B$ contained in $\phi(m)B$. Let $Z \subset Y$ be the zero locus of $I$. If $B\phi(m)^n \subset \phi(m)B$ for some integer $n$, then $\phi$ induces an affine map

$$g : Y \setminus Z \to X.$$ 

Proof. The category of modules over $Y \setminus Z$ is $\text{Mod}Y/\text{Mod}_Z Y$. This is equivalent to the quotient category $\text{GrMod}B/\text{T}$, where $\text{T}$ consists of those modules $M$ with the property that every element of $M$ is killed by some power of $I$. Let $\pi' : \text{GrMod}B \to \text{GrMod}B/\text{T}$ be the quotient functor. We have functors $(f^*, f_*, f^!)$ between the graded module categories and a diagram

$$
\begin{array}{ccc}
\text{GrMod} B & \xrightarrow{f_*} & \text{GrMod} A \\
\pi' \downarrow & & \downarrow \pi \\
\text{Mod} Y \setminus Z & & \text{Mod} X.
\end{array}
$$

To check that $f^*$ sends $\text{Fdim} A$ to $\text{T}$, it suffices to check that $f^*(A/m) \in \text{T}$, because $f^*$ commutes with direct limits and with the degree twist (1). However, $f^*(A/m) = B/\phi(m)B$ is in $\text{T}$ because $I \subset \phi(m)B$. Hence there is a unique functor $g^* : \text{Tails} A \to (\text{GrMod}B)/\text{T}$ satisfying $g^* \pi = \pi' f^*$.

To check that $f_*$ sends $\text{T}$ to $\text{Fdim} A$, it suffices to check that $f_*(B/I)$ is in $\text{Fdim} A$. However, $(B/I,m^n) = B\phi(m)^n + I/I$, the hypothesis that $B\phi(m)^n \subset \phi(m)B$ ensures that $B\phi(m)^n \subset I$, so $(B/I,m^n = 0$. Hence there is an exact functor $g_* : \text{GrMod}B/\text{T} \to \text{Tails} A$ such that $g_* \pi' = \pi f_*$. 


By Lemma 3.1(2), \( g_* \) has a right adjoint \( g_! = \pi^* f_! \).
To show that \( g_* \) is faithful we must show that if \( M \) is a graded \( B \)-module such that \( g_! \pi^* M = 0 \), then \( M \in \mathcal{T} \). Since \( \pi f_! M = 0 \), as an \( A \)-module \( M \) is a direct limit \( \lim M_{\lambda} \) where each \( M_{\lambda} \) is a finite dimensional \( A \)-module. There is an epimorphism
\[
\lim (M_{\lambda} \otimes_A B) \cong (\lim M_{\lambda}) \otimes_A B \cong M \otimes_A B \to M
\]
of graded \( B \)-modules. Since \( M_{\lambda} \otimes_A B \) equals \( f^* M_{\lambda} \), it is in \( \mathcal{T} \); but \( \mathcal{T} \) is closed under direct limits and quotients, so \( M \) is in \( \mathcal{T} \). Thus \( g_* \) is faithful.

The following consequence of the theorem slightly extends a result of Van den Bergh [13 Proposition 3.9.11].

**Corollary 3.4.** Let \( \phi : A \to B \) be a homomorphism of graded rings such that \( B \) becomes a finitely presented graded left \( A \)-module. Then \( \phi \) induces an affine map \( g : \text{Proj}_{nc} B \to \text{Proj}_{nc} A \).

**Proof.** If we apply \( A/m \otimes_A \) to a finite presentation of \( B \) as a left \( A \)-module, we see that \( B/\phi(m)B \) has finite dimension. Thus, as a right \( A \)-module \( B/\phi(m)B \) is annihilated by \( m^n \) for some \( n > 0 \). Equivalently, \( B\phi(m)^n \subset \phi(m)B \). Thus the hypotheses of the theorem are satisfied. It remains to show that \( Z \) is empty.

Let \( I \) denote the right annihilator in \( B/\phi(m)B \). We have already observed that \( \phi(m)^n \subset I \). Since \( A/m^n \) is finite dimensional, so is \( A/m^n \otimes_A B \cong B/\phi(m)^n B \). Thus \( B/I \) is finite dimensional. Hence the zero locus of \( I \) in \( \text{Proj}_{nc} B \) is empty. \( \Box \)

**Remark.** If, in Theorem 3.3, \( B_{\lambda} \) is finitely presented, then we have the useful technical fact that \( f_! \pi = \pi^* f_\lambda \). This follows from Lemma 3.1(3) once we show that \( f_! \) sends \( \text{Fdim} A \) to \( \text{Fdim} B \). Let \( M = \lim M_{\lambda} \) be a direct limit of finite dimensional \( A \)-modules. If \( B \) is a finitely presented right \( A \)-module, then \( \text{Hom}_{\text{Gr} A}(B, -) \) commutes with direct limits, so \( \text{Hom}_{\text{Gr} A}(N, \lim M_{\lambda}) = \lim \text{Hom}_{\text{Gr} A}(B, M_{\lambda}) \): this is a direct limit of finite dimensional \( B \)-modules, because \( B_{\lambda} \) is finitely generated. Hence \( f_!(\text{Fdim} A) \subset \text{Fdim} B \).

