

THOMASON'S THEOREM FOR VARIETIES OVER ALGEBRAICALLY CLOSED FIELDS

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ABSTRACT. We present a novel proof of Thomason's theorem relating Bott inverted algebraic K -theory with finite coefficients and étale cohomology for smooth varieties over algebraically closed ground fields. Our proof involves first introducing a new theory, which we term algebraic K -homology, and proving it satisfies étale descent (with finite coefficients) on the category of normal, Cohen-Macaulay varieties. Then, we prove algebraic K -homology and algebraic K -theory (each taken with finite coefficients) coincide on smooth varieties upon inverting the Bott element.

1. INTRODUCTION

In this paper we present a novel proof of Thomason's celebrated theorem [27, 4.1] comparing the Bott inverted algebraic K -groups with finite coefficients and étale cohomology. Our result applies to smooth quasi-projective varieties defined over an algebraically closed ground field, and is thus not as general as Thomason's original theorem, which imposes no restriction on the ground field. However, the proof given in this paper is arguably more elementary than that in Thomason's, and in particular we do not rely on any of the complicated constructions in the category of spectra employed by Thomason.

Recently, a new proof of Thomason's theorem has been given by Levine [17, 13.1] building on the recently established motivic Atiyah-Hirzebruch spectral sequence developed in [3], [8], and [17]. Our proof also does not use in any fashion this spectral sequence.

Throughout this paper, we frequently assume the following condition on the integer n used for the coefficient group \mathbb{Z}/n :

(Condition 2-3) If $3 \mid n$, then $9 \mid n$, and if $2 \mid n$, then $16 \mid n$.

The main result of this paper is

Theorem 1.1 (see Theorem 10.2). *Let X be a smooth, quasi-projective variety defined over an algebraically closed ground field k . Assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k and such that n satisfies (Condition 2-3). Then there is a convergent spectral sequence (concentrated in the portion of the pq -plane satisfying $p \geq q \geq p - 2 \dim(X)$) having the form*

$$E_2^{p,q} = H_{\text{ét}}^{p-q}(X, \mathbb{Z}/n) \implies K_{-p-q}(X; \mathbb{Z}/n)[1/\beta],$$

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where $K_m(X; \mathbb{Z}/n)[1/\beta]$ denotes the m -th graded piece of the ring obtained by inverting the action of the Bott element $\beta \in K_2(\mathrm{Spec} k; \mathbb{Z}/n)$ on $K_*(X; \mathbb{Z}/n)$.

The techniques used to prove Theorem 1.1 also allow for a comparison result relating K -theory and Bott-inverted K -theory (with finite coefficients). The precise statement is

Theorem 1.2 (see Theorem 10.4). *Let X , d , k , and n be as in Theorem 1.1. Then the map*

$$K_*(X; \mathbb{Z}/n) \rightarrow K_*(X; \mathbb{Z}/n)[1/\beta]$$

is a split surjection in degrees $ \geq 2d$, and in all degrees the kernel of this map is killed by β^d .*

Theorem 1.1 itself is not new, but the techniques we use to prove it are novel and perhaps of independent interest. In particular, and as in the author's previous paper [36] (in which the main result of the current paper was proven in the special case where $k = \mathbb{C}$), our proof uses properties of a new theory which we term "algebraic K -homology", K_*^{hom} .

Loosely speaking, the algebraic K -homology of a variety Y is defined by considering parameterized families of coherent sheaves of finite length on Y . Algebraic K -homology is analogous to *connective* topological K -homology, but should not be confused with G -theory (also known as K' -theory, defined from the category of coherent sheaves). For one thing, algebraic K -homology is a true homology theory, being covariantly functorial for all morphisms of varieties, whereas G -theory enjoys the functoriality of a "Borel-Moore" type theory. However, it is not even the case that G -theory represents the Borel-Moore theory associated to K -homology, as seen by the calculation

$$K_0^{\mathrm{hom}}(Y) \cong CH_0(Y)$$

given in Proposition 3.7 below. Thus, K -homology and G -theory do not coincide even for projective varieties (except for the trivial case of a point).

The distinction between algebraic K -homology and G -theory is explained, at least heuristically, by observing that K -homology is a truncated form of G -theory, involving only that portion of G -theory having weight at least equal to the dimension of a given variety (i.e., that portion of G -theory arising from coherent sheaves with zero-dimensional support). If such a truncation strikes the reader as odd, we note that the construction of K -homology is highly analogous to a certain construction of connective topological K -theory — the author's previous paper [36] makes this connection very precise. Moreover, since the Bott element β has weight one, such a truncation should not affect the theory upon inversion of β . The utility of truncating G -theory in this manner is hinted at by a theorem of Suslin [26] (see also Theorem 7.3), which states that for a smooth variety X defined over an algebraically closed ground field, the motivic cohomology with finite coefficients of weight at least $\dim(X)$ of X coincides with the étale cohomology of X . Since K -theory is related to motivic cohomology (in a manner which is made precise by the motivic spectral sequence, but which we do not use in this paper), it is reasonable to expect algebraic K -homology to enjoy étale descent. In fact, this theorem of Suslin not only provides motivation for the results of this paper but is indeed utilized in the proofs of Theorems 7.6 and 7.8.

To prove our main theorem for all quasi-projective varieties, we also utilize the “Borel-Moore theory” associated to K_*^{hom} , written K_*^{BM} , which coincides with K_*^{hom} on projective varieties and fits into the expected localization long exact sequence (see Corollary 5.6).

Using broad strokes, our proof of Theorem 1.1 is given by establishing the following ingredients. Let k , X , and n satisfy the hypotheses of Theorem 1.1.

- (A) The graded abelian group $K_*^{\text{BM}}(X; \mathbb{Z}/n)$ is a graded module (under cap product \cap) over the graded ring (under cup product \cup) $K_*(X; \mathbb{Z}/n)$ (see Section 3).
- (B) There exists a “fundamental class” $[X]$ belonging to $K_{2 \cdot \dim(X)}^{\text{BM}}(X; \mathbb{Z}/n)$ and a “duality map” $\gamma_X : K_*^{\text{BM}}(X; \mathbb{Z}/n) \rightarrow K_*(X; \mathbb{Z}/n)$ such that the compositions

$$K_*^{\text{BM}}(X; \mathbb{Z}/n) \xrightarrow{\gamma_X} K_*(X; \mathbb{Z}/n) \xrightarrow{- \cap [X]} K_{*+2 \cdot \dim(X)}^{\text{BM}}(X; \mathbb{Z}/n)$$

and

$$K_*(X; \mathbb{Z}/n) \xrightarrow{- \cap [X]} K_{*+2 \cdot \dim(X)}^{\text{BM}}(X; \mathbb{Z}/n) \xrightarrow{\gamma_X} K_{*+2 \cdot \dim(X)}^{\text{BM}}(X; \mathbb{Z}/n)$$

are given by multiplication by $\beta^{2 \cdot \dim(X)}$ under cap and cup product, respectively (see Theorem 8.13). In particular, $K_*^{\text{BM}}(X; \mathbb{Z}/n)[1/\beta]$ is a graded free module of rank one over the graded ring $K_*(X; \mathbb{Z}/n)[1/\beta]$.

- (C) Borel-Moore K -homology with finite coefficients, $K_*^{\text{BM}}(-; \mathbb{Z}/n)$, satisfies étale descent (without having to invert the Bott element) on the category of normal, Cohen-Macaulay varieties over k (see Theorems 7.6 and 7.8).

The main theorem follows more or less directly from these three results. The proofs of results (A) and (B) are rather formal in nature, but do require a fair amount of technical machinery. The proof of (C) ultimately amounts to an application of Suslin “rigidity” (although, it too requires a significant amount of technical machinery).

We now briefly outline the contents of each of the sections of this paper. Section 2 is devoted merely to recording formal definitions of the rather large number of exact categories and associated K -theory spaces used throughout the remainder of the paper. The reader might treat this section as a sort of glossary, not to be read in a linear fashion but rather referred to as the need arises. In Section 3 we introduce the main objects of study: algebraic K -homology \mathcal{K}^{hom} , its associated Borel-Moore theory \mathcal{K}^{BM} , and two bivariant theories $\mathcal{K}_{\text{geom}}(- \rightarrow -)$ and $\mathcal{K}_{\text{gqf}}(- \rightarrow -)$ which specialize to give K -theory (at least for smooth varieties), K -homology, and Borel-Moore K -homology. Also in this section basic properties like functoriality and the existence of pairings are developed. In particular, we prove that each of the theories $\mathcal{K}_{\text{geom}}(- \rightarrow -)$ and $\mathcal{K}_{\text{gqf}}(- \rightarrow -)$ satisfies the axioms of a “bivariant theory” in the sense of Fulton-MacPherson [10]. (The proof of this is relegated to an appendix.)

In Section 4 we develop a technical tool for use later in the paper, but which may also have utility outside the context of this paper — namely, we extend Voevodsky’s notion of a “pretheory” to that of a “presheaf with K_0^\oplus transfers”. As an example of the utility of these results, we mention that Andrei Suslin employs (a closely related version of) this notion in his forthcoming paper “On the Grayson Spectral Sequence”, and indeed he uses some of the properties of presheaves with K_0^\oplus transfers proven in Section 4. (The results Suslin requires were also proven in the author’s unpublished thesis [32].) In broad terms, we prove that many of the

desirable properties of pretheories remain valid for presheaves with K_0^\oplus transfers, and we prove that many functors arising in K -theory admit K_0^\oplus transfers (although they are not pretheories). The specific key properties enjoyed by presheaves with K_0^\oplus transfers used in this paper are listed in Corollary 4.23.

Beginning with Section 5, the major theorems of this paper are proven. The main goal of Section 5 itself is to prove Borel-Moore algebraic K -homology, \mathcal{K}^{BM} , satisfies the expected localization property (see Theorem 5.5). This is followed in Section 6 with a proof that the bivariant theory \mathcal{K}_{gqf} introduced in Section 3 satisfies a suitable version of the so-called “Gersten Conjecture” (see Theorem 6.2). Also of note in this section is a proof that the motivic complexes involving K_0 groups of automorphisms defined by Grayson [14] are locally equivalent to those of Voevodsky (see Theorem 6.5).

The tools developed in Sections 5 and 6 are employed in Section 7 to prove that Borel-Moore algebraic K -homology with finite coefficients, $\mathcal{K}^{BM}(-; \mathbb{Z}/n)$, satisfies étale descent on the category of normal, Cohen-Macaulay quasi-projective k -varieties (see Theorems 7.6 and 7.8). As usual, we require that n be relatively prime to the exponential characteristic of k . To relate \mathcal{K}^{BM} with Bott-inverted K -theory with finite coefficients, a form of Poincaré duality is needed. Such a duality result is proven in Section 8 for all smooth varieties that have a “fundamental class” relative to \mathcal{K}^{BM} -theory with \mathbb{Z}/n coefficients (see Theorem 8.13). In Section 9 we prove that all smooth, quasi-projective k -varieties have such fundamental classes (see Theorem 9.3), so that the duality result applies to all such varieties. Finally, using the results developed in the previous sections, we state and prove in Section 10 the main theorem of the paper, Theorem 10.2 — Thomason’s theorem for varieties over algebraically closed fields.

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2. DEFINITIONS

Throughout this paper, k denotes a ground field and by a k -scheme we mean a separated scheme essentially of finite type over k . We let Sch/k denote the category whose objects are k -schemes and whose morphisms are separated morphisms of k -schemes.

In this paper we consider a large number of exact categories and their associated K -theory spectra. We list many of these categories in the following two definitions so as to provide a handy reference for the remainder of the paper. The reader may wish to skip ahead to Section 3, referring back to the definitions given here as necessary.

Definition 2.1. For a k -scheme X , let $\mathcal{M}(X)$ denote the abelian category of coherent \mathcal{O}_X -modules and let $\mathcal{P}(X)$ denote the exact full subcategory of $\mathcal{M}(X)$ consisting of all locally free \mathcal{O}_X -modules.

Suppose $f : Y \rightarrow X$ is a morphism of k -schemes and let $U \subset Y$ be a locally closed subscheme (i.e., U is an open subscheme of a closed subscheme of Y). We define the following exact categories:

- $\mathcal{M}^{fin}(Y \xrightarrow{f} X)$ is the abelian full subcategory of $\mathcal{M}(Y)$ consisting of those M such that the induced morphism from the support of M , written $\text{supp}(M)$, to X is a finite morphism. (Briefly, we say M is *finite over* X .)
- $\mathcal{P}^{fin}(Y \xrightarrow{f} X)$ is the exact full subcategory of $\mathcal{M}^{fin}(Y \xrightarrow{f} X)$ consisting of those M which are \mathcal{O}_X -flat. (Briefly, we say such an M is *finite and flat over* X .) Equivalently, an object M of $\mathcal{M}^{fin}(Y \xrightarrow{f} X)$ belongs to $\mathcal{P}^{fin}(Y \xrightarrow{f} X)$ if and only if $f_* M$ is locally free on X .
- $\mathcal{M}^{qf}(Y \xrightarrow{f} X)$ is the abelian full subcategory of $\mathcal{M}(Y)$ consisting of those M such that the morphism $\text{supp}(M) \rightarrow X$ is quasi-finite (i.e., has finite fibers).
- $\mathcal{P}^{qf}(Y \xrightarrow{f} X)$ is the exact full subcategory of $\mathcal{M}^{qf}(Y \xrightarrow{f} X)$ consisting of those M which are \mathcal{O}_X -flat.
- $\mathcal{M}^{qf}((U, Y) \xrightarrow{f} X)$ is full abelian subcategory of $\mathcal{M}^{qf}(U \xrightarrow{f|_U} X)$ consisting of those M such that the map from the closure of $\text{supp}(M)$ in Y to X is a quasi-finite morphism. Note that $\mathcal{M}^{qf}((U, Y) \xrightarrow{f} X) = \mathcal{M}^{qf}((U, \overline{U}) \xrightarrow{f|_{\overline{U}}} X)$, where \overline{U} is the closure of U in Y .
- $\mathcal{P}^{qf}((U, Y) \xrightarrow{f} X)$ is the exact full subcategory of $\mathcal{M}^{qf}((U, Y) \xrightarrow{f} X)$ consisting of those M which are \mathcal{O}_X -flat.

When the morphism from Y to X is clear from context, we write each of the above categories more simply as $\mathcal{M}^{fin}(Y/X)$, $\mathcal{P}^{fin}(Y/X)$, $\mathcal{M}^{qf}(Y/X)$, $\mathcal{P}^{qf}(Y/X)$, $\mathcal{M}^{qf}((U, Y)/X)$, and $\mathcal{P}^{qf}((U, Y)/X)$.

To each of these exact categories we define its associated K -theory spectrum using Quillen's Q construction [20]. These spectra are written as

$$\begin{aligned} & \mathcal{G}^{fin}(Y \xrightarrow{f} X), \quad \mathcal{K}^{fin}(Y \xrightarrow{f} X), \quad \mathcal{G}^{qf}(Y \xrightarrow{f} X), \\ & \mathcal{K}^{qf}(Y \xrightarrow{f} X), \quad \mathcal{G}^{qf}((U, Y) \xrightarrow{f} X), \quad \mathcal{K}^{qf}((U, Y) \xrightarrow{f} X), \end{aligned}$$

or, more briefly, as

$$\mathcal{G}^{fin}(Y/X), \mathcal{K}^{fin}(Y/X), \mathcal{G}^{qf}(Y/X), \mathcal{K}^{qf}(Y/X), \mathcal{G}^{qf}((U, Y)/X), \mathcal{K}^{qf}((U, Y)/X).$$

The corresponding n -th homotopy group for each of these spectra is written as $G_n^{fin}(Y/X)$, $K_n^{fin}(Y/X)$, $G_n^{qf}(Y/X)$, etc.

In this paper, the primary example of a morphism $f : Y \rightarrow X$ arising in the above definitions occurs when $Y = X \times Z$ and $f = \pi_1$ is the canonical projection map. To simplify the notation in this case, we make the following definitions.

Definition 2.2. Let X and Y be k -schemes, let $\pi : X \times Y \rightarrow X$ denote the canonical projection map, and let U be a locally closed subscheme of Y . Define exact categories as follows:

- $\mathcal{M}^{fin}(X, Y) = \mathcal{M}^{fin}(X \times Y \xrightarrow{\pi} X)$,
- $\mathcal{P}^{fin}(X, Y) = \mathcal{P}^{fin}(X \times Y \xrightarrow{\pi} X)$,
- $\mathcal{M}^{qf}(X, Y) = \mathcal{M}^{qf}(X \times Y \xrightarrow{\pi} X)$,
- $\mathcal{P}^{qf}(X, Y) = \mathcal{P}^{qf}(X \times Y \xrightarrow{\pi} X)$,
- $\mathcal{M}^{qf}(X, (U, Y)) = \mathcal{M}^{qf}((X \times U, X \times Y) \xrightarrow{\pi} X)$,
- $\mathcal{P}^{qf}(X, (U, Y)) = \mathcal{P}^{qf}((X \times U, X \times Y) \xrightarrow{\pi} X)$.

The associated K -theory spectra are written as:

- $G^{fin}(X, Y) = G^{fin}(X \times Y \xrightarrow{\pi} X)$,
- $\mathcal{K}^{fin}(X, Y) = \mathcal{K}^{fin}(X \times Y \xrightarrow{\pi} X)$,
- $\mathcal{G}^{qf}(X, Y) = \mathcal{G}^{qf}(X \times Y \xrightarrow{\pi} X)$,
- $\mathcal{K}^{qf}(X, Y) = \mathcal{K}^{qf}(X \times Y \xrightarrow{\pi} X)$,
- $\mathcal{G}^{qf}(X, (U, Y)) = \mathcal{G}^{qf}((X \times U, X \times Y) \xrightarrow{\pi} X)$,
- $\mathcal{K}^{qf}(X, (U, Y)) = \mathcal{K}^{qf}((X \times U, X \times Y) \xrightarrow{\pi} X)$.

The associated K -groups for each of these spectra are written using subscripts as usual.

The following lengthy proposition describes the functoriality of the various theories defined in Definition 2.1.

Proposition 2.3. *Let $Y \xrightarrow{h} Z \xrightarrow{j} X$ and $g : X' \rightarrow X$ be morphisms of k -schemes. Define $f : Y \rightarrow X$ by $f = j \circ h$, define $Y' = Y \times_{X'} X$, and let $f' : Y' \rightarrow X'$ be the induced map. We have the following natural maps of spectra which are induced by pullback or pushforward of coherent sheaves:*

- (1) $h_* : \mathcal{G}^{fin}(Y \xrightarrow{f} X) \rightarrow \mathcal{G}^{fin}(Z \xrightarrow{j} X)$,
- (2) $g^* : \mathcal{G}^{fin}(Y \xrightarrow{f} X) \rightarrow \mathcal{G}^{fin}(Y' \xrightarrow{f'} X')$, provided g is flat,
- (3) $h_* : \mathcal{K}^{fin}(Y \xrightarrow{f} X) \rightarrow \mathcal{K}^{fin}(Z \xrightarrow{j} X)$,
- (4) $g^* : \mathcal{K}^{fin}(Y \xrightarrow{f} X) \rightarrow \mathcal{K}^{fin}(Y' \xrightarrow{f'} X')$,
- (5) $h^* : \mathcal{K}^{fin}(Z \xrightarrow{j} X) \rightarrow \mathcal{K}^{fin}(Y \xrightarrow{f} Z)$, provided h is finite and flat,
- (6) $h_* : \mathcal{G}^{qf}(Y \xrightarrow{f} X) \rightarrow \mathcal{G}^{qf}(Z \xrightarrow{j} X)$, provided h is proper,
- (7) $g^* : \mathcal{G}^{qf}(Y \xrightarrow{f} X) \rightarrow \mathcal{G}^{qf}(Y' \xrightarrow{f'} X')$, provided g is flat,
- (8) $h_* : \mathcal{K}^{qf}(Y \xrightarrow{f} X) \rightarrow \mathcal{G}^{qf}(Z \xrightarrow{j} X)$, provided h is proper,
- (9) $g^* : \mathcal{K}^{qf}(Y \xrightarrow{f} X) \rightarrow \mathcal{K}^{qf}(Y' \xrightarrow{f'} X')$,
- (10) $h^* : \mathcal{K}^{qf}(Z \xrightarrow{j} X) \rightarrow \mathcal{K}^{qf}(Y \xrightarrow{f} X)$, provided h is quasi-finite and flat.

Moreover, the maps of type (3) determine a covariant functor up to homotopy $(Y \rightarrow X) \mapsto \mathcal{K}^{fin}(Y/X)$ from Sch/X (the category of k -schemes over X) to spectra, and the maps of type (4) give $(X' \rightarrow X) \mapsto \mathcal{K}^{fin}(Y' \xrightarrow{f'} X')$ the structure of a contravariant functor up to homotopy from Sch/X to spectra. Similarly, the maps of type (8) determine a covariant functor on the category with objects in Sch/X and morphism given by proper morphisms, and the maps of type (9) determine a contravariant functor on Sch/X .

Proof. Each of these maps of spectra is induced by the evident corresponding functor, which in each case may readily be verified to be exact. The fact that $(Y \rightarrow X) \mapsto \mathcal{K}^{fin}(Y/X)$ is functorial follows from the fact that given $X'' \xrightarrow{p} X' \xrightarrow{g} X$, the functors $p^* \circ g^*$ and $(g \circ p)^*$ from $\mathcal{P}^{fin}(Y \rightarrow X)$ to $\mathcal{P}^{fin}(Y'' \rightarrow X'')$ are naturally isomorphic. The other functoriality assertions hold for similar reasons. \square

Proposition 2.4. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of k -schemes and assume $U \hookrightarrow Z$ is a locally closed subscheme. The functors*

$$\begin{aligned} \mathcal{P}^{fin}(Z \xrightarrow{g} Y) \times \mathcal{P}^{fin}(Y \xrightarrow{f} X) &\rightarrow \mathcal{P}^{fin}(Z \xrightarrow{f \circ g} X), \\ \mathcal{P}^{qf}(Z \xrightarrow{g} Y) \times \mathcal{P}^{qf}(Y \xrightarrow{f} X) &\rightarrow \mathcal{P}^{qf}(Z \xrightarrow{f \circ g} X), \end{aligned}$$

and

$$\mathcal{P}^{qf}((U, Z) \xrightarrow{g} Y) \times \mathcal{P}^{qf}(Y \xrightarrow{f} X) \rightarrow \mathcal{P}^{qf}((U, Z) \xrightarrow{f \circ g} X)$$

each given by the formula $(M, N) \mapsto M \otimes_{\mathcal{O}_Z} g^*N$ are biexact and thus induce pairings

$$\begin{aligned} \mathcal{K}^{fin}(Z \xrightarrow{g} Y) \wedge \mathcal{K}^{fin}(Y \xrightarrow{f} X) &\rightarrow \mathcal{K}^{fin}(Z \xrightarrow{f \circ g} X), \\ \mathcal{K}^{qf}(Z \xrightarrow{g} Y) \wedge \mathcal{K}^{qf}(Y \xrightarrow{f} X) &\rightarrow \mathcal{K}^{qf}(Z \xrightarrow{f \circ g} X), \end{aligned}$$

and

$$\mathcal{K}^{qf}((U, Z) \xrightarrow{g} Y) \wedge \mathcal{K}^{qf}(Y \xrightarrow{f} X) \rightarrow \mathcal{K}^{qf}((U, Z) \xrightarrow{f \circ g} X).$$

of spectra.

Proof. It is easy to verify that the targets of these functors are correct as stated (for example, that $M \in \mathcal{P}^{qf}(Z \xrightarrow{g} Y)$ and $N \in \mathcal{P}^{qf}(Y \xrightarrow{f} X)$ implies $M \otimes_{\mathcal{O}_Z} g^*N \in \mathcal{P}^{qf}(Z \xrightarrow{f \circ g} X)$). The flatness hypotheses ensure that these functors are biexact. The fact that such biexact functors leads to pairings of spectra is well known (and is worked out in careful detail in [8, C.2]). \square

The pairings of Proposition 2.4 determine the related pairings

$$\begin{aligned} (1) \quad \mathcal{K}^{fin}(Y, Z) \wedge \mathcal{K}^{fin}(X, Y) &\rightarrow \mathcal{K}^{fin}(X, Z), \\ \mathcal{K}^{qf}(Y, Z) \wedge \mathcal{K}^{qf}(X, Y) &\rightarrow \mathcal{K}^{qf}(X, Z), \quad \text{if } Y \text{ is projective, and} \\ \mathcal{K}^{qf}(Y, (U, Z)) \wedge \mathcal{K}^{qf}(X, Y) &\rightarrow \mathcal{K}^{qf}(X, (U, Z)), \quad \text{if } Y \text{ is projective} \end{aligned}$$

by taking the composition of

$$\begin{aligned} \mathcal{K}^{fin}(Y \times Z/Y) \wedge \mathcal{K}^{fin}(X \times Y/X) &\xrightarrow{\pi_1^* \wedge id} \mathcal{K}^{fin}(X \times Y \times Z/X \times Y) \wedge \mathcal{K}^{fin}(X \times Y/X) \\ &\rightarrow \mathcal{K}^{fin}(X \times Y \times Z/X) \xrightarrow{(\pi_{13})_*} \mathcal{K}^{fin}(X \times Z/X), \end{aligned}$$

and similarly for the other two cases. (The projective hypothesis is needed in the last two cases to ensure $(\pi_{13})_*$ is defined.)

3. BIVARIANT K -THEORY AND ALGEBRAIC K -HOMOLOGY

In this section we define the algebraic K -homology of a variety. Both algebraic K -homology and algebraic K -theory, at least for smooth varieties, are each special cases of a bivariant theory in the sense of Fulton-MacPherson [10] which is written $\mathcal{K}_{geom}(X \rightarrow Y)$ and which is also defined in this section. Since the constructions below do not recover algebraic K -theory for singular varieties (due to lack of homotopy invariance), we use the subscript *geom* in our notation. The motivation for this notation is that the space $\mathcal{K}_{geom}(X \times Y \rightarrow X)$ can be described as the “algebraic mapping space” from X to a certain ind-variety parameterizing finite length coherent sheaves on Y (see [15]), and is thus “geometric” in nature. (Since this description in terms of algebraic mapping spaces is not needed in this paper, we do not include it.) We also define a related bivariant theory, written $\mathcal{K}_{gqf}(X \rightarrow Y)$, defined in terms of sheaves with quasi-finite support.

Definition 3.1. Given a morphism $f : Y \rightarrow X$ of k -schemes, define

$$\mathcal{K}_{geom}(Y \xrightarrow{f} X) = \mathcal{K}_{geom}(Y/X)$$

to be the spectrum defined as the geometric realization of the simplicial spectrum $d \mapsto \mathcal{K}^{fin}(Y \times \Delta^d / X \times \Delta^d)$.

We introduce the following shorthand for special cases of the above construction:

$$\begin{aligned}\mathcal{K}_{geom}(X) &= \mathcal{K}_{geom}(X/X) = |d \mapsto \mathcal{K}(X \times \Delta^d)|, \\ \mathcal{K}^{hom}(Y) &= \mathcal{K}_{geom}(Y/\text{Spec } k) = |d \mapsto \mathcal{K}^{fin}(Y \times \Delta^d / \Delta^d)|.\end{aligned}$$

The spectrum $\mathcal{K}^{hom}(Y)$ is called the *algebraic K-homology* of Y .

We index the homotopy groups of these spectra as follows:

$$\begin{aligned}K_{geom}^{-n}(X/Y) &= \pi_n \mathcal{K}_{geom}(X/Y), \\ K_{geom}^{-n}(X) &= \pi_n \mathcal{K}_{geom}(X), \\ K_n^{hom}(Y) &= \pi_n \mathcal{K}^{hom}(Y).\end{aligned}$$

More generally, if A is an abelian group, we let $K_{geom}^{-n}(X/Y; A)$, $K_{geom}^{-n}(X; A)$, $K_n^{hom}(Y; A)$ denote the homotopy groups with coefficients in A of the corresponding spectra.

Remark 3.2. Strictly speaking, the assignment $V \mapsto \mathcal{K}^{fin}(Y \times V / X \times V)$, as we have defined it above, is a contravariant functor only up to homotopy, since the assignment $V \mapsto \mathcal{P}^{fin}(Y \times V / X \times V)$ is not actually a functor, but merely a functor up to natural isomorphism. Here we describe how to make Definition 3.1 rigorous by constructing a degree-wise homotopy equivalent version of $\mathcal{K}^{fin}(Y \times - / X \times -)$ that is a strictly functorial assignment of spectra to k -schemes. This is accomplished using the technique due to Grayson [13, §10] (see also [8, App. C]) to replace the exact categories $\mathcal{P}^{fin}(Y \times V / X \times V)$ with equivalent ones so that the resulting assignment $V \mapsto \mathcal{P}^{fin}(Y \times V / X \times V)$ is strictly functorial.

Specifically, redefine $\mathcal{P}^{fin}(Y/X)$ to be the category in which an object is an assignment $(p : X' \rightarrow X) \mapsto M_p$, where p is an arrow in Sch/X and M_p is a coherent sheaf on $Y' = X' \times_X Y$ that is finite and flat over X (i.e., M_p is an object of the category we formerly wrote as $\mathcal{P}^{fin}(Y'/X')$), together with isomorphisms $\theta_{p,q} : (q \times id_Y)^* M_p \xrightarrow{\cong} M_{p \circ q}$, for every pair of composable morphisms $X'' \xrightarrow{q} X' \xrightarrow{p} X$. These isomorphisms are required to satisfy the conditions that $\theta_{p,id} = id$ and that given a triple of composable morphisms $X''' \xrightarrow{r} X'' \xrightarrow{q} X' \xrightarrow{p} X$, the maps $\theta_{p \circ q, r} \circ (r \times id_Y)^*(\theta_{p,q})$ and $\theta_{p,q \circ r}$ coincide under the canonical isomorphism

$$(r \times id_Y)^*(q \times id_Y)^* M_p \cong (q \circ r \times id_Y)^* M_p.$$

A morphism f in this newly defined category $\mathcal{P}^{fin}(Y/X)$ from $(p \mapsto M_p)$ to $(p \mapsto N_p)$ is given by a compatible family of morphisms $f_p : M_p \rightarrow N_p$. The resulting category $\mathcal{P}^{fin}(Y/X)$ is equivalent via the forgetful functor to the category formerly called by the same name, but now the assignment $X \mapsto \mathcal{P}^{fin}(Y/X)$ is strictly functorial in X , since the map $f^* : \mathcal{P}^{fin}(Y/X) \rightarrow \mathcal{P}^{fin}(Y'/X')$ associated to a morphism $f : X' \rightarrow X$ is given by merely restricting the domain of the assignment $p \mapsto M_p$.

This construction also suffices to make the pushforward maps of Proposition 2.3 (3) strictly functorial. Specifically, given morphisms $Y \xrightarrow{g} Z \rightarrow X$ and an object $M = (p \mapsto M_p)$ of $\mathcal{P}^{fin}(Y/X)$, define $g_* M$ to be the object of $\mathcal{P}^{fin}(Z/X)$ defined by the assignment $p \mapsto (id_{X'} \times g)_* M_p$ and the family of isomorphisms given by the

compositions of the form

$$(q \times 1_Z)^*(1_{X'} \times g)_* M_p \cong (1_{X''} \times g)_*(q \times 1_Y)^* M_p \cong (1_{X''} \times g)_* M_{p \circ q}.$$

(The first isomorphism here is the canonical one which results from the fact that M_p is $\mathcal{O}_{X'}$ -flat and that the higher derived images of $(1_{X'} \times g)_*$ and $(1_{X''} \times g)_*$ vanish due to the finiteness hypothesis.) One may verify that these rules define a functor $g_* : \mathcal{P}^{fin}(Y/X) \rightarrow \mathcal{P}^{fin}(Z/X)$. Finally, one may verify that given a morphism $f : X' \rightarrow X$, the diagram

$$\begin{array}{ccc} \mathcal{P}^{fin}(Y/X) & \xrightarrow{g_*} & \mathcal{P}^{fin}(Z/X) \\ f^* \downarrow & & \downarrow f^* \\ \mathcal{P}^{fin}(X' \times_X Y/X') & \xrightarrow{(id_{X'} \times g)_*} & \mathcal{P}^{fin}(X' \times_X Z/X') \end{array}$$

commutes exactly.

As shown in the following result, $\mathcal{K}_{geom}(X)$ is weakly equivalent to $\mathcal{K}(X)$, the usual algebraic K -theory of X as defined by Quillen, provided X is regular. More generally, there are natural maps $\mathcal{K}(X) \rightarrow \mathcal{K}_{geom}(X) \rightarrow \mathcal{K}H(X)$, where $\mathcal{K}H(X)$ denotes Weibel's "homotopy algebraic K -theory" [37], and the former map is a weak equivalence provided X is K_0 -regular, with the latter map a weak equivalence provided X is K_0 -regular and the negative algebraic K -groups of X vanish. In this paper, the subscript *geom* serves as a reminder that \mathcal{K}_{geom} deviates from \mathcal{K} for singular varieties, but in fact we are ultimately only concerned with smooth varieties. Note also that we have taken the opportunity to rectify the unfortunate convention that the homotopy groups of K -theory are written using subscripts ($K_n(X) = \pi_n \mathcal{K}(X)$), by using superscripts for the cohomology theory \mathcal{K}_{geom} .

Proposition 3.3. *If X is a regular k -scheme, then $\mathcal{K}_{geom}(X)$ is weakly equivalent to $\mathcal{K}(X)$, the usual algebraic K -theory spectrum of X .*

Proof. The map of simplicial spectra

$$\mathcal{K}(X) \rightarrow (d \mapsto \mathcal{K}^{fin}(X \times \Delta^d / X \times \Delta^d)),$$

where the source is constant in the simplicial direction, is a degree-wise weak equivalence since K -theory is homotopy invariant on regular schemes. \square

Since we will mostly use the construction $\mathcal{K}_{geom}(Y/X)$ in the special case where $Y = X \times Z$, we introduce special notation in this setting:

Definition 3.4. For k -schemes X and Y , define

$$\mathcal{K}_{geom}(X, Y) = \mathcal{K}_{geom}(X \times Y / X) = |d \mapsto \mathcal{K}^{fin}(X \times Y \times \Delta^d / X \times \Delta^d)|,$$

where $X \times Y \rightarrow X$ is the canonical projection, and let $K_{geom}^{-n}(X, Y; A)$ denote its n -th homotopy group with coefficients in the abelian group A .

Note that we have the following identifications:

$$\begin{aligned} \mathcal{K}^{hom}(Y) &= \mathcal{K}_{geom}(\text{Spec } k, Y), \\ \mathcal{K}_{geom}(X) &= \mathcal{K}_{geom}(X, \text{Spec } k). \end{aligned}$$

We now define a variation of K_{geom} , using coherent sheaves with quasi-finite support in place of coherent sheaves with finite support. One should view the resulting theory, \mathcal{K}_{gqf} , as the "Borel-Moore theory" associated to \mathcal{K}_{geom} .

Definition 3.5. Given a morphism $f : Y \rightarrow X$ of k -schemes, the spectrum

$$\mathcal{K}_{gqf}(Y \xrightarrow{f} X) = \mathcal{K}_{gqf}(Y/X)$$

is the geometric realization of the simplicial spectrum $d \mapsto \mathcal{K}^{qf}(Y \times \Delta^d / X \times \Delta^d)$.

The *Borel-Moore K-homology* (or *algebraic K-homology with quasi-finite supports*) of a quasi-projective k -variety Y is defined to be $\mathcal{K}_{gqf}(Y/\mathrm{Spec} k)$ and is written

$$\mathcal{K}^{BM}(Y) = \mathcal{K}_{gqf}(Y/\mathrm{Spec} k) = |d \mapsto \mathcal{K}^{qf}(Y \times \Delta^d / \Delta^d)|.$$

More generally, if $U \rightarrowtail Y$ is a locally closed subscheme, we define

$$\mathcal{K}_{gqf}((U, Y)/X) = |d \mapsto \mathcal{K}^{qf}((U \times \Delta^d, Y \times \Delta^d) / X \times \Delta^d)|.$$

Finally, with no map $Y \rightarrow X$ specified, we define

$$\mathcal{K}_{gqf}(X, Y) = \mathcal{K}_{gqf}(X \times Y / X)$$

and

$$\mathcal{K}_{gqf}(X, (U, Y)) = \mathcal{K}_{gqf}((X \times U, X \times Y) / X),$$

where $X \times Y \rightarrow X$ is the canonical projection.

The homotopy groups of the above spectra are written

$$\begin{aligned} K_n^{BM}(Y) &= \pi_n \mathcal{K}^{BM}(Y), \\ K_{gqf}^{-n}((U, Y)/X) &= \pi_n \mathcal{K}_{gqf}((U, Y)/X), \\ K_{gqf}^{-n}(X, Y) &= \pi_n \mathcal{K}_{gqf}(X, Y), \\ K_{gqf}^{-n}(X, (U, Y)) &= \pi_n \mathcal{K}_{gqf}(X, (U, Y)), \end{aligned}$$

and similarly for homotopy groups with coefficients in an abelian group A .

Note that the definition of $\mathcal{K}_{gqf}((U, Y)/X)$ subsumes $\mathcal{K}_{gqf}(Y/X)$ by taking $U = Y$. The theory $\mathcal{K}_{gqf}((U, Y)/X)$ is not actually a bivariant theory in the sense of Fulton-MacPherson, but rather serves as a useful technical construction for comparing related theories. (See especially Section 5.) As before, to render Definition 3.5 rigorous, one must use a strictly functorial model for $\mathcal{K}^{qf}((U \times -, Y \times -) / X \times -)$. This is accomplished by paralleling the construction of Remark 3.2 in the obvious manner.

