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RATIONALITY, REGULARITY, AND C_2-COFINITENESS

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Abstract. We demonstrate that, for vertex operator algebras of CFT type, C_2-cofiniteness and rationality is equivalent to regularity. For C_2-cofinite vertex operator algebras, we show that irreducible weak modules are ordinary modules and C_2-cofinite, V_L^+ is C_2-cofinite, and the fusion rules are finite.

1. Introduction

One of the most important conjectures in the theory of vertex operator algebra is the equivalence of rationality and C_2-cofiniteness. Rationality tells us that the category of admissible modules is semisimple (see Section 2). The C_2 condition is slightly more technical and deals with the co-dimension of a certain subspace of V. In the case of finite-dimensional Lie algebras, we have an internal characterization for semisimplicity. Namely, the maximum solvable radical is zero. This condition implies the all the finite-dimensional modules of such a Lie algebra will be completely reducible. For vertex operator algebras, C_2-cofiniteness is the conjectured internal condition that implies complete reducibility of modules.

The evidence for this conjecture is overwhelming. Independently, rationality and C_2-cofiniteness both imply that the number of irreducible admissible modules for a vertex operator algebra is finite [16], [6]. Also independently, these two notions imply that irreducible admissible modules are irreducible ordinary modules [16], [6]. Rationality implies that Zhu's algebra is finite dimensional and semisimple [19], [6], while C_2-cofiniteness implies that Zhu's algebra is finite dimensional. Also well-known rational vertex operator algebras are C_2-cofinite. It is not surprising that a lot of good results in the theory of vertex operator algebras need both rationality and C_2-cofiniteness (cf. [7], [9], [19]).

The notion of regularity is given in [5] as a generalization of rationality to weak modules for vertex operator algebras. Regularity says that any weak module is a direct sum of irreducible ordinary modules. It is proved in [5] that rational vertex operator algebras associated to highest weight modules for affine Kac-Moody algebras, Virasoro algebra, and positive definite even lattices are regular. Based on these results a stronger conjecture is proposed in [5]. That is, rationality, C_2-cofiniteness, and regularity are all equivalent. It is proved in [16] that regularity implies C_2 co-finiteness. Also, by definition regularity implies rationality.

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In this paper we prove that C_2-cofiniteness and rationality together imply regularity. The main idea in the proof of this result is to find a "lowest weight vector" in any weak module. In the case of rational affine, Virasoro, and lattice vertex operator algebras, either the singular vectors in the Verma modules or the lattice itself plays the crucial role in the search of lowest weight vectors [5]. It turns out that the right analogue of singular vectors in an arbitrary vertex operator algebra is C_2-cofiniteness. In the case of affine or Virasoro vertex operator algebras, the existence of singular vectors and C_2-cofiniteness are equivalent. The PBW type spanning set for weak modules obtained in [4] is the key for us to obtain a lowest weight vector.

The plan for this paper is as follows. In Section 2 we will fix notation and give some basic definitions. In particular we define various notions for modules for a vertex operator algebra. In Section 3, we review several results concerning the C_2 subspace of vertex operator algebras. Section 4 is the proof of our main result. In Section 5 we give some additional results about C_2-cofinite vertex operator algebras. In particular, we prove that V_L^+ is C_2-cofinite for any positive definite even lattice L, extending a recent result in [18].

We make the assumption that the reader is somewhat familiar with the theory of vertex operator algebras (VOAs). We assume the definition of a vertex operator algebra as well as some basic properties. For more information see [12], [13].

2. Modules for vertex operator algebras

In this section we recall the various notions of modules for a vertex operator algebra, and we also define the terms C_2-cofinite, rationality and regularity (see [5], [19]).

Throughout this paper, we will work under the assumption that V is of CFT type. That is, $V = \bigoplus_{n\geq 0} V_n$ and $V_0 = \mathbb{C}\mathbf{1}$.

Definition 2.1. V, a VOA, is called *C_n-cofinite* for $n \geq 2$ if $V/C_n(V)$ is finite dimensional, where $C_n(V) = \{v_{-n}w \mid v, w \in V\}$.

When $n = 2$, the hypothesis that V is C_2-cofinite is sometime referred to as Zhu's finiteness condition. This C_2 condition appears in [19], as one of the conditions needed to prove the modularity of certain trace functions. In later work [15], it was shown that C_2-cofiniteness is equivalent to C_n-cofiniteness for all $n \geq 2$. If we were to define $C_1(V)$ as in the previous definition, then $C_1(V) = V$, because of the creation axiom for VOAs. Li used an alternate definition for $C_1(V)$ in [16].