4. The Veronese Mapping

Throughout this section \( A \) is a locally finite \( \mathbb{N} \)-graded \( k \)-algebra and \( n \) is a positive integer. The \( n \)th Veronese subalgebra \( A^{(n)} \) is defined by
\[
A_i^{(n)} := A_{ni}.
\]
It is a classical result in algebraic geometry that if \( A \) is a finitely generated commutative connected graded \( k \)-algebra generated in degree one, then \( \text{Proj} A \cong \text{Proj} A^{(n)} \). This isomorphism is implemented by the Veronese embedding.

Verevkin proved a non-commutative version of this result when \( A \) is noetherian and generated in degree one [12 Theorem 4.4].

Theorem 4.1 and Proposition 4.8 show what happens when \( A \) need not be commutative and need not be generated in degree one.

**Theorem 4.1.** Let \( A \) be a left noetherian locally finite \( \mathbb{N} \)-graded \( k \)-algebra. Fix a positive integer \( n \). There is a map \( g : \text{Proj}_{nc} A \to \text{Proj}_{nc} A^{(n)} \). Furthermore, \( g_* \) is exact and \( g_! g_* \cong \text{id} \). If \( A \) is also right noetherian, then \( g_* \) has a right adjoint \( g_! \).

We will use the notation \( X := \text{Proj}_{nc} A^{(n)} \) and \( X' := \text{Proj}_{nc} A \).
We need two preliminary results before proving the theorem. First we explain how the functors defining the map \( g : \text{Proj}_{nc} A \to \text{Proj}_{nc} A^{(n)} \) in the theorem are induced by functors between the categories \( \text{GrMod} A \) and \( \text{GrMod} A^{(n)} \).

If \( L \) is a graded \( A \)-module, we define the graded \( A^{(n)} \)-module \( L^{(n)} \) by

\[
L_i^{(n)} := L_{ni}.
\]

The rule \( L \mapsto L^{(n)} \) extends to give an exact functor

\[
f_* : \text{GrMod} A \to \text{GrMod} A^{(n)}.
\]

The functor \( f_* \) is not faithful when \( n \geq 2 \), because \( f_*((A/m)(1)) = 0 \).

**Proposition 4.2.** Let \( A \) be a locally finite \( \mathbb{N} \)-graded \( k \)-algebra. Fix a positive integer \( n \). Let \( W \) and \( W' \) be the spaces with module categories

\[
\text{Mod} W = \text{GrMod} A^{(n)}
\]

and

\[
\text{Mod} W' = \text{GrMod} A.
\]

Then there is a map \( f : W' \to W \) with direct image functor given by \( f_* L = L^{(n)} \).

Furthermore, \( f_* \) has a right adjoint \( f^! \).

**Proof.** It is clear that \( f_* \) is an exact functor commuting with direct sums. By the graded version of Watt’s Theorem, \( f_* \cong - \otimes_A M \), where

\[
M := \bigoplus_{p \in \mathbb{Z}} A(p)^{(n)}
\]

with components \( pM_q = (A(p)^{(n)})_q = A(p)_{nq} \). The right action of \( A^{(n)} \) on \( M \) is given by right multiplication, and each \( A(p)^{(n)} \) is a right \( A^{(n)} \)-module. The left action of \( A \) is given by left multiplication, whereby \( a \in A_i \) acts by sending \( A(p)_{nq} \) to \( A(p+i)_{nq} \).

Define \( f^* : \text{GrMod} A^{(n)} \to \text{GrMod} A \) by \( f^* N = N \otimes_{A^{(n)}} A \) with the usual right action of \( A \), and grading given by

\[
(N \otimes_{A^{(n)}} A)_s = \sum_{n_i + j = s} N_i \otimes A_j.
\]

It is not hard to show that \( f^* \) is a left adjoint to \( f_* \). Therefore \( f^* \cong - \otimes_{A^{(n)}} Q \), where

\[
Q = \bigoplus_{p \in \mathbb{Z}} f^*(A^{(n)}(p)) \cong \bigoplus_{p \in \mathbb{Z}} A(np);
\]

multiplication \( A^{(n)}(p) \otimes_{A^{(n)}} A \to A(np) \) gives an isomorphism of graded right \( A \)-modules. Thus \( \mu Q_* \cong A(np) \) with its usual grading. One can verify directly that \( f_* \cong \text{Hom}_{A^{(n)}}(Q, -) \).

The right adjoint to \( f_* \) is the functor \( f^! = \text{Hom}_{A^{(n)}}(M, -) \). If \( N \) is a graded right \( A^{(n)} \)-module, then

\[
(f^! N)_s = \text{Hom}_{A^{(n)}}(-, M)_s, N) = \text{Hom}_{A^{(n)}}(A(-)^{(n)}, N).
\]

If \( N \) is a graded \( A^{(n)} \)-module, then \( f_* f^*(N) = N \), so \( f_* f^* \) is naturally equivalent to \( \text{id}_W \).

