SZEGÖ KERNELS AND FINITE GROUP ACTIONS

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ABSTRACT. In the context of almost complex quantization, a natural generalization of algebro-geometric linear series on a compact symplectic manifold has been proposed. Here we suppose given a compatible action of a finite group and consider the linear subseries associated to the irreducible representations of $G$, give conditions under which these are base-point-free and study properties of the associated projective morphisms. The results obtained are new even in the complex projective case.

1. Introduction

Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$, such that $[\omega] \in H^2(M, \mathbb{Z})$. Fix $J \in \mathcal{J}(M, \omega)$ (the contractible space of all almost complex structures on $M$ compatible with $\omega$), and let $h$ and $g = R(h)$ be the induced hermitian and riemannian structures. There exist an hermitian line bundle $(A, h)$ on $M$ and a unitary covariant derivative $\nabla_A$ on $A$, such that $-2\pi \omega$ is the curvature of $\nabla_A$.

In this set-up, the usual $\bar{\partial}$-complex from complex geometry can be replaced by a complex of pseudodifferential operators enjoying similar symbolic properties [BdM], [BdMG]: building on this foundational result, a theory of almost complex quantization has been developed and studied by several authors.

Namely, one can define spaces of quantized sections

$$H(M, A^\otimes k) \subseteq \mathcal{C}^\infty(M, A^\otimes k),$$

as the kernel of the first operator in the complex. A related approach is in terms of the asymptotic spectral properties of a suitable renormalized laplacian [GU], [BU1].

These linear series determine projective embeddings of $M$ enjoying the same metric and symplectic asymptotic properties as in the integrable projective case [BU2], [Z], [SZ2], [T]. In the integrable case the theory reduces to the usual classical constructions of complex algebraic geometry.

Suppose now that $G$ is a finite group with a symplectic action

$$\nu : G \times M \to M,$$

so that $J$ may be chosen $G$-invariant. Then $\nu$ preserves $g$ and $h$. Assume also that $\nu$ lifts to a linear action $\tilde{\nu} : G \times A \to A$, and that $\tilde{\nu}$ preserves $h_A$ and $\nabla_A$. Then $\tilde{\nu}$ preserves each of the spaces $H(M, A^\otimes N)$. Let

$$\rho_i : G \to \text{GL}(V_i) \quad (1 \leq i \leq c)$$
be the irreducible representations of $G$; we shall assume that $i = 1$ corresponds to the trivial one-dimensional representation. For each $N$, we have a $G$-equivariant decomposition

$$H(M, A^\otimes N) = \bigoplus_{i=1}^{c} H(M, A^\otimes N)_i,$$

where $H(M, A^\otimes N)_i$ consists of a direct sum of copies of $V_i$. It is natural to ask whether the linear series $|H(M, A^\otimes N)_i|$ are base-point-free and, if so, what about their asymptotic properties? In this note, we apply arguments from [BU2] and [Z], [SZ2] to these questions.

If $x \in M$, let $G_x = \{ g \in G : g \cdot x = x \}$ be its stabiliser. Let $\chi_i : G \to \mathbb{C}$ be the character of the $i$-th irreducible representation. Let $A_x$ be the fibre of $A$ over $x \in M$. Clearly, $G_x$ acts on $A_x$ and thus we have a unitary character $\alpha_x : G_x \to S^1 \subset \mathbb{C}^*$. Let

$$\gamma_{i,N}(x) := (\alpha_x^N, \chi_i)_{G_x} = \sum_{g \in G_x} \alpha_x(g)^N \cdot \chi_i(g) \quad (x \in M, \ 1 \leq i \leq c, \ N \in \mathbb{N}),$$

$$(,)_{G_x}$$

denoting the $L^2$-product with respect to the counting measure on $G_x$. Note that $\gamma_{i,N} = \gamma_{i,N+|G|}$ for every $i$ and $N$, where $|G|$ denotes the order of $G$. Set

$$B_{i,N} := \{ x \in M : \gamma_{i,N}(x) = 0 \} = B_{i,N+|G|} \quad (1 \leq i \leq c).$$

Clearly, $x \in B_{i,N}$ implies $G_x \neq \{e\}$.

Our first goal is to determine the base locus of the spaces of sections $H(M, A^\otimes k)_i$ for $k \gg 0$. In algebro-geometric terminology, the base locus of a vector subspace $W \subseteq C^\infty(M, A^\otimes N)$ is

$$\text{Bs}(|W|) := \{ x \in M : s(x) = 0 \forall s \in W \}.$$

To begin with, we shall prove:

**Theorem 1.1.** Suppose $1 \leq i \leq c$, $0 \leq r \leq |G| - 1$, $x \in M$ and $\gamma_{i,r}(x) \neq 0$. Then for $N \gg 0$, $N \equiv r(\text{mod } |G|)$ there exists a section $s \in H(M, A^\otimes N)_i$ such that $s(x) \neq 0$.