Definition 2.2. V, a VOA, is called *C_1-cofinite* if $V/C_1(V)$ is finite dimensional, where $C_1(V) = \{u_n v, L(-1)u : u, v \in \bigoplus_{n\geq 1} V_n\}$

Since $L(-1)u = (L(-1)u)_{-1}\mathbf{1} = u_{-2}\mathbf{1}$ we see immediately that $C_2(V) \subset C_1(V)$. So if V is C_2-cofinite, then V is C_1-cofinite.

Rationality and regularity are two different formulations of complete reducibility of VOA modules. In order to define these terms, we must first describe three different types of vertex operator algebra modules.

Definition 2.3. A *weak V-module* is a vector space M with a linear map

$$Y_M : \quad V \to \mathrm{End}(M)[[z, z^{-1}]], \tag{2.1}$$

$$v \mapsto Y_M(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, \quad v_n \in \mathrm{End}(M). \tag{2.2}$$

In addition Y_M satisfies the following:
1) $v_n w = 0$ for $n >> 0$ where $v \in V$ and $w \in M$.
2) $Y_M(\mathbf{1}, z) = Id_M$.
3) The Jacobi Identity holds:

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta(\frac{z_2 - z_1}{-z_0})Y_M(v, z_2)Y_M(u, z_1)$$
$$= z_2^{-1}\delta(\frac{z_1 - z_0}{z_2})Y_M(Y(u, z_0)v, z_2). \tag{2.3}$$

Of the three types of modules we mention, only weak modules have no grading assumptions. Two important consequences of this definition are that weak modules admit a Virasoro representation and satisfy the $L(-1)$ derivation property [12].

Definition 2.4. An *admissible V-module* is a weak V-module which carries a $\mathbb{Z}_+$-grading, $M = \bigoplus_{n\in\mathbb{Z}_+} M(n)$, such that if $v \in V_r$, then $v_m M(n) \subseteq M(n+r-m-1)$.

Definition 2.5. An *ordinary V-module* is a weak V-module which carries a $\mathbb{C}$-grading, $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, such that:
1) $dim(M_\lambda) < \infty$,
2) $M_{\lambda+n=0}$ for fixed λ and $n << 0$,
3) $L(0)w = \lambda w = \mathrm{wt}(w)w$, for $w \in M_\lambda$.

An ordinary module has a grading that matches the $L(0)$-action of the Virasoro representation as well as finite-dimensional graded pieces. The definition of of a $\mathbb{C}$-grading on ordinary modules may seem weaker than a $\mathbb{Z}$-grading on admissible modules. It turns out that the finite-dimensionality of graded pieces in ordinary modules is a strong condition, and ordinary modules are admissible [6]. So we have this set of inclusions:

$$\{\text{ordinary modules}\} \subseteq \{\text{admissible modules}\} \subseteq \{\text{weak modules}\}.$$

Definition 2.6. A vertex operator algebra is called *rational* if every admissible module is a direct sum of simple admissible modules.

That is, a VOA is rational if there is complete reducibility of admissible modules. It is proved in [6] that if V is rational, then there are only finitely many simple admissible modules up to isomorphism and any simple admissible module is an ordinary module.

Definition 2.7. A vertex operator algebra is called *regular* if every weak module is a direct sum of simple ordinary modules.

Regularity is a stronger form of complete reducibility of modules and was first introduced as a generalization of rationality to weak modules in [5]. The goal of this paper is to demonstrate the equivalence of rationality and regularity under the assumption of C_2 cofiniteness. In [16], it was shown that any regular VOA is C_2-cofinite. So by this and the definition of rationality, regularity implies both rationality and C_2-cofiniteness.

For a vertex operator algebra V and a weak V-module M, the weight of the operator v_n on M is defined as $\mathrm{wt}(v_n) = \mathrm{wt}(v) - n - 1$ if v is homogeneous. A vector $w \in M$ is a lowest weight vector if $v_n w = 0$ for any homogeneous $v \in V$ and $n \in \mathbb{Z}$ where $\mathrm{wt}(v_n) < 0$. The space of lowest weight vectors of a module M is denoted $\Omega(M)$. It is important to note that a lowest weight vector is not necessarily

a homogeneous vector, but it has lowest weight in the sense that an operator of negative weight acting on a lowest weight vector will kill it.

3. PBW type spanning set

In this section, we will talk about the important work leading up to the main theorem. In particular we review a result obtained in [4] constructing a spanning set of PBW type for weak modules.