\( \square \)
Let \( \pi' : \text{GrMod}A \rightarrow \text{Tails}A \) and \( \pi : \text{GrMod}A^{(n)} \rightarrow \text{Tails}A^{(n)} \) be the quotient functors. To prove Theorem 4.1, we must find functors \( g^* \), \( g_* \), and \( g^! \) making the following diagrams commute:

\[
\begin{array}{ccc}
\text{GrMod}A & \xrightarrow{f_*} & \text{GrMod}A^{(n)} \\
\downarrow{\pi'} & & \downarrow{\pi} \\
\text{Tails}A & \xrightarrow{g_*} & \text{Tails}A^{(n)}.
\end{array}
\quad
\begin{array}{ccc}
\text{GrMod}A & \xleftarrow{f^*f!} & \text{GrMod}A^{(n)} \\
\downarrow{\pi'} & & \downarrow{\pi} \\
\text{Tails}A & \xleftarrow{g^*g!} & \text{Tails}A^{(n)}.
\end{array}
\]

where \( f^*, f_* \), and \( f^! \), are the functors in the previous proposition.

Since \( f_* \) sends \( \text{Fdim}A \) to \( \text{Fdim}A^{(n)} \), there is a functor \( g_* : \text{Tails}A \rightarrow \text{Tails}A^{(n)} \) such that \( g_*\pi' = \pi f_* \). Because \( f_* \) is exact, \( g_* \) is too [6].

To ensure that \( f^* \) and \( f^! \) induce functors between the quotient categories, we must impose a noetherian hypothesis. Although there is no noetherian hypothesis in Proposition 4.2, in Theorem 4.1 it is assumed that \( A \) is right noetherian. This hypothesis ensures that \( f^* \) sends \( \text{Fdim}A^{(n)} \) to \( \text{Fdim}A \).

Recall that \( \text{GrMod}A \) denotes the category of graded right \( A \)-modules.

Lemma 4.3. Let \( A \) be a locally finite \( \mathbb{N} \)-graded \( k \)-algebra. Then:

1. \( f_* \) sends right noetherian \( A \)-modules to right noetherian \( A^{(n)} \)-modules;
2. if \( A \) is right noetherian so is \( A^{(n)} \), and \( A \) is a finitely generated right \( A^{(n)} \)-module.
3. if \( A \) is left noetherian, then \( f^* \) sends \( \text{Fdim}A^{(n)} \) to \( \text{Fdim}A \);
4. if \( A \) is left noetherian, then there is a functor \( g^* : \text{Tails}A^{(n)} \rightarrow \text{Tails}A \) such that \( g^*\pi = \pi f^* \);
5. if \( A \) is right noetherian, then \( f^! \) sends \( \text{Fdim}A^{(n)} \) to \( \text{Fdim}A \);
6. if \( A \) is right noetherian, then there is a functor \( g^! : \text{Tails}A^{(n)} \rightarrow \text{Tails}A \) such that \( g^!\pi = \pi f^! \).

Proof. (1) Let \( M \) be a right noetherian graded \( A \)-module. If \( N \) is a submodule of \( M^{(n)} \), then \( N = NA \cap M^{(n)} \). Hence any proper ascending chain of submodules in \( M^{(n)} \) would give a proper ascending chain of submodules of \( M \) by multiplying by \( A \). Since \( M \) contains no such chain, neither does \( M^{(n)} \).

(2) Applying (1) to \( M = A \) shows that \( A^{(n)} \) is right noetherian.

Applying (1) to \( M := A \oplus A(1) \oplus \cdots \oplus A(n-1) \) gives the result, because \( M^{(n)} \cong A \) as a right \( A^{(n)} \)-module.

(3) Because \( A \) is left noetherian, the left module version of (2) implies that \( A \) is a finitely generated left \( A^{(n)} \)-module. Hence if \( N \) is a finite dimensional right \( A^{(n)} \)-module, \( N \otimes_{A^{(n)}} A \) is a finite dimensional \( A \)-module. Thus \( f^* \) sends finite dimensional right \( A^{(n)} \)-modules to finite dimensional right \( A \)-modules. Since \( f^* \) is a left adjoint, it commutes with direct limits. The result follows.

(4) Because \( A \) is left noetherian, we may invoke (3). The existence of such \( g^* \) now exists by the universal property of the quotient functor \( \pi \).

(5) First we show that it suffices to prove that \( f^! \) send finite dimensional \( A^{(n)} \)-modules to finite dimensional \( A \)-modules. To this end, let \( N \in \text{Fdim}A^{(n)} \) and write \( N = \varinjlim N_\lambda \) as a direct limit of finite dimensional modules. Recall that
Lemma 4.3 and so \( \text{Hom}_{\text{Mod}} f \) categories.

Since \( A \) is right\- presenting, right \( A^{(n)} \)-module, so \( \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, -) \) commutes with direct limits. It follows that

\[
f^!(\lim N_\lambda) = \bigoplus_p \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, \lim N_\lambda)
= \lim_p \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, N_\lambda)
= \lim_p \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, N_\lambda)
= \lim_p f^!(N_\lambda).
\]

Hence, if each \( f^!(N_\lambda) \) is finite dimensional, \( f^! N \) is a direct limit of finite dimensional modules.

Now we show that \( f^! N \) is finite dimensional when \( N \) is a finite dimensional graded right \( A^{(n)} \)-module. It suffices to show that \( (f^! N)_p \), which is equal to \( \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, N) \), is zero for almost all \( p \) and is finite dimensional for all \( p \).

By the noetherian hypothesis, \( A(p)^{(n)} \) is a finitely generated right \( A^{(n)} \)-module, so \( \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, N) \) has finite dimension.