This has a number of consequences:

**Corollary 1.1.** Suppose that the action of $G$ on $M$ is effective. Then

$$\dim(H(M, A^\otimes k)_i) > 0$$

for every $i = 1, \ldots, c$ and every $k \gg 0$.

In fact, it is proved in [P] that under the same hypothesis

$$\dim(H(M, A^\otimes k)_i) = \frac{\dim(V_i)^2}{|G|} \cdot \frac{k^n}{n!} \cdot c_1(A)^n + o(k^n).$$

**Proposition 1.1.** Suppose $1 \leq i \leq c$, $0 \leq r \leq |G| - 1$, and $\gamma_{i,r}(x) \neq 0$ for every $x \in M$. Then $H(M, A^\otimes k)_i$ globally generates $A^\otimes k$ if $k \gg 0$ and $k \equiv r(\text{mod } |G|)$, that is, for every $x \in M$ there is $s \in H(M, A^\otimes k)_i$ such that $s(x) \neq 0$.

**Corollary 1.2.** If $k \gg 0$ and $i = 1, \ldots, c$, the subspace of $G$-invariant sections

$$H(M, A^\otimes k)^G \subseteq H(M, A^\otimes k)^{|G|}$$

globally generates $A^\otimes k^{|G|}$. 
Corollary 1.3. If $M$ is a complex projective manifold and $A$ is ample, for every $i = 1, \ldots, c$ and $r = 0, \ldots, |G| - 1$ the base loci $Bs\left( |H^0(M, A^{\otimes(r+k|G|)}_i)| \right)$ stabilize for $k \gg 0$. Furthermore, for every $k \gg 0$,
\[ Bs\left( |H^0(M, A^{\otimes(r+k|G|)}_i)| \right) \subseteq B_{i,r}. \]

In the reverse direction, it is easily seen that if $G_x = G$ and there exists $s \in C^\infty(M, A^{\otimes N})_i$ with $s(x) \neq 0$, then
\[ (\alpha^N_x, \chi_i)_G \neq 0. \]

Therefore,
Corollary 1.4. In the hypothesis of Corollary 1.3, suppose in addition that either $G_x = \{ e \}$ or $G_x = G$ for every $x \in G$. Then
\[ Bs\left( |H(M, A^{\otimes N})_i| \right) = B_{i,N} \]
for $i = 1, \ldots, c$ and $N \gg 0$.

In the almost complex case, for any $i = 1, \ldots, c$ and $r = 0, \ldots, |G| - 1$ we may still define the $(i,r)$-th equivariant asymptotic base locus of $A$ as
\[ Bs(A, i, r)_\infty =: \{ x \in M : \forall s > 0 \exists k > s, k \equiv r \pmod{|G|} \}
\]
such that $x \in Bs\left( |H(M, A^{\otimes k})_i| \right)$. The general case (symplectic, almost complex) of Corollary 1.3 is then

Corollary 1.5. In the above situation,
\[ Bs(A, i, r)_\infty \subseteq B_{i,r}. \]
If furthermore $K \subseteq M$ is any compact subset with $K \cap B_{i,r} = \emptyset$, then
\[ K \cap Bs\left( |H(M, A^{\otimes k})_i| \right) = \emptyset \]
for all $k \gg 0$ with $k \equiv r \pmod{|G|}$.

Next, if $Bs\left( |H(M, A^{\otimes N})_i| \right) = \emptyset$, there are associated projective morphisms
\[ \Phi_{i,r+k|G|} : M \to \mathbb{P}(H(M, A^{\otimes(r+k|G|)}_i)^*), \]
and we now consider their asymptotic properties as $k \to +\infty$.

Theorem 1.2. Suppose $Bs\left( |H(M, A^{\otimes N})_i| \right) = \emptyset$ for some $1 \leq i \leq c$ and $0 \leq r \leq |G| - 1$. Let $U \subseteq M$ be the open subset of $M$ where the order $|G_x|$ is locally constant. Suppose $U' \subset U$ is open with $\overline{U'} \subset U$. Then $\Phi_{i,r+k|G|}$ is an immersion on $U'$ for $k \gg 0$.

Corollary 1.6. $|H(M, A^{\otimes N})_G|$ is base-point-free and $\Phi_{1,N}$ is an immersion on compact subsets of $U$ if $N \gg 0$ and $\sum_{G_x} \alpha_x(g)^N \neq 0$ for every $x \in G$.