Let $\{\bar{x}^\alpha\}_{\alpha\in I}$ be a basis of $V/C_2(V)$, where $\bar{x}^\alpha = x^\alpha + C_2(V)$, and x^α is a homogeneous vector. Then define $\bar{X} = \{x^\alpha\}_{\alpha\in I}$, which is a set of elements in V which are representatives of a basis for $V/C_2(V)$. We can simplify $\bar{X}$ slightly. The vacuum, $\mathbf{1}$, is not in $C_2(V)$, since the vacuum is killed by all v_n for $n < \mathrm{wt}(v) - 1$. This means we can choose $\bar{X}$ such that $\mathbf{1} \in \bar{X}$. In addition, $\mathbf{1}_n = 0$, the zero endomorphism on V, for all integers $n \neq -1$, and $\mathbf{1}_{-1}$ is the identity endomorphism on V. This ensures that removing the vacuum from $\bar{X}$, will have no effect on our spanning set. So now define $X = \bar{X} - \{\mathbf{1}\}$. This set X will be the elements of V that we will construct our spanning set from.

Theorem 3.1 ([15])**.** *V, a vertex operator algebra, is spanned by elements of the form*

$$x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} \mathbf{1},$$

where $n_1 > n_2 > \cdots > n_k > 0$ and $x^i \in X$ for $1 \leq i \leq k$.

It is important to note that for this theorem, we need not assume that V is C_2-cofinite. However the result is more interesting when V is C_2-cofinite, since the generating set is finite. Henceforth, we assume that V is C_2-cofinite.

The key feature of this vertex operator algebra spanning set is that each element satisfies a no-repeat condition. Since in the expression of a spanning set element the modes are strictly decreasing, each mode appears only once. The next result will generalize this result so that it applies to modules of vertex operator algebras. For the module spanning set we will not have a no-repeat condition, but we will have a finite-repeat condition. The number of allowed repetitions will depend on how large $V/C_2(V)$ is.

Remark 3.2. If V is C_2-cofinite, then for some $N > 0$, $\bigoplus_{i\geq N} V_i \subset C_2(V)$. In fact, $N = \max_{x\in X}\{\mathrm{wt}(x) + 1\}$.

Set $Q = 2N - 2$. Then Q is the maximum number a times a mode can repeat in an element of the module spanning set [4]. Let W be an irreducible weak V-module, and $w \in W$. Since X is a finite set there is a smallest non-negative integer L such that $x_m w = 0$ for all $x \in X$ and $m \geq L$.

Theorem 3.3 ([4])**.** *Let V be a C_2-cofinite vertex operator algebra, and let W be an weak V-module generated by $w \in W$. Then W is spanned by elements of the form*

$$x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} w,$$

where $x^i \in X$ for $1 \leq i \leq k$, subject to the following restrictions: $n_1 \geq n_2 \geq \cdots \geq n_k > -L$; if $n_j > 0$, then $n_j > n_{j'}$ for $j < j'$; and if $n_j \leq 0$ then $n_j = n_{j'}$ for at most $Q - 1$ indices, j'.

Since W is generated by w, W is spanned by elements of the form

$$v^1_{n_1} v^2_{n_2} \cdots v^k_{n_k} w,$$

where $v^i \in V$, and $n_i \in \mathbb{Z}$. This theorem tell us that we can restrict the v^i's to elements of X, and we can impose some conditions on the n_i's. These conditions say the following: For an element of this spanning set, all the modes are decreasing and strictly less than L. If the modes are negative, then they are strictly decreasing. If the modes are non-negative, then they are not strictly decreasing. There may be repeats of non-negative modes, but there are at most $Q-1$ repetitions. Here is a sort of picture of the mode restrictions:

$$\text{strictly decreasing} < 0 \leq Q-1 \text{ repetitions} < L.$$

The last result we need is the following.

Proposition 3.4 ([5]). *Let V be a rational vertex operator algebra such that any non-zero weak V-module contains a simple ordinary V-submodule. Then V is regular.*

This result is used in [5] to show that rational vertex operator algebras associated to the Virasoro algebra, affine Lie algebras, and lattices, including the moonshine module, are all regular. In a similar fashion, we use this proposition to show that any C_2-cofinite, rational vertex operator algebra is regular.

4. Main theorem

In this section we first show that in any weak module we can find a lowest weight vector. We then use this to show the main result: C_2-cofiniteness and rationality is equivalent to regularity.

Lemma 4.1. *Let $x \in X$ and $y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l} w$ be an element of the spanning set of M. There exist module spanning set elements, $y^{r_1}_{-m_{r_1}} y^{r_2}_{-m_{r_2}} \cdots y^{r_l}_{-m_{r_l}} w$, such that*

$$x_i y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l} w = \sum_{r \in R} c_r y^{r_1}_{-m_{r_1}} y^{r_2}_{-m_{r_2}} \cdots y^{r_l}_{-m_{r_l}} w,$$

where

$$\mathrm{wt}(x_i y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l}) = \mathrm{wt}(y^{r_1}_{-m_{r_1}} y^{r_2}_{-m_{r_2}} \cdots y^{r_l}_{-m_{r_l}})$$

and $c_r \in \mathbb{C}$ for all $r \in R$, R a finite index set.