Proposition 3.6. *The spectrum $\mathcal{K}_{geom}(Y \rightarrow X)$ is a covariant functor on Sch/X via the pushforward maps of Proposition 2.3(3). For a fixed morphism $Y \rightarrow X$, sending a morphism $f : X' \rightarrow X$ to $\mathcal{K}_{geom}(X' \times_X Y \rightarrow X')$ defines a contravariant functor on Sch/X using the pullback maps of Proposition 2.3(4).*

In particular, $\mathcal{K}_{geom}(X, Y)$ is contravariantly functorial in X . The spectrum $\mathcal{K}_{geom}(X, Y)$ is also covariantly functorial in Y , where a morphism $g : Y \rightarrow Z$ is sent to the map

$$(id \times g)_* : \mathcal{K}_{geom}(X, Y) = \mathcal{K}_{geom}(X \times Y / X) \rightarrow \mathcal{K}_{geom}(X \times Z / X) = \mathcal{K}_{geom}(X, Z).$$

Proof. The results follow from Remark 3.2, which establishes that the squares

$$\begin{array}{ccc} \mathcal{K}^{fin}(Y \times \Delta^d / X \times \Delta^d) & \xrightarrow{(1 \times \alpha)^*} & \mathcal{K}^{fin}(Y \times \Delta^e / X \times \Delta^e) \\ (f \times 1)^* \downarrow & & \downarrow (f \times 1)^* \\ \mathcal{K}^{fin}(X' \times_X Y \times \Delta^d / X' \times \Delta^d) & \xrightarrow{(1 \times \alpha)^*} & \mathcal{K}^{fin}(X' \times_X Y \times \Delta^e / X' \times \Delta^e) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{K}^{fin}(Y \times \Delta^d / X \times \Delta^d) & \xrightarrow{(1 \times \alpha)^*} & \mathcal{K}^{fin}(Y \times \Delta^e / X \times \Delta^e) \\ (g \times 1)_* \downarrow & & \downarrow (g \times 1)_* \\ \mathcal{K}^{fin}(Z \times \Delta^d / X \times \Delta^d) & \xrightarrow{(1 \times \alpha)^*} & \mathcal{K}^{fin}(Z \times \Delta^e / X \times \Delta^e) \end{array}$$

commute on the nose for any face or degeneracy map $\alpha : \Delta^e \rightarrow \Delta^d$. Thus, the families of maps $(f \times 1_{\Delta^d})^*$ and $(g \times 1_{\Delta^d})_*$ determine maps of simplicial spectra

$$f^* : \mathcal{K}_{geom}(Y/X) \rightarrow \mathcal{K}_{geom}(X' \times_X Y/X')$$

and

$$g_* : \mathcal{K}_{geom}(Y/X) \rightarrow \mathcal{K}_{geom}(Z/X').$$

Functoriality of these maps is easily verified. \square

In particular $\mathcal{K}^{hom}(Y)$ is a homology theory in the sense that it is covariantly functorial with respect to all maps. Thus it obviously differs from G -theory (also known as K' -theory), since the latter has the same functoriality as Borel-Moore homology. But the two theories do not agree even on projective varieties, as the following result shows. (The proposition is proven by use of some of the results of Sections 4 and 5 below — we have included it here for the sake of motivation.)

Proposition 3.7. *For a smooth, projective k -variety Y , there is a natural isomorphism*

$$K_0^{hom}(Y) \cong CH_0(Y),$$

where $CH_0(Y)$ is the group of zero-cycles on Y modulo rational equivalence.

Proof. Note that $K_0^{hom}(Y)$ is isomorphic to the cokernel of

$$K_0^{fin}(Y \times \mathbb{A}^1 / \mathbb{A}^1) \xrightarrow{i_0^* - i_1^*} K_0^{fin}(Y / \text{Spec } k),$$

where $i_\epsilon : \text{Spec } k \rightarrowtail \mathbb{A}^1$ is inclusion at ϵ . As in Proposition 4.5 below, there is a natural cycle class map forming the horizontal maps of the commutative diagram

$$(2) \quad \begin{array}{ccc} K_0^{fin}(Y \times \mathbb{A}^1 / \mathbb{A}^1) & \xrightarrow{\text{cyc}} & c_{equi}^0(Y \times \mathbb{A}^1 / \mathbb{A}^1) \\ i_0^* - i_1^* \downarrow & & \downarrow i_0^* - i_1^* \\ K_0^{fin}(Y / \text{Spec } k) & \xrightarrow{\text{cyc}} & c_{equi}^0(Y / \text{Spec } k). \end{array}$$

(Here, $c_{equi}^0(Y \times X/X)$ is the free abelian group on closed, integral subschemes of $Y \times X$ which are finite and dominant over X .) Moreover, the proof of Proposition 5.4 below shows that the natural maps

$$K_0^{fin}(Y \times \mathbb{A}^1 / \mathbb{A}^1) \xrightarrow{\cong} G_0^{fin}(Y \times \mathbb{A}^1 / \mathbb{A}^1) \quad \text{and} \quad K_0^{fin}(Y / \text{Spec } k) \xrightarrow{\cong} G_0^{fin}(Y / \text{Spec } k)$$

are isomorphisms. An easy devissage argument shows that $G_0^{fin}(Y / \text{Spec } k)$ is isomorphic to $c_{equi}^0(Y / \text{Spec } k)$ (which is the free abelian group on the set of closed points of Y), so that the bottom arrow of (2) is an isomorphism. One may readily verify that $\text{cyc} : G_0^{fin}(Y \times \mathbb{A}^1 / \mathbb{A}^1) \rightarrow c_{equi}^0(Y \times \mathbb{A}^1 / \mathbb{A}^1)$ is split by the map sending $Z \subset Y \times \mathbb{A}^1$ to $[\mathcal{O}_Z]$, and thus is a surjection. It follows that the top arrow of (2) is surjective, and thus an easy diagram chase shows that $K_0^{hom}(Y)$ is isomorphic to the cokernel of the right-hand vertical map of (2). The fact that the latter is isomorphic to $CH_0(Y)$ follows from [9, 4.4 and 8.1] and [26, 2.1]. \square

Proposition 3.8. *The maps of Proposition 2.3(8) give the assignment $(Y \rightarrow X) \mapsto \mathcal{K}_{gqf}(Y \rightarrow X)$ the structure of a covariant functor on the category consisting of the objects of Sch/X and morphisms given by proper morphisms of varieties over X .*

Similarly, the maps of Proposition 2.3(10) give the assignment $(Y \rightarrow X) \mapsto \mathcal{K}_{gqf}(Y \rightarrow X)$ the structure of a contravariant functor on the category with objects Sch/X and morphisms given by quasi-finite and flat morphisms of varieties over X .

Finally, for a fixed $Y \rightarrow X$, sending $f : X' \rightarrow X$ to $\mathcal{K}_{gqf}(X' \times_X Y \rightarrow X')$ is a contravariant functor on Sch/X via the maps induced by those of Proposition 2.3(9).

In particular, $\mathcal{K}^{BM}(Y)$ is covariantly functorial in Y for proper morphisms and contravariantly functorial in Y for quasi-finite, flat morphisms.

Proof. The proof is the same as that of Proposition 3.6. \square

Proposition 3.9. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of k -varieties. The pairings of Proposition 2.4 induce pairings*

$$-\bullet- : \mathcal{K}_{geom}(Z \xrightarrow{g} Y) \wedge \mathcal{K}_{geom}(Y \xrightarrow{f} X) \rightarrow \mathcal{K}_{geom}(Z \xrightarrow{f \circ g} X)$$

and

$$-\bullet- : \mathcal{K}_{gqf}(Z \xrightarrow{g} Y) \wedge \mathcal{K}_{gqf}(Y \xrightarrow{f} X) \rightarrow \mathcal{K}_{gqf}(Z \xrightarrow{f \circ g} X)$$

which are associative in the following sense: If $W \xrightarrow{h} Z$ is another morphism of k -schemes, the diagrams

$$\begin{array}{ccc} \mathcal{K}_{geom}(W/Z) \wedge \mathcal{K}_{geom}(Z/Y) \wedge \mathcal{K}_{geom}(Y/X) & \longrightarrow & \mathcal{K}_{geom}(W/Z) \wedge \mathcal{K}_{geom}(Z/X) \\ \downarrow & & \downarrow \\ \mathcal{K}_{geom}(W/Y) \wedge \mathcal{K}_{geom}(Y/X) & \longrightarrow & \mathcal{K}_{geom}(W/X) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{K}_{gqf}(W/Z) \wedge \mathcal{K}_{gqf}(Z/Y) \wedge \mathcal{K}_{gqf}(Y/X) & \longrightarrow & \mathcal{K}_{gqf}(W/Z) \wedge \mathcal{K}_{gqf}(Z/X) \\ \downarrow & & \downarrow \\ \mathcal{K}_{gqf}(W/Y) \wedge \mathcal{K}_{gqf}(Y/X) & \longrightarrow & \mathcal{K}_{gqf}(W/X) \end{array}$$

commute up to homotopy.

Proof. The collection of biexact pairings

$$\mathcal{P}^{fin}(Z \xrightarrow{g} Y) \times \mathcal{P}^{fin}(Y \xrightarrow{f} X) \rightarrow \mathcal{P}^{fin}(Z \xrightarrow{f \circ g} X)$$

of Proposition 2.4 extend to strictly functorial pairings in X using the models for these categories described in Remark 3.2. Namely, if $p \mapsto M_p$ and $q \mapsto N_q$ are objects of $\mathcal{P}^{fin}(Z/Y)$ and $\mathcal{P}^{fin}(Y/X)$, then the assignment of $r : X' \rightarrow X$ to $M_{r'} \otimes_{\mathcal{O}_Z} g^* N_h$ (where $r' : Y' \rightarrow Y$ is the pullback of r) defines an object of $\mathcal{P}^{fin}(Z/X)$. It follows that the maps

$$\mathcal{K}^{fin}(Z \times \Delta^d / Y \times \Delta^d) \wedge \mathcal{K}^{fin}(Y \times \Delta^d / X \times \Delta^d) \rightarrow \mathcal{K}^{fin}(Z \times \Delta^d / X \times \Delta^d), \quad d \geq 0,$$

determine a map of simplicial spectra. The geometric realization of this map is the map we seek. The asserted functoriality follows from the fact that tensor product

is associative up to canonical isomorphism, which determines for each fixed $d \geq 0$ a homotopy joining the two evident maps of the form

$$\begin{aligned} \mathcal{K}^{fin}(W \times \Delta^d / Z \times \Delta^d) \wedge \mathcal{K}^{fin}(Z \times \Delta^d / Y \times \Delta^d) \wedge \mathcal{K}^{fin}(Y \times \Delta^d / X \times \Delta^d) \\ \longrightarrow \mathcal{K}^{fin}(W \times \Delta^d / X \times \Delta^d). \end{aligned}$$

The canonical nature of the isomorphism used to define these homotopies implies that these homotopies commute with face and degeneracy maps, thereby determining a homotopy joining the two evident simplicial maps of the form

$$\begin{aligned} \mathcal{K}^{fin}(W \times \Delta^\bullet / Z \times \Delta^\bullet) \wedge \mathcal{K}^{fin}(Z \times \Delta^\bullet / Y \times \Delta^\bullet) \wedge \mathcal{K}^{fin}(Y \times \Delta^\bullet / X \times \Delta^\bullet) \\ \longrightarrow \mathcal{K}^{fin}(W \times \Delta^\bullet / X \times \Delta^\bullet). \end{aligned}$$

The result involving \mathcal{K}_{gqf} is proved in a parallel fashion. \square

Taking homotopy groups with coefficients in \mathbb{Z}/n with n any integer such that $n \not\equiv 2 \pmod{4}$ (possibly equal to 0), the pairings of Proposition 3.9 induce pairing of abelian groups:

$$(3) \quad \begin{aligned} K_{geom}^p(Z/Y; \mathbb{Z}/n) \otimes K_{geom}^q(Y/X; \mathbb{Z}/n) &\rightarrow K_{geom}^{p+q}(Z/X; \mathbb{Z}/n), \\ K_{gqf}^p(Z/Y; \mathbb{Z}/n) \otimes K_{gqf}^q(Y/X; \mathbb{Z}/n) &\rightarrow K_{gqf}^{p+q}(Z/X; \mathbb{Z}/n). \end{aligned}$$

However, the mod- n Moore space fails have an appropriate coproduct rule if the prime factorization of n involves small powers of 2 and 3. (In fact, the above pairings fail to exist even if $n \equiv 2 \pmod{4}$.) Thus, when using the product pairings (3), we frequently assume that n satisfies the following technical condition mentioned also in the introduction:

(Condition 2-3) If $3 | n$, then $9 | n$, and if $2 | n$, then $16 | n$.

(See [27, A.6].) For such n , the pairings (3) are associative in the evident sense, and when $X = Y = Z$, these pairings are graded-commutative.

For any value of n , the maps of Propositions 3.6 and 3.8 induce functorial maps of abelian groups:

$$(4) \quad \begin{aligned} f_* : K_{geom}^p(Y/X; \mathbb{Z}/n) &\rightarrow K_{geom}^p(Z/X; \mathbb{Z}/n), \quad f : Y \rightarrow Z \text{ arbitrary}, \\ g^* : K_{geom}^p(Y/X; \mathbb{Z}/n) &\rightarrow K_{geom}^p(Y'/X'; \mathbb{Z}/n), \quad g : X' \rightarrow X \text{ arbitrary}, \\ f_* : K_{gqf}^p(Y/X; \mathbb{Z}/n) &\rightarrow K_{gqf}^p(Z/X; \mathbb{Z}/n), \quad f : Y \rightarrow Z \text{ proper}, \\ g^* : K_{gqf}^p(Y/X; \mathbb{Z}/n) &\rightarrow K_{gqf}^p(Y'/X'; \mathbb{Z}/n) \quad g : X' \rightarrow X \text{ arbitrary}. \end{aligned}$$

We also have special pullback maps of the form

$$(5) \quad f^* : K_{gqf}^p(Z/X; \mathbb{Z}/n) \rightarrow K_{gqf}^p(Y/X; \mathbb{Z}/n), \quad \text{if } f : Y \rightarrow Z \text{ is quasi-finite, flat.}$$

Using the results established above, it is not difficult to verify that the groups K_{geom}^* and K_{gqf}^* define bivariant theories in the sense of Fulton-MacPherson [10]. A bivariant theory assigns a graded abelian group to each morphism of the category Sch/k together with pullback maps for arbitrary morphism and pushforward maps for “confined” morphisms. The definition of the class of confined morphisms is part of the definition of a bivariant theory, as is the class of so-called “independent squares”, each of which is required to satisfy certain axioms (cf. [10, 2.1]). The proof of the following theorem along with a precise list of the Fulton-MacPherson axioms is contained in the appendix of this paper.

Theorem 3.10. *Let n be an integer (possibly 0) such that n satisfies (Condition 2-3).*

The collection of graded abelian groups $K_{geom}^(Y \xrightarrow{f} X; \mathbb{Z}/n)$, where $f : Y \rightarrow X$ ranges over all morphisms of Sch/k , together with the product pairings (3) and pushforward and pullback maps (4) satisfy the axioms of a skew-commutative bivariant theory with units in the sense of Fulton-MacPherson [10]. In this case, the confined morphisms are all morphisms of Sch/k and the independent squares are all pullback squares in Sch/k .*

Similarly, the collection of graded abelian groups $K_{gqf}^(X \xrightarrow{f} Y; \mathbb{Z}/n)$ forms a skew-commutative bivariant theory with units in the sense of Fulton-MacPherson, where the confined morphisms are all proper morphisms in Sch/k and the independent squares are all pullback squares.*

Moreover, the canonical maps

$$K_{geom}^*(X \xrightarrow{f} Y; \mathbb{Z}/n) \rightarrow K_{gqf}^*(X \xrightarrow{f} Y; \mathbb{Z}/n),$$

induced by the evident inclusions of categories, define a so-called “Grothendieck” transformation of bivariant theories.

As with any bivariant theory, we have cup and cap product operations. Explicitly, *cup product* for K_{geom}^* is the pairing

$$- \cup - : K_{geom}^p(X; \mathbb{Z}/n) \otimes K_{geom}^q(X; \mathbb{Z}/n) \rightarrow K_{geom}^{p+q}(X; \mathbb{Z}/n)$$

given by taking $X = Y = Z$ in (3). The analogous definition for K_{gqf}^* leads to the exact same pairing since $\mathcal{K}_{gqf}(X/X) = \mathcal{K}_{geom}(X/X) = \mathcal{K}_{geom}(X)$. The *cap product* pairings for K_{geom}^* and K_{gqf}^* are given by the maps

$$- \cap - : K_{geom}^p(Y; \mathbb{Z}/n) \otimes K_q^{\text{hom}}(Y; \mathbb{Z}/n) \rightarrow K_{q-p}^{\text{hom}}(Y; \mathbb{Z}/n)$$

and

$$K_{geom}^p(Y; \mathbb{Z}/n) \otimes K_q^{\text{BM}}(Y; \mathbb{Z}/n) \rightarrow K_{q-p}^{\text{BM}}(Y; \mathbb{Z}/n)$$

defined by taking $Z = Y$ and $X = \text{Spec } k$ in (3).

Additionally, as an immediate consequence of the axioms, we obtain natural Gysin maps. Namely, if $f : X \rightarrow Y$ is finite and flat, then \mathcal{O}_X determines a class $[f]$ in $K_{geom}^0(X \rightarrow Y)$ and pairing with this class defines a “wrong-way” map

$$f_! : K_{geom}^*(X) \rightarrow K_{geom}^*(Y)$$

defined by $f_!(x) = f_*(x \bullet [f])$. Similarly, $[f]$ defines a map

$$f^! : K_*^{\text{hom}}(Y) \rightarrow K_*^{\text{hom}}(X)$$

by $f^!(y) = [f] \bullet y$. If $f : X \rightarrow Y$ is merely quasi-finite and flat, we have a similarly defined map

$$f^! : K_*^{\text{BM}}(Y) \rightarrow K_*^{\text{BM}}(X).$$

These Gysin maps satisfy the expected axioms (cf. [10, 2.5]). Moreover, the last of these Gysin maps coincides with the map $f^* : K_*^{\text{BM}}(Y) \rightarrow K_*^{\text{BM}}(X)$ given in (5) (by taking $X = \text{Spec } k$, $Y = X$, and $Z = Y$).

As shown in [10, §2.3–2.6], the axioms of a bivariant theory directly imply a host of other useful properties. We list those such properties needed in this paper in the following result.

Corollary 3.11. *Let $n \in \mathbb{Z}$ be an integer (possibly 0) satisfying (Condition 2-3). Then*

- (1) $K_{geom}^*(X; \mathbb{Z}/n)$ is a graded-commutative ring under cup product, and for any morphism $f : X' \rightarrow X$, the pullback morphism f^* is a ring homomorphism;
 - (2) $K_*^{hom}(X; \mathbb{Z}/n)$ is a $K_{geom}^*(X; \mathbb{Z}/n)$ -module under cap product, and for any morphism $f : X \rightarrow Y$, the map
- $$f_* : K_*^{hom}(X; \mathbb{Z}/n) \rightarrow K_*^{hom}(Y; \mathbb{Z}/n)$$
- is a $K_{geom}^*(Y; \mathbb{Z}/n)$ -module homomorphism (where $K_*^{hom}(X; \mathbb{Z}/n)$ is viewed as a $K_{geom}^*(Y; \mathbb{Z}/n)$ -module via restriction of scalars along f^*).
- (3) $K_*^{BM}(X; \mathbb{Z}/n)$ is a $K_{geom}^*(X; \mathbb{Z}/n)$ -module under cap product, and if $f : X \rightarrow Y$ is proper,
- $$f_* : K_*^{BM}(X; \mathbb{Z}/n) \rightarrow K_*^{BM}(Y; \mathbb{Z}/n)$$
- is a $K_{geom}^*(Y; \mathbb{Z}/n)$ -module homomorphism.

- (4) If $f : X \rightarrow Y$ is quasi-finite and flat, then the pullback map

$$f^* : K_*^{BM}(Y; \mathbb{Z}/n) \rightarrow K_*^{BM}(X; \mathbb{Z}/n)$$

defined in Proposition 3.8 is a $K_{geom}^*(Y; \mathbb{Z}/n)$ -module homomorphism.

Proof. The first three of these properties is a formal consequence of the axioms, as explained in [10, §2.3]. To prove the final assertion, since the map f^* coincides with the “Gysin” homomorphism $f^!$ introduced above, the desired result is given by [10, G₃(iii)]. \square

For use in later sections, we record the following evident fact.

Proposition 3.12. *If $f : Y \rightarrow X$ is a proper morphism of k -schemes, we have $\mathcal{M}^{fin}(Y/X) = \mathcal{M}^{qf}(Y/X)$ and $\mathcal{P}^{fin}(Y/X) = \mathcal{P}^{qf}(Y/X)$ and so we have $\mathcal{G}^{fin}(Y/X) = \mathcal{G}^{qf}(Y/X)$ and $\mathcal{K}^{fin}(Y/X) = \mathcal{K}^{qf}(Y/X)$. In particular, $\mathcal{K}_{gqf}(Y/X) = \mathcal{K}_{geom}(Y/X)$ for such an f , and $\mathcal{K}^{BM}(Y) = \mathcal{K}^{hom}(Y)$ if Y is any projective k -scheme.*

Proof. A subscheme Z of Y which is quasi-finite over X is necessarily finite, since $Y \rightarrow X$ is a proper morphism. \square

4. TRANSFERS

In this section we develop and study a notion of transfers for presheaves on the category Sm/k of smooth, quasi-projective k -varieties. The notion of transfers studied here is analogous to (but slightly more general than) Voevodsky’s notion of a pretheory [29, §3]. Some of the ideas in this section appeared in the author’s thesis [32] and all of the ideas are motivated by the work of Voevodsky [29] and Suslin-Voevodsky [24]. With an eye toward possible future applications, we endeavor to define transfer maps so as to include the largest collection of presheaves possible. This leads to the notion of a “presheaf with K_0^\oplus transfers” defined below. In Suslin’s forthcoming preprint “On the Grayson Spectral Sequence”, he uses (a slightly weaker version of) presheaves with K_0^\oplus transfers as part of his proof that the E_2 terms of Grayson’s spectral sequence [14] coincide with the motivic cohomology groups. Thus, the results in this section also find applications beyond this paper. (See Remark 4.3.)

Given a morphism $f : Y \rightarrow X$ in Sch/k , let $K_0^{fin,\oplus}(Y/X)$ denote the Grothendieck group of $\mathcal{P}^{fin}(Y/X)$ regarded as merely an additive category. That is, $K_0^{fin,\oplus}(Y/X)$ is the group completion of the abelian monoid generated by isomorphism classes of objects of $\mathcal{P}^{fin}(Y/X)$ subject to the relation $[M] + [N] = [P]$ if $P \cong M \oplus N$. (Note that not every short exact sequence of $\mathcal{P}^{fin}(Y/X)$ splits even if X and Y are both affine varieties.) The groups $K_0^{fin,\oplus}(Y/X)$ enjoy the same functoriality as do the groups $K_0^{fin}(Y/X)$.

Definition 4.1 (Compare with Voevodsky's definition of a pretheory [29, 3.1]). A *presheaf with K_0^\oplus transfers* is a presheaf F of abelian groups defined on the category Sm/k equipped with the following additional structure: given a morphism $\pi : X \rightarrow S$ that is smooth of relative dimension one with X and S objects of Sm/k (for short, we will refer to π as a "smooth curve"), there is a pairing of abelian groups

$$\theta_{X/S} : K_0^{fin,\oplus}(X/S) \otimes_{\mathbb{Z}} F(X) \rightarrow F(S).$$

These pairings are required to satisfy the following axioms:

- (1) (Normalization) If $i : S \rightarrow X$ is a splitting of π and $[i] = [\mathcal{O}_{\Gamma_i}] \in K_0^{fin,\oplus}(X/S)$ is the class of the structure sheaf of the graph of i , then $\theta_{X/S}([i] \otimes -) : F(X) \rightarrow F(S)$ coincides with i^* .
- (2) (Naturality) If $f : S' \rightarrow S$ is any morphism of smooth varieties, $X' = X \times_S S'$, and $g : X' \rightarrow X$ and $f^* : K_0^{fin,\oplus}(X/S) \rightarrow K_0^{fin,\oplus}(X'/S')$ are the induced maps, then the diagram

$$\begin{array}{ccc} K_0^{fin,\oplus}(X/S) \otimes F(X) & \xrightarrow{f^* \otimes g^*} & K_0^{fin,\oplus}(X'/S') \otimes F(X') \\ \theta_{X/S} \downarrow & & \downarrow \theta_{X'/S'} \\ F(S) & \xrightarrow{f^*} & F(S') \end{array}$$

commutes.

- (3) (Additivity) For X and Y in Sm/k , the natural map $F(X \amalg Y) \rightarrow F(X) \oplus F(Y)$ is an isomorphism.

A morphism $f : F \rightarrow G$ of presheaves with K_0^\oplus transfers is a morphism of presheaves such that the diagram

$$\begin{array}{ccc} K_0^{fin,\oplus}(X/S) \otimes_{\mathbb{Z}} F(X) & \longrightarrow & F(S) \\ \downarrow & & \downarrow \\ K_0^{fin,\oplus}(X/S) \otimes_{\mathbb{Z}} G(X) & \longrightarrow & G(S) \end{array}$$

commutes for all smooth curves $X \rightarrow S$.

In general, given a presheaf F , a smooth curve $X \rightarrow S$ and an element $\alpha \in K_0^{fin,\oplus}(X/S)$, we let $\alpha_* : F(X) \rightarrow F(S)$ denote the map $\theta_{X/S}(\alpha \otimes -)$.

Definition 4.2. A presheaf with K_0 transfers is a presheaf F of abelian groups on Sm/k satisfying the same axioms as in Definition 4.1 with $K_0^{fin,\oplus}$ replaced everywhere by K_0^{fin} .

Remark 4.3. In Suslin's forthcoming preprint "On the Grayson Spectral Sequence", he uses the notion of a " K_0^\oplus -presheaf". This is a presheaf F on Sm/k equipped

with additional pairings

$$F(V) \otimes K_0^{fin,\oplus}(U, V) \rightarrow F(U)$$

satisfying certain evident axioms. (Here, $K_0^{fin,\oplus}(U, V) = K_0^{fin,\oplus}(U \times V/U)$.) In fact, these axioms amount to asserting that F is a presheaf on $K^\oplus(Sm/k)$, where $K^\oplus(Sm/k)$ is the category whose objects are the objects of Sm/k but whose morphisms are given by the abelian groups $K^\oplus(X, Y)$. One may readily verify that by associating to each smooth curve $X \rightarrow S$ the map $K_0^{fin,\oplus}(X/S) \rightarrow K^\oplus(S, X)$ induced by pushforward along $X \rightarrow S \times X$, every K_0^\oplus -presheaf is naturally a presheaf with K^\oplus transfers. Thus, all of the results in this section about presheaves with K_0^\oplus transfers can be employed in Suslin's context to K^\oplus -presheaves.

Observe that the canonical map $K_0^{fin,\oplus}(X/S) \rightarrow K_0^{fin}(X/S)$ is a surjection and represents a natural transformation of functors on the category of smooth schemes over S . (By a “smooth scheme over S ”, we mean here and elsewhere a morphism $Y \rightarrow S$, not necessarily smooth, such that $Y \in Sm/k$.) Consequently, we could equivalently define a presheaf with K_0 transfers to be a presheaf with K_0^\oplus transfers such that for all smooth curves $X \rightarrow S$, the structure map

$$\theta_{X/S} : K_0^{fin,\oplus}(X/S) \otimes F(X) \rightarrow F(S)$$

factors through the canonical surjection $K_0^{fin,\oplus}(X/S) \otimes F(X) \twoheadrightarrow K_0^{fin}(X/S) \otimes F(X)$.

Recall that a presheaf of abelian groups F defined on Sm/k is said to be *homotopy invariant* if the natural map $F(X) \rightarrow F(X \times \mathbb{A}^1)$ is an isomorphism for all X in Sm/k .

Proposition 4.4. *A presheaf with K_0 transfers is naturally a presheaf with K_0^\oplus transfers. Conversely, a homotopy invariant presheaf with K_0^\oplus transfers is naturally a presheaf with K_0 transfers.*

Proof. The first assertion follows from the naturality of the map $K_0^{fin,\oplus}(X/S) \rightarrow K_0^{fin}(X/S)$. For a smooth curve $X \rightarrow S$, define $[K_0^{fin}(X/S)]$ (resp., $[K_0^{fin,\oplus}(X/S)]$) to be the cokernel of the map

$$i_1^* - i_0^* : K_0^{fin}(X \times \mathbb{A}^1/S \times \mathbb{A}^1) \rightarrow K_0^{fin}(X/S)$$

(resp., the map obtained by replacing K_0^{fin} with $K_0^{fin,\oplus}$), where $i_\epsilon : S \rightarrow S \times \mathbb{A}^1$ is inclusion at $\epsilon \in \mathbb{A}^1$. (Note that $[K_0^{fin}(X/S)]$ is naturally isomorphic to $K_{geom}^0(X/S)$.) Then the groups $[K_0^{fin}(X/S)]$ enjoy the same functoriality as do $K_0^{fin}(X/S)$ and the canonical surjection $K_0^{fin}(X/S) \twoheadrightarrow [K_0^{fin}(X/S)]$ is natural in S , and similarly for $K_0^{fin,\oplus}(X/S) \twoheadrightarrow [K_0^{fin,\oplus}(X/S)]$. If F is a homotopy invariant presheaf with K_0 transfers, then the structure maps induce natural pairings

$$[K_0^{fin}(X/S)] \otimes F(X) \rightarrow F(S).$$

The result will follow once we show the natural surjection $[K_0^{fin,\oplus}(X/S)] \rightarrow [K_0^{fin}(X/S)]$ is actually an isomorphism for all smooth curves $X \rightarrow S$. To show this, suppose

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is a short exact sequence in $\mathcal{P}^{fin}(X/S)$. Let $N[t] \in \mathcal{P}^{fin}(X \times \mathbb{A}^1/S \times \mathbb{A}^1)$, for $N \in \mathcal{P}^{fin}(X/S)$, denote the pullback of N along the projection $S \times \mathbb{A}^1 \rightarrow S$, where

$\mathbb{A}^1 = \text{Spec } k[t]$. Then we have an exact sequence in $\mathcal{P}^{fin}(X \times \mathbb{A}^1 / S \times \mathbb{A}^1)$ of the form

$$0 \rightarrow M'[t] \rightarrow M[t] \rightarrow M''[t] \rightarrow 0.$$

Form the diagram of coherent sheaves on $X \times \mathbb{A}^1$

$$\begin{array}{ccccccc} 0 & \longrightarrow & M'[t] & \longrightarrow & \tilde{M} & \longrightarrow & M''[t] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cdot t \\ 0 & \longrightarrow & M'[t] & \longrightarrow & M[t] & \longrightarrow & M''[t] \longrightarrow 0 \end{array}$$

with exact rows and such that the right-hand square is Cartesian. Then \tilde{M} belongs to $\mathcal{P}^{fin}(X \times \mathbb{A}^1 / S \times \mathbb{A}^1)$ and satisfies $i_0^* \tilde{M} \cong M' \oplus M''$ and $i_1^* \tilde{M} \cong M$. Since the kernel of $K_0^{fin, \oplus}(X/S) \twoheadrightarrow K_0^{fin}(X/S)$ is generated by elements of the form $[M] - [M'] - [M'']$, it follows that $[K_0^{fin, \oplus}(X/S)] \cong [K_0^{fin}(X/S)]$. \square

Recall that for a smooth curve $\pi : X \rightarrow S$, Voevodsky defines $c_{equi}^0(X/S)$ to be the group of relative Cartier divisors (cf. [29, §2]). Equivalently, $c_{equi}^0(X/S)$ is the free abelian group on closed, integral subschemes $Z \subset X$ such that the induced map $Z \rightarrow S$ is finite and dominates a connected component of S . (Note that we have not assumed S is connected.) Given a map $f : S' \rightarrow S$ of smooth varieties, there is an induced map

$$f^* : c_{equi}^0(X/S) \rightarrow c_{equi}^0(X'/S'),$$

where $X' = X \times_S S'$, given by pullback of cycles as defined, for example, in [22, V.C.7]. Specifically, if $Z \subset X$ is a closed, integral subscheme finite and dominant over a component of S , then $f^*([Z])$ is given by the formula

$$f^*([Z]) = \sum_w \text{length}((f^* \mathcal{O}_Z)_w)[\bar{w}] = \sum_w \text{length}(\mathcal{O}_{Z \times_X X', w})[\bar{w}],$$

where w ranges over points of X' whose closures are finite and dominant over a component of S' .

A *pretheory*, as defined by Voevodsky in [29, 3.1], is a presheaf satisfying the hypotheses of Definition 4.1 with $K_0^{fin, \oplus}(-)$ replaced by $c_{equi}^0(-)$ everywhere.

Proposition 4.5. *Given a smooth curve $\pi : X \rightarrow S$, there is a well-defined cycle map*

$$\text{cyc}_{X/S} : K_0^{fin}(X/S) \rightarrow c_{equi}^0(X/S)$$

given on a generator $[M]$, $M \in \mathcal{P}^{fin}(X/S)$, by $\text{cyc}([M]) = \sum_z \text{length}(M_z)$, where the sum ranges over points z of X whose closures are finite and dominant over some component of S . Moreover, the map $\text{cyc}_{X/S}$ is natural in the sense that given $S' \rightarrow S$ with $S' \in Sm/k$, the diagram

$$\begin{array}{ccc} K_0^{fin}(X/S) & \xrightarrow{\text{cyc}_{X/S}} & c_{equi}^0(X/S) \\ \downarrow & & \downarrow \\ K_0^{fin}(S' \times_S X/S') & \xrightarrow{\text{cyc}_{X'/S'}} & c_{equi}^0(S' \times_S X/S') \end{array}$$

commutes.

Proof. The function length_z , for any z , and hence the map $\text{cyc}_{X/S}$ are additive for short exact sequences in $\mathcal{P}^{fin}(X/S)$, so that $\text{cyc}_{X/S}$ is well defined. The naturality is immediate from the definition of cycle pullback. \square

Corollary 4.6. *Every pretheory is a presheaf with K_0 transfers and hence a presheaf with K_0^\oplus transfers.*

Suppose D is a generator of $c_{equi}^0(X/S)$. Then the sheaf of ideals \mathcal{I}_D cutting out D in X is locally principle. Using the local criterion of flatness [18, 22.6], it follows that the natural map $D \rightarrow S$ is flat, and so the coherent sheaf \mathcal{O}_D on X belongs to $\mathcal{P}^{fin}(X/S)$ and hence determines a class $[\mathcal{O}_D]$ in $K_0^{fin,\oplus}(X/S)$.

Definition 4.7. Given a smooth curve $\pi : X \rightarrow S$, where X and S are regular schemes essentially of finite type over k , let

$$cl_{X/S} : c_{equi}^0(X/S) \rightarrow K_0^{fin,\oplus}(X/S)$$

be the homomorphism defined on a closed, integral subscheme Z of X which is finite and dominant over S by the formula $cl_{X/S}(Z) = [\mathcal{O}_Z]$. By abuse of notation, we also write cl for the composition of the above map with the canonical surjection $K_0^{fin,\oplus}(X/S) \rightarrow K_0^{fin}(X/S)$.

Note that $cl_{X/S}$ is defined in terms of the standard basis of $c_{equi}^0(X/S)$, and in general $cl_{X/S}(D) \neq [\mathcal{O}_D]$ for an arbitrary Cartier divisor D . Additionally, the map $cl_{X/S}$ is not, in general, natural in S (but see the proof of Proposition 4.8 for an important situation in which a form of naturality holds).

Under the cycle map, the element $[\mathcal{O}_Z]$ is sent to $[Z] \in c_{equi}^0(X/S)$, and thus $cl_{X/S}$ gives a (non-natural) splitting of $\text{cyc}_{X/S}$. In particular, the map $\text{cyc}_{X/S}$ is surjective, and hence a pretheory can equivalently be defined as a presheaf F with K_0^\oplus transfers such that for all smooth curves $X \rightarrow S$, the structure map $K_0^{fin,\oplus}(X/S) \otimes F(X)$ factors through the natural surjection

$$K_0^{fin,\oplus}(X/S) \otimes F(X) \xrightarrow{\text{cyc} \otimes \text{id}} c_{equi}^0(X/S) \otimes F(X).$$

Corollary 4.6 admits a partial converse:

Proposition 4.8. *Suppose F is a presheaf with K_0 transfers such that F is separated for the Zariski topology (i.e., the natural map $F(X) \rightarrow \bigoplus_i F(U_i)$ is injective for every finite covering $X = \bigcup_i U_i$ by Zariski open subschemes). Then F admits the structure of a pretheory in a natural manner.*

Proof. The map $cl_{X/S}$ is not natural in S in general, but we do have the following fact: If $f : T \rightarrow S$ is any morphism of k -schemes, where $T = \text{Spec } \mathcal{O}_{U;u_1, \dots, u_m}$ is the spectrum of the semi-localization of a smooth variety U at a finite set of points $\{u_1, \dots, u_m\}$, and $Y = X \times_S T$, then the diagram

$$(6) \quad \begin{array}{ccc} c_{equi}^0(X/S) & \xrightarrow{cl_{X/S}} & K_0^{fin}(X/S) \\ f^* \downarrow & & f^* \downarrow \\ c_{equi}^0(Y/T) & \xrightarrow{cl_{Y/T}} & K_0^{fin}(Y/T) \end{array}$$

commutes. To see this, it suffices to let $Z \subset X$ be a closed, integral subscheme which is finite and dominant over a component of S and show that $\sum_w \text{length}(\mathcal{O}_{W,w})[\mathcal{O}_w] = [\mathcal{O}_W]$ in $K_0^{fin}(Y/T)$, where $W = Z \times_X Y$ and w ranges over the generic points

of W . Since T is the spectrum of a semi-local ring, so is W . It follows that $W = \text{Spec } R/f$, where R is the UFD obtained by semi-localizing Y at the closed points of W and f is a non-zero divisor of R such that $\text{Spec } R/f \rightarrow T$ is finite. Moreover, the irreducible components of W are cut out by the terms p_i in an irreducible factorization $f = p_1 \dots p_l$. Using the short exact sequence of objects in $\mathcal{P}^{fin}(Y/T)$

$$0 \rightarrow R/p_1 \xrightarrow{p_2 \dots p_l} R/f \rightarrow R/(p_2 \dots p_l) \rightarrow 0$$

and induction, the desired equation is seen to hold.

Given F as in the statement of the proposition, define pairings

$$c_{equi}^0(X/S) \otimes F(X) \rightarrow F(S),$$

for $X \rightarrow S$ a smooth curve in Sm/k , by taking the composition of

$$c_{equi}^0(X/S) \otimes F(X) \xrightarrow{\text{cl}_{X/S} \otimes id} K_0^{fin}(X/S) \otimes F(X) \rightarrow F(S).$$

The normalization and additivity axioms of a pretheory are readily seen to hold. To establish naturality, suppose we are given a smooth curve $X \rightarrow S$ and a morphism $S' \rightarrow S$ in Sm/k . Since F is Zariski separated, the map $F(S') \rightarrow \prod_{s'} F(\text{Spec } \mathcal{O}_{S',s'})$ is injective, where s' ranges over all closed points of S' and $F(\text{Spec } \mathcal{O}_{S',s'})$ is the stalk of F at s' (defined as $\varinjlim_{s' \in V \subset S'} F(V)$ with V ranging over all Zariski open neighborhoods of s' in V). Fixing such an $s' \in S'$, set $T = \text{Spec } \mathcal{O}_{S',s'}$ and $Y = T \times_S X$. To show the commutativity of the left-hand square in

$$\begin{array}{ccccc} c_{equi}^0(X/S) \otimes F(X) & \longrightarrow & c_{equi}^0(X'/S') \otimes F(X') & \longrightarrow & c_{equi}^0(Y/T) \otimes F(T) \\ \downarrow & & \downarrow & & \downarrow \\ F(S) & \longrightarrow & F(S') & \longrightarrow & F(T), \end{array}$$

it suffices to show the commutativity of the outer rectangle for all such T . But this follows from the commutativity of (6) and the fact that F is a presheaf with K_0 transfers. \square

Remark 4.9. Suppose F is a pretheory whose underlying presheaf is Zariski separated. By Corollary 4.6, F has the structure of a presheaf with K_0 transfers and by Proposition 4.8 it has the induced structure of a pretheory. These two potentially different pretheory structures on the presheaf F coincide, since the composition

$$c_{equi}^0(X/S) \rightarrow K_0^{fin}(X/S) \rightarrow c_{equi}^0(X/S)$$

is readily seen to be the identity.