We now show that \( \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, N) \) is zero if \( |p| \) is sufficiently large. Fix \( p \). For every integer \( j \) we have

\[
A(p + nj)^{(n)} \cong A(p)^{(n)}(j),
\]

so

\[
\text{Hom}_{\text{Gr}^{(n)}}(A(p + nj)^{(n)}, N) \cong \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, N(-j)).
\]

Since \( A(p)^{(n)} \) is finitely generated and \( N \) is finite dimensional, when \( |j| \) is sufficiently large \( \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, N(-j)) \) is zero. Hence \( \text{Hom}_{\text{Gr}^{(n)}}(A(p)^{(n)}, N) \) is zero for \( |p| \) sufficiently large. This completes the proof that \( f^! \) sends finite dimensional modules to finite dimensional modules.

(6) follows from (5) in the same way that (4) follows from (3).

**Proof of Theorem 4.1.** By Lemma 3.3 there are functors \( g^* \) and \( g_* \) between the categories \( \text{Mod Proj}_{nc} A^{(n)} = \text{Tails} A^{(n)} \) and \( \text{Mod Proj}_{nc} A = \text{Tails} A \) satisfying

\[
g^* \pi = \pi' f^*, \quad g_* \pi' = \pi f_*, \quad g^! \pi = \pi' f^!.
\]

Applying Lemma 3.1 to \( f^* \), we see that \( g^* \) has \( \pi f_* \omega' \) as a right adjoint. But \( \pi f_* \omega' = g_* \pi' \omega' = g_* \), so \( g_* \) is a right adjoint to \( g^* \). We have already remarked that \( g_* \) is exact because \( f_* \) is. Furthermore,

\[
\text{id} \cong \pi \omega \cong \pi f_* f^* \omega = g_* \pi' f^* \omega = g_* g^* \pi \omega \cong g_* g^*.
\]

Since \( f^* A^{(n)} = A \), \( g^* \mathcal{O}_{\text{Proj}_{nc} A^{(n)}} = \mathcal{O}_{\text{Proj}_{nc} A} \).

Now suppose that \( A \) is also right noetherian. Then \( g_* \) has a right adjoint \( g^! \) by Lemma 4.3 and \( g^! \pi = \pi' f^! \). This completes the proof of Theorem 4.1. \( \square \)
In the next result \( \text{Proj} A \) is the usual commutative scheme viewed as a non-commutative space with module category \( \text{Qcoh}(\text{Proj} A) \).

**Corollary 4.4.** If \( A \) is a finitely generated commutative connected graded \( k \)-algebra, there is a map \( g : \text{Proj} \nc A \to \text{Proj} A \). Furthermore, \( g \) has a right adjoint \( g' \).

**Proof.** For some sufficiently large \( n \), \( A^{(n)} \) is generated in degree one; so

\[
\text{Tails} A^{(n)} \cong \text{Qcoh} \text{Proj} A^{(n)} = \text{Qcoh} \text{Proj} A.
\]

Hence \( \text{Proj} \nc A^{(n)} \cong \text{Proj} A \). Therefore Theorem 4.1 gives the result. \( \square \)

**Remarks.** 1. Suppose that \( A \) is both left and right noetherian, as in Theorem 4.1. Since \( g \) has both a left and a right adjoint, it is exact; its right adjoint \( g' \) therefore preserves injectives. Hence there is a convergent spectral sequence

\[
\text{Ext}_{\text{Proj} \nc A}^{p+q}(M, Rq g^! N) \Rightarrow \text{Ext}_{\text{Proj} \nc A}^{p+q}(g_* M, N)
\]

for \( M \) and \( N \) modules over \( \text{Proj} \nc A \) and \( \text{Proj} \nc A^{(n)} \) respectively.

2. If \( J \) is a two-sided ideal of \( A \), then the natural map \( A \to A/J \) induces an isomorphism \( A^{(n)}/J^{(n)} \to (A/J)^{(n)} \), so there is a commutative diagram

\[
\begin{array}{ccc}
\text{Proj} \nc A/J & \longrightarrow & \text{Proj} \nc A \\
\downarrow & & \downarrow g \\
\text{Proj} \nc A^{(n)}/J^{(n)} & \longrightarrow & \text{Proj} \nc A^{(n)}
\end{array}
\]

where the horizontal maps are the natural closed immersions.

If \( A \) is prime, right noetherian, we define

\[
\text{Fract}_{gr} A := \{ ab^{-1} \mid a, b \in A \text{ are homogeneous and } b \text{ is regular}\}.
\]

**Proposition 4.5.** Let \( A \) be a right noetherian, locally finite, \( \mathbb{N} \)-graded \( k \)-algebra. Suppose that \( A \) is prime and \( \text{Fract}_{gr} A \) contains a copy of \( A(n) \) for all \( n \in \mathbb{Z} \). Then

1. \( \text{Proj} \nc A \) and \( \text{Proj} \nc A^{(n)} \) are integral spaces in the sense of [10], and
2. \( g : \text{Proj} \nc A \to \text{Proj} \nc A^{(n)} \) is a birational isomorphism in the sense that it induces an isomorphism between the function fields.