In general $\Phi_{i,N}$ is not injective; for example it is constant on every orbit for any $G$ if $i$ corresponds to the trivial representation, or for any $i$ if $G$ is abelian. We may still ask, however, if in these cases points in different orbits have different images under $\Phi_{i,N}$.

Let $d_G : M \times M \to \mathbb{R}$ be the orbit distance:
\[ d_G(x, y) := \min\{d(gx, y) : g \in G\} \quad (x, y \in M). \]
Clearly, $d_G(x, y) > 0$ if and only if $x \notin G \cdot y$. 
Proposition 1.2. Assume that either $G$ is abelian, or $G$ is arbitrary and $i = 1$. Let $U \subseteq M$ be as in Theorem [11] $N \in \mathbb{N}$ and suppose that $B_s(H(M, A^\otimes N)) = \emptyset$ and that $\gamma_{i,N}$ is constant on $W$. Let $K \subseteq W$ be a compact subset. There exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, $x, y \in K$ and $d_G(x, y) > 0$, then

$$\Phi_{i,N+k|G}(x) \neq \Phi_{i,N+k|G}(y).$$

Corollary 1.7. If the action of $G$ on $M$ is free, then $\Phi_{i,N}$ is well defined and is an embedding $M/G \hookrightarrow \mathcal{F}(H(M, A^\otimes k)^* G^*)$ for any $i = 1, \ldots, c$ and $N \gg 0$.

Similar statements hold for the asymptotic metric and almost complex structures, in the vein of Theorem 1.1 of [BU2].

2. Proofs

Proof of Theorem [11]. We recall some notation from [BU2, Z, SZ2]. Let $A^* = A^{-1}$ be the dual line bundle with the induced hermitian structure $h_{A^*}$, and let $A^* \supset S^1 \to M$ be the unit circle bundle, a strictly pseudoconvex domain. Given the connection, $S$ has natural riemannian and almost CR structures. We shall identify functions and half-forms throughout.

As $S$ is a principal $S^1$-bundle, $C^\infty(S) = \bigoplus_{N \in \mathbb{Z}} C^\infty(S)_N$, where $C^\infty(S)_N$ is the $N$-th isotype for the $S^1$-action. We shall identify $C^\infty(M, A^\otimes N)$ and $C^\infty(S)_N$ in the standard manner. Set $H(S) := \bigoplus_{N \in \mathbb{N}} H(S)_N$, where $H(S)_N \cong H(M, A^\otimes N)$ under this identification; in the integrable projective case, $H(S)$ is the Hardy space of boundary values of holomorphic functions on $A^*$. Let $\Pi : L^2(S) \to H(S)$ be the orthogonal projector and $\tilde{\Pi} \in \mathcal{D}(S \times S)$ its Schwartz kernel; decompose it as $\tilde{\Pi} = \bigoplus_{N \in \mathbb{N}} \tilde{\Pi}_N$, where $\tilde{\Pi}_N \in C^\infty(S \times S)$ is the $N$-th Fourier coefficient. We have $\tilde{\Pi}_N(x, y) = \sum_{i=0}^{d_N} s_i^N(x) \otimes s_i^N(y)$, where $\{s_0^N, \ldots, s_{d_N}^N\}$ is an orthonormal basis of $H(S)_N$. Let $\Phi_{i,N} : S \to H(M, A^\otimes N)^*$ be the coherent state map, given by evaluation, which is a lifting of $\Phi_{i,N}$ when the latter is defined. Then $\Phi_{i,N}(p, q) = (\Phi_{i,N}(p), \Phi_{i,N}(q)) (p, q \in S)$, where $(\cdot, \cdot)$ denotes the $L^2$-hermitian product on $H(M, A^\otimes N)^*$.

The induced action of $G$ on $A^*$ preserves $S$ and the riemannian and almost CR structures on $S$, and the isomorphisms $H(S)_N \cong H(M, A^\otimes N)$ are $G$-equivariant. For $N \gg 0$, we have $G$-equivariant decompositions $H(S)_N = \bigoplus_i H(S)_{i,N}$, where $H(S)_{i,N}$ is the factor consisting of a direct sum of copies of $V_i$, $1 \leq i \leq c$. Similarly, $H(S) = \bigoplus_i H(S)_i$. We shall implicitly identify $H(S)_N$ and $H(S)_{i,N}$ with $H(M, A^\otimes N)$ and $H(M, A^\otimes N)_i$, respectively. For each $i$, let $\Pi_i : L^2(S) \to H(S)_i$, denote the orthogonal projection and let $\tilde{\Pi}_i \in \mathcal{D}(S \times S)$ be its Schwartz kernel. For each $i$ and $N$, let $\Pi_{i,N} : L^2(S) \to H(S)_{i,N}$ be the orthogonal projection and $\tilde{\Pi}_{i,N}$ its Schwartz kernel, the $N$-th Fourier coefficient of $\tilde{\Pi}_i$: if $\{s_{0}^{(i,N)}, \ldots, s_{d_{i,N}}^{(i,N)}\}$ is an orthonormal basis of $H(S)_{i,N}$, then