Suppose now that G is a Zariski separated presheaf with K_0 transfers. Then G admits the structure of a pretheory by Proposition 4.8 and hence the structure of a presheaf with K_0 transfers by Corollary 4.6. It is unclear whether these potentially different structures of K_0 transfer map on the presheaf G coincide, since the composition of

$$K_0^{fin}(X/S) \rightarrow c_{equi}^0(X/S) \rightarrow K_0^{fin}(X/S)$$

is not known to be the identity map.

Our next goal is to show that a Zariski sheafification of a homotopy invariant presheaf F with K_0 transfers has the natural structure of a homotopy invariant presheaf with K_0 transfers. In fact, we will show in Theorem 4.17 below that the Zariski separated presheaf $s(F)$ associated to F is a homotopy invariant presheaf

with K_0 transfers such that the natural map $F \rightarrow s(F)$ is a map of presheaves with K_0 transfers. In light of Proposition 4.8, this suffices to show that the Zariski separated presheaf associated to a homotopy invariant presheaf with K_0 transfers is in fact a pretheory (see Corollary 4.19 below). Thus, Theorem 4.17 represents the central result in the theory of presheaves with K_0 transfers, since it allows one to apply all of Voevodsky's results on pretheories to presheaves with K_0 transfers. However, in order to prove Theorem 4.17, we must first reproduce several of the basic results on pretheories in the context of presheaves with K_0 transfers.

We will be using the concept of a *standard triple* as defined in [29, 4.1], except that we find it convenient to add two additional hypotheses. Namely, define a *nice standard triple* to be a triple $(\pi : \overline{X} \rightarrow S, X_\infty, Z)$ such that S belongs to Sm/k , the scheme \overline{X} is a normal, quasi-projective k -variety, and Z and X_∞ are reduced, closed subschemes of \overline{X} . Such a triple is required to satisfy the conditions that

- (1) the morphism π is proper having fibers of dimension one,
- (2) $Z \cap X_\infty = \emptyset$,
- (3) the closed subscheme $X_\infty \amalg Z$ of \overline{X} has an open neighborhood V such that $V \rightarrow S$ is an affine morphism and V is the complement of an effective divisor of \overline{X} , and
- (4) the map $\overline{X} - X_\infty \rightarrow S$ is smooth and quasi-affine (i.e., can be factored as an open immersion followed by an affine morphism).

For such a triple, we will customarily write X for $\overline{X} - X_\infty$, so that by hypothesis $X \rightarrow S$ is a smooth, quasi-affine curve. The definition of a nice standard triple differs from Voevodsky's definition of a standard triple in that Voevodsky does not require V to be the complement of a divisor or that \overline{X} be quasi-projective. (These modifications turn out to be insignificant, since the proof of Voevodsky's key lemma [29, 4.9], which is sketched in [33, 4.3], actually results in the construction of nice standard triples.)

Definition 4.10. Given a nice standard triple $(\overline{X} \rightarrow S, X_\infty, Z)$, define

$$\mathcal{K}(\overline{X} \text{ near } X_\infty)$$

to be the direct limit of K -theory spectra

$$\varinjlim_W \mathcal{K}(\overline{X} - W),$$

where W ranges over all closed subscheme of \overline{X} that are contained in X (equivalently, that miss X_∞).

Assume now that S is affine. Then the hypotheses of a nice standard triple imply that X_∞ is contained in an affine open subscheme $V = \text{Spec } R$ of \overline{X} , and so $X_\infty = \text{Spec } R/I$ for some ideal $I \subset R$. In this case, we have the identity:

$$\mathcal{K}(\overline{X} \text{ near } X_\infty) \cong \mathcal{K}((1+I)^{-1}R),$$

where $(1+I)$ is the multiplicatively closed subset of R consisting of all elements of the form $1+a$ for $a \in I$. To simplify the notation, we define

$$R_{\langle I \rangle} = (1+I)^{-1}R,$$

so that $\mathcal{K}(\overline{X} \text{ near } X_\infty) \cong \mathcal{K}(R_{\langle I \rangle})$.

In the situation of Definition 4.10, the saturation of $1 + I$ in R is precisely the set of elements of R which map to units in R/I . In particular, the ideal $I \cdot R_{\langle I \rangle}$ is the Jacobson radical of $R_{\langle I \rangle}$ and $R_{\langle I \rangle}/I \cdot R_{\langle I \rangle} \cong R/I$.

Proposition 4.11. *Suppose $(\pi : \overline{X} \rightarrow S, X_\infty, Z)$ is a nice standard triple such that S is a smooth, affine k -variety and let $X = \overline{X} - X_\infty$. Then there is a homotopy fibration sequence of spectra*

$$\mathcal{K}^{fin}(X/S) \rightarrow \mathcal{K}(\overline{X}) \rightarrow \mathcal{K}(\overline{X} \text{ near } X_\infty)$$

and thus a long exact sequence of K -groups of the form

$$\cdots \rightarrow K_1(\overline{X} \text{ near } X_\infty) \rightarrow K_0^{fin}(X/S) \rightarrow K_0(\overline{X}) \rightarrow K_0(\overline{X} \text{ near } X_\infty) \rightarrow 0.$$

Proof. Assume \overline{X} is embedded in \mathbb{P}_k^N . We claim that for each $W \subset X$ appearing in the direct limit defining $K(\overline{X} \text{ near } X_\infty)$, there is an effective divisor D in \overline{X} such that $W \subset D \subset X$. To see this, let \mathcal{L} be a line bundle on \overline{X} and g be an element of $\Gamma(\overline{X}, \mathcal{L})$ such that $V = \overline{X}_g$ (the locus of points on which g does not vanish) is an open, affine neighborhood of X_∞ . (The existence of such an \mathcal{L} and g is part of the hypothesis of a nice standard triple.) Say $V = \text{Spec } R$ so that $X_\infty = \text{Spec } R/J$ for some ideal J . Observe that $W \cap X_\infty = \emptyset$ implies that $J + \Gamma(V, \mathcal{I}_W) = R$, where \mathcal{I}_W is the sheaf of ideals defining W . So there is $h \in \Gamma(V, \mathcal{I}_W) \subset R$ which is congruent to 1 mod J . By [16, II.5.14], we have that $g^N f \in \Gamma(V, \mathcal{I}_W \otimes L)$ lifts to an element of $\Gamma(\overline{X}, \mathcal{I}_W \otimes L)$, for $N \gg 0$, which determines a divisor D . One may readily verify that $D \cap X_\infty = \emptyset$ and $W \subset D$.

Thus in the direct limit defining $K(\overline{X} \text{ near } X_\infty)$, we may assume each $W = D$ is an effective divisor. For a such a D , we have by [11, Theorem on p. 229] that the sequence of exact functors

$$\mathbb{H}(\overline{X}, D) \rightarrow \mathcal{P}(\overline{X}) \rightarrow \mathcal{P}(\overline{X} - D)$$

induces a homotopy fibration sequence of associated K -theory spectra. Here, $\mathbb{H}(\overline{X}, D)$ is the category consisting of coherent sheaves M on \overline{X} such that $M|_{\overline{X} - D} = 0$ and M admits a resolution of length one by locally free sheaves on \overline{X} . Taking limits, the sequence of exact functors

$$\varinjlim_D \mathbb{H}(\overline{X}, D) \rightarrow \mathcal{P}(\overline{X}) \rightarrow \varinjlim_D \mathcal{P}(\overline{X} - D)$$

induces a homotopy fibration sequence on K -theory spectra. Note that the singular locus of \overline{X} is contained in X_∞ , and so $K_0(\overline{X}) \rightarrow K_0(\overline{X} \text{ near } X_\infty)$ is surjective (cf. [38, 1.4]). For each D , since $D \subset X$, the category $\mathbb{H}(\overline{X}, D)$ may naturally be regarded as a full subcategory of $\mathcal{M}(X)$. We claim that $\varinjlim_D \mathbb{H}(\overline{X}, D)$ coincides with $\mathcal{P}^{fin}(X/S)$ as subcategories of $\mathcal{M}(X)$, which suffices to finish the proof.

Suppose $M \in \mathbb{H}(\overline{X}, D)$ for some such effective divisor D . Then as $D \rightarrow S$ is finite and quasi-affine, it is finite, and so M is finite over S . Moreover, by hypothesis there is an exact sequence of coherent sheaves on X , $0 \rightarrow E_1 \rightarrow E_0 \rightarrow M \rightarrow 0$, with E_0 and E_1 locally free, and hence an exact sequence

$$(7) \quad 0 \rightarrow \text{Tor}_1^{\mathcal{O}_X}(M, \mathcal{O}_D) \rightarrow E_1 \otimes_{\mathcal{O}_X} \mathcal{O}_D \rightarrow E_0 \otimes_{\mathcal{O}_X} \mathcal{O}_D \rightarrow M \otimes_{\mathcal{O}_X} \mathcal{O}_D \rightarrow 0.$$

Since M is supported on D , we have that $M \otimes_{\mathcal{O}_X} \mathcal{O}_D \cong M$ and $\text{Tor}_1^{\mathcal{O}_X}(M, \mathcal{O}_D) \cong M \otimes_{\mathcal{O}_D} \mathcal{I}/\mathcal{I}^2$, where \mathcal{I} is the sheaf of ideals defining D . Since $\mathcal{I}/\mathcal{I}^2$ is an invertible sheaf on D , M and $\text{Tor}_1^{\mathcal{O}_X}(M, \mathcal{O}_D)$ have the same Tor dimension when regarded as sheaves on D . Since \mathcal{O}_D is a finite, flat \mathcal{O}_S -module, these modules also have the

same Tor dimension as \mathcal{O}_S -modules. Since $X \rightarrow S$ is smooth, M has Tor dimension at most one as an \mathcal{O}_S -module. But now the sequence (7) shows $\text{Tor}_1^{\mathcal{O}_Y}(M, \mathcal{O}_D)$, and hence M , is \mathcal{O}_S -flat. We have thus proven M belongs to $\mathcal{P}^{fin}(X/S)$.

Conversely, suppose N belongs to $\mathcal{P}^{fin}(X/S)$. As argued above, N is supported on an effective divisor D of \overline{X} which misses X_∞ . It suffices to show N has Tor dimension one locally on \overline{X} (for then, since \overline{X} is quasi-projective, N admits a resolution of length one by locally free $\mathcal{O}_{\overline{X}}$ -modules). Let $x \in X$ be any point in the support of N , and let s be its image in S . Then there is a formally smooth map of relative dimension one $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ of local rings, and N_x is an $\mathcal{O}_{X,x}$ -module which is finite and flat as an $\mathcal{O}_{S,s}$ -module. Letting \mathfrak{m} and \mathfrak{n} denote the maximal ideals of $\mathcal{O}_{S,s}$ and $\mathcal{O}_{X,x}$, respectively, we have the spectral sequence

$$\text{Tor}_p^{\mathcal{O}_{X,x}/\mathfrak{m} \cdot \mathcal{O}_{X,x}}(\text{Tor}_q^{\mathcal{O}_{S,s}}(N_x, \mathcal{O}_{S,s}/\mathfrak{m}), \mathcal{O}_{X,x}/\mathfrak{n}) \implies \text{Tor}_{p+q}^{\mathcal{O}_{X,x}}(N_x, \mathcal{O}_{X,x}/\mathfrak{n}).$$

Observe that $\text{Tor}_q^{\mathcal{O}_{S,s}}(N_x, \mathcal{O}_{S,s}/\mathfrak{m}) = 0$ for $q > 0$ since N_x is $\mathcal{O}_{S,s}$ -flat. On the other hand, $\mathcal{O}_{X,x}/\mathfrak{m} \cdot \mathcal{O}_{X,x}$ is a local ring for a curve over the field $\mathcal{O}_{S,s}/\mathfrak{m}$, and so $\text{Tor}_p^{\mathcal{O}_{X,x}/\mathfrak{m} \cdot \mathcal{O}_{X,x}}(-, \mathcal{O}_{X,x}/\mathfrak{n}) = 0$ for $p > 1$. It follows that N_x has Tor dimension at most one (and hence exactly one) as an $\mathcal{O}_{X,x}$ -module. \square

Corollary 4.12. *Under the hypothesis of Proposition 4.11, there is a long exact sequence of the form*

$$\begin{aligned} \cdots &\rightarrow K_1(\overline{X} \text{ near } X_\infty \amalg Z) \rightarrow K_0^{fin}((X - Z)/S) \oplus K_1(\overline{X} \text{ near } X_\infty) \rightarrow K_0^{fin}(X/S) \\ &\rightarrow K_0(\overline{X} \text{ near } X_\infty \amalg Z) \rightarrow K_0(\overline{X} \text{ near } X_\infty) \rightarrow 0. \end{aligned}$$

Proof. Observe that $(\overline{X} \rightarrow S, X_\infty \amalg Z, \emptyset)$ is also a nice standard triple with $\overline{X} - (X_\infty \amalg Z) = X - Z$. Moreover, there are natural maps $\mathcal{K}^{fin}((X - Z)/S) \rightarrow \mathcal{K}^{fin}(X/S)$ (induced by pushforward along $X - Z \hookrightarrow X$) and $\mathcal{K}(\overline{X} \text{ near } X_\infty \amalg Z) \rightarrow \mathcal{K}(\overline{X} \text{ near } X_\infty)$ (induced by restriction) such that the diagram

$$\begin{array}{ccccc} \mathcal{K}^{fin}((X - Z)/S) & \longrightarrow & \mathcal{K}(\overline{X}) & \longrightarrow & \mathcal{K}(\overline{X} \text{ near } X_\infty \amalg Z) \\ \downarrow & & = \downarrow & & \downarrow \\ \mathcal{K}^{fin}(X/S) & \longrightarrow & \mathcal{K}(\overline{X}) & \longrightarrow & \mathcal{K}(\overline{X} \text{ near } X_\infty) \end{array}$$

commutes up to homotopy. By Proposition 4.11 the rows of this diagram are homotopy fibration sequences, from which the result follows. \square

The following theorem is the key technical result concerning presheaves with K_0 transfers and should be compared with Voevodsky's notion of a "splitting" of a standard triple [29, 4.4]. To set up the statement, recall from the proof of Proposition 4.4 that $[K_0^{fin, \oplus}(X/S)] = [K_0^{fin}(X/S)]$ is the cokernel of

$$K_0^{fin}(X \times \mathbb{A}^1/S \times \mathbb{A}^1) \xrightarrow{i_1^* - i_0^*} K_0^{fin}(X/S).$$

Given $\alpha \in K_0^{fin}(X/S)$, let $[\alpha]$ denote its image in $[K_0^{fin}(X/S)]$ under the canonical surjection. Also, given a nice standard triple $(\overline{X} \rightarrow S, X_\infty, Z)$ with $X = \overline{X} - X_\infty$, observe that there is a natural map $K_0^{fin}(X/S) \rightarrow K_0(\overline{X})$ and hence a map $K_0^{fin}(X/S) \rightarrow K_0(Z)$ induced by composition.

Theorem 4.13. *Let $(\pi : \overline{X} \rightarrow S, X_\infty, Z)$ be a nice standard triple such that S is a smooth, affine k -variety. If α lies in the kernel of the natural map $K_0^{fin}(X/S) \rightarrow K_0(Z)$, then $[\alpha] \in [K_0^{fin}(X/S)]$ lies in the image of the natural map*

$$[K_0^{fin}(X - Z/S)] \rightarrow [K_0^{fin}(X/S)].$$

Proof. From the hypotheses of a nice standard triple, $X_\infty \amalg Z$ is contained in an affine open subscheme $V = \text{Spec } R$ of \overline{X} , so that $X_\infty = \text{Spec } R/I$ and $Z = \text{Spec } R/J$ with $I + J = R$. Hence $K_i(\overline{X} \text{ near } X_\infty) = K_i(R_{\langle I \rangle})$ and $K_i(\overline{X} \text{ near } X_\infty \amalg Z) = K_i(R_{\langle I \cap J \rangle})$. For any ideal $L \subset R$, the Jacobson radical of $R_{\langle L \rangle}$ is $L \cdot R_{\langle L \rangle}$ and we have $R_{\langle L \rangle}/L \cdot R_{\langle L \rangle} \cong R/L$. Thus, using [2, IX.1.3], the natural map

$$(8) \quad K_0(R_{\langle L \rangle}) \rightarrow K_0(R/L)$$

is an injection and the natural map

$$(9) \quad K_1(R_{\langle L \rangle}) \rightarrow K_1(R/L)$$

is a surjection with kernel $(1 + L \cdot R_{\langle L \rangle})^\times$. Here, $(1 + L \cdot R_{\langle L \rangle})^\times$ is the multiplicative group of units congruent to 1 mod L , which for the sake of brevity we will write as $U(R, L)$. Note that $U(R, L)$ is the subgroup of the multiplicative group of non-zero elements of the field of fractions of R consisting of those elements of the form $\frac{1+a}{1+b}$, where $a, b \in L$.

The injectivity of (8) implies the map

$$K_0(\overline{X} \text{ near } X_\infty \amalg Z) \rightarrow K_0(X_\infty) \oplus K_0(Z)$$

is injective. Since $K_0^{fin}(X/S) \rightarrow K_0(X_\infty)$ is the zero map by Corollary 4.12, we see that the kernel of

$$K_0^{fin}(X/S) \rightarrow K_0(\overline{X} \text{ near } X_\infty \amalg Z)$$

coincides with the kernel of $K_0^{fin}(X/S) \rightarrow K_0(Z)$. Thus α lifts to an element of $K_0^{fin}((X - Z)/S) \oplus K_1(\overline{X} \text{ near } X_\infty)$ under the map of Corollary 4.12.

Let $[K_1(\overline{X} \text{ near } X_\infty)]$ denote the cokernel of

$$i_1^* - i_0^* : K_1(\overline{X} \times \mathbb{A}^1 \text{ near } X_\infty \times \mathbb{A}^1) \rightarrow K_1(\overline{X} \text{ near } X_\infty)$$

and similarly for $[K_1(\overline{X} \text{ near } X_\infty \amalg Z)]$. A diagram chase shows that the theorem holds provided the natural map

$$[K_1(\overline{X} \text{ near } X_\infty \amalg Z)] \rightarrow [K_1(\overline{X} \text{ near } X_\infty)]$$

is surjective. From (9), we have a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
U(R, I \cap J) & \longrightarrow & U(R, I) \\
\downarrow & & \downarrow \\
K_1(\overline{X} \text{ near } X_\infty \amalg Z) & \longrightarrow & K_1(\overline{X} \text{ near } X_\infty) \\
\downarrow & & \downarrow \\
K_1(R/I) \oplus K_1(R/J) & \longrightarrow & K_1(R/I) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

whose columns are exact. Since the bottom arrow of this diagram is surjective, it suffices to show $[U(R, I)]$ is the trivial group, where $[U(R, I)]$ is by definition the cokernel of the map

$$i_1^*/i_0^* : U(R[t], I[t]) \rightarrow U(R, I),$$

where i_ϵ^* is given by evaluation of t at ϵ . (Recall that the U 's are multiplicative groups.) Given $\frac{1+a}{1+b}$ in $U(R, I)$, where $a, b \in I$, the element $\frac{1+ta}{1+tb} \in U(R[t], I[t])$ maps to $\frac{1+a}{1+b}$ under i_1^*/i_0^* , which implies $[U(R, I)] = 1$. \square

Given a nice standard triple $(\overline{X} \rightarrow S, X_\infty, Z)$ and an open subscheme $j : U \rightarrow X$ (where $X = \overline{X} - X_\infty$), by pulling back along the induced map $U \rightarrow S$, we obtain the triple $(U \times_S \overline{X}, U \times_S X_\infty, U \times Z)$ which is readily verified to satisfy the hypothesis of a nice standard triple. In this situation, we define $\delta_U \in K_0^{fin}(U \times_S X/U)$ to be the class of the structure sheaf of the graph of $(1, j) : U \rightarrow U \times_S X$. The following theorem is the analogue of a result of Voevodsky concerning pretheories (cf. [29, 4.5]).

Theorem 4.14. *Suppose $(\overline{X} \rightarrow S, X_\infty, Z)$ is a nice standard triple such that S is a smooth, affine k -variety. Set $X = \overline{X} - X_\infty$ and suppose $i : U \rightarrow X$ is an affine open subscheme. Assume the image of δ_U under the natural map $K_0^{fin}(U \times_S X/U) \rightarrow K_0(U \times_S Z)$ is zero. Then for any homotopy invariant presheaf with K_0 transfers, there is a map $\psi : F(X - Z) \rightarrow F(U)$ such that the diagram*

$$\begin{array}{ccc}
F(X) & \xrightarrow{i^*} & F(U) \\
& \searrow j^* & \swarrow \psi \\
& F(X - Z) &
\end{array}$$

commutes, where $j : X - Z \rightarrow X$ is the canonical embedding.

Proof. By Theorem 4.13 the element $[\delta_U] \in [K_0^{fin}(U \times_S X/U)]$ lifts to an element $\gamma \in [K_0^{fin}(U \times_S (X - Z)/U)]$. We define ψ to be the composition of

$$F(X - Z) \xrightarrow{\pi^*} F(U \times_S (X - Z)) \xrightarrow{\gamma_*} F(U).$$

By the naturality axiom of presheaves with K_0 transfers, the map $\psi \circ j^*$ coincides with the composition of

$$F(X) \xrightarrow{\pi^*} F(U \times_S X) \xrightarrow{\delta_*} F(U).$$

The normalization axiom implies that $\delta_* = (1, i)^*$ and hence $\psi \circ j^* = i^*$. \square

The following result is the analogue for presheaves with K_0 transfers of [29, 4.18].

Corollary 4.15. *Let F be a homotopy invariant presheaf with K_0 transfers, X a connected, quasi-projective smooth k -variety, and $x = \{x_1, \dots, x_m\}$ a finite set of closed points on X . Write $\mathcal{O}_{X,x}$ for the semi-localization of X at the points of x and let E denote the field of fractions of X . Then the natural map*

$$F(\mathrm{Spec} \mathcal{O}_{X,x}) \rightarrow F(\mathrm{Spec} E)$$

is an injection (where $F(\mathrm{Spec} \mathcal{O}_{X,x})$ and $F(\mathrm{Spec} E)$ are each defined in terms of the evident direct limits).

Proof. It suffices to show that given an open neighborhood V of the finite set x in X , an arbitrarily small non-empty open subset W of X , and an element $\alpha \in F(V)$ lying in the kernel of $F(V) \rightarrow F(V \cap W)$, there exists a smaller neighborhood $x \subset V' \subset V$ such that α vanishes in $F(V')$. Moreover, we may assume $W \cap x = \emptyset$, and it suffices to prove the result upon replacing V with any smaller open neighborhood of x .

Let $Z = V - (V \cap W)$ (the reduced closed complement). Then by [29, 4.9], upon replacing V with a suitably small neighborhood of x , we may assume there is a nice standard triple $(\pi : \bar{Y} \rightarrow S, Y_\infty, Z)$, with S smooth and affine, such that $V = Y = \bar{Y} - Y_\infty$. (As mentioned above, the proof of [29, 4.9], which is sketched in [33, 4.3], results in the construction of a nice standard triple.) Observe that $\mathrm{Spec} \mathcal{O}_{V,x} \times_S Z$ is finite over $\mathrm{Spec} \mathcal{O}_{V,x}$ and is thus the spectrum of a semi-local ring. In particular, $K_0(\mathrm{Spec} \mathcal{O}_{V,x} \times_S Z)$ is isomorphic, via the rank map, to the free abelian group on the set of connected components of $\mathrm{Spec} \mathcal{O}_{V,x} \times_S Z$. It follows that for a sufficiently small neighborhood V' of x in X , the image of $\delta_{V'} \in K_0^{fin}(V' \times_S X/V')$ (with $\delta_{V'}$ defined as in Theorem 4.14) under the map $K_0^{fin}(V' \times_S X/V') \rightarrow K_0(V' \times_S X)$ is zero. Applying Theorem 4.14 with $X = V$ and $U = V'$, we see that the image of α in $F(V')$ must be zero by the commutativity of the diagram. \square

Lemma 4.16. *Let F be a homotopy invariant presheaf with K_0 transfers, $X \rightarrow S$ a smooth curve, and $j : U \hookrightarrow X$ and open embedding. Let $j_* : K_0^{fin}(U/S) \rightarrow K_0^{fin}(X/S)$ denote the map induced by the inclusion of categories $\mathcal{P}^{fin}(U/S) \rightarrow \mathcal{P}^{fin}(X/S)$. Then given any $\alpha' \in K_0^{fin}(U/S)$ and setting $\alpha = j_*(\alpha')$, we have $\alpha_* = \alpha'_* \circ j^*$ as maps from $F(U) \rightarrow F(S)$.*

Proof. The proof is identical to the proof of [29, 3.12]. \square

Theorem 4.17. *Let F be a homotopy invariant presheaf with K_0 transfers. Then $s(F)$, the Zariski separated presheaf associated to F , admits the unique structure of a homotopy invariant presheaf with K_0 transfers in such a manner that the natural map $F \rightarrow s(F)$ is a morphism of presheaves with K_0 transfers.*

Proof. Observe that the natural map $F \rightarrow s(F)$ is a surjection of presheaves, from which it follows that $s(F)$ is homotopy invariant. Given a smooth curve $X \rightarrow S$

and an element $\alpha \in K_0^{fin}(X/S)$, we have a diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_*} & F(S) \\ \downarrow & & \downarrow \\ s(F)(X) & \dashrightarrow & s(F)(S), \end{array}$$

which admits a unique completion provided α_* maps the kernel of the left-hand map into the kernel of the right-hand map. Once we prove such a completion exists for all $X \rightarrow S$ and α , the axioms of a presheaf with K_0 transfers are readily verified for $s(F)$.

Suppose $\gamma \in \ker(F(X) \rightarrow s(F)(X))$. Since $s(F)$ is separated, it suffices to prove $\alpha_*(\gamma)$ vanishes under the map $F(S) \rightarrow F(S_s)$, where s is an arbitrary closed point of S . Note that $\alpha \in K_0^{fin}(X/S)$ is supported on a closed subscheme $Z \subset X$ which is finite over S , in the sense that α lies in the image of $K_0^{fin}(Z/S) \rightarrow K_0^{fin}(X/S)$. Let z be the finite set of points of Z lying over $s \in S$. It follows from Corollary 4.15 that γ vanishes in a suitably small open neighborhood U of z in X . Shrinking S as we may, we may assume $Z \subset U \subset X$, and so $\alpha = j_*(\alpha')$ for some $\alpha' \in K_0^{fin}(U/S)$, where $j : U \rightarrow X$ is the inclusion map. From Lemma 4.16 we have that $\alpha_*(\gamma) = \alpha'_*(j^*(\gamma)) = 0$. \square

Corollary 4.18. *The Zariski separated presheaf $s(F)$ associated to a homotopy invariant presheaf with K_0 transfers F admits a unique structure of a pretheory so that $F \rightarrow s(F)$ is a morphism of presheaves with K_0 transfers.*

Proof. This follows immediately from the theorem and Proposition 4.8. \square

Corollary 4.19. *The Zariski sheafification F_{Zar} of a homotopy invariant presheaf with K_0 transfers F admits a unique structure of a pretheory so that $F \rightarrow F_{Zar}$ is a morphism of presheaves with K_0 transfers.*

Proof. This follows from Corollary 4.18 and [29, 4.25]. \square

Corollary 4.20. *Suppose F is a homotopy invariant presheaf with K_0 transfers such that $n \cdot F = 0$ for some integer n relatively prime to the exponential characteristic of the ground field k . Let $\mathcal{O}_{X,x}^{hen}$ denote the henselization of a smooth k -variety X at a closed point x . Then the natural map*

$$F(\mathrm{Spec} \mathcal{O}_{X,x}^{hen}) \rightarrow F(\mathrm{Spec} k(x))$$

is an isomorphism.

Proof. This follows by applying [29, 5.25] to the Zariski sheafification of F , which is a pretheory by Corollary 4.19. \square

A common method of producing homotopy invariant presheaves with K_0 transfers from presheaves with K_0^\oplus transfers is given in the following proposition.

Proposition 4.21. *Let F be a presheaf with K_0^\oplus transfers. Then the homology presheaves of the complex $F(- \times \Delta^\bullet)$ are homotopy-invariant presheaves with K_0 transfers.*

Proof. For any X , the presheaf $F(- \times X)$ is a presheaf with K_0^\oplus transfers, so that $F(- \times \Delta^\bullet)$ is a complex of presheaves with K_0^\oplus transfers. Moreover, the kernel and cokernel of a map of presheaves with K_0^\oplus transfers is again a presheaf with K_0^\oplus transfers. This proves the homology presheaves of $F(- \times \Delta^\bullet)$ have K_0^\oplus transfers. A standard argument shows these homology presheaves are homotopy invariant, and so the result follows from Proposition 4.4. \square

The following theorem and accompanying corollary describe the key properties, at least for the purposes of this paper, exhibited by presheaves with K_0^\oplus transfers.

Theorem 4.22 (cf. [9, 5.5] and [24, 7.6]). *Let F be a presheaf of abelian groups, simplicial abelian groups, group-like H -spaces, or connective spectra on Sch/k . Assume, for all $q \geq 0$, the restriction of the presheaf of abelian groups $X \mapsto \pi_q F(X)$ to Sm/k has the structure of a presheaf with K_0^\oplus transfers. (In case F is a presheaf of abelian groups, we interpret this hypothesis as meaning F is a presheaf with K_0^\oplus transfers.)*

- (1) *If k is an arbitrary field and $\pi_q F(-)_{(Sm/k)_{Nis}}^\sim$ (the sheafification in the Nisnevich topology of the restriction of the presheaf $\pi_q F(-)$ to the category Sm/k) is trivial, then the chain complex $F(- \times \Delta^\bullet)$ is locally acyclic for the Zariski topology on Sm/k (i.e., the sheaf homology groups of the chain complex of Zariski sheaves $F(- \times \Delta^d)_{Zar}^\sim$ are all trivial). In particular, we have*

$$\pi_q |d \mapsto F(\Delta^d)| = 0,$$

for all $q \geq 0$.

- (2) *If k admits resolutions of singularities and $\pi_q F(-)_{cdh}^\sim$ (the sheafification of the presheaf $\pi_q F(-)$ in the cdh topology) is trivial, then the chain complex $F(- \times \Delta^\bullet)$ is locally acyclic for the Zariski topology on Sm/k . In particular, we have*

$$\pi_q |d \mapsto F(\Delta^d)| = 0,$$

for all $q \geq 0$.

- (3) *If k is algebraically closed, $n \geq 1$ is an integer relatively prime to the exponential characteristic of k , and $\pi_q F(-)_h^\sim$ (the sheafification of the presheaf $\pi_q F(-)$ in the h topology) is trivial, then*

$$\pi_q (|d \mapsto F(\Delta^d)|, \mathbb{Z}/n) = 0,$$

for all $q \geq 0$.

Proof. Using the convergent spectral sequence (cf. [5, B.5])

$$\pi_p (d \mapsto \pi_q (F(\Delta^d); A)) \implies \pi_{p+q} (|d \mapsto F(\Delta^d)|; A),$$

where A is \mathbb{Z}/n or \mathbb{Z} , it suffices (by considering $X \mapsto \pi_q F(X)$, for all $q \geq 0$) to prove each result in the case where F is a presheaf of abelian groups.

The proofs of the first and second cases are identical to the proofs of [29, 5.9] and [9, 5.5], respectively, observing that each of these cited proofs only requires that the Zariski sheafification of $h_i(F(- \times \Delta^\bullet))$ be a pretheory, for all $i \geq 0$. This property holds in our context by Corollary 4.19.

To prove the third case, we assume $F_h^\sim = 0$ and we must prove the chain complex $F(\Delta^\bullet) \otimes^{\mathbb{L}} \mathbb{Z}/n$ is acyclic. In fact, it suffices to show $\text{Ext}_{Ab}^*(F(\Delta^\bullet), \mathbb{Z}/n) = 0$. By Corollary 4.20 we have

$$\text{Ext}_{Ab}^*(F(\Delta^\bullet), \mathbb{Z}/n) \cong \text{Ext}_{et}^*(F(- \times \Delta^\bullet)_{et}^\sim, \mathbb{Z}/n),$$

and thus by using [24, 7.1 and 10.10], we have the string of isomorphisms

$$\mathrm{Ext}_{et}^*(F(- \times \Delta^\bullet)_{et}, \mathbb{Z}/n) \cong \mathrm{Ext}_{et}^*(F_{et}, \mathbb{Z}/n) \cong \mathrm{Ext}_h^*(F_h, \mathbb{Z}/n) = 0.$$

□

Corollary 4.23. *Let $\psi : F \rightarrow G$ be a natural transformation of presheaves of abelian groups, simplicial abelian groups, group-like H -spaces, or connective spectra on Sch/k . Assume that for all $q \geq 0$ the restriction to Sm/k of the natural transformation of presheaves of abelian groups*

$$(X \mapsto \pi_q F(X)) \rightarrow (X \mapsto \pi_q G(X))$$

is a morphism of presheaves with K_0^\oplus transfers.

- (1) *If k is an arbitrary field and $\pi_q F(-)_{(Sm/k)_{Nis}} \xrightarrow{\sim} \pi_q G(-)_{(Sm/k)_{Nis}}$ is an isomorphism, then the natural map*

$$F(- \times \Delta^\bullet) \rightarrow G(- \times \Delta^\bullet)$$

is locally a weak homotopy equivalence for the Zariski topology on Sm/k . In particular, the natural map

$$\pi_q|d \mapsto F(\Delta^d)| \rightarrow \pi_q|d \mapsto G(\Delta^d)|$$

is an isomorphism, for all $q \geq 0$.

- (2) *If k admits resolutions of singularities and $\pi_q F(-)_{cdh} \rightarrow \pi_q G(-)_{cdh}$ is an isomorphism, then the natural map*

$$F(- \times \Delta^\bullet) \rightarrow G(- \times \Delta^\bullet)$$

is locally a weak homotopy equivalence for the Zariski topology on Sm/k . In particular, the natural map

$$\pi_q|d \mapsto F(\Delta^d)| \rightarrow \pi_q|d \mapsto G(\Delta^d)|$$

is an isomorphism, for all $q \geq 0$.

- (3) *If k is algebraically closed, $n \geq 1$ is an integer relatively prime to the exponential characteristic of k , and $\pi_q F(-)_h \rightarrow \pi_q G(-)_h$ is an isomorphism, then the natural map*

$$\pi_q(|d \mapsto F(\Delta^d)|, \mathbb{Z}/n) \rightarrow \pi_q(|d \mapsto G(\Delta^d)|, \mathbb{Z}/n)$$

is an isomorphism, for all $q \geq 0$.

Proof. There is a map of convergent spectral sequences from

$$\pi_p(d \mapsto \pi_q(F(\Delta^d); A)) \Longrightarrow \pi_{p+q}(|d \mapsto F(\Delta^d)|; A)$$

to

$$\pi_p(d \mapsto \pi_q(G(\Delta^d); A)) \Longrightarrow \pi_{p+q}(|d \mapsto G(\Delta^d)|; A),$$

where A is \mathbb{Z}/n or \mathbb{Z} . As in the proof of the theorem, this allows us to consider only the case where F and G are presheaves of abelian groups. In this case, the result follows by applying the theorem to the cone of the morphism $F \rightarrow G$ (which may be interpreted as a presheaf of simplicial abelian groups). □

The following theorem gives most of the instances in which presheaves with K_0 transfers arise in the remainder of this paper.

Theorem 4.24. *Let Y be a k -scheme, U a locally closed subscheme of Y , A an abelian group, and q an integer. The presheaves*

$$K_q^{qf}(-, (U, Y); A), \quad K_q^{qf}(-, Y; A), \quad \text{and} \quad K_q^{fin}(-, Y; A)$$

are presheaves with K_0 transfers. The requisite structure maps for $K_q^{qf}(-, (U, Y); A)$ are induced by the pairings

$$\mathcal{P}^{qf}(X, (U, Y)) \times \mathcal{P}^{fin}(X/S) \rightarrow \mathcal{P}^{qf}(S, (U, Y))$$

sending (M, N) to $(\pi \times id_U)_(M \otimes_{\mathcal{O}_{X \times U}} p^*N)$, where $\pi : X \rightarrow S$ is any smooth curve and $p : X \times U \rightarrow X$ is the canonical projection, and similarly for the other two presheaves. Likewise,*

$$K_{gqf}^q(-, (U, Y); A), \quad K_{gqf}^q(-, Y; A), \quad \text{and} \quad K_{geom}^q(-, Y; A)$$

are presheaves with K_0 transfers.

Moreover, the natural maps $K_q^{qf}(-, (U, Y); A) \rightarrow K_q^{qf}(-, U; A)$, $K_q^{fin}(-, U; A) \rightarrow K_q^{qf}(-, U; A)$, and $g_ : K_q^{fin}(-, Y; A) \rightarrow K_q^{fin}(-, Y'; A)$, where $g : Y \rightarrow Y'$ is any morphism, as well as the natural maps $K_{gqf}^q(-, (U, Y); A) \rightarrow K_{gqf}^q(-, U; A)$, $K_{gqf}^q(-, U; A) \rightarrow K_{geom}^q(-, U; A)$, and $g_* : K_{geom}^q(-, Y; A) \rightarrow K_{geom}^q(-, Y'; A)$ are all morphism of presheaves with K_0 transfers.*

Proof. Given $M \in \mathcal{P}^{qf}(X, (U, Y))$ and $N \in \mathcal{P}^{fin}(X/S)$, the support of $M \otimes_{\mathcal{O}_{X \times U}} p^*N$ in $X \times U$ is finite over $S \times U$. Thus, the functor described in the statement is biexact, thereby inducing a pairing on homotopy groups of associated spectra of the form

$$K_q^{qf}(X, (U, Y); A) \otimes K_0^{fin}(X/S) \rightarrow K_q^{qf}(S, (U, Y); A).$$

The axioms of a presheaf with K_0 transfers are readily verified.