Proof. First, we must look back to the proof of Theorem 3.3 [4]. In the proof, three identities are used to rearrange modes to put them in the proper form. To show that

$$\mathrm{wt}(x_i y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l}) = \mathrm{wt}(y^{r_1}_{-m_{r_1}} y^{r_2}_{-m_{r_2}} \cdots y^{r_l}_{-m_{r_l}})$$

we will show that the identities used to rewrite the expression

$$x_i y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l} w$$

as $\sum_{r \in R} c_r y^{r_1}_{-m_{r_1}} y^{r_2}_{-m_{r_2}} \cdots y^{r_l}_{-m_{r_l}} w$ preserve the weight of the operators.

The first of the identities we need to look at is

$$[u_{-k}, v_{-q}] = \sum_{i \geq 0} \binom{-k}{i} (u_i v)_{-k-q-i}, \tag{4.1}$$

where $u, v \in V$ and $k, q \in \mathbb{Z}$. The second identity is

$$\begin{aligned}(u_{-r}v)_{-q} &= \sum_{i\geq 0}(-1)^i\binom{-r}{i}u_{-r-i}v_{-q+i}\\ &\quad -\sum_{i\geq 0}(-1)^{i-r}\binom{-r}{i}v_{-r-q-i}u_i,\end{aligned} \tag{4.2}$$

where $u, v \in V$ and $r, q \in \mathbb{Z}$. The third identity is more complicated. Let $x^1_{-1}\cdots x^Q_{-1}\mathbf{1} = \sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}}\cdots x^{r_l}_{-n_{r_l}}\mathbf{1}$, where $x^i, x^{r_t} \in X$ for $1 \leq i \leq Q$ and $1 \leq t \leq l$, $l < Q$, and $n_{r_1} > n_{r_2} > \cdots > n_{r_l} > 0$ for fixed r. Then

$$\begin{aligned}&x^Q_{L-k}x^{Q-1}_{L-k}\cdots x^1_{L-k}\\ &= Res_z\{Y(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}}\cdots x^{r_l}_{-n_{r_l}}\mathbf{1}, z)z^{Q(-L-1+k)-1}\}\\ &\quad -\sum_{i=1}^{Q}\sum_{\lambda\in\Lambda^i_Q}\sum_{m_i\geq 0}(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}})(\prod_{j=i+1}^{Q} x^{\bar{\lambda}_j}_{m_{\bar{\lambda}_j}})\end{aligned} \tag{4.3}$$

$$-\sum_{m_j\geq 0, 1\leq j\leq Q} x^Q_{m_Q}x^{Q-1}_{m_{Q-1}}\cdots x^1_{m_1}, \tag{4.4}$$

where in (4.3), $\sum_{j=1}^{i}(-1-m_{\lambda_j}) + \sum_{j=i+1}^{Q}(m_{\bar{\lambda}_j}) = Q(L-k)$. Also, in (4.4), we have $\sum_{j=1}^{Q} m_j = Q(L-k)$ and $m_j \neq L-k$ for some j. It is clear that the both sides of the three identities have the same weights. □

Recall that $C_2(V) \subset C_1(V)$, so if V is C_2-cofinite, then V is C_1-cofinite. Let $Y \subset V$ be a set of homogeneous coset representatives of $V/C_1(V)$.

Lemma 4.2. *Let M be a weak module for a vertex operator algebra V. Then we have*

$$\Omega(M) = \{w \in M | y_m w = 0, y \in Y, \mathrm{wt}(y_m) < 0\}.$$

Proof. Let $w \in M$ such that $y_m w = 0$ if $\mathrm{wt}(y_m) < 0$ for $y \in Y$. We prove by induction on $\mathrm{wt}(v)$ that $v_m w = 0$ if $\mathrm{wt}(v_m) < 0$ for homogeneous $v \in V$. If $v \in V_0 \oplus V_1$, then v is in the span of Y. The result is clear.

Now we assume that $\mathrm{wt}(v) > 1$. In fact we can assume that $v \in C_1(V)$. Then $v = \sum_{i=1}^{s} u^i_{-1}v^i + L(-1)u$ for some homogeneous $u^i, v^i, u \in \sum_{j\geq 1} V_j$. Since $(L(-1)u)_m = -mu_{m+1}$ and $\mathrm{wt}(u) < \mathrm{wt}(v)$ it is by induction assumption that $(L(-1)u)_m w = 0$ when $\mathrm{wt}((L(-1)u)_m) < 0$. So it is enough to show that for homogeneous $a, b \in V$, $(a_{-1}b)_m w = 0$ if $\mathrm{wt}(a), \mathrm{wt}(b) < \mathrm{wt}(v)$ and $\mathrm{wt}((a_{-1}b)_m) < 0$. For short we set $p = \mathrm{wt}(a)$ and $q = \mathrm{wt}(b)$. Then $\mathrm{wt}(a_{-1}b) = p + q$ and $\mathrm{wt}((a_{-1}b)_m) = p + q - m - 1$. So $\mathrm{wt}((a_{-1}b)_m) < 0$ if and only if $m \geq p + q$.