**Proof.** That \( \text{Proj} \nc A \) is an integral space is proved in [10] Th. 4.5. It is also shown there that the function field of \( \text{Proj} \nc A \) is isomorphic to \( (\text{Fract}_{gr} A)_0 \). It is clear that \( (\text{Fract}_{gr} A^{(n)})_0 \subset (\text{Fract}_{gr} A)_0 \), and the reverse inclusion follows from the observation that \( ab^{-1} = ab^{n-1}b^{-n} \). \( \square \)

**Remarks.** 1. If \( A \) is prime noetherian and has a regular element of degree \( d \) for all \( d \gg 0 \), then \( \text{Fract}_{gr} A \) contains a copy of \( A(n) \) for all \( n \in \mathbb{Z} \), so the previous result applies.

2. If \( z \) is a normal regular element, then the complement in \( \text{Proj} \nc A \) to the zero locus of \( z \) is the open subspace \( U := \text{Proj} \nc A[z^{-1}] \). Now \( \text{Mod} U \) is equivalent to \( \text{GrMod} A[z^{-1}] \). If \( d \) is the smallest positive integer such that \( A[z^{-1}] \) has a unit of degree \( d \), then \( U \) is an affine space with coordinate ring

\[
\begin{pmatrix}
R_0 & R_1 & \cdots & R_{d-1} \\
R_{-1} & R_0 & \cdots & R_{d-2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{-d+1} & R_{-d+2} & \cdots & R_0
\end{pmatrix}.
\]
where \( R = \mathbb{A}[z^{-1}] \). The reason for this is that \( \bigoplus_{i=0}^{d-1} \mathbb{A}[z^{-1}](i) \) is a generator in \( \text{GrMod}\mathbb{A}[z^{-1}] \) and the tiled matrix ring is the endomorphism of this generator.

3. In the situation of Proposition 4.6, if \( s \) and \( t \) are homogeneous regular elements of relatively prime degrees in \( A \), and \( st \) is normal, meaning that \( stA = Ast \), then \( A[(st)^{-1}] \) has a unit of degree one; so, if \( U \) is the open complement to the zero locus of \( st \) in \( \text{Proj}_{nc} A \), then \( U \) is the affine space with coordinate ring \( A[(st)^{-1}]_{0} \). This ring is equal to \( A[(n)(st)^{-n}]_{0} \), so the open complement is isomorphic to the open complement to the zero locus of \( (st)^{n} \) in \( \text{Proj}_{nc} A^{(n)} \).

4. If \( \text{Fract}_{gr} A \) fails to contain a copy of every \( A(n) \), the map \( \text{Proj}_{nc} A \to \text{Proj}_{nc} A^{(n)} \) need not be a birational isomorphism. For example, take \( A = k[x] \) with \( \deg x = 2 \).

**Example 4.6.** If \( A \) is not generated in degree one, then \( g_{*} \) need not be faithful.

Let \( A = k[x, z] \) be the polynomial ring with \( \deg x = 1 \) and \( \deg z = n \geq 2 \).

The image under \( \pi \) of \( M = A(x) \) is a simple module \( \mathcal{O}_{p} \) in \( \text{Proj}_{nc} A \). We have \( \mathcal{O}_{p}(1) ≠ 0 \), but \( (M(1))^{(n)} = 0 \), so \( g_{*}(\mathcal{O}_{p}(1)) = 0 \).

One might anticipate that \( g : \text{Proj}_{nc} A \to \text{Proj}_{nc} A^{(n)} \) is an isomorphism on suitable open subspaces: in the previous example, \( g \) restricts to an isomorphism from the complement to the zero locus of \( x \) in \( \text{Proj}_{nc} A \) to the complement to the zero locus of \( x^{n} \) in \( \text{Proj}_{nc} A^{(n)} \). We prove a general result of this type in Proposition 4.7. First we need a lemma.

For each integer \( r \), define

\[
A^{(n)+r} := \sum_{j \in \mathbb{Z}} A_{nj+r}.
\]

Obviously \( A^{(n)+r} A^{(n)+s} \subset A^{(n)+r+s} \), so each \( A^{(n)+r} \) is an \( (A^{(n)}, A^{(n)}) \)-bimodule, and these bimodules depend only on \( r (\text{mod} n) \). Define

\[
I_{r} := A^{(n)+r} A = \sum_{j \in \mathbb{Z}} A_{nj+r} A
\]

and

\[
I := \bigcap_{r \in \mathbb{Z}} I_{r} = I_{1} \cap I_{2} \cap \cdots \cap I_{n}.
\]

Although \( I_{r} \) is in general only a right ideal of \( A \), \( I^{(n)}_{r} \) is a two-sided ideal of \( A^{(n)} \). Since \( A_{q} I_{r} \subset I_{q+r} \), \( I \) is a two-sided ideal of \( A \).

Notice that \( A^{(n)+r} A^{(n)-r} = I^{(n)}_{r} \).