The presheaf $K_q^{qf}(-, Y; A)$ coincides with $K_q^{qf}(-, (Y, Y); A)$. The assertion for the presheaf $K_q^{fin}(-, Y; A)$ is proven in exactly the same manner, using \mathcal{P}^{fin} and K^{fin} in place of \mathcal{P}^{qf} and K^{qf} .

Using strictly functorial models for the exact categories involved as in Remark 3.2, the pairing

$$\mathcal{K}^{qf}(X, (U, Y)) \wedge \mathcal{K}^{fin}(X/S) \rightarrow \mathcal{K}^{qf}(S, (U, Y))$$

is strictly natural on the category Sch/S . In particular, we have a pairing of simplicial spectra which at degree d is given as

$$\mathcal{K}^{qf}(X \times \Delta^d, (U, Y)) \wedge \mathcal{K}^{fin}(X \times \Delta^d/S \times \Delta^d) \rightarrow \mathcal{K}^{qf}(S \times \Delta^d, (U, Y)).$$

This gives the pairing

$$K_{gqf}^q(X, (U, Y); A) \otimes K_{geom}^0(X/S) \rightarrow K_{gqf}^q(S, (U, Y); A),$$

and the axioms of a presheaf with K_0 transfers are readily verified. (Note that $K_{geom}^0(X/S)$ is naturally a quotient of $K_0^{fin}(X/S)$.) The result for $K_{geom}^q(-, Y; A)$ is proven in exactly the same way.

The final assertion is readily verified. □

5. LOCALIZATION FOR K -HOMOLOGY

The goal of this section is to prove $\mathcal{K}^{BM}(-)$ admits a “localization” long exact sequence, analogous to that of G -theory. Specifically, we prove that the push forward along a closed immersion $Z \rightarrow X$ and the pullback along the open complement $X - Z \rightarrow X$ fit into the expected homotopy fibration sequence. Our proof technique requires us to assume Z and X are Cohen-Macaulay and that either k admits resolutions of singularities or k is algebraically closed, and we use spectra with finite coefficients $\mathcal{K}^{BM}(-; \mathbb{Z}/n)$ for some $n \geq 1$ relatively prime to the exponential characteristic of k . We know of no obstruction to the more general result involving a closed immersion of arbitrary varieties defined over an arbitrary base field. However, the result proven here is sufficient for the purposes of this paper.

Lemma 5.1. *Let $f : Y \rightarrow X$ be a morphism of quasi-projective varieties, $i : W \subset Y$ a closed subvariety, and $j : U \rightarrow Y$ the open complement of W in Y . Then the sequence of exact functors*

$$\mathcal{M}^{qf}(W/X) \xrightarrow{i_*} \mathcal{M}^{qf}(Y/X) \xrightarrow{j^*} \mathcal{M}^{qf}((U, Y)/X)$$

induces a homotopy fibration sequence of spectra

$$\mathcal{G}^{qf}(W/X) \xrightarrow{i_*} \mathcal{G}^{qf}(Y/X) \xrightarrow{j^*} \mathcal{G}^{qf}((U, Y)/X).$$

More generally, suppose $Y \subset \overline{Y}$ is an open embedding and f is the restriction of a morphism $\overline{Y} \rightarrow X$. Then the natural maps

$$\mathcal{G}^{qf}((W, \overline{Y})/X) \rightarrow \mathcal{G}^{qf}((Y, \overline{Y})/X) \rightarrow \mathcal{G}^{qf}((U, \overline{Y})/X)$$

form a homotopy fibration sequence.

Proof. The first assertion follows from the second by taking $Y = \overline{Y}$.

The abelian categories $\mathcal{M}^{qf}((W, \overline{Y})/X)$, $\mathcal{M}^{qf}((Y, \overline{Y})/X)$, and $\mathcal{M}^{qf}((U, \overline{Y})/X)$ are given by the filtered direct limits of abelian categories

$$\varinjlim_Z \mathcal{M}(Z \cap W), \quad \varinjlim_Z \mathcal{M}(Z \cap Y), \quad \text{and} \quad \varinjlim_Z \mathcal{M}(Z \cap U),$$

where Z ranges over all closed subschemes of \overline{Y} which are quasi-finite over X . For each fixed Z , the natural maps

$$\mathcal{G}(Z \cap W) \rightarrow \mathcal{G}(Z) \rightarrow \mathcal{G}(Z \cap U)$$

form a homotopy fibration sequence of spectra by Quillen’s Localization Theorem [20, Theorem 3]. Since a filtered direct limit of homotopy fibration sequences is again a homotopy fibration sequence, the result is proven. \square

The following proposition allows one (in certain contexts) to replace $\mathcal{K}_{gqf}(U/X)$ with $\mathcal{K}_{gqf}((U, Y)/X)$, where $U \rightarrow Y$ is a locally closed embedding (see Corollary 5.3 below). This result will be used to prove the localization sequence we seek (see Theorem 5.5 below).

Proposition 5.2. *Suppose $Y \rightarrow X$ is a flat morphism of quasi-projective k -varieties and let $U \subset Y$ be a locally closed subvariety. Let*

$$K_m^{qf}(U/X)(-) \quad \text{and} \quad K_m^{qf}((U, Y)/X)(-)$$

denote the presheaves of abelian groups on Sch/X given by the formulas

$$(V \rightarrow X) \mapsto K_m^{qf}(U \times_X V/V) \quad \text{and} \quad (V \rightarrow X) \mapsto K_m^{qf}((U \times_X V, Y \times_X V)/V).$$

Then the natural map of presheaves

$$K_m^{qf}((U, Y)/X)(-) \rightarrow K_m^{qf}(U/X)(-)$$

becomes an isomorphism upon sheafification in the cdh-topology.

Proof. It suffices to show that given any morphism $V \rightarrow X$ in Sch/k and an element α in the kernel or the cokernel of

$$K_m^{qf}((U, Y)/X)(V) \rightarrow K_m^{qf}(U/X)(V),$$

there is a cdh covering $\{p_i : V_i \rightarrow V\}$ of V such that $p_i^*(\alpha) = 0$ for all i . Note that if $\dim(V) = 0$, the above map is an isomorphism. Proceeding by induction on $\dim(V)$, we claim that it suffices to construct a blow-up $\pi : W \rightarrow V$ with center C of dimension strictly smaller than V such that $\pi^*(\alpha) = 0$. For if such a blow-up exists, then, by the induction hypothesis, there is a cdh covering $\{q_i : C_i \rightarrow C\}$ of C such that $q_i^*(\alpha|_C) = 0$, for all i , and the collection $\{C_i \rightarrow C \rightarrow V\} \cup \{W \rightarrow V\}$ forms a cdh covering of V with the desired property.

For brevity, write U_V for $U \times_X V$ and Y_V for $Y \times_X V$. We have

$$\mathcal{P}^{qf}(U_V/V) = \varinjlim_Z \mathcal{P}^{qf}(Z/V),$$

where the limit ranges over closed subschemes $Z \subset U_V$ which are quasi-finite over V . Similarly, $\mathcal{P}^{qf}((U_V, Y_V)/V)$ is given by the subdirect limit indexed by those Z 's satisfying the additional condition that the closure of Z in Y_V is quasi-finite over V .

To prove the proposition, it suffices to show that for each closed subscheme $Z \subset U_V$ which is quasi-finite and dominant over V , there is a blow-up $\pi : W \rightarrow V$ so that the composition of

$$\mathcal{P}^{qf}(Z/V) \xrightarrow{\subseteq} \mathcal{P}^{qf}(U_V/V) \xrightarrow{\pi^*} \mathcal{P}^{qf}(U_W/W)$$

lands in $\mathcal{P}^{qf}((U_W, Y_W)/W)$. To see that this suffices, if $\beta \in K_m^{qf}(U_V/V)$, then β is in the image of $K_m^{qf}(Z/V) \rightarrow K_m^{qf}(U_V/V)$ for some such Z . But then clearly $\pi^*(\beta)$ lies in the image of $K_m^{qf}((U_W, Y_W)/W) \rightarrow K_m^{qf}(U_W/W)$ as desired. Similarly, if α lies in the kernel of $K_m^{qf}((U_V, Y_V)/V) \rightarrow K_m^{qf}(U_V/V)$, then α lies in the image of $K_m^{qf}(Z_1/V) \rightarrow K_m^{qf}(U_V/V)$ for some $Z_1 \subset U_V$ whose closure in Y_V is quasi-finite over V , and there is a larger closed subscheme $Z_1 \subset Z \subset U_V$, with $Z \rightarrow V$ quasi-finite, such that α belongs to the kernel of the natural map $K_m^{qf}(Z_1/V) \rightarrow K_m^{qf}(Z/V)$. But then clearly $\pi^*(\alpha) = 0$ in $K_m^{qf}((U_W, Y_W)/W)$.

To construct such a blow-up π , let $\overline{Z} \subset Y_V$ denote the scheme-theoretic closure of Z in Y_V . By the platicification par éclatement theorem [21], there is a blow-up $\pi : W \rightarrow V$ such that the strict transform $\overline{Z}' \rightarrow W$ of $\overline{Z} \rightarrow V$ is flat and hence quasi-finite. Set $Z' = \overline{Z}' \cap U_W$, so that $Z' \rightarrow W$ is the strict transform of $Z \rightarrow V$. Since, by hypothesis, $Y_V \rightarrow V$ and $U_V \rightarrow V$ are flat, their strict transforms along $W \rightarrow V$ coincide with the pullbacks $Y_W \rightarrow W$ and $U_W \rightarrow W$. Similarly, given $M \in \mathcal{P}^{qf}(Z/V) \subset \mathcal{P}^{qf}(U_V/V)$, since M is \mathcal{O}_V -flat, its strict transform along $W \rightarrow V$ coincides with the pullback π^*M as coherent sheaves on U_W . Moreover, the support of π^*M is a subscheme of Z' , since the strict transform of a coherent sheaf supported on Z is supported on Z' . In particular, the closure of the support of π^*M in Y_W is contained in \overline{Z}' , which is quasi-finite over W . Thus the functor $\pi^* : \mathcal{P}^{qf}(Z/V) \rightarrow \mathcal{P}^{qf}(U_W/W)$ lands $\mathcal{P}^{qf}((U_W, Y_W)/W)$, as desired. \square

Corollary 5.3. *Let $Y \rightarrow X$ be a morphism of quasi-projective k -varieties and let $U \subset Y$ be a locally closed subvariety.*

- (1) *If k is an algebraically closed field and n is an integer relatively prime to the exponential characteristic of k , then the natural map*

$$\mathcal{K}_{gqf}(\mathrm{Spec} k, (U, Y)) \rightarrow \mathcal{K}_{gqf}(\mathrm{Spec} k, U) = \mathcal{K}^{BM}(U)$$

induces an isomorphism

$$K_{gqf}^{-q}(\mathrm{Spec} k, (U, Y); \mathbb{Z}/n) \cong K_q^{BM}(U; \mathbb{Z}/n),$$

for all $q \geq 0$.

- (2) *If k is any field admitting resolutions of singularities (for example, if k is any field of characteristic zero), then the natural map*

$$\mathcal{K}_{gqf}(\mathrm{Spec} k, (U, Y)) \rightarrow \mathcal{K}_{gqf}(\mathrm{Spec} k, U) = \mathcal{K}^{BM}(U)$$

is a weak homotopy equivalence.

Proof. This follows immediately from Proposition 5.2 (for $X = \mathrm{Spec} k$), Theorem 4.24, and Corollary 4.23. \square

Recall from Definition 2.2 the simplified bivariant notations $\mathcal{G}^{qf}(X, Y)$, $\mathcal{K}^{qf}(X, Y)$, etc., which stand for, respectively, $\mathcal{G}^{qf}(X \times Y/X)$, $\mathcal{K}^{qf}(X \times Y/X)$, etc.

Proposition 5.4. *Let Y be a projective variety, let $U \subset Y$ be a locally closed subvariety, and assume U is Cohen-Macaulay. If X is a smooth, quasi-projective variety, then the natural map*

$$\mathcal{K}^{qf}(X, (U, Y)) \rightarrow \mathcal{G}^{qf}(X, (U, Y))$$

is a weak equivalence of spectra.

Proof. Replacing Y with the closure of U , we may assume U is an open, dense subscheme of Y . Additionally, we may assume U is connected and hence equidimensional, so that Y is as well.

As the proof of [34, 2.2], Quillen's Resolution Lemma [20, Theorem 3] applies to the inclusion of categories $\mathcal{P}^{qf}(X, (U, Y)) \subset \mathcal{M}^{qf}(X, (U, Y))$, provided that given M in the larger category, there is surjection $E \twoheadrightarrow M$ with E in the smaller category. Given such an M , assume M is supported on $Z \subset X \times U$ such that the closure \overline{Z} of Z in $X \times Y$ is quasi-finite, and hence finite, over X .

Recall Y has been assumed equi-dimensional, say of dimension d , and so there exists a finite morphism $\pi : Y \rightarrow \mathbb{P}^d$. Since U is Cohen-Macaulay, the map $\pi|_U : U \rightarrow \mathbb{P}^d$ is necessarily flat by [18, 23.1]. Let $W = (1_X \times \pi)(\overline{Z})$ (the scheme-theoretic image), a closed subscheme of $X \times \mathbb{P}^d$ finite over X . As shown in the proof of [34, 2.3], there exists a larger closed subscheme $W' \subset X \times \mathbb{P}^d$ with $W \subset W'$ such that $W' \rightarrow X$ is finite and flat. Now let $\overline{Z}' = (1_X \times \pi)^{-1}(W') = W' \times_{\mathbb{P}^d} Y$ and observe that \overline{Z} is contained in \overline{Z}' and the morphism $\overline{Z}' \rightarrow X$ is finite, since it is the composition $\overline{Z}' \rightarrow W' \rightarrow X$ of finite morphisms. Set $Z' = \overline{Z}' \cap X \times U = W' \times_{\mathbb{P}^d} U$, so that we have the containment $Z \subset Z'$ and so that the morphism $Z' \rightarrow X$ is quasi-finite. The morphism $Z' \rightarrow X$ is flat as well, since it is the composition $Z' \rightarrow W' \rightarrow X$ of flat morphism. Regarding M as a coherent $\mathcal{O}_{Z'}$ -module as we may, we can find a locally free $\mathcal{O}_{Z'}$ -module E and a surjection $E \twoheadrightarrow M$. Since $E \in \mathcal{P}^{qf}(X, (U, Y))$, the result follows. \square

In the following theorem, we write $\mathcal{K}^{BM}(Y; \mathbb{Z}/n)$ for the spectrum obtained by smashing $\mathcal{K}^{BM}(Y)$ with the mod- n Moore spectrum, so that the q -th homotopy group of $\mathcal{K}^{BM}(Y; \mathbb{Z}/n)$ is naturally isomorphic to $K_q^{BM}(Y; \mathbb{Z}/n)$.

Theorem 5.5. *Let $W \subset Y$ be a closed embedding of Cohen-Macaulay, quasi-projective k -varieties and let $U = Y - Z$ denote the open complement.*

- (1) *If k is an algebraically closed field and $n \geq 1$ is an integer relatively prime to the exponential characteristic of k , then the natural maps of mod- n spectra*

$$\mathcal{K}^{BM}(W; \mathbb{Z}/n) \rightarrow \mathcal{K}^{BM}(Y; \mathbb{Z}/n) \rightarrow \mathcal{K}^{BM}(U; \mathbb{Z}/n)$$

form a homotopy fibration sequence of spectra.

- (2) *If k is any field admitting resolutions of singularities, then the natural maps of spectra*

$$\mathcal{K}^{BM}(W) \rightarrow \mathcal{K}^{BM}(Y) \rightarrow \mathcal{K}^{BM}(U)$$

form a homotopy fibration sequence of spectra.

Proof. Let A denote \mathbb{Z}/n or \mathbb{Z} . Embed Y as an open subvariety of a projective k -variety \overline{Y} . By Corollary 5.3, it suffices to show

$$\mathcal{K}_{gqf}((W, \overline{Y}) / \text{Spec } k; A) \rightarrow \mathcal{K}_{gqf}((Y, \overline{Y}) / \text{Spec } k; A) \rightarrow \mathcal{K}_{gqf}((U, \overline{Y}) / \text{Spec } k; A)$$

is a homotopy fibration sequence of spectra. Since a degree-wise fibration sequence of simplicial spectra induces a homotopy fibration sequence on geometric realizations (cf. [5]), it suffices to show that for any smooth, quasi-projective k -variety X , the natural maps

$$\mathcal{K}^{qf}(X, (W, \overline{Y}); A) \rightarrow \mathcal{K}^{qf}(X, (Y, \overline{Y}); A) \rightarrow \mathcal{K}^{qf}(X, (U, \overline{Y}); A)$$

form a homotopy fibration sequence. Using Proposition 5.4, it suffices to show

$$\mathcal{G}^{qf}(X, (W, \overline{Y}); A) \rightarrow \mathcal{G}^{qf}(X, (Y, \overline{Y}); A) \rightarrow \mathcal{G}^{qf}(X, (U, \overline{Y}); A)$$

is a homotopy fibration sequence. This follows from Lemma 5.1. \square

The following result shows that \mathcal{K}^{BM} really is the Borel-Moore theory associated to \mathcal{K}^{hom} (at least on the category of Cohen-Macaulay varieties).

Corollary 5.6. *Let $W \subset Y$ be a closed embedding of Cohen-Macaulay, projective k -varieties and set $U = Y - W$.*

- (1) *If k is an algebraically closed field and $n \geq 1$ is an integer relatively prime to the exponential characteristic of k , then the natural maps of mod- n spectra*

$$\mathcal{K}^{hom}(W; \mathbb{Z}/n) \rightarrow \mathcal{K}^{hom}(Y; \mathbb{Z}/n) \rightarrow \mathcal{K}^{BM}(U; \mathbb{Z}/n)$$

form a homotopy fibration sequence of spectra.

- (2) *If k is any field admitting resolutions of singularities, then the natural maps of spectra*

$$\mathcal{K}^{hom}(W) \rightarrow \mathcal{K}^{hom}(Y) \rightarrow \mathcal{K}^{BM}(U)$$

form a homotopy fibration sequence of spectra.

Proof. This is an immediate consequence of the theorem and Proposition 3.12. \square

6. GERSTEN CONJECTURE

In this section we establish the analogue of the so-called Gersten conjecture in K -theory for the collection of contravariant functors $K_m^{qf}(-, (U, Y))$, $m \geq 0$. The main result of this section will be used to understand the local behavior of the presheaf $K_m^{qf}(-, (U, Y))$ in the proof of Theorem 7.1 below. To state the result in its natural level of generality, we need the following new definition. Throughout this section, the reader might wish to consider initially the special case when $U = Y = \text{Spec } k$, when all of the results, and indeed all of the proofs, reduce to those pertaining to the classical Gersten conjecture.

Definition 6.1. Let Y be a projective k -variety, and let $U \subset Y$ be an open immersion. For any k -scheme X , let $\mathcal{M}^{qf,t}(X, (U, Y))$ denote the abelian category consisting of those coherent sheaves M on $X \times U$ which are supported on closed subschemes Z satisfying (1) the closure \overline{Z} of Z in $X \times Y$ is finite over X and (2) the image of $\pi : \overline{Z} \rightarrow X$ has codimension at least t in X (equivalently, the closure of the image of Z in X has codimension at least t). Let $\mathcal{G}^{qf,t}(X, (U, Y))$ denote the associated K -theory spectrum.

Let $M \in \mathcal{M}^{qf,t}(X, (U, Y))$ with Y an *equidimensional* projective k -variety, say of dimension d . Then M is supported on a closed subscheme $Z \subset X \times U$ such that the closure \overline{Z} of Z in $X \times Y$ is finite over X . As shown in the proof of Proposition 5.4, for any choice of a finite, dominant morphism $\pi : Y \rightarrow \mathbb{P}^d$, the image of \overline{Z} in $X \times \mathbb{P}^d$ is contained in a larger closed subscheme $(1 \times \pi)(Z) \subset W \subset X \times \mathbb{P}^d$ such that $W \rightarrow X$ is finite and flat. Upon pulling back W to $X \times Y$, we see that Z is contained in a larger closed subscheme $Z \subset Z' \subset X \times Y$ such that $Z' \rightarrow X$ is finite and dominant. Thus, in this situation, the category $\mathcal{M}^{qf,t}(X, (U, Y))$ can be described as the direct limit

$$\varinjlim_Z \mathcal{M}^t(Z),$$

where the indexing ranges over those closed subschemes $Z \subset X \times U$ whose closures in $X \times Y$ are finite and dominant over X . Here, $\mathcal{M}^t(Z)$ denotes the abelian category of coherent sheaves on Z whose supports have codimension at least t . This observation will be used in the proof of the following theorem.

Theorem 6.2. *Let X be a smooth, quasi-projective k -variety, $x = \{x_1, \dots, x_\nu\} \subset X$ a finite set of closed points, Y an equidimensional projective k -variety, and $U \subset Y$ an open subscheme. Write X_x for $\text{Spec } \mathcal{O}_{X,x}$, where $\mathcal{O}_{X,x}$ is the semi-localization of X at the finite set x . Then the map of K -theory spectra associated to the inclusion of abelian categories*

$$\mathcal{M}^{qf,t+1}(X_x, (U, Y)) \rightarrow \mathcal{M}^{qf,t}(X_x, (U, Y))$$

is weakly null-homotopic.

In particular, if U is Cohen-Macaulay and A is any abelian group, then for all $m \geq 0$, the natural map

$$K_m^{qf}(X_x, (U, Y); A) \rightarrow \bigoplus_{z \in X_x \times U} K_m(k(z); A)$$

is an injection, where the direct sum is indexed by all points of $X_x \times U$ whose closures in $X_x \times Y$ are finite and dominant over X_x . If additionally $m = 0$, this map is an isomorphism.

Proof. The proof is essentially a souped-up version of Quillen's original proof of the Gersten conjecture (cf. [20, 5.11]). We follow the modification of Quillen's proof presented in [12]. In particular, we first assume k is infinite.

As explained above, for any d we have

$$\mathcal{M}^{qf,d}(X_x, (U, Y)) = \varinjlim_Z \mathcal{M}^d(Z),$$

where the direct limit ranges over all those closed subschemes of $X_x \times U$ whose closures in $X_x \times Y$ are finite and dominant over X_x . To prove the theorem, it therefore suffices to fix such a subscheme Z and prove the natural exact functor

$$\mathcal{M}^{t+1}(Z) \rightarrow \mathcal{M}^t(X_x, (U, Y))$$

induces a weak null-homotopic map on K -theory spectra. In fact, since $\mathcal{M}^{t+1}(Z) = \varinjlim_{Z'} \mathcal{M}^t(Z')$ where the limit ranges over all closed subvarieties Z' of Z of codimension one, it suffices to fix such a subvariety $Z' \subset Z$ and prove that the natural map $\mathcal{G}^t(Z') \rightarrow \mathcal{G}^{qf,t}(X_x, (U, Y))$ is null-homotopic. (Here, $\mathcal{G}^t(Z')$ denotes the K -theory spectrum associated to $\mathcal{M}^t(Z')$.)

Upon replacing X with a sufficiently small open neighborhood of x in X , we may assume that $Z \rightarrow X$ is quasi-finite and that its closure in $X \times Y$ is finite and dominant over X . Modifying our previous notation for simplicity by replacing X with a suitably small open neighborhood of x , let us now write $Z \subset X \times U$ for a closed subscheme whose closure \overline{Z} in $X \times Y$ is finite and dominant over X . Let $Z' \subset Z$ be a codimension one subscheme and let \overline{Z}' denote its closure in $X \times Y$. Finally, we will write Z_x , Z'_x , \overline{Z}_x , and \overline{Z}'_x for the pullbacks of the evident schemes along $X_x \rightarrow X$. To prove the theorem, it suffices to show the natural functor $j_* : \mathcal{M}^t(Z'_x) \rightarrow \mathcal{M}^{qf,t}(X_x, (U, Y))$, where $j : Z'_x \rightarrow X_x \times U$ is the canonical closed immersion, induces a weak null-homotopy on associated K -theory spectra.

Observe that the image of \overline{Z}' in X has codimension one. As shown in [12, 1.3], since X is smooth (say of dimension d), there exists a map $\pi : X \rightarrow \mathbb{A}^{d-1}$ which is smooth near x and such that the composition of $\overline{Z}' \rightarrow X \rightarrow \mathbb{A}^{d-1}$ is a finite map. Let $\overline{W}' = X \times_{\mathbb{A}^{d-1}} \overline{Z}'$ and $W' = X \times_{\mathbb{A}^{d-1}} Z'$. Then there is a pullback square of the form

$$\begin{array}{ccc} W' & \longrightarrow & X \times U \\ \downarrow & & \downarrow \\ \overline{W}' & \longrightarrow & X \times Y. \end{array}$$

The bottom map in this diagram is finite since its composition with the projection $X \times Y \rightarrow X$ is finite. Thus the map $W' \rightarrow X \times U$ is also finite. By construction, there is a natural map $q : W' \rightarrow Z'$ which is split by a closed immersion $i : Z' \rightarrow W'$. Moreover, the map q is smooth near each of the finitely many points $Q = \{w'_1, \dots, w'_m\}$ comprising the fiber of $W' \rightarrow X$ over x . It follows that the closed subscheme $j(Z') \subset W'$ is a Cartier divisor near each point of $Q \cap j(Z')$, and is thus defined by a single non-zero divisor in some neighborhood of Q in W' . We thus have a short exact sequence of $\mathcal{O}_{W', Q}$ -modules

$$0 \rightarrow \mathcal{O}_{W', Q} \rightarrow \mathcal{O}_{W', Q} \rightarrow \mathcal{O}_{Z', Q \cap Z'} \rightarrow 0.$$

The composition $Z' \rightarrow W' \rightarrow X \times U$ coincides with the original closed immersion $j : Z' \rightarrow X \times U$, and thus $\mathcal{O}_{Z', Q \cap Z'} = \mathcal{O}_{Z', x}$. Note that $\mathcal{O}_{W', Q}$ is $\mathcal{O}_{Z', x}$ -flat and

consequently we have a short exact sequence of exact functors from $\mathcal{M}^t(Z'_x)$ to $\mathcal{M}^t(W'_Q)$ given on an object $M \in \mathcal{M}^t(Z'_x)$ by

$$0 \rightarrow M \otimes_{\mathcal{O}_{Z',x}} \mathcal{O}_{W',Q} \rightarrow M \otimes_{\mathcal{O}_{Z',x}} \mathcal{O}_{W',Q} \rightarrow i_* M \rightarrow 0.$$

From above, the map $W' \rightarrow X \times U$ is finite and thus so is the localization $g : W'_Q \rightarrow X_x \times U$. Taking pushforwards we have a short exact sequence of exact functors $\mathcal{M}^t(Z_x) \rightarrow \mathcal{M}^{qf,t}(X_x, (U, Y))$ given on an object M by

$$0 \rightarrow g_*(M \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{W',Q}) \rightarrow g_*(M \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{W',Q}) \rightarrow j_* M \rightarrow 0.$$

(Note that $g_*(M \otimes_{\mathcal{O}_{Z',x}} \mathcal{O}_{W',Q})$ is supported on the image of \overline{W}' and hence lies in $\mathcal{M}^{qf,t}(X_x, (U, Y))$.) Using Quillen's Additivity Theorem [20, Theorem 2], this proves the map $j_* : \mathcal{G}^t(Z'_x) \rightarrow \mathcal{G}^{qf,t}(X_x, (U, Y))$ is null-homotopic.

The case of a finite field k is deduced from the infinite case in precisely the same fashion as in [12].

To prove the final two assertions, define an exact functor

$$(10) \quad F : \mathcal{M}^{qf}(X_x, (U, Y)) \rightarrow \bigoplus_z \mathcal{M}^{fin.len.}(\mathcal{O}_z)$$

by sending M to $(M_z)_z$, where z ranges over points of $X_x \times U$ as in the statement of the theorem and $\mathcal{M}^{fin.len.}$ denotes the abelian category of finitely generated modules of finite length. We claim that the functor (10) is “surjective up to isomorphism”, in the sense that for every object (M_z) of $\bigoplus_z \mathcal{M}^{fin.len.}(\mathcal{O}_z)$, there is an object N of $\mathcal{M}^{qf}(X_x, (U, Y))$ and isomorphisms $M_z \cong N_z = N \otimes_{\mathcal{O}_{X_x \times U}} \mathcal{O}_{X_x \times U, z}$ and, furthermore, for every map $(M_z) \rightarrow (M'_z)$ in $\bigoplus_z \mathcal{M}^{fin.len.}(\mathcal{O}_z)$ there is a map $N \rightarrow N'$ in $\mathcal{M}^{qf}(X_x, (U, Y))$ such that $M_z \cong N_z$, $M'_z \cong N'_z$, and the evident triangles commute. To see this, note that if M_z is in $\mathcal{M}^{fin.len.}(\mathcal{O}_{X_x \times U, z})$, then M_z is supported on $\mathcal{O}_{\tilde{Z}, z}$ where \tilde{Z} is an infinitesimal thickening of Z (i.e., if \mathcal{I} is the sheaf of ideals definition $Z \subset X \times U$, then let \tilde{Z} be defined by \mathcal{I}^n for $n \gg 0$). Since the closure of \tilde{Z} in $X \times Y$ is also finite over X , the desired surjectivity holds.

It follows that the functor F identifies the abelian category $\bigoplus_z \mathcal{M}^{fin.len.}(\mathcal{O}_z)$ with the quotient abelian category of $\mathcal{M}^{qf}(X_x, (U, Y))$ by the Serre subcategory $\mathcal{M}^{qf,1}(X_x, (U, Y))$. Thus by [20, Theorem 5] there is a homotopy fibration sequence of spectra

$$\mathcal{G}^{qf,1}((U_x, Y_x)/X_x) \rightarrow \mathcal{G}^{qf}((U_x, Y_x)/X_x) \rightarrow \bigoplus_z \mathcal{G}^{fin.len.}(\mathcal{O}_z).$$

Since the exact functor defined by pushforward

$$\bigoplus_z \mathcal{M}(k(z)) \rightarrow \bigoplus_z \mathcal{M}^{fin.len.}(\mathcal{O}_z)$$

induces a weak equivalence on K -theory spectra by [20, Theorem 4], the injectivity of

$$(11) \quad K_m^{qf}(X_x, (U, Y); A) \rightarrow \bigoplus_{z \in X_x \times U} K_m(k(z); A)$$

follows from the first part of the theorem and Proposition 5.4. Finally, when $m = 0$ and $A = \mathbb{Z}$, the target of (11) is isomorphic to the free abelian group on the set of closed, integral subschemes of $X_x \times U$ whose closures in $X_x \times Y$ are finite and dominant over X_x . Given such a subscheme Z , its structure sheaf \mathcal{O}_Z determines a class in $K_0^{qf}(X_x, (U, Y))$ which clearly maps to the basis element $[Z]$

of this free abelian group, and thus (11) is surjective in this case. For a general A with $m = 0$, observe that $K_0^{qf}(X_x, (U, Y); A) = K_0^{qf}(X_x, (U, Y)) \otimes A$ and that $\bigoplus_z K_0(k(z); A) \cong (\bigoplus_z K_0(k(z))) \otimes A$, since the underlying spectra are connective. \square

Remark 6.3. The spectra $\mathcal{G}^{qf,t}(X, (U, Y))$ form a decreasing tower starting with $\mathcal{G}^{qf}(X, (U, Y))$

$$\mathcal{G}^{qf,d}(X, (U, Y)) \rightarrow \cdots \rightarrow \mathcal{G}^{qf,1}(X, (U, Y)) \rightarrow \mathcal{G}^{qf,0}(X, (U, Y)) = \mathcal{G}^{qf}(X, (U, Y)),$$

where $d = \dim(X)$. Letting $\mathcal{M}^{qf,t/t+1}(X, (U, Y))$ denote the quotient of the abelian category $\mathcal{M}^{qf,t}(X, (U, Y))$ by the Serre subcategory $\mathcal{M}^{qf,t+1}(X, (U, Y))$ and letting $\mathcal{G}^{qf,t/t+1}(X, (U, Y))$ denote the associated K -theory spectrum, we obtain the existence of a complex of abelian groups

$$\begin{aligned} 0 \rightarrow G_m^{qf}(X, (U, Y)) &\rightarrow G_m^{qf,0/1}(X, (U, Y)) \rightarrow G_{m-1}^{qf,1/2}(X, (U, Y)) \rightarrow \cdots \\ &\rightarrow G_0^{qf,m/m+1}(X, (U, Y)) \rightarrow 0 \end{aligned}$$

(where $G_i^{qf,t/t+1}(X, (U, Y)) = 0$ if $t > \dim(X)$). Moreover, this complex is natural for quasi-finite, flat maps and in particular may be regarded as a complex of presheaves for the small Zariski (or étale) site on a fixed X . For X smooth, the theorem implies that this complex is exact locally in Zariski topology.

As was also proven in the theorem, when U is Cohen-Macaulay, the presheaf $G_m^{qf,0/1}(-, (U, Y))$ (defined on the small Zariski site over a smooth, connected variety X) is locally isomorphic to $\bigoplus_{z \in X \times U} K_m(k(z))$, where z ranges over points whose closures in $X \times Y$ are finite and dominant over X . More generally, write $P^t(X, (U, Y))$ for the collection of points $z \in X \times U$ such that the closure of z in $X \times Y$ is finite over X and the image of z in X has codimension t . Then it is not hard to prove $G_m^{qf,t/t+1}(-, (U, Y))$ is locally isomorphic to $\bigoplus_{z \in P^t(X, (U, Y))} K_m(k(z))$. The latter presheaf is a sheaf on the small Zariski site over X for all t and m , a fact which is seen to hold by observing that the covariant set-valued functor $(V \subset X) \mapsto P^t(V, (U, Y))$ satisfies the dual of the sheaf axiom. Thus we obtain a resolution of the sheafification of $K_m^{qf}(-, (U, Y))$ of the form

$$\begin{aligned} 0 \rightarrow K_m^{qf}(-, (U, Y))_{\text{Zar}}^{\sim} &\rightarrow \bigoplus_{z \in P^0(X, (U, Y))} K_m(k(z)) \rightarrow \bigoplus_{z \in P^1(X, (U, Y))} K_{m-1}(k(z)) \\ &\rightarrow \cdots \rightarrow \bigoplus_{z \in P^{m-1}(X, (U, Y))} K_1(k(z)) \rightarrow \bigoplus_{z \in P^m(X, (U, Y))} K_0(k(z)) \rightarrow 0. \end{aligned}$$

Taking $U = Y = \text{Spec } k$, this recovers Quillen's well-known resolution of $K_m(-)_{\text{Zar}}^{\sim}$ (cf. [20, 5.9]), which leads to a proof of Bloch's formula:

$$H_{\text{Zar}}^m(X, K_m(-)_{\text{Zar}}^{\sim}) \cong CH^m(X).$$

Unfortunately, unlike in the special case $U = Y = \text{Spec } k$, the sheaf

$$X \mapsto \bigoplus_{z \in P^t(X, (U, Y))} K_m(k(z))$$

is in general *not* flasque, since for a open subscheme $V \subset X$ (with $V \neq X$), the set $P^t(V, (U, Y))$ is strictly larger than $P^t(X, (U, Y))$. Thus one apparently cannot obtain, in general, a formula for the m -th Zariski cohomology group of $K_m^{qf}(-, (U, Y))_{\text{Zar}}^{\sim}$ analogous to Bloch's formula.

As an application (unrelated to the remainder of this paper) of the machinery developed in this section and the previous ones, we provide the following two theorems. Together they relate the motivic complexes of Voevodsky with certain complexes defined by Grayson [14] involving K_0 groups of automorphisms (see Theorem 6.5). For any k -variety Y , define $c_{equi}^0(-, Y)$ to be the presheaf on Sch/k whose value at X is $c_{equi}^0(X, Y) = c_{equi}^0(X \times Y/X)$, the free abelian group on the collection of closed, integral subschemes of $X \times Y$ that are finite and dominant over some irreducible component of X .

Theorem 6.4. *Let P be a Cohen-Macaulay, projective k -variety, where k is an arbitrary field. Let*

$$K_0^{fin}(-, P) \rightarrow c_{equi}^0(-, P)$$

be the map of presheaves on Sch/k defined at a variety X by sending the class of a module $M \in \mathcal{P}^{fin}(X, P)$ to the cycle $\sum_z \text{length}_{\mathcal{O}_{X \times P, z}}(M_z)[\bar{z}]$, where the sum ranges over points z of $X \times P$ whose closures are finite and dominant over an irreducible component of X . Then the induced map

$$K_0^{fin}(- \times \Delta^\bullet, P) \rightarrow c_{equi}^0(- \times \Delta^\bullet, P)$$

is a homotopy equivalence locally in the Zariski topology on Sm/k .

Proof. By Theorems 4.24 and 4.22, it suffices to prove the natural map

$$K_0^{fin}(-, P) \rightarrow c_{equi}^0(-, P)$$

is an isomorphism locally for the Nisnevich topology on Sm/k . (One may readily verify that this morphism is a morphism of presheaves with K_0 transfers.) In fact, we show it is so for the Zariski topology. Let X belong to Sm/k , let x be a point in X , and define $X_x = \text{Spec } \mathcal{O}_{X,x}$, the spectrum of the local ring of X at x . It suffices to prove $K_0^{fin}(X_x, P) \rightarrow c_{equi}^0(X_x, P)$ is an isomorphism. This is precisely the final assertion of Theorem 6.2 (for $Y = U = P$ and $A = \mathbb{Z}$). \square

Define $K_0^{fin}(-, \mathbb{G}_m^{\wedge t})$ to be the quotient presheaf of $K_0^{fin}(-, \mathbb{G}_m^t)$ given by factoring out the images of

$$K_0^{fin}(-, \mathbb{G}_m^i \times \{1\} \times \mathbb{G}_m^{t-i-1}) \rightarrow K_0^{fin}(-, \mathbb{G}_m^t),$$

for $i = 0, \dots, t-1$, induced by the closed immersions $\mathbb{G}_m^i \times \{1\} \times \mathbb{G}_m^{t-i-1} \rightarrow \mathbb{G}_m^t$. One may readily verify that the t projection maps of the form $\mathbb{G}_m^t \rightarrow \mathbb{G}_m^{t-1}$ induce a family of compatible splitting so that $K_0^{fin}(-, \mathbb{G}_m^{\wedge t})$ is a summand of $K_0^{fin}(-, \mathbb{G}_m^t)$. Define $c_{equi}^0(-, \mathbb{G}_m^{\wedge t})$ similarly, and observe that the same splittings show it is a summand of $c_{equi}^0(-, \mathbb{G}_m^t)$.