Let $m \geq p + q$. By (4.2) we see that

$$(a_{-1}b)_m = \sum_{i\geq 0} a_{-1-i}b_{m+i} + \sum_{i\geq 0} b_{-1+m-i}a_i.$$

Since $m \geq p + q$, $b_{m+i}w = 0$ for all $i \geq 0$ by the induction assumption. Also if $i \geq p$, then $\mathrm{wt}(a_i) < 0$ and $a_i w = 0$. So

$$(a_{-1}b)_m w = \sum_{i=0}^{p-1} b_{-1+m-i}a_i w.$$

By (4.1) we have

$$b_{-1+m-i}a_iw = a_ib_{-1+m-i}w + \sum_{t\geq 0}\binom{-1+m-i}{t}(b_ta)_{-1+m-t}w.$$

Since $i < p$, $\mathrm{wt}(b_{-1+m-i}) < 0$, we conclude that $a_ib_{-1+m-i}w = 0$. Note that $\mathrm{wt}(b_ta) < p+q = \mathrm{wt}(v)$ for $t \geq 0$ and $\mathrm{wt}((b_ta)_{-1+m-t}) = \mathrm{wt}((a_{-1}b)_m) < 0$. Again by induction assumption, $(b_ta)_{-1+m-t}w = 0$. The proof is complete. □

Let $w \in M$, consider the submodule W generated by w. By Theorem 3.3, W is spanned by elements of the form

$$y^1_{-m_1}y^2_{-m_2}\cdots y^l_{-m_l}w$$

with $y^i \in X$ for $1 \leq i \leq l$, and $m_1 \geq m_2 \geq \cdots \geq m_l > -L$. In addition, if $m_j > 0$, then $m_j > m_{j'}$ for $j < j'$, and if $m_j \leq 0$ then $m_j = m_{j'}$ for at most $Q-1$ indices, j'.

Now these repetition restrictions allow for only finitely many non-negative modes. Operators with large enough positive modes have negative weight. The idea here is that we can "push down" w only so far. So we make the following definition.

Definition 4.3. $B = min\{\mathrm{wt}(y^1_{-m_1}y^2_{-m_2}\cdots y^l_{-m_l})\}$, where $y^1_{-m_1}y^2_{-m_2}\cdots y^l_{-m_l}w$ is a non-zero spanning set element.

So this B is the furthest we can "push down" w without killing w. Again we know that there is a minimal weight, because the repetition restrictions only allow for finitely many positive modes. Next we look at how VOA spanning set elements act on module spanning set elements.

Lemma 4.4. *Let V be C_2-cofinite. Then any weak V-module has a non-zero lowest weight vector.*

Proof. Let $v = y^1_{-m_1}\cdots y^l_{-m_l}w$ be a non-zero module spanning set element such that $\mathrm{wt}(y^1_{-m_1}\cdots y^l_{-m_l}) = B$. We show that v lies in $\Omega(M)$. By Lemma 4.2 we only need to prove that $x_mv = 0$ for $x \in X$ and $m \geq \mathrm{wt}(x)$.

By Lemma 4.1

$$x_mv = \sum_{r\in R} y^{r_1}_{-m_{r_1}}y^{r_2}_{-m_{r_2}}\cdots y^{r_l}_{-m_{r_l}}w,$$

where we have

$$\mathrm{wt}(y^{r_1}_{-m_{r_1}}y^{r_2}_{-m_{r_2}}\cdots y^{r_l}_{-m_{r_l}}) = \mathrm{wt}(x_{m-n+1}y^1_{-m_1}y^2_{-m_2}\cdots y^l_{-m_l}) < B.$$

This means that $y^{r_1}_{-m_{r_1}}y^{r_2}_{-m_{r_2}}\cdots y^{r_l}_{-m_{r_l}}w = 0$ for all $r \in R$ and $x_mv = 0$. □

We are now in a position to prove the main theorem of this paper.

Theorem 4.5. *A vertex operator algebra, V, of CFT type is regular if and only if V is C_2-cofinite and rational.*

Proof. Regularity implies rationality by definition and C_2-cofiniteness by the work of Li [16]. So one direction is already known. We only need to prove that if V is rational and C_2-cofinite, then V is regular. From Proposition 3.4 it is enough to prove that any weak module M contains an irreducible ordinary module. By Lemma 4.4, $\Omega(M)$ is not empty.