$$\tilde{\Pi}_{i,N}(p, q) = \sum_{j=0}^{d_{i,N}} s_j^{(i,N)}(p) \otimes \overline{s_j^{(i,N)}(q)} (p, q \in S).$$

Clearly, $\tilde{\Pi} = \sum_{i=1}^{c} \tilde{\Pi}_i$. Notice that the Fourier components of the total and equivariant Szegő kernels, $\Pi_N$ and $\Pi_{i,N}$, when restricted to the diagonal in $S \times S$, descend to well-defined smooth functions on the diagonal in $M \times M$, that is, with some abuse of language we may write $\Pi_N(p, p) = \Pi_N(x, x)$ and $\Pi_{i,N}(p, p) = \Pi_{i,N}(x, x)$ for any $p \in S$ and $x \in M$ with $\pi(p) = x$. This will be done implicitly below.
Thus the former term is
\[ \dim (\mathcal{P}) \] where \( \rho : G \to \text{GL}(H(\mathbb{S})_N) \) is the induced representation; explicitly, \( \rho(g) = \sigma \circ g^{-1} \) (\( g \in G, \sigma \in H(\mathbb{S})_N \)), where we view \( g^{-1} \) as a contactomorphism of \( \mathbb{S} \). Thus,
\[
\tilde{\Pi}_{i,N}(p,q) = (\dim(V_i))/|G| \cdot \sum_g \sum_j \overline{\chi}_i(g)s_j^N(g^{-1}p)s_j^N(q)
\]
\[
= (\dim(V_i))/|G| \cdot \sum_g \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,q).
\]

On the diagonal, \( \tilde{\Pi}_{i,N}(p,p) = (\dim(V_i))/|G| \cdot \sum_g \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,p) \). Let \( d \) be the geodesic distance function on \( M \) and also its pull-back \( d \circ \pi \) to \( S \). If \( x \in M \) and \( G \cdot x \neq \{ x \} \), set \( a_x := \min \{ d(gx,x) : g \in G \setminus G_x \} \). Suppose \( p \in \mathbb{S}, x = \pi(p) \). Then
\[
\tilde{\pi}_{i,N}(p,p) = (\dim(V_i))/|G| \cdot \sum_{g \in G_x} \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,p)
\]
\[
+ (\dim(V_i))/|G| \cdot \sum_{g \notin G_x} \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,p).
\]

By virtue of Lemma 4.5 of [BU2], the latter term is bounded in absolute value by \( C\tilde{\Pi}_N(p,p)e^{-a_x^2N/2} + O(N^{n-1}/2) \), where \( C \) is independent of \( x \) and \( N \). By (13) of [SZ1] and the definition of dual action, \( \tilde{\Pi}_N(g^{-1}p,p) = \alpha_x(g)\tilde{\Pi}_N(p,p) \) if \( g \in G_x \). Thus the former term is
\[
(\dim(V_i))/|G| \cdot \left[ \sum_{g \in G_x} \overline{\chi}_i(g)\alpha_x(g)^N \right] \tilde{\Pi}_N(p,p)
\]
\[
= (\dim(V_i))/|G| \cdot (\alpha_x^N, \chi_i)G_x \cdot \tilde{\Pi}_N(p,p).
\]

Given the asymptotic expansion for \( \tilde{\Pi}_N(p,p) \) in [BU2] and [Z], \( \tilde{\Pi}_{i,N}(p,p) \neq 0 \) if \( N \gg 0, x \notin B_{i,N} \). This clearly implies the statement.

**Proof of Corollary [LJ]** Let \( V \subseteq M \) be the locus of points with non-trivial stabilizer. By Theorem 8.1 on page 213 of [S] and because the action is effective, \( V \) is a union of proper submanifolds of \( M \). If \( x \in M \setminus V \), then \( G_x = \{ e \} \) and therefore \( \gamma_{i,k}(x) = \dim(V_i) \neq 0 \) for every \( i \) and \( N \). By the theorem, there exists \( s \in H(M, A^{\otimes k}) \), with \( s(x) \neq 0 \) if \( k > 0 \).