Theorem 6.5. *The natural map*

$$K_0^{fin}(- \times \Delta^\bullet, \mathbb{G}_m^{\wedge t}) \rightarrow c_{equi}^0(- \times \Delta^\bullet, \mathbb{G}_m^{\wedge t})$$

is a homotopy equivalence locally in the Zariski topology on Sm/k .

Proof. Since the given map is a summand of the map

$$(12) \quad K_0^{fin}(- \times \Delta^\bullet, Y_1 \times \cdots \times Y_t) \rightarrow c_{equi}^0(- \times \Delta^\bullet, Y_1 \times \cdots \times Y_t),$$

where $Y_i = \mathbb{G}_m^1$ for all i , it suffices to prove (12) is a homotopy equivalence locally for the Zariski topology on Sm/k , where each Y_i is \mathbb{A}^1 , \mathbb{G}_m^1 or \mathbb{P}^1 .

As a preliminary step, we contend that if $V = U_0 \cup U_1$ is an open covering of a quasi-projective variety V , then each of the squares of chain complexes of sheaves

$$\begin{array}{ccc} K_0^{fin}(- \times \Delta^\bullet, U_0 \cap U_1)_{\text{Zar}} & \longrightarrow & K_0^{fin}(- \times \Delta^\bullet, U_0)_{\text{Zar}} \\ \downarrow & & \downarrow \\ K_0^{fin}(- \times \Delta^\bullet, U_1)_{\text{Zar}} & \longrightarrow & K_0^{fin}(- \times \Delta^\bullet, V)_{\text{Zar}} \end{array}$$

and

$$\begin{array}{ccc} c_{equi}^0(- \times \Delta^\bullet, U_0 \cap U_1)_{\text{Zar}} & \longrightarrow & c_{equi}^0(- \times \Delta^\bullet, U_0)_{\text{Zar}} \\ \downarrow & & \downarrow \\ c_{equi}^0(- \times \Delta^\bullet, U_1)_{\text{Zar}} & \longrightarrow & c_{equi}^0(- \times \Delta^\bullet, V)_{\text{Zar}} \end{array}$$

has the property that the induced map on the cones of the columns is a homotopy equivalence. By Theorems 4.24 and 4.22, it suffices to prove the induced map on the cones of the columns of

$$\begin{array}{ccc} K_0^{fin}(-, U_0 \cap U_1) & \longrightarrow & K_0^{fin}(-, U_0) \\ \downarrow & & \downarrow \\ K_0^{fin}(-, U_1) & \longrightarrow & K_0^{fin}(-, V) \end{array}$$

is locally an isomorphism for the Nisnevich topology, and similarly for the diagram obtained by replacing K_0^{fin} by c_{equi}^0 . Let X be a smooth variety, let x be a point of X , and let X_x^{hen} denote the spectrum of the henselization of the local ring of X at x . Observe that, for any open subscheme W of V , $K_0^{fin}(X_x^{hen}, W) = \varinjlim_Z K_0^{fin}(Z/X_x^{hen})$, where Z ranges over closed subschemes of $X_x^{hen} \times W$ that are finite and dominant over X_x^{hen} . Since any such Z decomposes as a disjoint union of spectra of local rings, we have

$$K_0^{fin}(X_x^{hen}, W) \cong \varinjlim_Z K_0^{fin}(Z \cap W/X_x^{hen})$$

where the limit ranges over closed subschemes Z of $X_x^{hen} \times V$ that are finite and dominant over X_x^{hen} . The desired result follows. A similar (in fact, simpler) argument applies for c_{equi}^0 .

Returning our attention to the map (12), note that it is locally a homotopy equivalence if each Y_i is \mathbb{P}^1 by Theorem 6.4. Moreover, for any W , a standard argument shows that the maps

$$K_0^{fin}(- \times \Delta^\bullet, W \times \mathbb{A}^1) \rightarrow K_0^{fin}(- \times \Delta^\bullet, W)$$

and

$$c_{equi}^0(- \times \Delta^\bullet, W \times \mathbb{A}^1) \rightarrow c_{equi}^0(- \times \Delta^\bullet, W)$$

defined by pushforward along $W \times \mathbb{A}^1 \rightarrow W$ are homotopy equivalences. Thus (12) is a homotopy equivalence if each Y_i is either \mathbb{P}^1 or \mathbb{A}^1 . We proceed by induction on the number N of copies of \mathbb{G}_m^1 appearing in the list Y_1, \dots, Y_t . For $N > 0$, after re-indexing, we may assume that $Y_1 = \mathbb{G}_m^1$. Let $Y = Y_2 \times \dots \times Y_t$ and let $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$ denote the standard open cover. Then the general result proven above

shows that the induced map on the cones of the columns of

$$\begin{array}{ccc} K_0^{fin}(- \times \Delta^\bullet, \mathbb{G}_m^1 \times Y) & \longrightarrow & K_0^{fin}(- \times \Delta^\bullet, \mathbb{A}^1 \times Y) \\ \downarrow & & \downarrow \\ K_0^{fin}(- \times \Delta^\bullet, \mathbb{A}^1 \times Y) & \longrightarrow & K_0^{fin}(- \times \Delta^\bullet, \mathbb{P}^1 \times Y) \end{array}$$

is locally a homotopy equivalence for the Zariski topology, and similarly for c_{equi}^0 replacing K_0^{fin} . Using the naturality of $K_0^{fin}(-, W) \rightarrow c_{equi}^0(-, W)$ for open immersions $W \rightarrow W'$, the proof is complete. \square

7. ÉTALE DESCENT FOR K -HOMOLOGY

In this section we establish the existence of a natural convergent spectral sequence abutting to $K_*^{BM}(U; \mathbb{Z}/n)$ whose E_2 terms are given by étale cohomology groups (see Theorem 7.6). Here, U is assumed to be a Cohen-Macaulay variety defined over an algebraically closed field k and $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . This spectral sequence should be thought of as the analogue for \mathcal{K}^{BM} with finite coefficients of the motivic Atiyah-Hirzebruch spectral sequence [3, 8, 17]. We also show $\mathcal{K}^{BM}(-; \mathbb{Z}/n)$ satisfies étale descent in the sense described in Theorem 7.8 below.

A key ingredient in both of these theorems is Suslin's theorem relating motivic cohomology of weight at least $\dim(U)$ with étale cohomology. In fact, we need a slight extension of Suslin's result involving a naturality assertion not present in his original formulation. This is given in Theorem 7.3. Another key ingredient is given by the following result.

Theorem 7.1. *Let U be a Cohen-Macaulay, quasi-projective variety over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Then for all $m \geq 0$, the map of presheaves*

$$K_0^{qf}(-, U) \otimes K_m(k; \mathbb{Z}/n) \rightarrow K_m^{qf}(-, U; \mathbb{Z}/n)$$

induced by a cup product (i.e., the pairing given by (1)) becomes an isomorphism upon sheafification in the h -topology. Moreover, this isomorphism is contravariantly natural for quasi-finite, flat morphisms of Cohen-Macaulay varieties.

Proof. Let $U \subset Y$ be an open dense embedding with Y projective. Using Proposition 5.2, it suffices to show the map defined by cup product

$$(13) \quad \theta : K_0^{qf}(-, (U, Y)) \otimes K_m(k; \mathbb{Z}/n) \rightarrow K_m^{qf}(-, (U, Y); \mathbb{Z}/n)$$

becomes an isomorphism upon sheafification in the h -topology. For any integral k -scheme X , we have a commutative diagram of the form

$$\begin{array}{ccc} K_0^{qf}(X, (U, Y)) \otimes K_m(k; \mathbb{Z}/n) & \longrightarrow & K_m^{qf}(X, (U, Y); \mathbb{Z}/n) \\ \downarrow & & \downarrow \\ (14) \quad \bigoplus_z K_m(k; \mathbb{Z}/n) & \longrightarrow & \bigoplus_z G_m^{fin.len.}(\mathcal{O}_{X \times U, z}; \mathbb{Z}/n) \\ \uparrow = & & \uparrow \cong \\ \bigoplus_z K_m(k; \mathbb{Z}/n) & \longrightarrow & \bigoplus_z K_m(k(z); \mathbb{Z}/n), \end{array}$$

where z ranges over all points of $X \times U$ whose closures in $X \times Y$ are finite and dominant over X . The upper-left vertical map in this diagram is defined by tensoring $K_m(k; \mathbb{Z}/n)$ with the cycle class map $K_0^{qf}(X, (U, Y)) \rightarrow \bigoplus_z \mathbb{Z}$ taking M to $\sum_z \text{length}_z(M_z)$. The upper-right vertical map is given by the evident localization functor, and the bottom horizontal map is induced by the field extensions $k \rightarrow k(z)$. If X is the localization of a smooth variety at a closed point, then by Theorem 6.2 the upper-right vertical map of (14) is an injection and the upper-left vertical map is an isomorphism.

Let $f : X' \rightarrow X$ be a dominant map of smooth, connected varieties of the same dimension and let x and x' denote the generic points of X and X' . In this situation, we define two pullback maps

$$f^* : \bigoplus_z K_m(k; \mathbb{Z}/n) \rightarrow \bigoplus_{z'} K_m(k; \mathbb{Z}/n)$$

and

$$f^* : \bigoplus_z K_m(k(z); \mathbb{Z}/n) \rightarrow \bigoplus_{z'} K_m(k(z'); \mathbb{Z}/n)$$

with z (resp., z') ranging over points of $X \times Y$ (resp., $X' \times Y$) whose closures in $X \times Y$ (resp., $X' \times Y$) are finite and dominant over X (resp., X'). To define the latter map f^* , given such a z , let z'_1, \dots, z'_l be the finitely many points which belong to the fiber of $X' \times U \rightarrow X \times U$ over z and whose closures in $X' \times Y$ are finite and dominant over X' . (That there are only finitely many such points follows from the fact that $k(x) \rightarrow k(z)$ and $k(x) \rightarrow k(x')$ are finite field extensions.) Then the map $f^* : K_m(k(z); \mathbb{Z}/n) \rightarrow \bigoplus_{z'} K_m(k(z'); \mathbb{Z}/n)$ is defined by the diagram

$$K_m(k(z); \mathbb{Z}/n) \longrightarrow G_m^{fin.len.}(\mathcal{O}_{\text{Spec } k(z) \times_X X'}; \mathbb{Z}/n) \xleftarrow{\cong} \bigoplus_i K_m(k(z'_i); \mathbb{Z}/n).$$

The map $f^* : \bigoplus_z K_m(k; \mathbb{Z}/n) \rightarrow \bigoplus_{z'} K_m(k; \mathbb{Z}/n)$ is defined by taking $m = 0$ in the above situation and tensoring over \mathbb{Z}/n by $K_m(k; \mathbb{Z}/n)$.

We claim that the diagram (14) is functorial with respect to dominant morphisms of smooth, connected varieties of the same dimension. That is, given such a morphism $f : X' \rightarrow X$, we claim that the cube

(15)

$$\begin{array}{ccccc} K_0^{qf}(X, (U, Y)) \otimes K_m(k; \mathbb{Z}/n) & \xrightarrow{\quad} & K_m^{qf}(X, (U, Y); \mathbb{Z}/n) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ K_0^{qf}(X', (U, Y)) \otimes K_m(k; \mathbb{Z}/n) & \xrightarrow{\quad} & K_m^{qf}(X, (U, Y); \mathbb{Z}/n) & & \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_z K_m(k; \mathbb{Z}/n) & \xrightarrow{\quad} & \bigoplus_z K_m(k(z); \mathbb{Z}/n) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \bigoplus_{z'} K_m(k; \mathbb{Z}/n) & \xrightarrow{\quad} & \bigoplus_{z'} K_m(k(z'); \mathbb{Z}/n) & & \end{array}$$

commutes. The front (resp., back) square of this cube are given by the outer square of (14) for the variety X (resp., X'), and the diagonal arrows are given by functoriality. The front, back, top, and bottom squares of (15) are readily seen to commute. The left-hand square commutes by the functoriality of the cycle class map. To establish the commutativity of the right-hand square, let $z \in X \times U$ and

$z'_1, \dots, z'_l \in X \times U$ be given as above. We claim that the diagram of spectra

$$(16) \quad \begin{array}{ccccc} \mathcal{K}^{qf}(X, (U, Y)) & \longrightarrow & \mathcal{G}^{f.l., \mathcal{O}_X\text{-flat}}(\mathcal{O}_{X \times U, z}) & \xleftarrow{\sim} & \mathcal{K}(k(z)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}^{qf}(X', (U, Y)) & \longrightarrow & \bigoplus_i \mathcal{G}^{f.l., \mathcal{O}_{X'}\text{-flat}}(\mathcal{O}_{X' \times U, z'_i}) & \xleftarrow{\sim} & \bigoplus_i \mathcal{K}(k(z'_i)) \end{array}$$

commutes. Here, $\mathcal{G}^{f.l., \mathcal{O}_X\text{-flat}}(\mathcal{O}_{X \times U, z})$ is the K -theory spectrum associated to the exact subcategory $\mathcal{M}^{f.l., \mathcal{O}_X\text{-flat}}(\mathcal{O}_{X \times U, z})$ of $\mathcal{M}^{fin.len.}(\mathcal{O}_{X \times U, z})$ consisting of coherent sheaves which are \mathcal{O}_X -flat. The spectrum $\mathcal{G}^{f.l., \mathcal{O}_X\text{-flat}}(\mathcal{O}_{X' \times U, z'})$ is similarly defined. (Note, in particular, that the middle vertical arrow of (16) is well-defined, since the underlying functor $-\otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is exact on $\mathcal{M}^{f.l., \mathcal{O}_X\text{-flat}}(\mathcal{O}_{X \times U, z})$.) Indeed, the commutativity of (16) is an immediate consequence of the commutativity of the diagram of schemes

$$\begin{array}{ccccc} \coprod_i \mathrm{Spec} k(z'_i) & \longrightarrow & \coprod_i \mathrm{Spec} \mathcal{O}_{X' \times U, z'_i} & \longrightarrow & X' \times U \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec} k(z) & \longrightarrow & \mathrm{Spec} \mathcal{O}_{X \times U, z} & \longrightarrow & X \times U. \end{array}$$

The fact that horizontal arrows on the right in (16) are weak equivalences as indicated follows from the fact that the natural map

$$G^{f.l., \mathcal{O}_X\text{-flat}}(\mathcal{O}_{X \times U, z}) \rightarrow G^{fin.len.}(\mathcal{O}_{X \times U, z})$$

is a weak equivalence by Quillen's Resolution Theorem [20, Theorem 3], and similarly for X' and z' in place of X and z . Indeed, given a coherent sheaf M in $\mathcal{M}^{fin.len.}(\mathcal{O}_{X \times Y, z})$, as in the proof of Theorem 6.2, we have that M is the localization sheaf \tilde{M} in $\mathcal{M}^{qf}(X, (U, Y))$. As shown in the proof of Proposition 5.4, there exists a sheaf $E \in \mathcal{P}^{qf}(X, (U, Y))$ and a surjection $E \twoheadrightarrow \tilde{M}$. Localizing at z , we have a surjection $E_z \twoheadrightarrow M$ with $E_z \in \mathcal{M}^{f.l., \mathcal{O}_X\text{-flat}}(\mathcal{O}_{X \times U, z})$. Since X is smooth, every object of $\mathcal{M}^{fin.len.}(\mathcal{O}_{X \times U, z})$ thus admits a finite resolution by objects of $\mathcal{M}^{f.l., \mathcal{O}_X\text{-flat}}(\mathcal{O}_{X \times U, z})$. The commutativity of the right-hand square of (15) follows from (16).

We now show the map (13) becomes an injection upon h sheafification. In fact, it is injective locally in the Zariski topology on the category of smooth varieties. Let X be the localization of a smooth variety at a closed point. Then the upper vertical arrows of (14) are injections. The bottom arrow of (14) is injective since $k \rightarrow k(z)$ has finite transcendence degree and k is algebraically closed. By de Jong's Theorem on alterations [7], every k -variety admits an h covering by smooth varieties, and hence (13) is injective locally in the h -topology.

To prove the map (13) becomes a surjection upon h sheafification, it suffices to pick a smooth, connected k -variety X and an element $\alpha \in K_m^{qf}(X, (U, Y); \mathbb{Z}/n)$ and show that the pullback of α along some h covering $\tilde{X} \rightarrow X$ lies in the image of $\theta_{\tilde{X}}$.

Referring to diagram (14), the image of α in $\bigoplus_z K_m(k(z); \mathbb{Z}/n)$ involves only finitely many z 's — say z_1, \dots, z_k — and let this image be $\sum_i \alpha_i$, where $\alpha_i \in K_m(k(z_i); \mathbb{Z}/n)$. By Suslin's theorem on the K -theory of algebraically closed fields [23], the map $K_m(k; \mathbb{Z}/n) \rightarrow K_m(L; \mathbb{Z}/n)$ is an isomorphism for any field extension $k \subset L$ with L algebraically closed. Let x denote the generic point of X and choose an algebraic closure $\overline{k(x)}$ of $k(x)$ with $k(z_i) \subset \overline{k(x)}$ for all i . Then for each $\alpha_i \in K_m(k(z_i); \mathbb{Z}/n)$, there is a $\delta_i \in K_m(k; \mathbb{Z}/n)$ such that the images of α_i and δ_i

in $K_m(\overline{k(x)}; \mathbb{Z}/n)$ are equal. There is thus a finite field extension $k(x) \subset F$ such that the images of α_i and δ_i in $K_m(F; \mathbb{Z}/n)$ coincide, for all i .

We claim there is a dominant map $f : X' \rightarrow X$ of smooth, connected k -varieties of the same dimension such that $F \subset k(x')$, where x' is the generic point of X' . To see this, let X'' denote the integral closure of X in F . By de Jong's Theorem [7], there is a dominant map $X' \rightarrow X''$ with X' smooth and such that $\dim(X'') = \dim(X')$, and we take f to be the evident composition. Using the commutativity of (15), we see that the pullback $f^*(\alpha)$ lies in the image of

$$\bigoplus_{z' \in X' \times U} K_m(k; \mathbb{Z}/n) \rightarrow \bigoplus_{z' \in X' \times U} K_m(k(z'); \mathbb{Z}/n).$$

Using the facts that the upper-left (resp., upper-right) arrow of (14) is an isomorphism (resp., injective) locally in the Zariski topology on the smooth variety X' , we may take $\tilde{X} = \coprod_i V_i$ for some finite Zariski open covering $X' = \bigcup_i V_i$ of X' .

The naturality assertion is evident since cupping with elements of $K_m(k; \mathbb{Z}/n)$ commutes with flat pullback. \square

Theorem 7.1 together with Corollary 4.23 and Suslin's theorem identifying $K_g(k; \mathbb{Z}/n)$ with $\mu_n^{\otimes q/2}$ implies the existence of a spectral sequence abutting to $K_*^{\text{BM}}(U; \mathbb{Z}/n)$ whose E_2 terms are given by the homotopy groups of the family of simplicial abelian groups $d \mapsto K_0^{qf}(\Delta^d, U) \otimes \mu_n^{\otimes q/2}$, $q \geq 0$. To relate these E_2 terms with étale cohomology, we use another theorem of Suslin, involving the presheaf $z_{\text{equi}}^0(-, Y)$ defined as follows.

Definition 7.2. For a quasi-projective k -variety Y , let $z_{\text{equi}}^0(-, Y)$ denote the presheaf on Sch/X defined in [25, §3] whose value at a smooth variety X , $z_{\text{equi}}^0(X, Y)$, is the free abelian group of closed integral subschemes Z of $X \times Y$ which are equi-dimensional of relative dimension 0 (i.e., quasi-finite and dominant) over a connected component of X (cf. [25, 3.3.15]).

We write $c_{\text{equi}}^0(-, Y)$ for the sub-presheaf of $z_{\text{equi}}^0(-, Y)$ also defined in [25, §3] whose value at a smooth variety X is a free abelian group of closed integral subschemes Z of $X \times Y$ which are finite and dominant over a connected component of X .

For a morphism $f : X' \rightarrow X$ of smooth varieties, the maps $f^* : z_{\text{equi}}^0(X, Y) \rightarrow z_{\text{equi}}^0(X', Y)$ and $f^* : c_{\text{equi}}^0(X, Y) \rightarrow c_{\text{equi}}^0(X', Y)$ are given by a pullback of cycles in the expected manner. Moreover, the collection of pairings

$$c_{\text{equi}}^0(X/S) \otimes z_{\text{equi}}^0(X, Y) \rightarrow z_{\text{equi}}^0(S, Y)$$

for $\pi : X \rightarrow S$ a smooth curve, defined by $z \otimes w \mapsto (\pi \times \text{id}_Y)_*(p^*(z) \cap w)$ (where $p : X \times Y \rightarrow X$ is the canonical projection) endow $z_{\text{equi}}^0(-, Y)$ with the structure of a pretheory (see [25, 3.7.5]) and hence with the structure of a presheaf with K_0 transfers by Corollary 4.6. Moreover, these pairings restrict to pairings involving $c_{\text{equi}}^0(-, Y)$, so that $c_{\text{equi}}^0(-, Y)$ is a sub-pretheory of $z_{\text{equi}}^0(-, Y)$. Finally, using [25, 3.6.4], if $g : Y' \rightarrow Y$ is a quasi-finite, flat morphism of varieties, there is a morphism of presheaves

$$(17) \quad g^* : z_{\text{equi}}^0(-, Y) \rightarrow z_{\text{equi}}^0(-, Y'),$$

which upon restriction to a map of presheaves on Sm/k is given by a pullback of cycles.

The presheaf $z_{equi}^0(-, U)$ is related to the étale cohomology of U as shown in the following result which is due to Suslin. For an abelian group A , we write $A^\#$ for $\text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. Of course, if A is an \mathbb{Z}/n -module for some $n \geq 1$, the group $A^\#$ coincides with $\text{Hom}_{\mathbb{Z}/n}(A, \mathbb{Z}/n)$.

Theorem 7.3 (Suslin). *Let U be a quasi-projective variety defined over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Let F be a finite \mathbb{Z}/n -module. Then there is an isomorphism*

$$(18) \quad \pi_n(z_{equi}^0(\Delta^\bullet, U) \otimes F) \cong H_c^n(U, F^\#)^\#.$$

This isomorphism is natural for quasi-finite, flat morphisms of normal varieties, where the functoriality of $H_c^n(U, F^\#)^\#$ for such morphisms is given by applying $(-)^{\#}$ to the “trace” map in étale cohomology with compact supports (cf. [1, XVII. 6.2.7.5]).

Moreover, if U is smooth of pure dimension d , then there is an isomorphism

$$\pi_n(z_{equi}^0(\Delta^\bullet, U) \otimes F) \cong H_{\text{ét}}^{2d-n}(U, F \otimes_{\mathbb{Z}/n} \mu_n^{\otimes d})$$

which is natural for quasi-finite, flat morphisms of smooth varieties.

Proof. Suslin’s proof of [26, 4.1] applies directly to establish the existence of the asserted isomorphism if we use $F^\#$ coefficients in lieu of \mathbb{Z}/n coefficients. Since we need the details of this proof in order to establish the asserted naturality, we recall Suslin’s proof here. Begin by choosing an open, dense embedding $j : U \hookrightarrow \overline{U}$ with \overline{U} projective and let $i : U_\infty \hookrightarrow \overline{U}$ be the (reduced) closed complement of U in \overline{U} . Recall that $H_c^p(U, F^\#)$ is by definition $H_{\text{ét}}^p(U, j_! F^\#)$, where $j_!$ denotes extension by zero. By [25, 4.2.9 and 4.3.1], we have that

$$0 \rightarrow z_{equi}^0(-, U_\infty)[1/p] \xrightarrow{i_*} z_{equi}^0(-, \overline{U})[1/p] \xrightarrow{j^*} z_{equi}^0(-, U)[1/p] \rightarrow 0$$

is an exact sequence of h sheaves, where p is the exponential characteristic of the ground field k . Moreover, $z_{equi}^0(-, Y)[1/p] \cong \mathbb{Z}[1/p]_h(Y)$ for any projective variety Y , where $\mathbb{Z}[1/p]_h(Y)$ denotes the h sheafification of $\mathbb{Z}[1/p] \text{Hom}(-, Y)$. An h sheaf \mathcal{G} on Sch/k determines a collection of étale sheaves \mathcal{G}_X on the small étale site $X_{\text{ét}}$, for all X in Sch/k , together with canonical maps $\mathcal{F}_X \rightarrow f_* \mathcal{F}_Y$ and $f^* \mathcal{F}_X \rightarrow \mathcal{F}_Y$ for each morphism $f : Y \rightarrow X$. (The étale sheaf \mathcal{G}_X is by definition the sheaf sending an étale map $W \rightarrow X$ to $\Gamma(W, \mathcal{G})$.) If \mathcal{G} is any h sheaf of $\mathbb{Z}[1/p]$ -modules, we therefore have the isomorphisms

$$(19) \quad \begin{aligned} \text{Hom}_h(z_{equi}^0(-, U)[1/p], \mathcal{G}) &\cong \ker(\Gamma(\overline{U}, \mathcal{G}) \rightarrow \Gamma(U_\infty, \mathcal{G})) \\ &\cong \ker(\Gamma(\overline{U}, \mathcal{G}_{\overline{U}}) \rightarrow \Gamma(\overline{U}, i_*(\mathcal{G}_{U_\infty}))). \end{aligned}$$

If \mathcal{G} is constant, then we have in addition the isomorphism

$$\text{Hom}_h(z_{equi}^0(-, U)[1/p], \mathcal{G}) \cong \ker(\Gamma(\overline{U}, \mathcal{G}) \rightarrow \Gamma(\overline{U}, j_! \mathcal{G})) \cong \Gamma(\overline{U}, j_! \mathcal{G}).$$

Now choose an injective resolution $F^\# \rightarrow I^\bullet$ of the h sheaf $F^\#$ consisting of h sheaves of $\mathbb{Z}[1/p]$ -modules. Since, for all Y , the map $F^\# \rightarrow I_Y^\bullet$ defines an injective resolution of the constant sheaf $F^\#$ for the small étale site $Y_{\text{ét}}$, by using (19) we obtain that the canonical map from $j_! F^\#$ to the total complex associated to the surjection of chain complexes $I_{\overline{U}}^\bullet \twoheadrightarrow i_* I_{U_\infty}^\bullet$ is a quasi-isomorphism and hence determines an injective resolution of $j_! F^\#$:

$$j_! F^\# \xrightarrow{\sim} \text{Tot}(I_{\overline{U}}^\bullet \twoheadrightarrow i_* I_{U_\infty}^\bullet).$$

Combining these observations gives the isomorphism

$$\begin{aligned}
 \text{Ext}_h^q(z_{equi}^0(-, U)_{\sim}, F^\#) &= H^q \text{Hom}_h(z_{equi}^0(-, U)_{\sim}, I^\bullet) \\
 (20) \quad &\cong H^q(\Gamma(\overline{U}, \text{Tot}(I_{\overline{U}}^\bullet \rightarrow i_* I_{U_\infty}^\bullet))), \\
 &= H_{\text{ét}}^p(\overline{U}, j_!(F^\#)) \\
 &= H_c^p(U, F^\#).
 \end{aligned}$$

Finally, using [24, 7.6], we obtain

$$\begin{aligned}
 H_c^p(U, F^\#) &\cong \text{Ext}_h^p(z_{equi}^0(-, U)_{\sim}, F^\#) \cong \text{Ext}_{Ab}^p(z_{equi}^0(\Delta_{top}^\bullet, U), F^\#) \\
 (21) \quad &\cong \pi_p(z_{equi}^0(\Delta^\bullet, U) \otimes F)^\#,
 \end{aligned}$$

where the last isomorphism holds since $z_{equi}^0(\Delta^\bullet, U)$ is a complex of free abelian groups. Since $H_c^p(X, F^\#)$ is a finitely generated \mathbb{Z}/n -module, it follows that $\pi_p(z_{equi}^0(\Delta^\bullet, U) \otimes F)^\#$ and hence $\pi_p(z_{equi}^0(\Delta^\bullet, U) \otimes F)$ are finitely generated too, so that applying $(-)^{\#}$ to (21) gives

$$H_c^p(X, F^\#)^\# \cong \pi_p(z_{equi}^0(\Delta^\bullet, U) \otimes F),$$

as desired.

To establish the (contravariant) naturality of (18) with respect to quasi-finite, flat morphisms of normal varieties, it suffices to establish (covariant) naturality of the isomorphism (20) with respect to such morphisms, since the remaining isomorphisms of (21) are clearly natural. Suppose $f : U' \rightarrow U$ is a quasi-finite, flat morphism of normal, connected varieties. Choose a projective closure of U and take its normalization, so that we have an open, dense embedding $U \hookrightarrow \overline{U}$ with \overline{U} normal and projective. By Zariski's Main Theorem, the quasi-finite map $U' \rightarrow \overline{U}$ factors as an open embedding $U' \hookrightarrow V$ followed by a finite morphism $V \rightarrow \overline{U}$. Letting \overline{U}' be the normalization of V , we obtain a commutative square

$$\begin{array}{ccc}
 U' & \xrightarrow{j'} & \overline{U}' \\
 f \downarrow & & \overline{f} \downarrow \\
 U & \xrightarrow{j} & \overline{U}
 \end{array}
 \quad (22)$$

whose horizontal arrows j, j' are open immersions into normal, projective varieties and such that the right-hand vertical arrow \overline{f} is finite. Let $i : U_\infty \hookrightarrow \overline{U}$ and $i' : U'_\infty \hookrightarrow \overline{U}'$ be the reduced, closed complements of U in \overline{U} and U' in \overline{U}' .

With this set up, the trace map in cohomology with compact supports of the constant sheaf $F^\#$ is given as follows. Choose an injective resolution $j'_! F^\# \rightarrow J^\bullet$ of sheaves on $\overline{U}'_{\text{ét}}$. Then the canonical map $\overline{f}_* j'_! F^\# \rightarrow \overline{f}_*(J^\bullet)$ is an injective resolution of $\overline{f}_* j'_! F^\#$ of sheaves on $\overline{U}_{\text{ét}}$. There is a natural isomorphism $\overline{f}_* j'_! F^\# \cong j_! f_! F^\#$, where $f_!$ is the functor defined in [1, XVII.6]. The trace map is a natural transformation

$$Tr_f : f_! F^\# \rightarrow F^\#$$

uniquely characterized by the properties of [1, XVII.6.2.3]. (Note that we identify the constant sheaf $F^\# = F_{U'}^\#$ on $U'_{\text{ét}}$ with $f^*(F_U^\#)$.) Applying $j_!$ to Tr_f and using the natural isomorphism above, we obtain a map $Tr : \overline{f}_* j'_! F^\# \rightarrow j_! F^\#$. Now choose

an injective resolution $j_!F^\# \rightarrow I^\bullet$, and form the commutative diagram

$$\begin{array}{ccc} \overline{f}_*j'_!F^\# & \longrightarrow & \overline{f}_*J^\bullet \\ Tr \downarrow & & \downarrow \\ j_!F^\# & \longrightarrow & I^\bullet. \end{array}$$

Here, the right-hand vertical map is unique up to homotopy, and upon applying $\Gamma(\overline{U}, -)$ to it and taking cohomology groups, one obtains the trace map in cohomology with compact supports

$$Tr_f : H_c^*(U', F^\#) = H^*\Gamma(\overline{U}', J^\bullet) \cong H^*\Gamma(\overline{U}, \overline{f}_*J^\bullet) \rightarrow H^*\Gamma(\overline{U}, I^\bullet) \cong H_c^*(U, F^\#).$$

To complete the proof of naturality, we need a general framework for transfer type maps arising from *qfh* sheaves. For brevity, throughout the remainder of this proof, for any variety Z we write $c^0(Z)$ and $z^0(Z)$ for the *qfh* sheaves of $\mathbb{Z}[1/p]$ -modules $c_{equi}^0(-, Z)[1/p]$ and $z_{equi}^0(-, Z)[1/p]$, respectively. Note that there is a natural isomorphism $c^0(Z) \cong \mathbb{Z}[1/p]_{qfh}(Z) = \mathbb{Z}[1/p] \text{Hom}(-, Z)_{qfh}^\sim$ by [25, 4.2.7 and 4.2.12].

Given a closed subscheme $i : Z \hookrightarrow X$, we may consider the quotient *qfh* sheaf $c^0(X)/c^0(Z)$ given as the cokernel of the inclusion $c^0(Z) \rightarrow c^0(X)$. For any *qfh* sheaf of $\mathbb{Z}[1/p]$ -modules \mathcal{G} , we have

$$\text{Hom}_{qfh}(c^0(X)/c^0(Z), \mathcal{G}) = \ker(i^* : \Gamma(X, \mathcal{G}) \rightarrow \Gamma(Z, \mathcal{G})).$$

Pulling back cycles along the complementary open subscheme $j : U \subset X$ defines a map

$$j^* : c^0(X)/c^0(Z) \rightarrow z^0(U)$$

of *qfh* sheaves. (This map becomes an isomorphism upon *h* sheafification if X is projective by [25, 4.2.9 and 4.3.1].)

Now suppose $g : X' \rightarrow X$ is a morphism such that X' is a disjoint union of integral varieties, X is a disjoint union of normal varieties, and each connected component of X' maps finitely and dominantly onto a connected component of X . Suppose also that $i' : Z' \hookrightarrow X'$ is a closed subscheme such that $g^{-1}(Z) \subset Z'$ and such that the induced map on open complements

$$f : U' = X' - Z' \xrightarrow{g} X - Z = U$$

is flat. Since each component of X is normal, the graph of g defines an element of $c_{equi}^0(X, X')$, which in light of the natural isomorphisms

$$\text{Hom}_{qfh}(c^0(X), c^0(X')) \cong \text{Hom}_{qfh}(\mathbb{Z}[1/p](X), c^0(X')) \cong c_{equi}^0(X, X')[1/p]$$

defines a map $g^* : c^0(X) \rightarrow c^0(X')$ of *qfh* sheaves. (If g happens to be flat, the map g^* is given on normal varieties by pullback of cycles.) Moreover, the composition $c^0(Z) \rightarrow c^0(X) \xrightarrow{g^*} c^0(X')$ factors through $c^0(Z')$ and hence we obtain a map $g^* : c^0(X)/c^0(Z) \rightarrow c^0(X')/c^0(Z')$ of *qfh* sheaves. Since $f : U' \rightarrow U$ is assumed to be flat, we have a commutative diagram of *qfh* sheaves of the form

$$\begin{array}{ccc} c^0(X)/c^0(Z) & \xrightarrow{j^*} & z^0(U) \\ g^* \downarrow & & f^* \downarrow \\ c^0(X')/c^0(Z') & \xrightarrow{(j')^*} & z^0(U') \end{array}$$

where j' denotes the open immersion $U' \rightarrow X'$ and f^* is given by flat pullback of cycles. Consequently, for any qfh sheaf of $\mathbb{Z}[1/p]$ -modules, we have a commutative diagram of the form

$$(23) \quad \begin{array}{ccc} \mathrm{Hom}_{qfh}(z^0(U'), \mathcal{G}) & \longrightarrow & \ker(\Gamma(X', \mathcal{G}) \rightarrow \Gamma(Z', \mathcal{G})) \\ f_* \downarrow & & g_* \downarrow \\ \mathrm{Hom}_{qfh}(z^0(U), \mathcal{G}) & \longrightarrow & \ker(\Gamma(X, \mathcal{G}) \rightarrow \Gamma(Z, \mathcal{G})) \end{array}$$

Let $\alpha : V \rightarrow X$ be an étale map, and write $Z_V = V \times_X Z$, $X'_V = V \times_X X'$, etc. We claim that pushforward along α (and the evident maps induced by α) defines a commutative square of qfh sheaves

$$(24) \quad \begin{array}{ccc} c^0(V)/c^0(Z_V) & \xrightarrow{\alpha_*} & c^0(X)/c^0(Z) \\ g^* \downarrow & & g^* \downarrow \\ c^0(X'_V)/c^0(Z'_V) & \xrightarrow{\alpha_*} & c^0(X')/c^0(Z') \end{array}$$

To see this, it suffices to observe that the pullback of the graph of g along α (which gives an element of $c_{equi}^0(V, X')$) coincides with the image of the graph of the canonical map $X'_V \rightarrow V$ (which is an element of $c_{equi}^0(V, X'_V)$) along the map $X'_V \rightarrow X'$.

If \mathcal{G} is any qfh sheaf of $\mathbb{Z}[1/p]$ -modules, applying $\mathrm{Hom}_{qfh}(-, \mathcal{G})$ to the top arrow of the square (24) for each étale morphism $V \rightarrow X$, we see that the assignment

$$(V \rightarrow X) \mapsto \mathrm{Hom}_{qfh}(c^0(V)/c^0(Z_V), \mathcal{G}) = \ker(\Gamma(V, \mathcal{G}) \rightarrow \Gamma(Z_V, \mathcal{G}))$$

defines an étale sheaf on the small site $X_{\acute{e}t}$ — let us write this sheaf as $\mathcal{G}_{X, Z}$. Note that $\mathcal{G}_{X, Z}$ is merely the étale sheaf $\ker(\mathcal{G}_X \rightarrow i_* \mathcal{G}_Z)$ defined on $X_{\acute{e}t}$. However, this description of the sheaf gives the existence of the transfer map we seek — namely, since the square (24) commutes, the maps

$$g_* : \mathrm{Hom}_{qfh}(c^0(X'_V)/c^0(Z'_V), \mathcal{G}) \rightarrow \mathrm{Hom}_{qfh}(c^0(V)/c^0(Z_V), \mathcal{G})$$

induced by g^* determine a “transfer” morphism of étale sheaves on $X_{\acute{e}t}$ of the form

$$g_* : g_* \mathcal{G}_{X', Z'} \rightarrow \mathcal{G}_{X, Z}.$$

Observe that since $f : U' \rightarrow U$ is flat, for any étale map $V \rightarrow X$, the commutativity of (23) gives the commutative diagram

$$(25) \quad \begin{array}{ccc} \mathrm{Hom}_{qfh}(z^0(U'_V), \mathcal{G}) & \longrightarrow & \mathrm{Hom}_{qfh}(c^0(X'_V)/c^0(Z'_V), \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{qfh}(z^0(U_V), \mathcal{G}) & \longrightarrow & \mathrm{Hom}_{qfh}(c^0(V)/c^0(Z_V), \mathcal{G}) \end{array}$$

whose left-hand vertical arrow is induced by a flat pullback of cycles. (It is perhaps worth noting that the assignment sending $V \rightarrow X$ to $\mathrm{Hom}_{qfh}(z^0(U_V), \mathcal{G})$ does not obviously define an étale sheaf, since there is not a reasonable map $z^0(V_U) \rightarrow z^0(U)$ unless $V_U \rightarrow U$ is finite. In particular, we are not asserting that the rows of (25) define morphisms of étale sheaves.)