In order to finish the proof we need to recall the theory of the associative algebra $A(V)$ (cf. [6] and [19]). For homogeneous $u, v \in V$, we define products $u * v$ and $u \circ v$ as follows:

$$u * v = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(u)}}{z} Y(u,z)v \right) = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(u)}{i} u_{i-1} v,$$

$$u \circ v = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(u)}}{z^2} Y(u,z)v \right) = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(u)}{i} u_{i-2} v. \tag{4.5}$$

Extend (4.5) to linear products on V, and let $O(V)$ be the linear span of $u \circ v$ for $u, v \in V$. Set $A(V) = V/O(V)$. Then $A(V)$ is an associative algebra under multiplication $*$ and with identity $\mathbf{1}+O(V)$ and central element $\omega+O(V)$. Moreover, $\Omega(M)$ is an $A(V)$-module such that $u+O(V)$ acts as $o(u)$, where $o(u) = u_{\mathrm{wt}(u)-1}$ if u is homogeneous. Since our V is rational, $A(V)$ is a finite-dimensional semisimple associative algebra.

Since $A(V)$ is semisimple, we can choose a simple $A(V)$-submodule Z of $\Omega(M)$. Then the V-submodule generated by Z is an ordinary irreducible module (see [6] and [19]). □

5. C_2-cofinite vertex operator algebras

In this section we study weak modules for a C_2-cofinite vertex operator algebra. The result in Section 4 on the existence of lowest weight vectors in weak modules allow us to prove that weak modules are admissible. We also prove that V_L^+ is C_2-cofinite for any even positive definite lattice L.

Definition 5.1. Let W be a weak V-module. Then define $C_n(W) = \{u_{-n}w \mid u \in V, w \in W\}$ for $n > 0$. We say that W is *C_n-cofinite* if $\dim(W/C_n(W)) < \infty$.

Since there is no creation axiom for VOA modules, we do not have to worry about a special case for $n = 1$.

Proposition 5.2. *If V is C_2-cofinite and W is an irreducible weak V-module, then W is C_2-cofinite.*

Proof. Let $w \in W$. By Theorem 3.3, W is spanned by elements of the form $y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l} w$, with the modes subject to some repetition restrictions and $y^i \in X$. Consider a module spanning set element with $\mathrm{wt}(y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l}) > N$. If N is large enough, the mode restrictions given in Theorem 3.3 ensure that $m_1 \geq 2$. Thus $y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l} w \in C_2(W)$. Recall Definition 4.3. Given $y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l} w$, a modules spanning set element, B is the minimum weight of $y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l}$ such that the element is non-zero. Now $W/C_n(W)$ is spanned by module spanning set elements of the form

$$y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l} w + C_2(V),$$

where $B \leq \mathrm{wt}(y^1_{-m_1} y^2_{-m_2} \cdots y^l_{-m_l}) \leq N$. This is a finite set because of the mode restrictions on the module spanning set. □

Next we use the result in Proposition 5.2 together with a recent result in [18] to prove that V_L^+ is C_2-cofinite for any positive definite even lattice L. Before we can prove this result, we need a lemma about the tensor product of C_2-cofinite VOAs.

Lemma 5.3. *If vertex operator algebras $V^1, \dots, V^n$ are C_2-cofinite, then $V^1 \otimes \cdots \otimes V^n$ is C_2-cofinite.*

Proof. In order to see this, we write $V^i = C_2(V) + W^i$ for a finite-dimensional subspace W^i. It is obvious that $V^1 \otimes \cdots V^{i-1} \otimes C_2(V^i) \otimes V^{i+1} \otimes \cdots V^n$ is contained in $C_2(V^1 \otimes \cdots \otimes V^n)$. This shows that $V^1 \otimes \cdots \otimes V^n = C_2(V^1 \otimes \cdots \otimes V^n) + W^1 \otimes \cdots \otimes W^n$. Thus $V^1 \otimes \cdots \otimes V^n$ is C_2-cofinite, as desired. □

We recall from [3] and [13] the vertex operator algebra V_L associated to an even positive definite lattice L. So L is a free Abelian group of finite rank with a positive definite $\mathbb{Z}$-bilinear form $(,)$ such that $(\alpha, \alpha) \in 2\mathbb{Z}$ for $\alpha \in L$. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the corresponding affine Lie algebra. Let $M(1) = \mathbb{C}[h(-n) | h \in \mathfrak{h}, n > 0]$ be the unique irreducible module for $\hat{\mathfrak{h}}$ such that c acts as 1 and $\mathfrak{h} \otimes t^0$ acts trivially. Then as a vector space,

$$V_L = M(1) \otimes \mathbb{C}[L],$$

where $\mathbb{C}[L]$ is the group algebra of L. Then V_L is a rational vertex operator algebra.