**Lemma 4.7.** With the above notation, \( I^{2n} \subset I^{(n)} A \).

**Proof.** From the containment

\[
I^{2} \subset I_{r} I = A^{(n)+r} I \subset A^{(n)+r} I_{n-r} = A^{(n)+r} A^{(n)+n-r} A = I^{(n)}_{r} A,
\]

it follows that

\[
I^{2n} \subset I^{(n)}_{1} I^{2n-2} \subset I^{(n)}_{2} I^{2n-4} \subset \cdots \subset I^{(n)}_{1} \cdots I^{(n)}_{n} A.
\]

But this last term is contained in

\[
(I^{(n)}_{1} \cap \cdots \cap I^{(n)}_{n}) A = I^{(n)} A,
\]

which completes the proof. \( \square \)
Proposition 4.8. With the hypotheses of Theorem [4.4], the map \( g \) restricts to an isomorphism \( g : \text{Proj}_{nc} A\setminus Z' \to \text{Proj}_{nc} A^{(n)}\setminus Z \), where \( Z' \) and \( Z \) are the zero loci of \( I \) and \( I^{(n)} \) respectively.

Proof. Write \( X' = \text{Proj}_{nc} A, X = \text{Proj}_{nc} A^{(n)}, U' = X' \setminus Z' \) and \( U = X \setminus Z \). Write \( \alpha : U \to X \) and \( \beta : U' \to X' \) for the inclusions. We will use Lemma 3.1 to show that there is an isomorphism \( h : U' \to U \) such that the diagram

\[
\begin{array}{ccc}
U' & \xrightarrow{\beta} & X' = \text{Proj}_{nc} A \\
\downarrow h & & \downarrow g \\
U & \xrightarrow{\alpha} & X = \text{Proj}_{nc} A^{(n)}
\end{array}
\]

commutes.

Let \( T \) be the localizing subcategory of \( \text{GrMod} A \) consisting of those modules \( L \) such that every element of \( L \) is killed by a suitably large power of \( I \). Let \( S \) be the localizing subcategory of \( \text{GrMod} A^{(n)} \) consisting of those modules \( N \) such that every element of \( N \) is killed by a suitably large power of \( I^{(n)} \). These two localizing subcategories contain all the finite dimensional modules. The spaces \( U \) and \( U' \) are defined by

\[
\text{Mod} U := (\text{GrMod} A^{(n)})/S \quad \text{and} \quad \text{Mod} U' := (\text{GrMod} A)/T.
\]

Let \( f_* \) and \( f^* \) be the functors defined in Proposition 4.2. We will show that \( f_*(T) \subset S \) and \( f^*(S) \subset T \). The first of these inclusions is obvious: if every element of an \( A \)-module \( L \) is annihilated by a power of \( I \), then every element of \( L^{(n)} \) is annihilated by a power of \( I^{(n)} \). To show that \( f^*(S) \subset T \), it suffices to show that \( f^*(A^{(n)}/I^{(n)}) \) belongs to \( T \). But \( f^*(A^{(n)}/I^{(n)}) \cong A/I^{(n)} A \), and by Lemma 4.7 \( A/I^{(n)} A \) is annihilated by \( I^{2n} \) so belongs to \( T \).

We now use Lemma 3.1 in the context of the following diagram:

\[
\begin{array}{ccc}
\text{GrMod} A^{(n)} & \xrightarrow{f^*} & \text{GrMod} A \\
\downarrow \alpha^* \pi & & \downarrow \beta^* \pi' \\
\text{Mod} U & & \text{Mod} U'.
\end{array}
\]

Because \( f^*(S) \subset T \), there exists a functor \( h^* : \text{Mod} U \to \text{Mod} U' \) such that \( h^* \alpha^* \pi = \beta^* \pi' f^* \). Because \( f_* \) is right adjoint to \( f^* \), \( h_* := \alpha^* \pi f_* \omega' \beta_* \) is right adjoint to \( h^* \).

Thus, \( h^* \) and \( h_* \) define a map \( h : U' \to U \). Since \( g_* \pi' = \pi f_* \), a computation gives \( \alpha_* h_\alpha \cong g_\beta \). Therefore \( \alpha h = g \beta \).

It remains to show that \( h \) is an isomorphism.

The unit \( \text{id}_{\text{GrMod} A^{(n)}} \to f_* f^* \) is an isomorphism because the natural map \( L \to (L \otimes_{A^{(n)}} A)^{(n)} \) is an isomorphism for all \( L \in \text{GrMod} A^{(n)} \). Because \( f_*(T) \subset S \), part (3) of Lemma 3.1 gives \( h_* \beta^* \pi' \cong \alpha^* \pi f_* \). Therefore,

\[
h_* h^* \cong h_* \alpha^* \pi \omega \alpha_* = h_* \beta^* \pi' f^* \omega \alpha_* \cong \alpha^* \pi f_* \omega \alpha_* \cong \text{id}_U.
\]

To show that the natural transformation \( h^* h_* \to \text{id}_U \) is an isomorphism, we first consider the natural transformation \( f^* f_* \to \text{id}_{\text{GrMod} A} \). For an \( A \)-module \( M \) this is the multiplication map

\[
f^* f_* M = M^{(n)} \otimes_{A^{(n)}} A \to M.
\]

We claim that the kernel and cokernel of this map belong to \( T \).
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Suppose that $\sum m_i \otimes a_i \in M^{(n)} \otimes_{A^{(n)}} A$ is in the kernel. Then $\sum m_i a_i = 0$. By taking homogeneous components we can reduce to the case where each $a_i$ belongs to $A_{n-r} + A_{2n-r} + \cdots$ for some $r \in \{1, \ldots, n\}$. Then, if $b \in A_{nj+r}$ for some $j$, then

$$\left(\sum_i m_i \otimes a_i\right)b = \sum_i m_i \otimes a_ib = \sum_i m_i a_ib \otimes 1 = 0.$$ 

Thus $(\sum_i m_i \otimes a_i)I_r = 0$. Hence the kernel is annihilated by $I$, so belongs to $T$. The cokernel of $f^*f_* M \to M$ is $M/M^{(n)}A$. If $r \in \{1, \ldots, n\}$, then $M_{n-j}I_r \subset M^{(n)}A$. Hence $I$ annihilates the cokernel.