Now observe that the map $\mathcal{G}_X \rightarrow i_* \mathcal{G}_Z$ factors as $\mathcal{G}_X \rightarrow i_* i^* \mathcal{G}_X \rightarrow i_* \mathcal{G}_X$, and hence there are natural maps

$$j_! \mathcal{G}_U \rightarrow \mathcal{G}_{X, Z} \quad \text{and} \quad j'_! \mathcal{G}_{U'} \rightarrow \mathcal{G}_{X', Z'},$$

since $j_! \mathcal{G}_U = j_! j^* \mathcal{G}_X$ is the kernel of $\mathcal{G}_X \rightarrow i_* i^* \mathcal{G}_X$ and similarly for $j'_! \mathcal{G}_{U'}$. Using the canonical map $f^* \mathcal{G}_U \rightarrow \mathcal{G}_{U'}$, we let $j'_! f^* \mathcal{G}_U \rightarrow \mathcal{G}_{X', Z'}$ be the composition of

$$j'_! f^* \mathcal{G}_{U'} \rightarrow j'_! \mathcal{G}_{U'} \rightarrow \mathcal{G}_{X', Z'}.$$

Lemma 7.4. *In the situation above, the diagram*

$$\begin{array}{ccc} g_* j'_! f^* \mathcal{G}_U & \longrightarrow & g_* \mathcal{G}_{X', Z'} \\ Tr \downarrow & & g_* \downarrow \\ j_! \mathcal{G}_U & \longrightarrow & \mathcal{G}_{X, Z} \end{array}$$

commutes, where Tr is defined as the composition of

$$g_* j'_! f^* \mathcal{G}_U \cong j_! f_! f^* \mathcal{G}_U \xrightarrow{j_! (Tr_f)} j_! \mathcal{G}_U.$$

Proof. Since $\mathcal{G}_{X, Z}$ and $\mathcal{G}_{X', Z'}$ map to \mathcal{G}_X and $\mathcal{G}_{X'}$ injectively via maps commuting with g_* , it suffices to prove

$$(26) \quad \begin{array}{ccc} g_* j'_! f^* \mathcal{G}_U & \longrightarrow & g_* \mathcal{G}_{X'} \\ Tr \downarrow & & g_* \downarrow \\ j_! \mathcal{G}_U & \longrightarrow & \mathcal{G}_X \end{array}$$

commutes. Moreover, since (26) is a diagram of étale sheaves on $X_{\text{ét}}$ and the sheaves on the left are supported on $U \subset X_{\text{ét}}$, it suffices to establish that this diagram commutes at the stalk of an arbitrary point of U . Let U_u denote the spectrum of the strict henselization of U at a point $u \in U$. Then since g is finite, the scheme $U_u \times_X X'$ decomposes as $V_0 \amalg V_1 \amalg \cdots \amalg V_t$, such that for each $1 \leq i \leq t$ we have (1) the scheme V_i is the spectrum of a strictly hensel local ring, (2) the map $V_i \rightarrow U_u$ is finite and flat, and (3) the scheme V_i contained in the open subscheme $U_u \times_U U'$. Moreover, we have that the closed points of the scheme V_0 (which is a product of spectra of local rings) are not contained in $U_u \times_U U'$. Noting that $g_* j'_! f^* \mathcal{G}_U \cong j_! f_! f^* \mathcal{G}_U$, applying $\Gamma(U_u, -)$ to (26) (i.e., taking stalks at u) gives the diagram

$$\begin{array}{ccc} \Gamma(U_u, g_* j'_! f^* \mathcal{G}_U) = \bigoplus_{i=1}^t \Gamma(V_i, f^* \mathcal{G}_U) & \longrightarrow & \Gamma(U_u, g_* \mathcal{G}_{X'}) = \bigoplus_{i=0}^t \Gamma(V_i, \mathcal{G}) \\ Tr \downarrow & & g_* \downarrow \\ \Gamma(U_u, \mathcal{G}) & \xrightarrow{\cong} & \Gamma(U_u, \mathcal{G}), \end{array}$$

where the top horizontal arrow is a composition of the canonical map and the evident summand inclusion:

$$\bigoplus_{i=1}^t \Gamma(V_i, f^* \mathcal{G}_U) \rightarrow \bigoplus_{i=1}^t \Gamma(V_i, \mathcal{G}_U) \hookrightarrow \bigoplus_{i=0}^t \Gamma(V_i, \mathcal{G}_U).$$

Moreover, for each $i = 1, \dots, t$, since U_u is the spectrum of a strictly henselian ring, the natural map $\Gamma(U_u, \mathcal{G}_U) \rightarrow \Gamma(V_i, f^* \mathcal{G}_U)$ is an isomorphism, and the composition of

$$\Gamma(U_u, \mathcal{G}) \xrightarrow{\cong} \Gamma(V_i, f_* \mathcal{G}_U) \xrightarrow{Tr} \Gamma(U_u, \mathcal{G})$$

is multiplication by the degree of the finite, flat map $V_i \rightarrow U_u$ (cf. [1, XVII.6.2.3]). It therefore suffices to prove that for $i = 1, \dots, t$, the composition of

$$\Gamma(U_u, \mathcal{G}) \xrightarrow{g^*} \Gamma(V_i, \mathcal{G}) \xrightarrow{g_*} \Gamma(U_u, \mathcal{G})$$

is also multiplication by the degree of $V_i \rightarrow U_u$. Since $V_i \rightarrow U_u$ is an inverse limit of finite, flat maps of normal varieties, it suffices to prove that the composition of

$$\Gamma(S, \mathcal{G}) \xrightarrow{h^*} \Gamma(T, \mathcal{G}) \xrightarrow{h_*} \Gamma(S, \mathcal{G})$$

is multiplication by $\deg(h)$ for any finite, flat morphism $h : T \rightarrow S$ of normal varieties. This in turn holds since the composition of

$$c^0(S) \xrightarrow{g^*} c^0(T) \xrightarrow{g_*} c^0(S)$$

is given by composing pullback and pushforward along the flat morphism h , and is thus multiplication by $\deg(h)$. \square

We now return to the main argument of the proof of Theorem 7.3. The square (22) satisfies the needed hypotheses so that, by Lemma 7.4, given any *qfh* sheaf of $\mathbb{Z}[1/p]$ -modules, we have a commutative square

$$\begin{array}{ccc} \overline{f}_* j'_! f^* \mathcal{G}_U & \longrightarrow & \overline{f}_* \mathcal{G}_{\overline{U}', U'_\infty} \\ Tr \downarrow & & \overline{f}_* \downarrow \\ j_! \mathcal{G}_U & \longrightarrow & \mathcal{G}_{\overline{U}, U_\infty} \end{array}$$

of *qfh* sheaves. In particular, we apply this to each sheaf in the injective resolution $F^\# \rightarrow I^\bullet$ consisting of *h*-sheaves of $\mathbb{Z}[1/p]$ -modules to obtain a commutative diagram

$$\begin{array}{ccc} \overline{f}_* j'_! F^\# & \longrightarrow & \overline{f}_* I_{\overline{U}', U'_\infty}^\bullet \\ Tr \downarrow & & \downarrow \\ j_! F^\# & \longrightarrow & I_{\overline{U}, U_\infty}^\bullet \end{array}$$

of complexes of étale sheaves on $\overline{U}_{\text{ét}}$. (Note that $f^* F_U^\# = F_{U'}^\# = F^\#$, since $F^\#$ is a constant sheaf.) Since $I_{\overline{U}, U_\infty}^\bullet$ coincides with the total complex of $I_{\overline{U}}^\bullet \rightarrow I_{U_\infty}^\bullet$ (and similarly for $I_{\overline{U}', U'_\infty}^\bullet$), the rows of this diagram represent injective resolutions of $\overline{f}_* j'_! F^\#$ and $j_! F^\#$. Thus, from our description given earlier, applying $\Gamma(\overline{U}, -)$ to the right-hand column of this diagram and taking cohomology gives the trace map in étale cohomology with compact supports. On the other hand, the diagram

$$\begin{array}{ccc} \text{Hom}_{qfh}(z^0(U'), I^m) & \longrightarrow & \Gamma(\overline{U}, \overline{f}_* I_{\overline{U}', U'_\infty}^m) = \Gamma_{U'_\infty}(\overline{U}', I^m) \\ f_* \downarrow & & \downarrow \\ \text{Hom}_{qfh}(z^0(U), I^m) & \longrightarrow & \Gamma(\overline{U}, I_{\overline{U}, U_\infty}^m) = \Gamma_{U_\infty}(\overline{U}, I^m) \end{array}$$

commutes for each m by (25), from which it follows that the isomorphism (20) is natural as claimed. (Note that $\text{Hom}_{qfh}(z^0(Y), \mathcal{G})$ is naturally isomorphic to $\text{Hom}_h(z^0(Y)_h^\sim, \mathcal{G})$ if \mathcal{G} is an *h* sheaf of $\mathbb{Z}[1/p]$ -modules.) The isomorphism (18) is therefore natural for quasi-finite, flat morphisms of normal varieties.

Finally, if U is smooth of pure dimension d , then Poincaré duality for étale cohomology (cf. [1, XVIII.3.2.5]) gives the isomorphism

$$H_c^p(U, \mu_n^{\otimes m/2}(k)^\#)^\# \cong H_{\text{ét}}^{2d-p}(U, \mu_n^{\otimes(d+m/2)}).$$

The naturality of this isomorphism with respect to a quasi-finite, flat morphism follows from the functoriality and compatibility with composition of the trace map as shown in [1, XVIII.2.9]. \square

The following theorem relates K -groups with quasi-finite support to the presheaves $z_{equi}^0(-, U)$ and hence to étale cohomology.

Theorem 7.5. *Let U be a Cohen-Macaulay quasi-projective variety defined over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Then there is a homotopy equivalence of simplicial abelian groups*

$$\left(d \mapsto z_{equi}^0(\Delta^d, U) \otimes_{\mathbb{Z}} \mu_n^{\otimes m/2}(k) \right) \sim \left(d \mapsto K_m^{qf}(\Delta^d, U; \mathbb{Z}/n) \right),$$

where $\mu_n^{\otimes m/2}(k)$ is defined to be the zero group for m odd. Consequently, we have an isomorphism

$$\pi_p \left(d \mapsto K_m^{qf}(\Delta^d, U; \mathbb{Z}/n) \right) \cong H_c^p(U, \mu_n^{\otimes m/2}(k)^{\#}),$$

which is contravariantly natural with respect to quasi-finite, flat morphisms of normal, Cohen-Macaulay varieties.

Moreover, if U is smooth of pure dimension d , we have in addition the isomorphism

$$\pi_p \left(d \mapsto K_m^{qf}(\Delta^d, U; \mathbb{Z}/n) \right) \cong H_{\text{ét}}^{2d-p}(U, \mu_n^{\otimes(d+m/2)}),$$

which is contravariantly natural with respect to quasi-finite, flat morphisms of smooth varieties.

Proof. Theorem 6.2 and Proposition 5.2 imply that the h sheafification of $K_0^{qf}(-, U)$ is isomorphic to $z_{equi}^0(-, U)$. It follows from Corollary 4.23 that the cycle class map

$$K_0^{qf}(\Delta^\bullet, U; \mathbb{Z}/n) \xrightarrow{\sim} z_{equi}^0(\Delta^\bullet, U; \mathbb{Z}/n)$$

defines a homotopy equivalence of simplicial abelian groups. (The map $K_0^{qf}(-, U) \rightarrow z_{equi}^0(-, U)$ is readily verified to be a morphism of presheaves with K_0 transfers.) Applying Theorem 4.24 and Corollary 4.23 to the map

$$(27) \quad K_0^{qf}(-, U) \otimes K_m(k; \mathbb{Z}/n) \rightarrow K_m^{qf}(-, U; \mathbb{Z}/n)$$

and using Theorem 7.1, we obtain homotopy equivalences

$$z_{equi}^0(\Delta^\bullet, U) \otimes K_m(k; \mathbb{Z}/n) \sim K_0^{qf}(\Delta^\bullet, U) \otimes K_m(k; \mathbb{Z}/n) \sim K_m^{qf}(\Delta^\bullet, U; \mathbb{Z}/n)$$

of simplicial abelian groups. (The map (27) is a morphism of presheaves with K_0 transfers since the two evident maps of the form

$$\mathcal{K}^{fin}(X/S) \wedge \mathcal{K}^{qf}(X, U) \wedge \mathcal{K}(k) \rightarrow \mathcal{K}^{qf}(S, U)$$

coincide up to homotopy.) The cup product operation in K -theory defines a natural map

$$\mu_n(k)^{\otimes t} \rightarrow K_{2t}(k; \mathbb{Z}/n),$$

which by Suslin's theorem [23, 3.13] is an isomorphism. Moreover, Suslin's theorem also shows that $K_{2t+1}(k; \mathbb{Z}/n) = 0$.

The remaining assertions follow directly from Theorem 7.3. \square

In Section 5 we used $\mathcal{K}^{qf}(X, U; \mathbb{Z}/n)$ to denote the spectrum obtained from $\mathcal{K}^{qf}(X, U)$ by smashing with the mod- n Moore spectrum. For the purposes of this section, we need a more explicit and functorial model for the space whose homotopy groups are $K_*^{qf}(X, U; \mathbb{Z}/n)$. Specifically, let $M = M_n^2$ denote the singular simplicial set associated to the mapping cone of $S^1 \xrightarrow{\sim} S^1$ and let $\Omega^{-2}\mathcal{K}^{qf}(-, U)$ denote a functor taking values in Kan complexes that gives a functorial 2-fold delooping of $\mathcal{K}^{qf}(-, U)$. Namely, we define

$$\Omega^{-2}\mathcal{K}^{qf}(-, U) = \text{Ex}(\text{diag}(S_\bullet S_\bullet \mathcal{P}^{qf}(-, U)))$$

where S_\bullet is the Waldhausen S construction (which may be iterated to form a bisimplicial set as above), diag denotes taking the diagonal of a bisimplicial set, and Ex is the Kan extension functor taking simplicial sets to Kan complexes. (Recall also that $\mathcal{P}^{qf}(-, U)$ is made strictly functorial as discussed in Section 3.) Let $\underline{\text{Maps}}_*(-, -)$ denote the simplicial mapping space functor. Then the geometric realization of $\underline{\text{Maps}}_*(M, \Omega^{-2}\mathcal{K}^{qf}(X, U))$ is weakly equivalent to the total space of the spectrum we wrote as $\mathcal{K}^{qf}(X, U; \mathbb{Z}/n)$ before, so that we have

$$\pi_t \underline{\text{Maps}}_*(M, \Omega^{-2}\mathcal{K}^{qf}(X, U)) \cong K_t^{qf}(X, U; \mathbb{Z}/n),$$

for all $t \geq 0$. Indeed, in the remainder of this section, we will write $\mathcal{K}^{qf}(X, U; \mathbb{Z}/n)$ for the Kan complex $\underline{\text{Maps}}_*(M, \Omega^{-2}\mathcal{K}^{qf}(X, U))$. In particular, $\mathcal{K}^{qf}(X, Y; \mathbb{Z}/n)$ is strictly functorial in X , taking values in Kan complexes, and there is a natural isomorphism

$$\pi_q \mathcal{K}^{qf}(X, Y; \mathbb{Z}/n) \cong K_q^{qf}(X, Y; \mathbb{Z}/n).$$

Theorem 7.6. *Let U be a Cohen-Macaulay, quasi-projective variety defined over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Then there is a convergent spectral sequence*

$$E_{s,t}^2 = H_c^s(U, \mu_n^{\otimes t/2}(k)^\#)^\# \Longrightarrow K_{s+t}^{BM}(U; \mathbb{Z}/n),$$

where $\mu_n^{\otimes t/2}(k) = 0$ for t odd. Here, the indexing is homological (so that the d_2 differential has bidegree $(-2, 1)$) and the E_2 terms are concentrated in the portion of the first quadrant of the st -plane with $s \leq 2\dim(U)$. Moreover, the spectral sequence is natural with respect to all quasi-finite, flat morphisms of normal, Cohen-Macaulay varieties.

When U is smooth, we may write this as a cohomologically indexed convergent spectral sequence (with d_2 differential having bidegree $(2, -1)$)

$$E_2^{p,q} = H_{\text{ét}}^{p-q}(U, \mu_n^{\otimes(-q)}(k)) \Longrightarrow K_{-p-q}^{BM}(U; \mathbb{Z}/n)$$

which is concentrated in the portion of the pq -plane defined by $p \geq q \geq p - 2d$ and $q \leq -d$ and which is natural for quasi-finite, flat morphisms of smooth varieties.

Proof. The first spectral sequence is the standard spectral sequence associated to the simplicial space $d \mapsto \mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)$. Since we will need the details of this construction later, we recall them here.

If $Y = (d \mapsto Y_d)$ is a pointed Kan complex, let $P_t(Y)$, $t \geq 0$, denote the Postnikov truncation of Y , so that $P_t(Y)$ is a quotient of Y defined by the equivalence relation where two n -simplices of Y are equivalent if their t -skeleta coincide. The natural

surjection $Y \rightarrow P_t(Y)$ induces an isomorphism on π_n for $n \leq t$, and $\pi_n P^t(Y) = 0$ for $n > t$. Define $Q_t(Y)$ to be the fiber of the Kan fibration

$$P_t(Y) \rightarrow P_{t-1}(Y),$$

so that $Q_t(Y)$ is a Eilenberg-MacLane Kan complex whose only non-vanishing homotopy group is $\pi_t Q_t(Y) \cong \pi_t(Y)$. Applying $P_t(-)$ to the Kan complexes $\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)$, for $d \geq 0$, and then taking diagonals gives a tower of fibration sequences

$$\begin{aligned} \cdots &\rightarrow \text{diag}(d \mapsto P_{t+1}\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)) \rightarrow \text{diag}(d \mapsto P_t\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)) \\ &\quad \rightarrow \cdots \rightarrow \text{diag}(d \mapsto P^0\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)) \rightarrow * \end{aligned}$$

such that the fiber of the t -th map is $\text{diag}(d \mapsto Q_t\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n))$. The definitions

$$D_{s,t}^2 = \pi_{s+t+1}|d \mapsto P_{t-1}\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)|$$

and

$$E_{s,t}^2 = \pi_{s+t}|d \mapsto Q_t\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)| \cong \pi_s|d \mapsto K_t^{qf}(\Delta^d, U; \mathbb{Z}/n)|$$

and the maps appearing in the long exact sequences associated to these fibration sequences determine an exact couple and thus a spectral sequence abutting to $\pi_*|d \mapsto \mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)|$. Theorem 7.5 shows that the E^2 terms are isomorphic to the asserted étale cohomology groups. Finally, to show that the abutment of this spectral sequence is isomorphic to $K_*^{\text{BM}}(U; \mathbb{Z}/n)$, we observe that we have the fibration sequence

$$\begin{array}{c} |d \mapsto \underline{\text{Maps}}_*(S^1, \Omega^{-2}\mathcal{K}^{qf}(\Delta^d, U))| \\ \downarrow \\ |d \mapsto \underline{\text{Maps}}_*(S^1, \Omega^{-2}\mathcal{K}^{qf}(\Delta^d, U))| \\ \downarrow \\ |d \mapsto \underline{\text{Maps}}_*(M, \Omega^{-2}\mathcal{K}^{qf}(\Delta^d, U))| = |d \mapsto \mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)|. \end{array}$$

Moreover, using [30, 5.2], we have homotopy equivalences

$$\begin{aligned} \Omega|d \mapsto \underline{\text{Maps}}_*(S^1, \Omega^{-2}\mathcal{K}^{qf}(\Delta^d, U))| &\sim |d \mapsto \underline{\text{Maps}}_*(S^2, \Omega^{-2}\mathcal{K}^{qf}(\Delta^d, U))| \\ &\sim \mathcal{K}^{\text{BM}}(U), \end{aligned}$$

and it follows from the five lemma that we have

$$\pi_*|d \mapsto \mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)| \cong K_*^{\text{BM}}(U; \mathbb{Z}/n).$$

(That is, to attain $\mathcal{K}^{\text{BM}}(Y; \mathbb{Z}/n)$ up to weak homotopy, we may apply $\underline{\text{Maps}}_*(M, -)$ degree-wise.)

Thus the first spectral sequence exists. The inequality of $s \leq 2\dim(U)$ holds since U has étale cohomological dimension of at most $2d$. This bound also establishes the convergence of the spectral sequence. The naturality assertion follows from the contravariant naturality for quasi-finite morphisms of $\Omega^{-2}\mathcal{K}^{qf}(X, -)$ and the naturality assertion of Theorem 7.5.

The second spectral sequence is constructed by using Poincaré duality for étale cohomology and re-indexing the first spectral sequence. That is, if U is smooth of pure dimension d , then we have the isomorphism

$$H_c^p(U, \mu_n^{\otimes q/2}(k)^\#)^\# \cong H_{\text{ét}}^{2d-p}(U, \mu_n^{\otimes(d+q/2)}(k)),$$

which is natural with respect to quasi-finite, flat morphisms. Now perform the linear change of indexing $q = -d-t/2$ and $p = d-s-t/2$ to form a spectral sequence with E_2 terms $\tilde{E}_2^{p,q}$. Note that $\tilde{E}_2^{p,q} = 0$ for q odd, and so we can progress through the pages of the spectral sequence in steps of two without losing any information. That is, we form a new spectral sequence by writing $E_r^{p,q}$ for $\tilde{E}_{2r-1}^{p,q}$, (and similarly for the D terms), which gives us the desired spectral sequence. The case of a general smooth U follows. \square

Corollary 7.7. *Let U be a Cohen-Macaulay, quasi-projective variety defined over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Then $K_m^{BM}(U; \mathbb{Z}/n)$ is a finite abelian group for all $m \geq 0$.*

Proof. This follows from the theorem and the fact that for any constructible sheaf F , the group $H_c^p(U; F)$ is finite and vanishes unless $0 \leq p \leq 2\dim(U)$ (cf. [19, VI.1.1 and VI.2.1]). \square

A slight extension of the proofs of Theorems 7.6 and 7.3 gives a bit more than we have asserted — namely, the functor $\mathcal{K}^{BM}(-; \mathbb{Z}/n)$ satisfies “étale descent” in a manner made precise in the following statement. To clarify the terminology of this result, recall that an *étale hypercovering* of a variety U is an augmented simplicial variety $U_\bullet \rightarrow U$ such that the canonical map $U_n \rightarrow \text{cosk}_{n-1}^U(U_\bullet)_n$ is an étale covering for all $n \geq 0$. Here, cosk_{n-1}^U denotes the $(n-1)$ -st coskeleton of a simplicial scheme over U . For example, given an étale covering $U_0 \rightarrow U$, its nerve $U_n = U_0 \times_U \cdots \times_U U_0$, $n \geq 0$, defines an étale hypercovering. A *cosimplicial space* \mathcal{X}^\bullet is a functor from the standard simplicial category Δ (consisting of ordered sets $[n] = \{0 < \cdots < n\}$, $n \geq 0$, and order preserving functions) to the category of spaces, written $d \mapsto \mathcal{X}^d$. Observe that $\mathcal{K}^{BM}(-; \mathbb{Z}/n)$ (or any contravariant functor from varieties to spaces) sends an étale hypercovering $U_\bullet \rightarrow U$ to a cosimplicial space $d \mapsto \mathcal{K}^{BM}(U_d; \mathbb{Z}/n)$ equipped with an augmentation map $\mathcal{K}^{BM}(U; \mathbb{Z}/n) \rightarrow \mathcal{K}^{BM}(U_0; \mathbb{Z}/n)$. Finally, the *homotopy inverse limit* of a cosimplicial space \mathcal{X}^\bullet , written $\varprojlim_{\Delta} \mathcal{X}^\bullet$, is the space defined by [6, XI.4.4].

Theorem 7.8. *Let U be a normal, Cohen-Macaulay, quasi-projective variety defined over an algebraically closed field k , let $U_\bullet \rightarrow U$ be an étale hypercovering, and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Then the canonical map from $\mathcal{K}^{BM}(U; \mathbb{Z}/n)$ to the homotopy inverse limit of the cosimplicial space [6, XI.3.2] $d \mapsto \mathcal{K}^{BM}(U_d; \mathbb{Z}/n)$ is a weak homotopy equivalence:*

$$\mathcal{K}^{BM}(U; \mathbb{Z}/n) \xrightarrow{\sim} \varprojlim_{\Delta} \mathcal{K}^{BM}(U_\bullet; \mathbb{Z}/n).$$

Proof. Summarizing the constructions of the proof of Theorem 7.6, we have a tower of Kan complexes, which we write here as

$$\cdots \rightarrow G_t(V) \rightarrow G_{t-1}(V) \rightarrow \cdots \rightarrow G_0(V) \rightarrow *,$$

together with a family of Kan fibration sequences of the form

$$F_t(V) \rightarrow G_t(V) \rightarrow G_{t-1}(V), \quad t \geq 0.$$

(By definition, $G_{-1}(V) = *$.) Specifically, $G_t(V) = \text{diag}(d \mapsto P_t \mathcal{K}^{qf}(\Delta^d, V; \mathbb{Z}/n))$ and $F_t(V) = \text{diag}(d \mapsto Q_t \mathcal{K}^{qf}(\Delta^d, V; \mathbb{Z}/n))$. Moreover, the canonical map

$$\mathcal{K}^{BM}(V; \mathbb{Z}/n) \rightarrow \varprojlim_t G_t(V) \sim \varprojlim_t G_t(V)$$

is a homotopy equivalence. All of these maps are natural in V for quasi-finite, flat morphisms. Thus, for all $t \geq 0$, we have a diagram

$$(28) \quad \begin{array}{ccc} F_t(U) & \longrightarrow & \varprojlim F_t(U_\bullet) \\ \downarrow & & \downarrow \\ G_t(U) & \longrightarrow & \varprojlim G_t(U_\bullet) \\ \downarrow & & \downarrow \\ G_{t-1}(U) & \longrightarrow & \varprojlim G_{t-1}(U_\bullet). \end{array}$$

Since for each U_i and $t \geq 0$, the map $G_t(U_i) \rightarrow G_{t-1}(U_i)$ is a Kan fibration inducing a surjection on π_0 , and the right-hand column of this diagram is also a Kan fibration sequence by [6, XI.5.5].

Suppose $F_t(U) \rightarrow \varprojlim F_t(U_\bullet)$ were a homotopy equivalence for all $t \geq 0$. Then using diagram (28) and induction on t , we would have that $G_t(U) \rightarrow \varprojlim G_t(U_\bullet)$ is a homotopy equivalence for all $t \geq 0$. Hence, the canonical map

$$\mathcal{K}^{BM}(U; \mathbb{Z}/n) \xrightarrow{\sim} \varprojlim_t G_t(U) \longrightarrow \varprojlim_t \varprojlim_\Delta G_t(U_\bullet)$$

would be a homotopy equivalence. Then by the “Fubini Theorem” for homotopy limits (cf. [6, XI.4.3]), we would obtain a natural homotopy equivalence

$$\mathcal{K}^{BM}(U; \mathbb{Z}/n) \xrightarrow{\sim} \varprojlim_\Delta \varprojlim_t G_t(U_\bullet) \sim \varprojlim_\Delta \mathcal{K}^{BM}(U_\bullet; \mathbb{Z}/n),$$

as desired.

It therefore remains to show $F_t(U) \rightarrow \varprojlim F_t(U_\bullet)$ is a homotopy equivalence for all $t \geq 0$. Observe that there is a homotopy equivalence of bisimplicial sets

$$(d \mapsto Q_t \mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)) \rightarrow (d \mapsto \Gamma(K_t^{qf}(\Delta^d, U; \mathbb{Z}/n), t)),$$

where for any abelian group A and integer t , we write $\Gamma(A, t)$ for the simplicial abelian group associated to the (homological) chain complex $A[-t]$ consisting of the group A concentrated in homological degree t . Moreover, this homotopy equivalence is natural in U . Since $F_t(U)$ is by definition the diagonal of the first of these bisimplicial sets, it suffices to prove

$$(29) \quad \text{diag} \left(d \mapsto \Gamma(K_t^{qf}(\Delta^d, U; \mathbb{Z}/n), t) \right) \rightarrow \varprojlim_\Delta \text{diag} \left(d \mapsto \Gamma(K_t^{qf}(\Delta^d, U_\bullet; \mathbb{Z}/n), t) \right),$$

which is a map of simplicial abelian groups, is a homotopy equivalence. Every cosimplicial simplicial abelian group $A = A_\bullet^\bullet$ is fibrant [6, X.4.9] and hence $\varprojlim_\Delta (d \mapsto A_\bullet^d)$ coincides up to a natural homotopy equivalence with the total space $\text{Tot } A_\bullet^\bullet$ [6, XI.4.2]. Moreover, the normalized chain complex of $\text{Tot } A_\bullet^\bullet$ (which is a simplicial abelian group) coincides up to a natural quasi-isomorphism with the total complex of the normalized bicomplex $\mathcal{N}^\bullet \mathcal{N}_\bullet A_\bullet^\bullet$ associated to A_\bullet^\bullet [4, 2.2]. Finally, for any simplicial abelian group B_\bullet (such as $d \mapsto K_q^{qf}(\Delta^d, V; \mathbb{Z}/n)$), the normalized chain complex associated to $\text{diag}(d \mapsto \Gamma(B_d, t))$ coincides up to a natural quasi-isomorphism with $\mathcal{N}_\bullet(B_\bullet)[-t]$, where $[-t]$ denotes shifting t degrees to the right.

Combining these facts, we see that upon taking normalized chain complexes associated to (29), applying Theorem 7.5, and reindexing, it suffices to prove

$$(30) \quad \mathcal{N}_\bullet(z_{equi}^0(\Delta^\bullet, U) \otimes F) \rightarrow \text{Tot } \mathcal{N}^\bullet \mathcal{N}_\bullet(z_{equi}^0(\Delta^\bullet, U_\bullet) \otimes F)$$

is a quasi-isomorphism of chain complexes for any finite \mathbb{Z}/n -module F . (In this context, Tot refers to taking the total complex associated to a bicomplex.)

We now use the notation and results of the proof of Theorem 7.3. By repeatedly applying Zariski's main theorem and taking normalizations, we can form a augmented "semi-simplicial variety" $(d \mapsto X_d) \rightarrow X$ (i.e., an augmented simplicial variety without degeneracy maps) and a map of augmented semi-simplicial varieties

$$\begin{array}{ccc} U_\bullet & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow \\ U & \longrightarrow & X, \end{array}$$

such that (1) X and each $X_d, d \geq 0$, are disjoint unions of projective, normal varieties, (2) the maps $U \rightarrow X$ and $U_d \rightarrow X_d, d \geq 0$ are open, dense immersions, and (3) all of the face maps $X_d \rightarrow X_{d-1}, d \geq 1$, and the map $X_0 \rightarrow X$ send connected components of the source finitely and dominantly onto connected components of the target. Let $j : U \rightarrow X$ and $j_d : U_d \rightarrow X_d, d \geq 0$, denote these open immersions, and let $i : Z \rightarrow X$ and $i_d : Z_d \rightarrow X_d, d \geq 0$, be the complementary closed embeddings. For each $d \geq 0$, let $f_d : X_d \rightarrow X$ be the unique map induced by the augmentation map. Then, as in the proof of Theorem 7.3, for any h sheaf of $\mathbb{Z}[1/p]$ modules, we have étale sheaves $\mathcal{G}_{X,Z}$ (defined on $X_{\text{ét}}$) and \mathcal{G}_{X_d,Z_d} (defined on $(X_d)_{\text{ét}}$), for $d \geq 0$. Moreover, the augmentation map $f_0 : X_0 \rightarrow X$ and the various face maps $d_i : X_d \rightarrow X_{d-1}$ define transfer morphisms of étale sheaves $(f_0)_* : (f_0)_* \mathcal{G}_{X_0,Z_0} \rightarrow \mathcal{G}_{X,Z}$ and $(d_i)_* : (d_i)_* \mathcal{G}_{X_d,Z_d} \rightarrow \mathcal{G}_{X_{d-1},Z_{d-1}}$. Upon pushing all of these maps forward to maps of étale sheaves on $X_{\text{ét}}$, we obtain a chain complex of sheaves on $X_{\text{ét}}$ of the form

$$\cdots \rightarrow (f_d)_* \mathcal{G}_{X_d,Z_d} \rightarrow \cdots \rightarrow (f_0)_* \mathcal{G}_{X_0,Z_0} \rightarrow \mathcal{G}_{X,Z}$$

with maps given by taking alternating sums of transfer maps associated to the face maps of the semi-simplicial variety X_\bullet .

In particular, we apply this construction to each sheaf in an injective resolution $F^\# \rightarrow I^\bullet$ by h sheaves of $\mathbb{Z}[1/p]$ -modules, to obtain the bicomplex

$$(31) \quad \cdots \rightarrow (f_d)_* I_{X_d,Z_d}^\bullet \rightarrow \cdots \rightarrow (f_0)_* I_{X_0,Z_0}^\bullet \rightarrow I_{X,Z}^\bullet$$

of sheaves on $X_{\text{ét}}$. The isomorphisms (20) and (21) show that applying $\Gamma(X, -)$ to (31) yields a bicomplex whose total complex is quasi-isomorphic to the complex associated to the map (30), and so it suffices to prove the total complex of (31) is exact.

Using Lemma 7.4, there is a quasi-isomorphism of chain complexes of sheaves from

$$(32) \quad \cdots \rightarrow (f_d)_*(j_d)_! F^\# \rightarrow \cdots \rightarrow (f_0)_*(j_0)_! F^\# \rightarrow j_! F^\#$$

to (31), where the maps of (32) are given by the alternating sums of trace maps associated to the face maps of the simplicial variety $d \mapsto U_d$. (Note that these face maps are étale morphisms.) Thus, it suffices to show (32) is exact.

Let $g_d : U_d \rightarrow U$ be the unique map induced by the augmentation map. We have $(f_d)_*(j_d)_! F^\# \cong j_!(g_d)_! F^\#$, and, in particular, each sheaf appearing in (32)

is supported on U . For an arbitrary point u of U , let U_u denote the spectrum of the strict henselization of U at u , and define $U_{d,u} = U_d \times_U U_u$, for $d \geq 0$. Since each map $g_d : U_d \rightarrow U$ is étale, the scheme $U_{d,u}$ decomposes as a disjoint union of copies of U_u and components whose images in U_u do not contain the unique closed point u . Let $U'_{d,u}$ denote the open and closed subscheme of $U_{d,u}$ consisting of those components that are isomorphic to U_u . Then, by definition of the maps $(g_d)_!$, we have

$$\Gamma(U_u, (f_d)_*(j_d)_!F^\#) \cong \Gamma(U_u, j_!(g_d)_!F^\#) \cong \Gamma(U_u, (g_d)_!F^\#) = \Gamma(U'_{d,u}, F^\#).$$

One may readily verify that

$$(d \mapsto U'_{d,u}) \rightarrow U_u$$

is an augmented simplicial scheme and defines an étale hypercovering of U_u . Hence the complex

$$\cdots \rightarrow \Gamma(U'_d, F^\#) \rightarrow \cdots \rightarrow \Gamma(U'_0, F^\#) \rightarrow \Gamma(U_u, F^\#)$$

is exact, from which it follows that (32) is exact locally at u . \square

Remark 7.9. Using the general spectral sequence associated to a cosimplicial space described in [6, XI.7.5], we have a spectral sequence whose E_2 terms are given by

$$E_2^{p,q} = \check{H}_{\text{ét}}^p(U_\bullet, K_q^{\text{BM}}(-; \mathbb{Z}/n))$$

with the differential d_2 having bidegree $(2, 1)$. Here, for any presheaf F defined on the small étale site of U , the group $\check{H}_{\text{ét}}^p(U_\bullet, F)$ denotes the p -th cohomology group of the complex

$$F(U_0) \rightarrow F(U_1) \rightarrow \cdots,$$

where the maps are given by the alternating sum of coboundary maps in the usual manner. Ideally, Theorem 7.8 would imply that this spectral sequence converges to $K_{q-p}^{\text{BM}}(U; \mathbb{Z}/n)$; however, the convergence of the spectral sequence is uncertain in general (cf. [6, IX.5]).

8. DUALITY FOR SMOOTH VARIETIES

In this section we define a map

$$\gamma_Y : \bigoplus_p K_p^{\text{BM}}(Y; \mathbb{Z}/n) \rightarrow \bigoplus_p K_{\text{geom}}^{-p}(Y; \mathbb{Z}/n)$$

of graded abelian groups, for any smooth, quasi-projective k -variety Y , which is natural with respect to quasi-finite morphisms. (The functor $K_*^{\text{BM}}(-; \mathbb{Z}/n)$ is shown to be contravariantly functorial with respect to such morphisms below.) Here, we assume k is algebraically closed and $n \geq 1$ is relatively prime to the exponential characteristic of k . (One can extend the definition of γ_Y to theories with integral coefficients and to non-algebraically closed fields k , provided one assumes k admits resolutions of singularities.) Moreover, we prove γ_Y is a map of $K_{\text{geom}}^*(Y)$ -modules and establish the key formula

$$\gamma_Y(a) \cap b = \gamma_Y(b) \cap a$$

for all $a, b \in K_*^{\text{BM}}(Y; \mathbb{Z}/n)$. We then define the notion of a *fundamental class* for \mathbb{Z}/n coefficients of such a variety Y . By definition this is an element $[Y]$ of $K_{2d}^{\text{BM}}(Y; \mathbb{Z}/n)$ such that $\gamma_Y([Y]) \in K_{\text{geom}}^{2d}(Y; \mathbb{Z}/n)$ is equal to the d -th power β^d of

the so-called Bott element β , where $d = \dim(Y)$. Taking the cap product with a fundamental class $[Y]$ defines a map

$$K_{geom}^p(Y; \mathbb{Z}/n) \xrightarrow{- \cap [Y]} K_{2d-p}^{\text{BM}}(Y; \mathbb{Z}/n)$$

which, as we shall prove, is an inverse of γ_Y up to multiplication by β^d . This then gives a form of “duality” relating K -homology and K -theory with finite coefficients for varieties with fundamental classes. In the next section, we prove every smooth, quasi-projective variety defined over an algebraically closed field has a fundamental class in this sense. Taken together with the results proved in this section, the existence of fundamental classes leads directly to the main results of this paper.

In order to define the map γ_Y , we need to introduce models for the K -theory spectra under consideration that use perfect complexes and the results and constructions of Waldhausen [31] and Thomason-Trobaugh [28].