Let $\theta : V_L \to V_L$ be the order 2 automorphism such that

$$\theta(\alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^{\alpha}) = (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^{-\alpha}$$

for $\alpha_i \in \mathfrak{h}, \alpha \in L$ and $n_i < 0$. Let $V_L^{\pm}$ be the eigenspaces of θ with eigenvalues ± 1. Then V_L^+ is a simple vertex operator algebra and V_L^- is an irreducible V_L^+-module.

Theorem 5.4. *V_L^+ is C_2-cofinite.*

Proof. In the case that the rank of L is one, this result has been proved recently in [18]. Let the rank of L be n. Then there exists a sublattice K of L such that the rank K is n and K is a direct sum of n orthogonal rank one lattices $L_1, ..., L_n$. Let $L = \bigcup_{i \in L/K} (K + \lambda_i)$ be a coset decomposition. Then

$$V_L = \bigoplus_{i \in L/K} V_{K+\lambda_i}$$

is a direct sum of irreducible V_K-modules. By Theorems 4.4 and 6.1 of [8], or Theorem 6.1 of [10], $V_{K+\lambda_i}$ is an irreducible V_K^+-module if $2\lambda_i \notin K$ and $V_{K+\lambda_i}$ is a direct sum of two irreducible V_K^+-modules otherwise. So by Proposition 5.2 it is enough to prove that V_K^+ is C_2-cofinite.

Note that

$$\begin{aligned} V_K &= V_{L_1} \otimes \cdots \otimes V_{L_n} \\ &= \sum_{\epsilon_i = \pm} V_{L_1}^{\epsilon_1} \otimes \cdots \otimes V_{L_n}^{\epsilon_n} \end{aligned}$$

(see [12] for the definition of tensor product of vertex operator algebras) and

$$V_K^+ = \sum_{\epsilon_i = \pm, \prod_i |\epsilon_i| = 1} V_{L_1}^{\epsilon_1} \otimes \cdots \otimes V_{L_n}^{\epsilon_n},$$

where $|\pm| = \pm 1$. Since each $V_{L_1}^{\epsilon_1} \otimes \cdots \otimes V_{L_n}^{\epsilon_n}$ is an irreducible $V_{L_1}^+ \otimes \cdots \otimes V_{L_n}^+$-module, by Proposition 5.2 again it suffices to prove that $V_{L_1}^+ \otimes \cdots \otimes V_{L_n}^+$ is C_2-cofinite.

By Corollary 5.17 of [18], each $V_{L_i}^+$ is C_2-cofinite. So by Lemma 5.3, $V_{L_1}^+ \otimes \cdots \otimes V_{L_n}^+$ is C_2-cofinite. □

Corollary 5.5. *Let $L = \mathbb{Z}\alpha$ such that $\frac{(\alpha,\alpha)}{2}$ is prime. Then V_L^+ is regular.*

Proof. This follows from Theorem 4.5, Theorem 5.17 of [18], and Theorem 4.12 of [1], which says that $V^+_{\mathbb{Z}\alpha}$ is rational if $\frac{(\alpha,\alpha)}{2}$ is prime. □

In order to discuss other consequences, we need a result on admissible modules.

Lemma 5.6. *Let V be C_2-cofinite. Then a weak module is admissible if and only if it is a direct sum of generalized eigenspaces for $L(0)$.*

Proof. First let $M = \bigoplus_{n=0}^{\infty} M(n)$ be an admissible V-module with $M(0) \neq 0$. Then $M(0)$ is an $A(V)$-module. Since V is C_2-cofinite, $A(V)$ is a finite-dimensional associative algebra. Let U be a simple $A(V)$-submodule $M(0)$. Then $L(0)$ acts on U as a constant. As a result, the V-module generated by U is an ordinary module by Theorem 3.3. Let W be a maximal admissible submodule of M such that W is a direct sum of generalized eigenspaces for $L(0)$. We assert that $M = W$. Otherwise, M/W contains an ordinary module $\bar{M}/W$ for some admissible submodule $\bar{M}$ of M. Clearly, $\bar{M}$ is a sum of generalized eigenspaces for $L(0)$, a contradiction.

Conversely, if a weak module M is a direct sum of generalized eigenspaces for $L(0)$, then it is enough to prove that for any $\lambda \in \mathbb{C}$, the subspace $M^\lambda = \sum_{n\in\mathbb{Z}} M_{\lambda+n}$ is an admissible submodule, where $M_{\lambda+n}$ is the generalized eigenspace for $L(0)$ with eigenvalue $\lambda+n$. We assume that $M^\lambda \neq 0$. Clearly, M^λ is a weak submodule of M. We claim that $M_{\lambda+n} = 0$ if n is sufficiently small. For any non-zero $w \in M_{\lambda+n}$, the submodule generated by w is an admissible V-module by Theorem 3.3. From the proof of Lemma 4.4, there is a non-zero lowest weight vector whose weight is $\lambda + m$ with $m \leq n$. As a result we have a simple $A(V)$-module on which $L(0)$ acts as a scalar $\lambda+m$. C_2 cofiniteness implies $A(V)$ is finite dimensional [7]. Since there are only finitely many simple $A(V)$-modules up to isomorphism [16], we can repeat this process only a finite number times, proving the claim. So $M^\lambda = \sum_{n\geq N} M_{\lambda+n}$ for some N. Set $M^\lambda(n) = M_{\lambda+N+n}$ for $n \geq 0$. Then $M^\lambda = \oplus_{n\geq 0} M^\lambda(n)$ is an admissible module. □