Since the kernel and cokernel of $f^*f_* \to id$ belong to $T$, there is an isomorphism $\beta^*\pi'f^*f_* \to \beta^*\pi'$. Hence

$$h^*h_* = h^*\alpha^*\pi f_*\omega'\beta_* \cong \beta^*\pi'f^*f_*\omega'\beta_* \cong id_U'.$$

□

In Example 4.8, $I_1 = (x)$ and $I_2 = A$, so $I = (x)$, whence $Z'$ is the zero locus of $x$. This explains why we need to remove the zero locus of $x$ to get the isomorphism.

If $A$ is generated in degree one, then $I_r = A_{\geq r}$ for $r \in \{0, 1, \ldots, n-1\}$, so $A/I_r \in \text{Fdim}A$, whence $A/I \in \text{Fdim}A$. It follows that $Z$ and $Z'$ are empty, and therefore $U = X$ and $U' = X'$. We therefore recover Verevkin’s result $X \cong X'$ when $A$ is generated in degree one over $A_0$.

Example 4.9. Let $A$ be a weighted polynomial ring. That is, $A = k[x_0, \ldots, x_n]$, where $\deg x_i = q_i \geq 1$. Write $Q = (q_0, \ldots, q_n)$. Then $\mathbb{P}_Q := \text{Proj} A$ is called a weighted projective space. It is isomorphic to the quotient variety $\mathbb{P}/\mu_Q$, where $\mu_Q = \mu_{q_0} \times \cdots \times \mu_{q_n}$. There is a sufficiently divisible integer $d$ such that $A^{(d)}$ is generated in degree one. Hence

$$\mathbb{P}_Q = \text{Proj} A = \text{Proj} A^{(d)} \cong \text{Proj}_{nc} A^{(d)}.$$ 

By Theorem 4.1 there is a map

$$g : \text{Proj}_{nc} A \to \text{Proj}_{nc} A^{(d)} \cong \mathbb{P}_Q.$$

This is an isomorphism on an open subspace by Proposition 4.8. Since $A$ has global homological dimension $n + 1$, $\text{Proj}_{nc} A$ has global homological dimension $n$. We therefore think of $\text{Proj}_{nc} A$ as a smooth space of dimension $n$ and the map $g$ as a non-commutative resolution of $\mathbb{P}_Q$. Let $X \subset \mathbb{P}_Q$ be the closed subscheme cut out by an ideal $J$ in $A$. Then there is a commutative diagram

$$\begin{array}{ccc}
\text{Proj}_{nc} A / J & \overset{i}{\longrightarrow} & \text{Proj}_{nc} A \\
f \downarrow & & \downarrow g \\
X & \longrightarrow & \mathbb{P}_Q
\end{array}$$

in which $f$ is a birational isomorphism and $i$ is a closed immersion. It can happen that $\text{Proj}_{nc} A / J$ is smooth even when $X$ is singular. Thus $\text{Proj}_{nc} A$ is a “non-commutative resolution” of $X$. An interesting case to examine in some detail is that where $X$ is an orbifold of a Calabi-Yau three-fold.

If $A = k[x]$ with $\deg x = 2$, and $n = 2$, then $\text{Proj}_{nc} A \cong \text{Spec} k \times k$ and $\text{Proj} A \cong \text{Spec} k$. Furthermore, $Z' = X'$ and $Z = X$. This is a special case of the next result, the truth of which was suggested by Darin Stephenson.
Proposition 4.10. Let $A$ be a locally finite $\mathbb{N}$-graded $k$-algebra such that $A_i = 0$ whenever $i \neq 0 \pmod{n}$. Then $\text{Proj}_{nc}A$ is isomorphic to the disjoint union of $n$ copies of $\text{Proj}_{nc}A^{(n)}$.

Proof. Let $p_r: \text{GrMod}A \to \text{GrMod}A$ be the functor defined by

$$p_r(M) = \bigoplus_{i \in \mathbb{N}} M_{r+i}$$

on objects, and $p_r(\theta) = \theta|_{p_r(M)}$ whenever $\theta \in \text{Hom}_{\text{Gr}A}(M,N)$. The hypothesis on $A$ ensures that each $p_r(M)$ is a graded $A$-submodule of $M$, so $p_r$ is indeed a functor from $\text{GrMod}A$ to itself. It is clear that $\text{id}_{\text{GrMod}A} = p_0 \oplus \cdots \oplus p_{n-1}$, where this direct sum is taken in the abelian category of $k$-linear functors from $\text{GrMod}A$ to itself; essentially, this is the observation that $M = p_0(M) \oplus \cdots \oplus p_{n-1}(M)$, and that any map $\theta : M \to N$ of graded $A$-modules respects this decomposition. Furthermore, each $p_r$ is idempotent and the $p_r$s are mutually orthogonal. It follows from this that there is a decomposition of $\text{GrMod}A$ as a product of categories, each component being the full subcategory on which $p_r$ is the identity.