Definition 8.1. Let Y and X be quasi-projective k -varieties and $U \hookrightarrow Y$ a locally closed subscheme. We define $\mathcal{P}^{qfW}(X, (U, Y))$ to be the category consisting of perfect complexes (cf. [28, 2.2.10]) of coherent $\mathcal{O}_{X \times U}$ -modules M_\bullet on $X \times U$ such that each M_n is \mathcal{O}_X -flat and which satisfy the condition that the closure of the support of M_\bullet in $X \times Y$ is quasi-finite over X . (Recall that the support of a complex M_\bullet is defined by those points $z \in X \times U$ such that the localization of M_\bullet at z is not exact.) We endow $\mathcal{P}^{qfW}(X, (U, Y))$ with the structure of a “complicial biWaldhausen category” [28, 1.2.11] by viewing it as a subcategory of the category of all perfect complexes M_\bullet on $X \times U$ [28, 3.5]. We write $\mathcal{K}^{qfW}(X, (U, Y))$ for the K -theory spectrum associated to $\mathcal{P}^{qfW}(X, (U, Y))$ as defined by Waldhausen [31] (see also [28, 1.5.2]).

If $U = Y$, we write $\mathcal{P}^{qfW}(X, U)$ and $\mathcal{K}^{qfW}(X, U)$ instead of $\mathcal{P}^{qfW}(X, (U, U))$ and $\mathcal{K}^{qfW}(X, (U, U))$.

We define $\mathcal{P}^{qfWfl}(X, (U, Y))$ to be the subcategory of $\mathcal{P}^{qfW}(X, (U, Y))$ consisting of complexes M^\bullet satisfying the additional property that each M^n is flat as an $\mathcal{O}_{U \times X}$ -module; we write $\mathcal{K}^{qfWfl}(X, (U, Y))$ for the corresponding K -theory spectrum. If $U = Y$, we write $\mathcal{P}^{qfWfl}(X, U)$ and $\mathcal{K}^{qfWfl}(X, U)$ for $\mathcal{P}^{qfWfl}(X, (U, Y))$ and $\mathcal{K}^{qfWfl}(X, (U, Y))$.

Observe that an object of $\mathcal{P}^{qf}(X, U)$ may be regarded as a perfect complex concentrated in degree zero provided X and U are smooth varieties. In other words, for such varieties we have a natural functor

$$\mathcal{P}^{qf}(X, U) \rightarrow \mathcal{P}^{qfW}(X, U)$$

which determines a natural map of spectra

$$\mathcal{K}^{qf}(X, U) \rightarrow \mathcal{K}^{qfW}(X, U).$$

Also, the evident inclusion of categories induces a natural map of spectra

$$\mathcal{K}^{qfWfl}(X, U) \rightarrow \mathcal{K}^{qfW}(X, U).$$

Proposition 8.2. Let U be a smooth, quasi-projective variety defined over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Then the maps of simplicial spectra

$$(d \mapsto \mathcal{K}^{qf}(\Delta^d, U)) \longrightarrow (d \mapsto \mathcal{K}^{qfW}(\Delta^d, U)) \longleftarrow (d \mapsto \mathcal{K}^{qfWfl}(\Delta^d, U))$$

induce isomorphisms upon taking homotopy groups with \mathbb{Z}/n coefficients. Moreover, these isomorphisms are contravariantly natural in U for quasi-finite, flat morphisms.

Proof. The right-hand arrow is shown to be a degree-wise weak equivalence as in the proof of [28, 3.5], and thus it induces an isomorphism with homotopy groups in arbitrary coefficients.

Choose a projective closure $U \rightarrow \overline{U}$ and consider the commutative diagram of simplicial spectra

$$\begin{array}{ccc} (d \mapsto \mathcal{K}^{qf}(\Delta^d, U)) & \longrightarrow & (d \mapsto \mathcal{K}^{qfW}(\Delta^d, U)) \\ \uparrow & & \uparrow \\ (d \mapsto \mathcal{K}^{qf}(\Delta^d, (U, \overline{U}))) & \longrightarrow & (d \mapsto \mathcal{K}^{qfW}(\Delta^d, (U, \overline{U}))). \end{array}$$

The left-hand vertical arrow induces an isomorphism upon taking $\pi_p(-; \mathbb{Z}/n)$, for all $p \geq 0$, by part (1) of Corollary 5.3. A nearly identical proof shows the right-hand vertical arrow also induces an isomorphism upon applying $\pi_p(-; \mathbb{Z}/n)$. It therefore suffices to prove the bottom horizontal arrow is a weak equivalence.

For each fixed d , we have a commutative diagram

$$\begin{array}{ccc} \mathcal{K}^{qf}(\Delta^d, (U, \overline{U})) & \longrightarrow & \mathcal{K}^{qfW}(\Delta^d, (U, \overline{U})) \\ \sim \downarrow & & \sim \downarrow \\ \mathcal{G}^{qf}(\Delta^d, (U, \overline{U})) & \longrightarrow & \widetilde{\mathcal{K}^{qfW}}(\Delta^d, (U, \overline{U})) \end{array}$$

of spectra. Here, $\widetilde{\mathcal{K}^{qfW}}(\Delta^d, (U, \overline{U}))$ denotes the K -theory of the complicial biWaldhausen category consisting of all perfect complexes of coherent $\mathcal{O}_{\Delta^d \times U}$ -modules F^\bullet such that the closure of the support of F^\bullet in $\Delta^d \times \overline{U}$ is quasi-finite over X . (In other words, we drop the hypothesis that each F^n be \mathcal{O}_{Δ^d} -flat.) Objects of $\mathcal{M}^{qf}(\Delta^d, (U, \overline{U}))$ are viewed as perfect complexes concentrated in degree 0, which justifies the existence of the bottom arrow of this diagram. The left-hand vertical arrow of this diagram is a weak equivalence by Proposition 5.4 and the right-hand vertical arrow is shown to be a weak equivalence as in the proof of [28, 3.5]. Finally, we claim that each of the spectra on the bottom row is weakly equivalent to $\varinjlim_Z \mathcal{G}(Z)$, where Z ranges over closed subschemes of $\Delta^d \times U$ whose closures in $\Delta^d \times \overline{U}$ are quasi-finite over Δ^d , and thus the bottom map is a weak equivalence. This is evident for $\mathcal{G}^{qf}(\Delta^d, (U, \overline{U}))$, and for $\widetilde{\mathcal{K}^{qfW}}(\Delta^d, (U, \overline{U}))$ the assertion follows from the Thomason Localization Theorem [28, 5.1].

The fact that the maps are natural for quasi-finite, flat morphisms is evident. \square

Definition 8.3. Given a quasi-projective k -variety U , we write $\mathcal{P}^W(U)$ (respectively, $\mathcal{P}^{Wfl}(U)$) for the complicial biWaldhausen category of all perfect complexes of coherent sheaves M^\bullet on U (respectively, all perfect complexes of coherent sheaves M^\bullet such that each M^n is \mathcal{O}_U -flat). (Thus, $\mathcal{P}^{Wfl}(U) = \mathcal{P}^{qfW}(U, \text{Spec } k)$.) We write $\mathcal{K}^W(U)$ and $\mathcal{K}^{Wfl}(U)$ for the associated K -theory spectra.

Note that there is a natural inclusion $\mathcal{P}(U) \rightarrow \mathcal{P}^{Wfl}(U)$ (given by regarding a vector bundle as a chain complex concentrated in degree zero) and hence a natural map $\mathcal{K}(U) \rightarrow \mathcal{K}^{Wfl}(U)$ of associated spectra.

Lemma 8.4. *For any quasi-projective variety U , the inclusion $\mathcal{P}(U) \rightarrow \mathcal{P}^{Wfl}(U)$ induces a natural weak equivalence*

$$\mathcal{K}_{geom}(U) \xrightarrow{\sim} |d \mapsto \mathcal{K}^{Wfl}(\Delta^d \times U)|.$$

Proof. The underlying map of simplicial spectra is a degree-wise weak equivalence by [28, 3.5 and 3.10]. \square

The inclusion $\mathcal{P}^{qfWfl}(X, U) \rightarrow \mathcal{P}^{Wfl}(X \times U)$ is a map of complicial biWaldhausen categories, and thus it induces a natural map of K -theory spectra. The map we seek, γ_U , is defined in terms of this map of spectra by using the weak equivalences established above.

Definition 8.5. Let U be a smooth, quasi-projective variety defined over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Define

$$\gamma_U : K_q^{BM}(U; \mathbb{Z}/n) \rightarrow K_{geom}^{-q}(U; \mathbb{Z}/n),$$

for any q , via the chain of maps

$$\begin{array}{ccc} K_q^{BM}(U; \mathbb{Z}/n) & \xrightarrow{\cong} & \pi_q(|d \mapsto \mathcal{K}^{qfW}(\Delta^d, U)|; \mathbb{Z}/n) \\ & & \cong \uparrow \\ & & \pi_q(|d \mapsto \mathcal{K}^{qfWfl}(\Delta^d, U)|; \mathbb{Z}/n) \\ & & \downarrow \\ K_{geom}^{-q}(U; \mathbb{Z}/n) & \xrightarrow{\cong} & \pi_q(|d \mapsto \mathcal{K}^{Wfl}(\Delta^d \times U)|; \mathbb{Z}/n), \end{array}$$

where the displayed isomorphisms are established in Proposition 8.2 and Lemma 8.4.

Remark 8.6. In the case where Y is smooth and projective and n satisfies (Condition 2-3), the map γ_Y of Definition 8.5 admits a simpler description in terms of standard cohomological operations. Namely, for such a Y , let

$$[\delta_Y] \in K_{geom}^0(Y \times Y) \cong K_0^{\text{alg}}(Y)$$

be the class of the structure sheaf of the diagonal closed subscheme $Y \rightarrowtail Y \times Y$, and recall that the slant product is the pairing

$$-/- : K_{geom}^p(X \times Y; \mathbb{Z}/n) \otimes K_q^{\text{hom}}(Y; \mathbb{Z}/n) \rightarrow K_{geom}^{p-q}(X; \mathbb{Z}/n)$$

defined by the formula $x/y = p_*(x \bullet q^*y)$, where $p : X \times Y \rightarrow Y$ is the canonical projection map and $q : X \rightarrow \text{Spec } k$ is the structure map (cf. [10, 2.4]). Then we have

$$\gamma_Y = [\delta_Y]/- : K_q^{\text{hom}}(Y; \mathbb{Z}/n) \rightarrow K_{geom}^{-q}(Y; \mathbb{Z}/n).$$

In addition to enabling the definition γ_U , the isomorphisms of Proposition 8.2 also allow us to define Gysin maps in a more general setting than before and, consequently, allows for the construction of a product rule for $K^{BM}(-; \mathbb{Z}/n)$. Concerning the Gysin maps, given a quasi-finite (but possibly not flat) map $f : V \rightarrow U$ of smooth, quasi-projective k -varieties, pullback along $id_X \times f$ determines a natural exact functor

$$f^* : \mathcal{P}^{qfWfl}(X, U) \rightarrow \mathcal{P}^{qfWfl}(X, V),$$

for any X , and hence a map of spectra

$$f^* : \mathcal{K}^{qfWfl}(X, U) \rightarrow \mathcal{K}^{qfWfl}(X, V).$$

Replacing X with Δ^\bullet and using the isomorphisms of Proposition 8.2, this natural map of spectra determines a map

$$f^* : K_q^{BM}(U; \mathbb{Z}/n) \rightarrow K_q^{BM}(V; \mathbb{Z}/n),$$

for all q , provided k is algebraically closed and $n \geq 1$ is relatively prime to the exponential characteristic of k . If f happens to be flat, the Gysin map defined here coincides with the pullback map defined in Proposition 3.8.

Proposition 8.7. *If $f : V \rightarrow U$ is a quasi-finite morphism of smooth, quasi-projective k -varieties, then the diagram*

$$\begin{array}{ccc} K_p^{BM}(U; \mathbb{Z}/n) & \xrightarrow{\gamma_U} & K_{geom}^{-p}(U; \mathbb{Z}/n) \\ f^* \downarrow & & \downarrow f^* \\ K_p^{BM}(V; \mathbb{Z}/n) & \xrightarrow{\gamma_V} & K_{geom}^{-p}(V; \mathbb{Z}/n) \end{array}$$

commutes. Similarly, if $i : Z \rightarrow X$ is a regular closed immersion of smooth, quasi-projective k -varieties, then the diagram

$$\begin{array}{ccc} K_p^{BM}(Z; \mathbb{Z}/n) & \xrightarrow{\gamma_Z} & K_{geom}^{-p}(Z; \mathbb{Z}/n) \\ i_* \downarrow & & \downarrow i_* \\ K_p^{BM}(X; \mathbb{Z}/n) & \xrightarrow{\gamma_X} & K_{geom}^{-p}(X; \mathbb{Z}/n) \end{array}$$

commutes, where the map i_* on the right is the usual pushforward map in K -theory for a regular closed immersion.

Proof. Using the weak equivalences of Proposition 8.2 and Lemma 8.4, the commutativity of the first diagram follows from the fact that the diagram

$$\begin{array}{ccc} \mathcal{K}^{qfWfl}(Y, U) & \longrightarrow & \mathcal{K}^{Wfl}(Y \times U) \\ (id_Y \times f)^* \downarrow & & \downarrow (id_Y \times f)^* \\ \mathcal{K}^{qfWfl}(Y, V) & \longrightarrow & \mathcal{K}^{Wfl}(Y \times V) \end{array}$$

commutes for all smooth Y and is natural in Y . Similarly, the commutativity and naturality of

$$\begin{array}{ccc} \mathcal{K}^{qfW}(Y, Z) & \longrightarrow & \mathcal{K}^W(Y \times Z) \\ (id_Y \times i)_* \downarrow & & \downarrow (id_Y \times i)_* \\ \mathcal{K}^{qfW}(Y, X) & \longrightarrow & \mathcal{K}^W(Y \times X) \end{array}$$

for all smooth Y implies the commutativity of the second diagram. (The fact that i is a regular closed immersion implies that the pushforward along i of a perfect complex on $Y \times Z$ is a perfect complex on $Y \times X$.) \square

Proposition 8.8. *Let U be a smooth, quasi-projective variety defined over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Then the map*

$$\gamma_U : K_m^{BM}(U; \mathbb{Z}/n) \rightarrow K_{geom}^{-m}(U; \mathbb{Z}/n)$$

is a homomorphism of $K_{\text{geom}}^*(U)$ -modules. If additionally n satisfies (Condition 2-3), it is a map of $K_{\text{geom}}^*(U; \mathbb{Z}/n)$ -modules.

Proof. Using the equivalences of Proposition 8.2 and Lemma 8.4, the first assertion follows from the commutativity of the diagram of spectra

$$\begin{array}{ccc} \mathcal{K}^{qfWfl}(X, U) \wedge \mathcal{K}^{Wfl}(X \times U) & \longrightarrow & \mathcal{K}^{qfWfl}(X, U) \\ \downarrow & & \downarrow \\ \mathcal{K}^{Wfl}(X \times U) \wedge \mathcal{K}^{Wfl}(X \times U) & \longrightarrow & \mathcal{K}^{Wfl}(X \times U) \end{array}$$

whose horizontal maps are induced by a tensor product of perfect complexes and whose vertical maps are induced by the evident inclusions of categories. The final assertion holds since the mod- n Moore space admits an appropriate coproduct rule for such values of n . \square

The isomorphisms of Proposition 8.2 also allows us to define certain pairings for K^{BM} . Namely, given smooth varieties U and V , there is a natural biexact pairing of categories

$$\mathcal{P}^{qfWfl}(X, U) \times \mathcal{P}^{qfWfl}(X, V) \rightarrow \mathcal{P}^{qfWfl}(X, U \times V),$$

given by taking external tensor product of complexes. That is, if M^\bullet (resp., N^\bullet) is a degree-wise flat perfect complex on $X \times U$ (resp., $X \times V$), then $p^*M^\bullet \otimes_{\mathcal{O}_{X \times U \times V}} q^*N^\bullet$ is a degree-wise flat perfect complex on $X \times U \times V$. (Here, p and q are the evident projections.) Moreover, the quasi-finiteness of support is preserved under this operation, and so we have a biexact pairing as indicated above. We thus obtain a natural (in X) pairing of spectra

$$\mathcal{K}^{qfWfl}(X, U) \wedge \mathcal{K}^{qfWfl}(X, V) \rightarrow \mathcal{K}^{qfWfl}(X, U \times V).$$

Replacing X with Δ^\bullet , taking homotopy groups with coefficients in \mathbb{Z}/n (with $n \not\equiv 2 \pmod{4}$), and using the isomorphisms of Proposition 8.2, we obtain the pairing

$$\mu_{U,V} : K_p^{\text{BM}}(U; \mathbb{Z}/n) \otimes K_p^{\text{BM}}(V; \mathbb{Z}/n) \rightarrow K_p^{\text{BM}}(U \times V; \mathbb{Z}/n).$$

This pairing is commutative and associative in the evident sense provided n satisfies (Condition 2-3).

The next result will be used in the next section to establish the existence of fundamental classes for affine space (cf. Lemma 9.1).

Proposition 8.9. *Let U and V be smooth, quasi-projective varieties defined over an algebraically closed field. Assume $n \geq 1$ is an integer relative prime to the exponential characteristic of k and that n satisfies (Condition 2-3). Then the pairing*

$$\mu_{U,V} : K_i^{\text{BM}}(U; \mathbb{Z}/n) \otimes K_j^{\text{BM}}(V; \mathbb{Z}/n) \rightarrow K_{i+j}^{\text{BM}}(U \times V; \mathbb{Z}/n)$$

defined above is natural U and V for quasi-finite morphisms, and it satisfies the equation

$$\pi_1^* \gamma_U(a) \cup \pi_2^* \gamma_V(b) = \gamma_{U \times V}(\mu_{U,V}(a \otimes b)).$$

Proof. Naturality holds since the pairing

$$\mathcal{K}^{qfWfl}(X, U) \wedge \mathcal{K}^{qfWfl}(X, V) \rightarrow \mathcal{K}^{qfWfl}(X, U \times V)$$

is natural in U and V with respect to quasi-finite pullback, for any X . To establish the identity, note there is a commutative diagram of spectra

$$(33) \quad \begin{array}{ccc} \mathcal{K}^{qfWfl}(X, U) \wedge \mathcal{K}^{qfWfl}(X, V) & \longrightarrow & \mathcal{K}^{qfWfl}(X, U \times V) \\ \downarrow & & \downarrow \\ \mathcal{K}^{Wfl}(X \times U) \wedge \mathcal{K}^{qfWfl}(X \times V) & \longrightarrow & \mathcal{K}^{Wfl}(X \times U \times V) \end{array}$$

in which the horizontal maps are given by taking the tensor product of perfect complexes and the vertical maps are induced by the evident inclusions of categories. Replacing X with Δ^\bullet and taking homotopy groups with \mathbb{Z}/n coefficients, it is clear that under the equivalences of Proposition 8.2 and Lemma 8.4 that the composition of the lower-left vertical arrow with the bottom horizontal arrow in this diagram induces the map

$$K_p^{\text{BM}}(U; \mathbb{Z}/n) \otimes K_q^{\text{BM}}(V; \mathbb{Z}/n) \rightarrow K_{geom}^{-p-q}(U \times V; \mathbb{Z}/n)$$

given by $a \otimes b \mapsto \pi_1^*(\gamma_U(a)) \cup \pi_2^*(\gamma_V(b))$. The other composition in this square induces the map $a \otimes b \mapsto \gamma_{U \times V}(\mu_{U,V}(a \otimes b))$. \square

Composing the pairing $\mu_{U,U}$ with pullback along the diagonal embedding $\Delta : U \rightarrow U \times U$, we obtain the pairing

$$\mu_U : K_p^{\text{BM}}(U; \mathbb{Z}/n) \otimes K_q^{\text{BM}}(U; \mathbb{Z}/n) \rightarrow K_{p+q}^{\text{BM}}(U; \mathbb{Z}/n).$$

Under the isomorphisms of Proposition 8.2, the pairing μ_U may equivalently be defined as the map induced by the natural biexact pairing

$$\mathcal{P}^{qfWfl}(X, U) \times \mathcal{P}^{qfWfl}(X, U) \rightarrow \mathcal{P}^{qfWfl}(X, U)$$

sending a pair of complexes (M^\bullet, N^\bullet) to $M^\bullet \otimes_{\mathcal{O}_{X \times U}} N^\bullet$.

Proposition 8.10. *Let U be a smooth, quasi-projective variety defined over an algebraically closed field k . Assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k and that n satisfies (Condition 2-3). Then the pairing*

$$\mu_U : K_p^{\text{BM}}(U; \mathbb{Z}/n) \otimes K_q^{\text{BM}}(U; \mathbb{Z}/n) \rightarrow K_{p+q}^{\text{BM}}(U; \mathbb{Z}/n)$$

defined above satisfies the equation

$$a \cap \gamma_U(b) = \mu_U(a \otimes b) = (-1)^{pq} b \cap \gamma_U(a).$$

Proof. Observe that there is a commutative diagram of spectra

$$\begin{array}{ccc} \mathcal{K}^{qfWfl}(X, U) \wedge \mathcal{K}^{Wfl}(X \times U) & & \\ \uparrow & \searrow & \\ \mathcal{K}^{qfWfl}(X, U) \wedge \mathcal{K}^{qfWfl}(X, U) & \xrightarrow{\quad} & \mathcal{K}^{qfWfl}(X, U), \\ \downarrow & \nearrow & \\ \mathcal{K}^{Wfl}(X \times U) \wedge \mathcal{K}^{qfWfl}(X, U) & & \end{array}$$

where the two diagonal arrows and the horizontal arrow are given by tensor product of perfect complexes and the vertical arrows are induced by the evident inclusion of categories. Under the isomorphisms given by Proposition 8.2 and Lemma 8.4, the

composition of the lower vertical map and the lower diagonal map in this diagram induces the pairing

$$K_p^{\text{BM}}(U; \mathbb{Z}/n) \otimes K_q^{\text{BM}}(U; \mathbb{Z}/n) \rightarrow K_{p+q}^{\text{BM}}(U; \mathbb{Z}/n)$$

which sends $a \otimes b$ to $\gamma_U(a) \cap b$. Similarly, the composition of the upper vertical map and upper diagonal map induces the pairing $a \otimes b \mapsto (-1)^{pq} \gamma_U(b) \cap a$, and the horizontal map induces $a \otimes b \mapsto \mu_U(a \otimes b)$. \square

We now define the notion of a fundamental class for a smooth, quasi-projective variety. The mere existence of such a class in conjunction with the preceding results immediately gives the desired “duality” result discussed above. In the next section, we establish the existence of such classes for all smooth, quasi-projective varieties.

Definition 8.11. Let k be an algebraically closed field, and $n \geq 1$ any integer relatively prime to the exponential characteristic of k . Choose $b \in \mu_n(k)$ to be a primitive n -th root of unity. Define the *Bott element* $\beta \in K_{\text{geom}}^2(k; \mathbb{Z}/n)$ to be the unique element mapping to b under the boundary map

$$K_{\text{geom}}^2(k; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^1(k) \cong k^\times.$$

We also write β for the image of β under the natural map $K_{\text{geom}}^2(k; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^2(Y; \mathbb{Z}/n)$ for any k -scheme Y .

Definition 8.12. Let k be an algebraically closed field, Y a smooth, quasi-projective k -variety of pure dimension d , and $n \geq 1$ an integer relatively prime to the exponential characteristic of k . A *fundamental class* of Y for \mathbb{Z}/n coefficients is an element $[Y] \in K_{2d}^{\text{BM}}(Y; \mathbb{Z}/n)$ such that $\gamma_Y([Y]) = \beta^d$.

Theorem 8.13. Let k be an algebraically closed field, Y a smooth, quasi-projective k -variety of pure dimension d , and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k and that n satisfies (Condition 2-3). If Y has a fundamental class $[Y] \in K_{2d}^{\text{BM}}(Y; \mathbb{Z}/n)$, then the composition of

$$K_i^{\text{BM}}(Y; \mathbb{Z}/n) \xrightarrow{\gamma_Y} K_{\text{geom}}^{-i}(Y; \mathbb{Z}/n) \xrightarrow{- \cap [Y]} K_{2d+i}^{\text{BM}}(Y; \mathbb{Z}/n)$$

coincides with $\beta^d \cap -$ and the composition of

$$K_{\text{geom}}^j(Y; \mathbb{Z}/n) \xrightarrow{- \cap [Y]} K_{2d-j}^{\text{BM}}(Y; \mathbb{Z}/n) \xrightarrow{\gamma_Y} K_{\text{geom}}^{j-2d}(Y; \mathbb{Z}/n)$$

coincides with $\beta^d \cup -$. Consequently, as a $K_{\text{geom}}^*(Y; \mathbb{Z}/n)[1/\beta]$ -module, the module $K_*^{\text{BM}}(Y; \mathbb{Z}/n)[1/\beta]$ is graded-free of rank one with basis given by the fundamental class.

Proof. For $x \in K_i^{\text{BM}}(Y; \mathbb{Z}/n)$ we have $\gamma_Y(x) \cap [Y] = \gamma_Y([Y]) \cap x = \beta^d \cap x$ by Proposition 8.10 (using that $[Y]$ has even degree). Similarly for $x \in K_{\text{geom}}^j(Y; \mathbb{Z}/n)$, we have $\gamma_Y(x \cap [Y]) = x \cup \gamma_Y([Y]) = \beta^d \cup x$ by Proposition 8.8 (using that β has even degree). The final assertion clearly follows. \square

9. FUNDAMENTAL CLASSES FOR SMOOTH VARIETIES

In this section we establish that every smooth, quasi-projective variety Y over an algebraically closed field k has a fundamental class for \mathbb{Z}/n coefficients, where $n \geq 1$ is relatively prime to the exponential characteristic of k .

Lemma 9.1. *Let k be an algebraically closed field and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k satisfying (Condition 2-3). Then for all $m \geq 0$ the variety \mathbb{A}^m has a fundamental class $[\mathbb{A}^m]$ for \mathbb{Z}/n coefficients. In fact, $K_*^{BM}(\mathbb{A}^m; \mathbb{Z}/n)$ is a graded-free $K_{geom}^{-*}(\text{Spec } k; \mathbb{Z}/n)$ -module of rank one generated by $[\mathbb{A}^m]$.*

Proof. For $m = 0$, the assertion is obvious, since $\mathcal{K}^{BM}(\text{Spec } k) = \mathcal{K}_{geom}(\text{Spec } k)$. Suppose there was a fundamental class $[\mathbb{A}^1]$ for \mathbb{A}^1 belonging to $K_2^{BM}(\mathbb{A}^1; \mathbb{Z}/n)$. Then by applying Proposition 8.9 repeatedly, we would have that the element of $K_{2m}^{BM}(\mathbb{A}^m; \mathbb{Z}/n)$ obtained by multiplying $[\mathbb{A}^1]$ with itself m times using the pairing μ defines a fundamental class for \mathbb{A}^m . (Note that we have not proven μ is associative — but any manner of multiplying $[\mathbb{A}^1]$ with itself suffices.) Thus, to prove \mathbb{A}^m has a fundamental class for all $m \geq 0$, it suffices to prove \mathbb{A}^1 has a fundamental class.

For any finitely generated smooth k -algebra R , we claim there is a homotopy fibration sequence of the form

$$\mathcal{K}^{qf}(\text{Spec } R, \mathbb{A}^1) \rightarrow \mathcal{K}(R[x]) \rightarrow \mathcal{K}(S^{-1}R[x]),$$

where S is the multiplicatively closed subset of $R[x]$ consisting of polynomials of unit content (i.e., those $r_n x^n + \dots + r_1 x + r_0$ such that r_0, \dots, r_n generate R as an ideal). To see this, one argues as in the proof of Proposition 4.11 above. Namely, we have a fibration sequence of the form

$$\varinjlim_f \mathcal{K}(\mathbb{H}(R[x], f)) \rightarrow \mathcal{K}(R[x]) \rightarrow \mathcal{K}(S^{-1}R[x])$$

by [11, Theorem on p. 229], where f ranges over elements of S (ordered by division). A nearly identical argument to that given in Proposition 4.11 shows that the category $\varinjlim_f \mathbb{H}(R[x], f)$ (which is a full subcategory of $\mathcal{M}(R[x])$) coincides with $\mathcal{P}^{qf}(\text{Spec } R, \mathbb{A}^1)$. Replacing R by the simplicial ring $d \mapsto k[\Delta^d]$, where $k[\Delta^d] = k[y_0, \dots, y_d]/\langle(\sum_i y_i) - 1\rangle$, and taking geometric realizations gives a fibration sequence of the form

$$|d \mapsto \mathcal{K}^{qf}(\Delta^d, \mathbb{A}^1)| \rightarrow |d \mapsto \mathcal{K}(\Delta^d \times \mathbb{A}^1)| \rightarrow |d \mapsto \mathcal{K}^{qf}(S_d^{-1}(k[\Delta^d])[x])|,$$

where S_d is the set of polynomials in $k[\Delta^d][x]$ of unit content. (Observe that the map $K_0(R[x]) \rightarrow K_0(S^{-1}R[x])$ is surjective since $R[x]$ is regular.) Since the simplicial ring $d \mapsto S_d^{-1}k[\Delta^d][x]$ satisfies the primitive criterion degree-wise, the space

$$|d \mapsto \mathcal{K}^{qf}(S_d^{-1}k[\Delta^d][x])|$$

is homotopy equivalent to the discrete space \mathbb{Z} by [35, 2.4]. It follows that the map

$$|d \mapsto \mathcal{K}^{qf}(\Delta^d, \mathbb{A}^1)| \rightarrow |d \mapsto \mathcal{K}(\Delta^d \times \mathbb{A}^1)|$$

induces an isomorphism on $\pi_n(-; A)$ for any $n \geq 1$ and any group of coefficients A . This map defines the map $\gamma_{\mathbb{A}^1}$ introduced in the previous section, and it follows that \mathbb{A}^1 has a fundamental class.

To prove the final assertion, observe that the spectral sequence of Theorem 7.6 for affine space degenerates at the E_2 terms to give

$$K_i^{BM}(\mathbb{A}^m; \mathbb{Z}/n) \cong H_{\text{ét}}^0(\mathbb{A}^m; \mu_n^{\otimes i/2}(k)) \cong \mu_n^{\otimes i/2}(k), \quad \text{for } i \geq 2m,$$

and $K_i^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n) = 0$ for $i < 2m$. (As usual, one interprets $\mu_n^{\otimes i/2}$ as being 0 for i odd.) By Proposition 8.8, the map

$$\gamma_{\mathbb{A}^m} : K_*^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-*}(\mathbb{A}^m; \mathbb{Z}/n) \cong K_{\text{geom}}^{-*}(\text{Spec } k; \mathbb{Z}/n) \cong \mathbb{Z}/n[\beta]$$

is a graded $\mathbb{Z}/n[\beta]$ -module homomorphism, and by Theorem 8.13, the composition of

$$K_{\text{geom}}^{-*}(\mathbb{A}^m; \mathbb{Z}/n) \xrightarrow{- \cap [\mathbb{A}^m]} K_*^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n) \xrightarrow{\gamma_{\mathbb{A}^m}} K_{\text{geom}}^{2m-*}(\mathbb{A}^m; \mathbb{Z}/n)$$

is multiplication by β^d . In particular, for $i \geq 2m$ and even, the map

$$\gamma_{\mathbb{A}^m} : K_i^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-i}(\mathbb{A}^m; \mathbb{Z}/n)$$

is a surjection of groups isomorphic to \mathbb{Z}/n , and hence must be an isomorphism. Since $K_i^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n) = 0$ for i odd or $i < 2m$, it follows that $\gamma_{\mathbb{A}^m}$ is an isomorphism onto the ideal $\beta^m \cdot \mathbb{Z}/n[\beta]$. The desired result now follows from Theorem 8.13. \square

Theorem 9.2. *Let k be an algebraically closed field and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k satisfying (Condition 2-3). Then for all $m \geq 0$, the map*

$$\gamma_{\mathbb{P}^m} : K_p^{\text{BM}}(\mathbb{P}^m; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-p}(\mathbb{P}^m; \mathbb{Z}/n)$$

is an isomorphism in degrees $p \geq 2m$. In particular, the variety \mathbb{P}^m has a fundamental class for \mathbb{Z}/n coefficients.

Proof. We proceed by induction on m , with the case $m = 0$ being obvious.

Let $j : \mathbb{A}^m \rightarrow \mathbb{P}^m$ be the canonical open embedding with closed complement $i : \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^m$. The spectral sequences of Theorem 7.6 for \mathbb{P}^m and \mathbb{A}^m both degenerate at the E_2 terms and, as we have seen, the degeneration of the spectral sequence for \mathbb{A}^m determines the isomorphism

$$K_{2m+2i}^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n) \cong H_{\text{ét}}^0(\mathbb{A}^m; \mu_n^{\otimes(m+i)}) = E_2^{m+i, -m-i},$$

for all $i \geq 0$. The generation of the spectral sequence of \mathbb{P}^m implies the existence of a filtration on $K_{2m+2i}^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n)$, for $i \geq 0$, one of whose associated quotient groups is

$$E_2^{m+i, -m-i} = H_{\text{ét}}^0(\mathbb{P}^m; \mu_n^{\otimes(m+i)}).$$

Since j^* induces a map of spectral sequences and the map

$$j^* : H_{\text{ét}}^0(\mathbb{P}^m; \mu_n^{\otimes(m+i)}) \rightarrow H_{\text{ét}}^0(\mathbb{A}^m; \mu_n^{\otimes(m+i)})$$

is an isomorphism, we have that

$$j^* : K_{2m+2i}^{\text{BM}}(\mathbb{P}^m; \mathbb{Z}/n) \rightarrow K_{2m+2i}^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n)$$

is surjective for all $i \geq 0$. Since $K_t^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n) = 0$ for t odd, the columns of the diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_{2m+t}^{\text{BM}}(\mathbb{P}^{m-1}; \mathbb{Z}/n) & \xrightarrow{\gamma_{\mathbb{P}^{m-1}}} & K_{\text{geom}}^{-2m-t}(\mathbb{P}^{m-1}; \mathbb{Z}/n) \\
i_* \downarrow & & i_* \downarrow \\
K_{2m+t}^{\text{BM}}(\mathbb{P}^m; \mathbb{Z}/n) & \xrightarrow{\gamma_{\mathbb{P}^m}} & K_{\text{geom}}^{-2m-t}(\mathbb{P}^m; \mathbb{Z}/n) \\
j^* \downarrow & & j^* \downarrow \\
K_{2m+t}^{\text{BM}}(\mathbb{A}^m; \mathbb{Z}/n) & \xrightarrow{\gamma_{\mathbb{A}^m}} & K_{\text{geom}}^{-2m-t}(\mathbb{A}^m; \mathbb{Z}/n) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

are exact. The diagram commutes by Proposition 8.7, and for all $t \geq 0$ the bottom horizontal map is an isomorphism by Lemma 9.1. Using induction and the Five Lemma, we have that

$$\gamma_{\mathbb{P}^m} : K_{2m+t}^{\text{BM}}(\mathbb{P}^m; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-2m-t}(\mathbb{P}^m; \mathbb{Z}/n)$$

is an isomorphism for all $t \geq 0$.

In particular, taking $t = 0$, the element $\beta^m \in K_{\text{geom}}^{-2m}(\mathbb{P}^m; \mathbb{Z}/n)$ lifts to a fundamental class $[\mathbb{P}^m] \in K_{2m}^{\text{BM}}(\mathbb{P}^m; \mathbb{Z}/n)$. \square

Theorem 9.3. *Let k be an algebraically closed field and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k satisfying (Condition 2-3). Then every smooth, quasi-projective k -variety of pure dimension has a fundamental class for \mathbb{Z}/n coefficients.*

Proof. Let U be any smooth, quasi-projective k -variety of pure dimension d . Let $U \subset Y$ be an open dense embedding into a projective variety Y and choose a finite map $\pi : Y \rightarrow \mathbb{P}^d$. Then the restriction $f : U \rightarrow \mathbb{P}^d$ of π to U is quasi-finite (and hence flat by [18, 23.1]). It follows immediately from the naturality property of γ_U established in Proposition 8.7 that $f^*([\mathbb{P}^d])$ is a fundamental class of U , where $[\mathbb{P}^d]$ is a fundamental class of \mathbb{P}^d whose existence is guaranteed by Theorem 9.2. \square

Corollary 9.4. *Let k be an algebraically closed field and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k satisfying (Condition 2-3). For any smooth, quasi-projective k -variety Y , the natural map*

$$K_*^{\text{BM}}(Y; \mathbb{Z}/n)[1/\beta] \xrightarrow{\gamma_Y} K_{\text{geom}}^{-*}(Y; \mathbb{Z}/n)[1/\beta]$$

is an isomorphism.

Proof. This follows immediately from Theorems 8.13 and 9.3. \square

10. THOMASON'S THEOREM FOR VARIETIES OVER ALGEBRAICALLY CLOSED FIELDS

In this section we deduce Thomason's theorem for smooth varieties defined over an algebraically closed base field k (with the usual conditions on the coefficients \mathbb{Z}/n). Namely, for such a variety U , we prove the existence of a spectral sequence

with E_2 terms given by the appropriate étale cohomology groups converging to $K_*(U; \mathbb{Z}/n)[1/\beta]$. The proof of this theorem is essentially given by combining Theorems 7.6, 8.13, and 9.3. Indeed, in order to establish the existence of the above-mentioned spectral sequence, the only additional component we require is that the Bott element acts on the spectral sequence of Theorem 7.6 and this action has the expected behavior. The formal assertion is given by the following result. Recall that $b \in \mu_n(k) \cong H_{\text{ét}}^0(\text{Spec } k, \mu_n)$ denotes a chosen primitive n -th root of unity and $\beta \in K_2(k; \mathbb{Z}/n) \cong \mathbb{Z}/n$ maps to b under the boundary map $K_2(k; \mathbb{Z}/n) \rightarrow K_1(k) \cong k^\times$.

Proposition 10.1. *Let k be an algebraically closed field and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k satisfying (Condition 2-3). For any Cohen-Macaulay, quasi-projective k -variety U , there is an endomorphism of bidegree $(0, 2)$ of the exact couple $(D_{s,t}^r, E_{s,t}^r)$ defining the first spectral sequence*

$$E_{s,t}^2 = H_c^s(U, \mu_n^{\otimes t/2}(k)) \Longrightarrow K_{s+t}^{BM}(U; \mathbb{Z}/n)$$

of Theorem 7.6 that is given by multiplication by b on E^2 terms and that is given by multiplication by β on the abutment.

If U is smooth, this determines an endomorphism of bidegree $(-1, -1)$ of the exact couple $(D_r^{p,q}, E_r^{p,q})$ defining the second spectral sequence

$$E_2^{p,q} = H_{\text{ét}}^{p-q}(U, \mu_n^{\otimes(-q)}(k)) \Longrightarrow K_{-p-q}^{BM}(U; \mathbb{Z}/n)$$

of Theorem 7.6 that is given by multiplication by b on E_2 terms and that is given by multiplication by β on the abutment.

These endomorphisms are natural with respect to quasi-finite, flat morphisms.