Proposition 5.7. *If V is a C_2-cofinite vertex operator algebra, then any weak V-module is admissible.*

Proof. In the proof of Theorem 4.5, all we needed to show that there existed a lowest weight vector in any weak module was C_2-cofiniteness of V. So given a weak module M, the space of lowest weight vectors $\Omega(M)$ is not zero. Since V is C_2-cofinite, $A(V)$ is a finite-dimensional associative algebra. Then the $A(V)$-module $\Omega(M)$ contains a simple $A(V)$-module and the V-submodule generated by the simple $A(V)$-module is an ordinary V-module by Theorem 3.3.

Let W be the maximal weak submodule of M which is a direct sum of generalized eigenspaces for the $L(0)$-action. If M/W is nontrivial, then M/W contains a non-zero ordinary module $\bar{M}/W$ for some weak module $\bar{M}$ contained in M. This follows by the argument in the previous paragraph, since M/W is again a weak V-module. Clearly, $\bar{M}$ is a direct sum of generalized eigenspaces for the $L(0)$-action. This is a contradiction. Thus $M = W$. By Lemma 5.6, M is an admissible module. □

Corollary 5.8. *If V is a C_2-cofinite vertex operator algebra, then any irreducible weak V-module is an irreducible ordinary V-module.*

Proof. Since any irreducible weak module is an irreducible admissible module, it follows from Theorem 3.3 that any irreducible admissible is ordinary. □

We now apply these new results to existing results about fusion rules for admissible modules to obtain results about the fusion rules for weak modules.

Definition 5.9. Let W_1, W_2, W_3 be weak V-modules. An *intertwining operator* of type $\binom{W_3}{W_1 W_2}$ is a linear map, $I : W_1 \to (\mathrm{Hom}(W_2, W_3))[[z, z^{-1}]]$ by $u \mapsto I(u, z) = \sum_{\alpha \in \mathbb{C}} u_\alpha z^{-\alpha-1}$, where the following hold for all $a \in V$, $u \in W_1$, and $v \in W_2$:

1) For all α, $u_{\alpha+n} v = 0$ for $n >> 0$.
2) $I(L(-1)u, z)v = \frac{d}{dz} I(u, z)v$.
3) The Jacobi Identity:

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y(a, z_1)I(u, z_2) - z_0^{-1}\delta(\frac{z_2 - z_1}{-z_0})I(u, z_2)Y(a, z_1)$$
$$= z_2^{-1}\delta(\frac{z_1 - z_0}{z_2})I(Y(a, z_0)u, z_2). \tag{5.1}$$

The set of intertwining operators of type $\binom{W_3}{W_1 W_2}$ forms a vector space. The dimension of this vector space, denoted $\dim\binom{W_3}{W_1 W_2}$, is called fusion rule or Clebsh-Gordon coefficient. We say that the fusion rules are finite if $\dim\binom{W_3}{W_1 W_2} < \infty$ for any three irreducible modules W_1, W_2, W_3. For more details see [12], [17].

Corollary 5.10. *Let V be a C_2-cofinite VOA. Then the fusion rules for irreducible weak V-modules are finite.*

Proof. By Corollary 5.8, any irreducible weak module is an ordinary module. For three irreducible weak modules W_i $(i = 1, 2, 3)$, we have $W_i = \bigoplus_{n \geq 0} W_i(n)$ with $W_i(0) \neq 0$. Clearly, $\dim(W_i(0)) < \infty$. By Proposition 5.2, we know that W_i is C_2-cofinite. Then $A(W_i)$ is finite dimensional (cf. [4]). Here $A(W) = W/O(W)$ is an $A(V)$-bimodule for any weak module W and $O(W)$ is spanned by elements of the form $\mathrm{Res}_z Y(u, z)\frac{(1+z)^{\mathrm{wt}(u)}}{z^2} w$ for $u \in V$ and $w \in W$. By Proposition 2.10 of [17],

$$\dim\binom{W_3}{W_1 W_2} \leq \dim(\mathrm{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0))).$$

Thus $\binom{W_3}{W_1 W_2}$ is finite dimensional. □

This corollary also appears in [2] with an additional assumption that the irreducible weak modules W_i are C_n-cofinite for all n. This assumption can be omitted due to Corollary 5.2.

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