It is clear that the shift functor (1) cyclicly permutes these subcategories, so they are all equivalent to one another and $(n)$ is an autoequivalence of each component. However, any one of these categories together with its autoequivalence $(n)$ is equivalent to $\text{GrMod}A^{(n)}$ with its autoequivalence (1). Thus $\text{GrMod}A$ is equivalent to the product of $n$ copies of $\text{GrMod}A^{(n)}$.

Finally, this decomposition descends to the Tails categories. \hfill \Box

5. An Ore Extension and an Example

The morphism

$$p : \mathbb{P}^n \setminus \{(0, \ldots, 0, 1)\} \to \mathbb{P}^{n-1},$$

$$(a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_{n-1}),$$

is called the projection with center $(0, \ldots, 0, 1)$. This section examines a non-commutative analogue of this basic operation.

Consider a connected graded $k$-algebra $R$ and a connected graded Ore extension

$$S = R[t; \sigma, \delta]$$

with respect to a graded automorphism $\sigma$ and a graded $\sigma$-derivation $\delta$ of degree $n \geq 1$. Thus $S = \bigoplus_{n=0}^{\infty} R t^n$ and $tr = r^\sigma t + \delta(r)$ for all $r \in R$. Since $\delta(R_i) \subset R_{i+n}$ for all $i$, by setting $\deg t = n$, $S$ becomes a connected graded algebra.

One expects that the inclusion map $R \to S$ induces a map $\text{Proj}_{nc}S \to \text{Proj}_{nc}R$. Indeed, the projection map above can be obtained as a special case of this.

Let $\mathfrak{m}$ denote the augmentation ideal of $R$. Since $\delta(\mathfrak{m}) \subset \mathfrak{m}$, $\mathfrak{m}S$ is a two-sided ideal of $S$. Furthermore, $S/\mathfrak{m}S \cong k[t]$ as graded rings.

Proposition 5.1. With the above notation, let $Z$ denote the zero locus of $\mathfrak{m}S$ in $\text{Proj}_{nc}S$.

1. $Z \cong \text{Spec } k^{\times n}$.

2. There is an affine map $g : \text{Proj}_{nc}S \setminus Z \to \text{Proj}_{nc}R$.

Proof. (2) The existence of $g$ is a special case of Theorem \cite{Smith}. That theorem applies because $S\mathfrak{m} \subset \mathfrak{m}S$. 

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We think of $\text{Proj}_{\text{nc}} S$ as a “cone over $\text{Proj}_{\text{nc}} R$ with vertex $Z$”. It would be interesting to describe the “fibers” of the map $g$.

When $\deg t > 1$, the Ore extension $S = R[t; \sigma, \delta]$ is not generated by its elements of degree one. This sometimes causes technical problems; however, if $R$ is generated in degree one, then the $n^{th}$ Veronese $S^{(n)}$ is generated in degree one. We can then combine Theorems [5.3] and [1.1] to analyze the space with homogeneous coordinate ring $S$ as follows.

**Proposition 5.2.** The inclusion of the $n$-Veronese subalgebras of $S$ and $S/mS$ gives a commutative diagram of rings and an induced commutative diagram of spaces as in the following diagrams:

$$
\begin{array}{cccc}
k[t]^{(n)} & \longrightarrow & k[t] = S/mS & \longrightarrow & k[t] = S/mS \\
\uparrow & & \uparrow & & \uparrow \\
S^{(n)} & \longrightarrow & S & \longrightarrow & S \\
\uparrow & & \uparrow & & \uparrow \\
R^{(n)} & \longrightarrow & R & \longrightarrow & R \\
\end{array}
$$

An application. In [11], a family of three-dimensional Artin-Schelter regular algebras $A$ is constructed and studied. Although the algebraic properties of $A$ are quite well understood, our understanding of the corresponding geometric object $\text{Proj}_{\text{nc}} A$ is rudimentary. The algebras are of the form $A = R[t; \sigma, \delta]$ with $\deg t = n$ and $R$ a two-dimensional Artin-Schelter regular algebra generated in degree one. It is well-known that $R$ and its Veronese subalgebras are (not necessarily commutative) homogeneous coordinate rings of $\mathbb{P}^1$. By Proposition 5.2, there is a commutative diagram of spaces and maps

$$
\begin{array}{cccc}
v = \text{Spec } k & \longrightarrow & Z' \cong \text{Spec } k^{\times n} & \longrightarrow & Z' \cong \text{Spec } k^{\times n} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Proj}_{\text{nc}} A^{(n)} & \longrightarrow & \text{Proj}_{\text{nc}} A & \longrightarrow & \text{Proj}_{\text{nc}} A \\
\uparrow & & \uparrow & & \uparrow \\
\text{Proj}_{\text{nc}} A^{(n)} \setminus \{v\} & \longrightarrow & \text{Proj}_{\text{nc}} A \setminus Z' & \longrightarrow & \text{Proj}_{\text{nc}} A \setminus Z' \\
\alpha \downarrow & & \beta \downarrow & & \beta \\
\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1.
\end{array}
$$

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References


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