Proof. Recall that the first spectral sequence of Theorem 7.6 arises from a tower of spaces depending functorially on U ,

$$\cdots \rightarrow G_{t+1}(U) \rightarrow G_t(U) \rightarrow \cdots \rightarrow G_0(U) \rightarrow *,$$

together with a family of fibration sequences of the form

$$F_t(U) \rightarrow G_t(U) \rightarrow G_{t-1}(U), \quad t \geq 0.$$

Specifically, we have

$$G_t(V) = \text{diag}(d \mapsto P_t \mathcal{K}^{qf}(\Delta^d, V; \mathbb{Z}/n))$$

and

$$F_t(V) = \text{diag}(d \mapsto Q_t \mathcal{K}^{qf}(\Delta^d, V; \mathbb{Z}/n)),$$

and the exact couple defining the spectral sequence is given by the formulas

$$D_{s,t}^2 = \pi_{s+t+1} G_{t-1}(U)$$

and

$$E_{s,t}^2 = \pi_{s+t} F_t(U) \cong H_c^s(U, \mu_n^{t/2}(k)^\#)^\#.$$

Recall additionally that, in this context, $\mathcal{K}^{qf}(-, U; \mathbb{Z}/n)$ is defined to be a functor from Sch/k to Kan complexes given at $V \in Sch/k$ by the formula

$$\mathcal{K}^{qf}(V, U; \mathbb{Z}/n) = \underline{\text{Maps}}_*(M, \Omega^{-2} \mathcal{K}^{qf}(V, U)),$$

where $\Omega^{-2} \mathcal{K}^{qf}(V, U) = \text{Ex}(\text{diag}(S_\bullet S_\bullet \mathcal{P}^{qf}(V, U)))$ and M is the mod- n Moore space. We extend the definition of $\Omega^{-2} \mathcal{K}^{qf}(V, U)$ by setting

$$\Omega^{-p} \mathcal{K}^{qf}(V, U) = \text{Ex}(\text{diag}(S_\bullet^{(p)} \mathcal{P}^{qf}(V, U))), \quad p \geq 1,$$

where $S^{(p)}$ denotes p iterates of the S construction of Waldhausen (so that $S^{(p)}\mathcal{P}$ is a p -fold multi-simplicial set). Observe that there are natural pairings

$$\Omega^{-p}\mathcal{K}^{qf}(V, U) \wedge \Omega^{-q}\mathcal{K}^{qf}(V, U) \rightarrow \Omega^{-p-q}\mathcal{K}^{qf}(V, U)$$

induced by a tensor product over \mathcal{O}_V .

The Bott element is an element of $K_2(k; \mathbb{Z}/n)$ and is given as the π_0 class of an element β of the space $\Omega^2\mathcal{K}^{qf}(k; \mathbb{Z}/n) = \underline{\text{Maps}}_*(S^2 \wedge M, \Omega^{-2}\mathcal{K}(k))$. Pairing with β defines a natural transformation

$$\underline{\text{Maps}}_*(M, \Omega^{-2}\mathcal{K}^{qf}(-, U)) \rightarrow \underline{\text{Maps}}_*(S^2 \wedge M \wedge M, \Omega^{-4}\mathcal{K}^{qf}(-, U)),$$

which, provided $n \not\equiv 2 \pmod{4}$, can be composed with precomposition along a “comultiplication” rule $S^2 \wedge M \rightarrow M \wedge M$ (cf. [8, C.3]) to give a natural transformation

$$\underline{\text{Maps}}_*(M, \Omega^{-2}\mathcal{K}^{qf}(-, U)) \rightarrow \underline{\text{Maps}}_*(S^4 \wedge M, \Omega^{-4}\mathcal{K}^{qf}(-, U)).$$

There is a natural weak equivalence

$$\underline{\text{Maps}}_*(S^2 \wedge M, \Omega^{-2}\mathcal{K}^{qf}(V, U)) \xrightarrow{\sim} \underline{\text{Maps}}_*(S^4 \wedge M, \Omega^{-4}\mathcal{K}^{qf}(V, U)),$$

which is induced by the natural transformation $S^2 \wedge S_\bullet^{(2)}\mathcal{P} \rightarrow S_\bullet^{(4)}\mathcal{P}$ associated to any exact category \mathcal{P} . Also, for any Kan complex Y , we have the identification

$$\underline{\text{Maps}}_*(S^2, P_{t+2}(Y)) = P_t(\underline{\text{Maps}}_*(S^2, Y)).$$

Combining the above natural maps together yields the following diagram of natural transformations of functors taking values in Kan complexes:

$$\begin{array}{c}
 P_t\mathcal{K}^{qf}(-, U; \mathbb{Z}/n) \\
 \downarrow = \\
 P_t\underline{\text{Maps}}_*(M, \Omega^{-2}\mathcal{K}^{qf}(-, U)) \\
 \downarrow \beta \wedge - \\
 P_t\underline{\text{Maps}}_*(S^4 \wedge M, \Omega^{-4}\mathcal{K}^{qf}(-, U)) \\
 \uparrow \sim \\
 P_t\underline{\text{Maps}}_*(S^2 \wedge M, \Omega^{-2}\mathcal{K}^{qf}(-, U)) \\
 \downarrow \cong \\
 \underline{\text{Maps}}_*(S^2, P_{t+2}\underline{\text{Maps}}_*(M, \Omega^{-2}\mathcal{K}^{qf}(-, U))) \\
 \downarrow = \\
 \underline{\text{Maps}}_*(S^2, P_{t+2}\mathcal{K}^{qf}(-, U; \mathbb{Z}/n)).
 \end{array} \tag{34}$$

Applying each of the functors in (34) to Δ^d , $d \geq 0$, we obtain a diagram of bisimplicial sets relating

$$d \mapsto P_t\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)$$

to

$$d \mapsto \underline{\text{Maps}}_*(S^2, P_{t+2}\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n))$$

via a natural “zig-zag” diagram. Finally, on the level of geometric realizations we have the natural map

$$\begin{array}{c} |d \mapsto \underline{\text{Maps}}_*(S^2, P_{t+2}\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n))| \\ \downarrow \\ \Omega^2 |d \mapsto P_{t+2}\mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)| = \Omega^2 G_{t+2}(U), \end{array}$$

so that we have a natural zig-zag diagram relating $G_t(U)$ to $\Omega^2 G_{t+2}(U)$. That is, there is a diagram of the form

$$G_t(U) \longrightarrow \bullet \xleftarrow{\sim} \bullet \longrightarrow \Omega^2 G_{t+2}(U)$$

in which the symbols \bullet denote explicit functors in U and the arrows are natural for quasi-finite, flat morphisms. Applying π_* to this diagram for each $t \geq 0$ yields a natural map which we write as

$$\beta \cap - : D_{*,*}^2 \rightarrow D_{*,*+2}^2.$$

The map $i : D_{s,t+1}^2 \rightarrow D_{s+1,t}^2$ is induced by the natural transformation $P_t \rightarrow P_{t-1}$, which clearly commutes with each of the natural transformations in (34). We therefore have a commutative diagram of the form

$$\begin{array}{ccccccc} G_t(U) & \longrightarrow & \bullet & \xleftarrow{\sim} & \bullet & \longrightarrow & \Omega^2 G_{t+2}(U) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G_{t-1}(U) & \longrightarrow & \bullet & \xleftarrow{\sim} & \bullet & \longrightarrow & \Omega^2 G_{t+1}(U). \end{array}$$

In particular, the square

$$\begin{array}{ccc} D_{s,t}^2 & \xrightarrow{\beta \cap -} & D_{s,t+2}^2 \\ i \downarrow & & i \downarrow \\ D_{s+1,t-1}^2 & \xrightarrow{\beta \cap -} & D_{s+1,t+1}^2 \end{array}$$

commutes, for all s, t . Moreover, when $t = 1$ we have $D_{s-2,1}^2 = K_s^{\text{BM}}(U; \mathbb{Z}/n)$, and it is apparent that the composition

$$K_s^{\text{BM}}(U; \mathbb{Z}/n) = D_{s-2,1}^2 \xrightarrow{\beta \cap -} D_{s-2,3}^2 \xrightarrow{i^2} D_{s,1}^2 = K_{s+2}^{\text{BM}}(U; \mathbb{Z}/n)$$

coincides with $\beta \cap -$ as defined in Section 3.

The exact same construction using Q_t in place of P_t produces a natural zig-zag map from $F_t(U)$ to $\Omega^2 F_{t+2}(U)$ of the form

$$F_t(U) \longrightarrow \bullet \xleftarrow{\sim} \bullet \longrightarrow \Omega^2 F_{t+2}.$$

We write the induced map on homotopy groups π_{s+t} as

$$b \cap - : E_{s,t}^2 \rightarrow E_{s,t+2}^2,$$

for all s, t . Moreover, we have a commutative diagram of the form

$$\begin{array}{ccccccc}
 F_t(U) & \longrightarrow & \bullet & \xleftarrow{\sim} & \bullet & \longrightarrow & \Omega^2 F_{t+2}(U) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 G_t(U) & \longrightarrow & \bullet & \xleftarrow{\sim} & \bullet & \longrightarrow & \Omega^2 G_{t+2}(U) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 G_{t-1}(U) & \longrightarrow & \bullet & \xleftarrow{\sim} & \bullet & \longrightarrow & \Omega^2 G_{t+1}(U),
 \end{array}$$

whose columns are fibration sequences and which is natural for quasi-finite, flat morphisms. This implies that the maps $i : D_2^{s,t} \rightarrow D_2^{s+1,t-1}$, $j : D_2^{s,t} \rightarrow E_2^{s,t}$, $k : E_2^{s,t} \rightarrow D_2^{s-2,t+1}$ determining the exact couple commute with the maps $\beta \cap -$ and $b \cap -$.

It remains to show that $b \cap - : E_{s,t}^2 \rightarrow E_{s,t+2}^2$ coincides with multiplication by b under the isomorphisms $\pi_{s+t} F_t(U) \cong H_c^s(U, \mu_n^{\otimes(t/2)}(k)^\#)^\#$ of Theorem 7.5. The map $b \cap - : E_{s,t}^2 \rightarrow E_{s,t+2}^2$ is induced by the map of simplicial abelian groups

$$\left(d \mapsto K_t^{qf}(\Delta^d, U; \mathbb{Z}/n) \right) \xrightarrow{\beta \cap -} \left(d \mapsto K_{t+2}^{qf}(\Delta^d, U; \mathbb{Z}/n) \right)$$

given by capping with $\beta \in K_{geom}^2(k; \mathbb{Z}/n)$ degree-wise. Recall that the cap product determines a family of homotopy equivalences of simplicial abelian groups

$$\left(d \mapsto K_t(k; \mathbb{Z}/n) \otimes K_0^{qf}(\Delta^d, U) \right) \xrightarrow{\sim} \left(d \mapsto K_t^{qf}(\Delta^d, U; \mathbb{Z}/n) \right), \quad t \geq 0,$$

by Theorem 7.1 (and Corollary 4.23). Under these homotopy equivalences, the map $b \cap -$ corresponds with the map

$$(35) \quad \left(d \mapsto K_t(k; \mathbb{Z}/n) \otimes K_0^{qf}(\Delta^d, U) \right) \xrightarrow{(\beta \cup -) \otimes id} \left(d \mapsto K_{t+2}(k; \mathbb{Z}/n) \otimes K_0^{qf}(\Delta^d, U) \right)$$

given by a degree-wise cup product of β on $K_*(k; \mathbb{Z}/n)$. By construction, β maps to $b \in \mu_n(k) \cong {}_n K_1(k)$ under the evident boundary map, which is an isomorphism. (The subscript n denotes taking a subgroup of n -torsion elements.) Thus, under the isomorphisms

$$\mu_n^{\otimes t/2}(k) \cong ({}_n K_1(k))^{\otimes t/2} \cong K_2(k; \mathbb{Z}/n)^{\otimes t/2} \cong K_t(k; \mathbb{Z}/n)$$

(where the last map is defined by the cup product) and using the homotopy equivalence established in Theorem 7.5, we see that the map (35) corresponds to the map

$$\left(d \mapsto \mu_n^{\otimes t/2}(k) \otimes z_{equi}^0(\Delta^d, U) \right) \xrightarrow{(b \bullet -) \otimes id} \left(d \mapsto \mu_n^{\otimes(t+2)/2}(k) \otimes z_{equi}^0(\Delta^d, U) \right).$$

Finally, for any finite \mathbb{Z}/n -module F , the isomorphism

$$\pi_p(d \mapsto z_{equi}^0(\Delta^d, U) \otimes F) \cong H_c^p(U, F^\#)^\#$$

due to Suslin [26] (cf. Theorem 7.3) is natural in F . In particular, applying it to $\mu_n^{\frac{t}{2}} \xrightarrow{b \bullet -} \mu_n^{\frac{(t+2)}{2}}$, we see that the map (35) corresponds to the map

$$b \bullet - : H_c^s(U, \mu_n^{\otimes(t/2)}(k)^\#)^\# \rightarrow H_c^s(U, \mu_n^{\otimes(t/2)+1}(k)^\#)^\#$$

under the given isomorphisms.

We have thus established the existence of a natural (with respect to finite, flat morphisms) endomorphism of the exact couple $(D_{s,t}^r, E_{s,t}^r)$ having the asserted properties. The result for the other exact couple now follows by reindexing as in the proof of Theorem 7.6, using the fact that the Poincaré duality isomorphism

$$H_c^s(U, F^\#)^\# \cong H_{\text{ét}}^{2d-s}(U, F \otimes_{\mathbb{Z}/n} \mu_n^{\otimes d})$$

is natural in F . \square

We now invert the action of b and β in the spectral sequence of Theorem 7.6 to obtain the main result of this paper.

Theorem 10.2. *Let U be a smooth, quasi-projective variety defined over an algebraically closed field k . Assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k and such that n satisfies (Condition 2-3). Then there is a convergent spectral sequence concentrated in the portion of the pq -plane satisfying $p \geq q \geq p - 2 \dim(U)$ having the form*

$$E_2^{p,q} = H_{\text{ét}}^{p-q}(U, \mathbb{Z}/n) \implies K_{-p-q}(U; \mathbb{Z}/n)[1/\beta],$$

where $K_m(U; \mathbb{Z}/n)[1/\beta]$ denotes the m -th graded piece of the result of inverting β in the graded ring $K_*(U; \mathbb{Z}/n)$. Moreover, this spectral sequence is natural with respect to quasi-finite, flat morphisms of smooth varieties.

Proof. Let $('D_2^{p,q}, 'E_2^{p,q})$ denote the exact couple obtained by inverting the endomorphism $\beta \cap -$ constructed in Proposition 10.1 of the exact couple $(D_2^{p,q}, E_2^{p,q})$ determining the second spectral sequence of Theorem 7.6. (Formally, we have

$$'D_2^{p,q} = \varinjlim \left(D_2^{p,q} \xrightarrow{\beta \cap -} D_2^{p-1,q-1} \xrightarrow{\beta \cap -} \dots \xrightarrow{\beta \cap -} D_2^{p-m,q-m} \xrightarrow{\beta \cap -} \dots \right)$$

and similarly for the $'E_2$ terms.) Since $D_2^{p,q}$ vanishes for $p - q > 2d$ and multiplication by β has degree $(-1, -1)$, we see that $'D_2^{p,q} = 0$ for $p - q > 2d$ as well. Similarly, the map $i : D_2^{p+1,q-1} \rightarrow D_2^{p,q}$ is an isomorphism if $p - q < 0$, and so $i : 'D_2^{p+1,q-1} \rightarrow 'D_2^{p,q}$ is also an isomorphism if $p - q < 0$. Thus the spectral sequence associated to $('D_2^{p,q}, 'E_2^{p,q})$ converges. The abutment of this spectral sequence is the graded abelian group

$$\bigoplus_m \varinjlim ('D_2^{-m,0} \xrightarrow{i} 'D_2^{-m-1,1} \xrightarrow{i} \dots),$$

which is given by the colimit of the $\mathbb{N} \times \mathbb{N}$ indexed diagram

$$\begin{array}{ccccccc} D_2^{-m,0} & \longrightarrow & D_2^{-m-1,1} & \longrightarrow & \dots \\ \downarrow & & \downarrow & & & & \\ D_2^{-m-1,-1} & \longrightarrow & D_2^{-m-2,0} & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \end{array}$$

We can therefore identify the abutment with the graded abelian group

$$\bigoplus_m \varinjlim (D_2^{-m-d,-d} \rightarrow D_2^{-m-2-d,-d} \rightarrow \dots)$$

where the transition maps are given by composing $\beta \cap -$ with i . Since $D_2^{-j-d,-d} = \pi_j|d \mapsto \mathcal{K}^{qf}(\Delta^d, U; \mathbb{Z}/n)| \cong K_j^{\text{BM}}(U; \mathbb{Z}/n)$, for all j , we see that the abutment of the spectral sequence is the graded abelian group obtained by inverting the action of β on $K_*^{\text{BM}}(U; \mathbb{Z}/n)$. By Theorems 8.13 and 9.3, the abutment is isomorphic to $K_*(U; \mathbb{Z}/n)[1/\beta]$, and so we have a convergent spectral sequence

$$'E_2^{p,q} \Longrightarrow K_*(U; \mathbb{Z}/n)[1/\beta].$$

Since $E_2^{p,q} = H_{\text{ét}}^{p-q}(U, \mu_n^{\otimes -q}(k))$ and the endomorphism we are inverting is multiplication by b , we have

$$'E_2^{p,q} = \varinjlim(H_{\text{ét}}^{p-q}(U, \mu_n^{\otimes -q}(k)) \xrightarrow{b \bullet -} H_{\text{ét}}^{p-q}(U, \mu_n^{\otimes -q+1}(k)) \xrightarrow{b \bullet -} \dots).$$

Since $b^j \bullet - : \mathbb{Z}/n \rightarrow \mu_n^{\otimes j}(k)$ is an isomorphism for all j , it follows that $'E_2^{p,q} \cong H_{\text{ét}}^{p-q}(U, \mathbb{Z}/n)$. \square

Remark 10.3. The spectral sequence of Theorem 10.2 is presumably natural with respect to arbitrary morphisms of smooth varieties. However, since it arises from the spectral sequence of Theorem 7.6, our techniques only imply the more limited naturality asserted above.

In fact, the results we have proven allow us to conclude a slightly stronger form of Thomason's theorem.

Theorem 10.4. *Let U be a smooth, quasi-projective variety of pure dimension d defined over an algebraically closed field k . Assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k and such that n satisfies (Condition 2-3). Then the canonical map*

$$K_*^{\text{BM}}(U; \mathbb{Z}/n) \rightarrow K_*^{\text{BM}}(U; \mathbb{Z}/n)[1/\beta]$$

is an isomorphism in degrees $ \geq 2d$ and thus the map*

$$\gamma_U : K_*^{\text{BM}}(U; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-*}(U; \mathbb{Z}/n)$$

is a split injection in degrees $ \geq 2d$.*

Consequently, the map

$$K_*(U; \mathbb{Z}/n) \rightarrow K_*(U; \mathbb{Z}/n)[1/\beta]$$

is a split surjection in degrees $ \geq 2d$, and in all degrees the kernel of this map is killed by β^d .*

Proof. The proof of Theorem 10.2 establishes the existence of a map of spectral sequences from

$$E_2^{p,q} = H_{\text{ét}}^{p-q}(U, \mu_n^{\otimes(-q)}(k)) \Longrightarrow K_{-p-q}^{\text{BM}}(U; \mathbb{Z}/n)$$

to

$$'E_2^{p,q} = H_{\text{ét}}^{p-q}(U, \mathbb{Z}/n) \Longrightarrow K_{-p-q}^{\text{BM}}(U; \mathbb{Z}/n)[1/\beta].$$

The E_2 terms vanish outside the portion of the pq -plane defined by $p \geq q \geq p-2d$ and $q \leq -d$, whereas the $'E_2$ terms vanish outside the portion of the pq -plane defined by $p \geq q \geq p-2d$. An easy induction argument shows that $E_r^{p,q} = 'E_r^{p,q}$ for all $r \geq 2$ such that for $p+q \leq -2d$. It follows that $E_\infty^{p,q} = 'E_\infty^{p,q}$ and hence

$$K_{-p-q}^{\text{BM}}(U; \mathbb{Z}/n) \cong K_{-p-q}^{\text{BM}}(U; \mathbb{Z}/n)[1/\beta]$$

whenever $-p - q \geq 2d$.

For any i , the composition of

$$K_{2d+i}^{\text{BM}}(U; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-2d-i}(U; \mathbb{Z}/n) \rightarrow K_{4d+i}^{\text{BM}}(U; \mathbb{Z}/n)$$

is given by multiplication by β^d . When $i \geq 0$, this composition is isomorphic to multiplication by β^d on $K_{2d+i}^{\text{BM}}(U; \mathbb{Z}/n)[1/\beta]$, and hence is an isomorphism. The asserted split injectivity therefore holds. Since

$$K_*^{\text{BM}}(U; \mathbb{Z}/n)[1/\beta] \cong K_{\text{geom}}^{-*}(U; \mathbb{Z}/n)[1/\beta] = K_*(U; \mathbb{Z}/n)[1/\beta],$$

the asserted split surjectivity also follows. Similarly, for $i \geq 0$, the map

$$\beta^d : K_{\text{geom}}^{-i}(U; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-i-2d}(U; \mathbb{Z}/n)$$

factors through $K_{2d+i}^{\text{BM}}(U; \mathbb{Z}/n) \cong K_{2d+i}^{\text{BM}}(U; \mathbb{Z}/n)[1/\beta]$ which implies $\ker(\beta^d) = \ker(\beta^{d+1}) = \ker(\beta^{d+2}) = \dots$, and thus

$$\begin{aligned} & \ker(K_{\text{geom}}^{-i}(U; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-i}(U; \mathbb{Z}/n)[1/\beta]) \\ &= \ker\left(K_{\text{geom}}^{-i}(U; \mathbb{Z}/n) \xrightarrow{\beta^d} K_{\text{geom}}^{-i-2d}(U; \mathbb{Z}/n)\right), \end{aligned}$$

for all $i \geq 0$. □

We close with the following conjecture.

Conjecture 10.5. Let U be a smooth, quasi-projective variety of pure dimension d defined over an algebraically closed field k and assume $n \geq 1$ is an integer relatively prime to the exponential characteristic of k . Then the map

$$\gamma_U : K_p^{\text{BM}}(U; \mathbb{Z}/n) \rightarrow K_{\text{geom}}^{-p}(U; \mathbb{Z}/n)$$

is a surjection for $p \geq 2d$ (and hence an isomorphism by Theorem 10.4).

In light of Theorem 10.4, Conjecture 10.5 implies (a slightly weakened form of) the Quillen-Lichtenbaum conjecture. Namely, it implies that the natural map

$$K_*(U; \mathbb{Z}/n) \rightarrow K_*(U; \mathbb{Z}/n)[1/\beta]$$

is an isomorphism in degrees $* \geq 2 \dim(U)$. (A modern way to express the strong form of the Quillen-Lichtenbaum conjecture is that this map is an isomorphism in degrees $* \geq \dim(U) - 1$.)

APPENDIX A. THE FULTON-MACPHERSON AXIOMS

In this appendix, we give a complete proof of Theorem 3.10, showing that the collection of graded abelian groups $K_{\text{geom}}^*(Y \rightarrow X)$ and $K_{\text{gqf}}^*(Y \rightarrow X)$ define “skew-commutative bivariant theories with units” in the sense of Fulton-MacPherson [10, 2.2]. Recall that part of the definition of a bivariant theory involves specifying classes of “independent squares” and “confined morphisms”. For each of the two theories considered here, the class of independent squares consists of all pullback squares. For K_{geom}^* every morphism is a confined morphism, while for K_{gqf}^* the confined morphisms are the proper morphisms. For the convenience of the reader, we list the Fulton-MacPherson axioms in full detail below.

Throughout this appendix, we write

$$\mathcal{P}(Y \rightarrow X), \quad \mathcal{K}(Y \rightarrow X), \quad \mathcal{K}_{\text{g}}(Y \rightarrow X),$$

for either of the two triples

$$\mathcal{P}^{\text{fin}}(Y \rightarrow X), \quad \mathcal{K}^{\text{fin}}(Y \rightarrow X), \quad \mathcal{K}_{\text{geom}}(Y \rightarrow X),$$

or

$$\mathcal{P}^{qf}(Y \rightarrow X), \quad \mathcal{K}^{qf}(Y \rightarrow X), \quad \mathcal{K}_{gqf}(Y \rightarrow X).$$

Moreover, we let $K^*(Y \rightarrow X)$ denote either

$$K_{geom}^*(Y \rightarrow X; \mathbb{Z}/n) \quad \text{or} \quad K_{gqf}^*(Y \rightarrow X; \mathbb{Z}/n),$$

where n is an integer (possibly 0) satisfying (Condition 2-3). The terms “independent square” and “confined morphism” will be used as indicated above.

The axioms required of the classes of independent squares and confined morphisms are given in [10, 2.1] and will not be recalled here, but the reader may readily verify that in each of the two cases above, these axioms are satisfied. The Fulton-MacPherson axioms of a bivariant theory assert that the pushforward maps (along confined morphisms), pullback maps (along arbitrary morphisms), and pairings (for arbitrary pairs of composable morphisms) given in (3) and (4) above satisfy the following conditions:

(A1) Associativity of Product: Given morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

and elements

$$\alpha \in K^p(X \rightarrow Y), \quad \beta \in K^q(Y \rightarrow Z), \quad \text{and } \gamma \in K^r(Z \rightarrow W),$$

we have

$$(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma) \in K^{p+q+r}(Z \rightarrow W).$$

(A2) Functoriality of Pushforward: Given morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

with f and g confined and an element $\alpha \in K^p(X \rightarrow W)$, we have

$$(gf)_*(\alpha) = g_*(f_*(\alpha)) \in K^p(Z \rightarrow W).$$

(A3) Functoriality of Pullback: Given a commutative diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ f'' \downarrow & & f' \downarrow & & f \downarrow \\ Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

consisting of two independent squares and an element $\alpha \in K^p(X \rightarrow Y)$, we have

$$(gh)^*(\alpha) = h^*(g^*(\alpha)) \in K^p(X'' \rightarrow Y'').$$

(A12) Commutativity of Pushforward with Product: Given morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

with f confined and given $\alpha \in K^p(X \rightarrow Z)$ and $\beta \in K^q(Z \rightarrow W)$, we have

$$f_*(\alpha \bullet \beta) = f_*(\alpha) \bullet \beta \in K^{p+q}(Y \rightarrow W).$$

(A₁₃) Commutativity of Pullback with Product: Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ f' \downarrow & & f \downarrow \\ Y' & \xrightarrow{h'} & Y \\ g' \downarrow & & g \downarrow \\ Z' & \xrightarrow{h} & Z \end{array}$$

of independent squares and given $\alpha \in K^p(X \rightarrow Y)$ and $\beta \in K^q(Y \rightarrow Z)$, we have

$$h^*(\alpha \bullet \beta) = (h')^*(\alpha) \bullet h^*(\beta) \in K^{p+q}(X' \rightarrow Z').$$

(A₂₃) Commutativity of Pullback with Pushforward: Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ f' \downarrow & & f \downarrow \\ Y' & \xrightarrow{h'} & Y \\ g' \downarrow & & g \downarrow \\ Z' & \xrightarrow{h} & Z \end{array}$$

of independent squares with f confined and given $\alpha \in K^p(X \rightarrow Z)$, we have

$$f'_*(h^*(\alpha)) = h^*f_*(\alpha) \in K^p(Y' \rightarrow Z').$$

(A₁₂₃) Projection Formula: Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & f \downarrow \\ Y' & \xrightarrow{g} & Y \xrightarrow{h} Z \end{array}$$

with an independent square and g confined and given $\alpha \in K^p(X \rightarrow Y)$ and $\beta \in K^q(Y' \rightarrow Z)$, we have

$$g'_*(g^*(\alpha) \bullet \beta) = \alpha \bullet g_*(\beta) \in K^{p+q}(X \rightarrow Z).$$

In addition, the bivariant theory K^* has units and is skew-commutative provided the following axioms hold:

(Units) For any X in Sch/k , there is an element $1_X \in K^0(X \rightarrow X)$ such that

$$\alpha \bullet 1_X = \alpha \quad \text{and} \quad 1_X \bullet \beta = \beta$$

for all $\alpha \in K^*(W \rightarrow X)$ and $\beta \in K^*(X \rightarrow Y)$ and such that

$$g^*(1_X) = 1_{X'}$$

for any morphism $g : X' \rightarrow X$.

(Skew-commutativity) For any independent square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & Y \end{array}$$

and any elements $\alpha \in K^m(X \rightarrow Z)$ and $\beta \in K^n(Y \rightarrow Z)$, we have

$$g^*(\alpha) \bullet \beta = (-1)^{mn} f^*(\beta) \bullet \alpha.$$

Proof of Theorem 3.10. Axiom (A1) follows immediately from Proposition 3.9 and the fact that the mod- n Moore space admits a suitable coproduct rule for such values of n . Axioms (A2) and (A3) follow immediately from Propositions 3.6 and 3.8.

To prove Axiom (A₁₂), we use the fact that the diagram

$$\begin{array}{ccc} \mathcal{P}(X \rightarrow Z) \times \mathcal{P}(Z \rightarrow W) & \longrightarrow & \mathcal{P}(X \rightarrow W) \\ \downarrow & & \downarrow \\ \mathcal{P}(Y \rightarrow Z) \times \mathcal{P}(Z \rightarrow W) & \longrightarrow & \mathcal{P}(Y \rightarrow W) \end{array}$$

of exact functors commutes up to the canonical isomorphisms

$$f_*(M \otimes_{\mathcal{O}_X} (g \circ f)^* N) \cong f_*(M \otimes_{\mathcal{O}_X} f^*(g^* N)) \cong f_* M \otimes_{\mathcal{O}_Y} g^* N$$

and that each of the two possible compositions in this square yields a biexact functor. (Recall from Remark 3.2 and the proof of Proposition 3.9 that the categories and functors appearing in this square can be made strictly functorial with respect to pullback along W .) Thus, for each $d \geq 0$, there is a homotopy joining the two possible maps of the form

$$\mathcal{K}(X \times \Delta^d \rightarrow Z \times \Delta^d) \wedge \mathcal{K}(Z \times \Delta^d \rightarrow W \times \Delta^d) \rightarrow \mathcal{K}(Y \times \Delta^d \rightarrow W \times \Delta^d),$$

and the canonical nature of the isomorphisms inducing these homotopies implies that they are compatible with the simplicial structure. Hence, there is a homotopy relating the two induced maps of simplicial spaces of the form

$$(d \mapsto \mathcal{K}(X \times \Delta^d \rightarrow Z \times \Delta^d) \wedge \mathcal{K}(Z \times \Delta^d \rightarrow W \times \Delta^d)) \rightarrow (d \mapsto \mathcal{K}(Y \times \Delta^d \rightarrow W \times \Delta^d)),$$

from which Axiom (A₁₂) follows.

To establish Axiom (A₁₃), note that the square

$$\begin{array}{ccc} \mathcal{P}(X \rightarrow Y) \times \mathcal{P}(Y \rightarrow Z) & \longrightarrow & \mathcal{P}(X \rightarrow Z) \\ \downarrow & & \downarrow \\ \mathcal{P}(X' \rightarrow Y') \times \mathcal{P}(Y' \rightarrow Z') & \longrightarrow & \mathcal{P}(X' \rightarrow Z') \end{array}$$

commutes up to the canonical isomorphisms

$$(h'')^*(M \otimes_{\mathcal{O}_X} f^* N) \cong (h'')^*(M) \otimes_{\mathcal{O}_{X'}} (h'')^* f^* N \cong (h'')^*(M) \otimes_{\mathcal{O}_{X'}} (f')^*(h')^* N,$$

and that each of the two possible compositions yields a biexact functor. As before, this leads to a homotopy joining the two possible maps of the form

$$\mathcal{K}_g(X \rightarrow Y) \wedge \mathcal{K}_g(Y \rightarrow Z) \rightarrow \mathcal{K}_g(X' \rightarrow Z').$$

To establish Axiom (A₂₃), note that

$$\begin{array}{ccc} \mathcal{P}(X \rightarrow Z) & \longrightarrow & \mathcal{P}(X' \rightarrow Z') \\ \downarrow & & \downarrow \\ \mathcal{P}(Y \rightarrow Z) & \longrightarrow & \mathcal{P}(Y' \rightarrow Z') \end{array}$$

commutes up to the canonical isomorphism

$$f'_*((h'')^*M) \cong h'^*(f_*M),$$

which holds since the higher derived functors of f_* and f'_* vanish and since the higher Tor sheaves $\text{Tor}_*^{\mathcal{O}_X}(\mathcal{O}_{X'}, -)$ and $\text{Tor}_*^{\mathcal{O}_Y}(\mathcal{O}_{Y'}, -)$ vanish for objects in $\mathcal{P}(X \rightarrow Y)$. (The vanishing of the higher derived functors of f_* and f'_* holds since objects in $\mathcal{P}(X \rightarrow Z)$ (respectively, $\mathcal{P}(X' \rightarrow Z')$) have support finite over Y (respectively, Y'). Note that in the case $\mathcal{P} = \mathcal{P}^{qf}$, the finiteness of supports holds since the supports are quasi-finite and f and f' are assumed proper.) As before, such canonical isomorphisms establish the homotopy commutativity of

$$\begin{array}{ccc} \mathcal{K}_g(X \rightarrow Z) & \longrightarrow & \mathcal{K}_g(X' \rightarrow Z') \\ \downarrow & & \downarrow \\ \mathcal{K}_g(Y \rightarrow Z) & \longrightarrow & \mathcal{K}_g(Y' \rightarrow Z'). \end{array}$$

To establish Axiom (A₁₂₃), we use the fact that the diagram of functors

$$\begin{array}{ccccc} \mathcal{P}(X \rightarrow Y) \times \mathcal{P}(Y' \rightarrow Z) & \longrightarrow & \mathcal{P}(X' \rightarrow Y') \times \mathcal{P}(Y' \rightarrow Z) & \longrightarrow & \mathcal{P}(X' \rightarrow Z) \\ \downarrow & & & & \downarrow \\ \mathcal{P}(X \rightarrow Y) \times \mathcal{P}(Y \rightarrow Z) & \longrightarrow & \mathcal{P}(X \rightarrow Z) & \xrightarrow{=} & \mathcal{P}(X \rightarrow Z) \end{array}$$

commutes up to a canonical isomorphism by using the chain of canonical isomorphisms

$$(g')_*((g')^*M \otimes_{\mathcal{O}_X} (f')^*N) \cong M \otimes_{\mathcal{O}_X} g'_*(f')^*N \cong M \otimes_{\mathcal{O}_X} f^*g_*(N).$$

As before, this chain of isomorphisms exists since all relevant higher derived images and Tor modules vanish. This gives the homotopy commutativity of the associated diagram of spaces

$$\begin{array}{ccccc} \mathcal{K}_g(X \rightarrow Y) \wedge \mathcal{K}_g(Y' \rightarrow Z) & \longrightarrow & \mathcal{K}_g(X' \rightarrow Y') \wedge \mathcal{K}_g(Y' \rightarrow Z) & \longrightarrow & \mathcal{K}_g(X' \rightarrow Z) \\ \downarrow & & & & \downarrow \\ \mathcal{K}_g(X \rightarrow Y) \wedge \mathcal{K}_g(Y \rightarrow Z) & \longrightarrow & \mathcal{K}_g(X \rightarrow Z) & \xrightarrow{=} & \mathcal{K}_g(X \rightarrow Z) \end{array}$$

We have thus shown that both \mathcal{K}_{geom}^* and \mathcal{K}_{gqf}^* determine bivariant theories. The unit elements for each theory are given by the classes $[\mathcal{O}_X] \in K_g^0(X \rightarrow X; \mathbb{Z}/n) \cong K_{geom}^0(X; \mathbb{Z}/n)$, which are readily seen to satisfy the required properties. To establish skew-commutativity, note that the square

$$\begin{array}{ccccc} \mathcal{P}(X \rightarrow Z) \times \mathcal{P}(Y \rightarrow Z) & \longrightarrow & \mathcal{P}(W \rightarrow Y) \times \mathcal{P}(Y \rightarrow Z) & \xrightarrow{=} & \mathcal{P}(W \rightarrow Y) \times \mathcal{P}(Y \rightarrow Z) \\ \downarrow & & & & \downarrow \\ \mathcal{P}(X \rightarrow Z) \times \mathcal{P}(W \rightarrow X) & \xrightarrow{\tau} & \mathcal{P}(W \rightarrow X) \times \mathcal{P}(X \rightarrow Z) & \longrightarrow & \mathcal{P}(W \rightarrow Z) \end{array}$$

commutes up to the canonical isomorphism

$$(g')^*M \otimes_{\mathcal{O}_W} (f')^*N \cong (f')^*N \otimes_{\mathcal{O}_W} (g')^*(M),$$

where τ is the evident transposition. As before, this gives the homotopy commutative diagram of spaces

$$\begin{array}{ccccc} \mathcal{K}_g(X \rightarrow Z) \wedge \mathcal{K}_g(Y \rightarrow Z) & \longrightarrow & \mathcal{K}_g(W \rightarrow Y) \wedge \mathcal{K}_g(Y \rightarrow Z) & \xrightarrow{=} & \mathcal{K}_g(W \rightarrow Y) \wedge \mathcal{K}_g(Y \rightarrow Z) \\ \tau \downarrow & & & & \downarrow \\ \mathcal{K}_g(W \rightarrow X) \wedge \mathcal{K}_g(X \rightarrow Z) & \longrightarrow & \mathcal{K}_g(X \rightarrow Z) \wedge \mathcal{K}_g(W \rightarrow X) & \longrightarrow & \mathcal{K}_g(W \rightarrow Z). \end{array}$$

The desired formula follows. (On the level of homotopy classes represented by maps $\alpha : S^m \rightarrow \mathcal{K}_g(X \rightarrow Z)$ and $\beta : S^n \rightarrow \mathcal{K}_g(Y \rightarrow Z)$, the map τ introduces a sign of $(-1)^{mn}$ since the endomorphism of S^{m+n} given as the composition of

$$S^{m+n} \cong S^m \wedge S^n \cong S^n \wedge S^m \cong S^{n+m}$$

has degree $(-1)^{mn}$.)

The inclusion of $\mathcal{P}^{fin}(X \rightarrow Y)$ into $\mathcal{P}^{qf}(X \rightarrow Y)$ is clearly compatible with the pushforward, pullback, and product operations defined in Propositions 2.3 and 2.4, and thus the collection of maps

$$K_{geom}^*(X \rightarrow Y) \rightarrow K_{gqf}^*(X \rightarrow Y)$$

determines a Grothendieck transformation of bivariant theories (cf. [10, 2.7.1]). \square

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