Before coming to the proof of Proposition [LJ] let us dwell on the previous description of the equivariant Szegő kernels \( \Pi_{i,k} \) restricted to the diagonal. As is well known, the wave front of the Szegő kernel \( \Pi \) is
\[
\Sigma = \left\{ ((p,D), (r\alpha_p, -r\alpha_p)) : p \in \mathbb{S}, r > 0 \} \right\} \subseteq T^*(\mathbb{S} \times \mathbb{S}) \setminus \{0\}
\]

In the notation of [BdMG], [BU2] we have in fact \( \Pi \in J^{1/2}(\mathbb{S} \times \mathbb{S}, \Sigma) \). Now we have seen that
\[
\tilde{\Pi}_{i,N}(p,p) = (\dim(V_i))/|G| \cdot \sum_{g \in G} \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,p).
\]

For any \( g \in G \) let \( \alpha_g : \mathbb{S} \times \mathbb{S} \to \mathbb{S} \times \mathbb{S} \) be the diffeomorphism \( (p,q) \mapsto (g p,q) \), and let \( \tilde{\Pi}_g = \Pi \circ \alpha_g^* \in \mathcal{D}'(\mathbb{S} \times \mathbb{S}) \), where \( \alpha_g^* \) denotes pull-back of functions under \( \alpha_g \). Then \( \tilde{\Pi}_g \in J^{1/2}(\mathbb{S} \times \mathbb{S}, \alpha_g^*(\Sigma)) \) and \( \tilde{\Pi}_k(p,q) \) is the \( k \)-th Fourier component of \( \tilde{\Pi}_g \),
for every integer \( k \). One can then see, arguing as in the proofs of Lemmas 3.5 and 3.6 of [BU2], that \( k^{-n} \Pi_k(g, p, p) \) is bounded in \( C^1 \) norm, say, for every \( g \in G \). The same then holds for \( k^{-n} \Pi_{i,k}(x, x) \).

**Proof of Proposition [1.1]** By the above, in the hypothesis of the proposition for every \( x \in M \) there exists \( k_x \in \mathbb{N} \) such that \( x \notin \text{Bs}(\{H(M, A^{\otimes k})\}) \) for every \( k \geq k_x \). We now make the stronger claim that for every \( x \in M \) there exist an open neighbourhood \( U_x \) of \( x \) and \( k_x \in \mathbb{N} \) such that \( U_x \cap \text{Bs}(\{H(M, A^{\otimes k})\}) = \emptyset \) for every \( k \geq k_x \). The statement will follow given the compactness of \( M \).

If the claim was false, there would exist \( x \in M \) and sequences \( k_j \in \mathbb{N} \) and \( x_j \in M \) \((j = 1, 2, \ldots)\) with \( k_j \equiv r \pmod{|G|} \), \( k_j \to +\infty \) and \( x_j \to x \), such that \( x_j \in \text{Bs}(\{H(M, A^{\otimes k})\}) \) for every \( j \). Thus,

\[
\Pi_{i,k_j}(x_j, x_j) = 0 \quad (j = 1, 2, \ldots)
\]

while

\[
\Pi_{i,k_j}(x, x) = \frac{\dim(V_i)}{|G|} \cdot \gamma_{i,r}(x) \cdot \Pi_{k_j}(x, x) + \text{L.O.T.},
\]

where L.O.T. denotes lower order terms in \( k_j \). Thus, \( k_j^{-n} \Pi_{i,k_j}(x, x) \) is bounded away from zero and therefore the derivatives in \( x \) of the sequence of functions \( k_j^{-n} \Pi_{i,k_j}(x', x') \) are unbounded, a contradiction.

**Proof of Corollary [1.2]** Let us agree that the irreducible representation corresponding to \( i = 1 \) is just the trivial representation, so that

\[
H(M, A^{\otimes N})_1 = H(M, A^{\otimes N})^G
\]

for every integer \( N \). Then \( \chi_1(g) = 1 \) for every \( g \in G \). Furthermore, for every \( x \in M \), \( g \in G_x \) and \( k \in \mathbb{N} \) we have \( \alpha_{x^G_k}(g) = 1 \). Thus

\[
\gamma_{1,k|G}(x) = |G_x| \neq 0 \quad \text{for every } x \in M,
\]

and the statement follows from Proposition [1.1].

**Proof of Corollary [1.3]** If \( M \) is a complex projective manifold and \( A \) is ample, we have section multiplication maps

\[
H^0(M, A^{\otimes \ell})^G \otimes H^0(M, A^{\otimes m})_i \to H^0(M, A^{\otimes (\ell + m)})_i
\]

for every \( i = 1, \ldots, c \) and integers \( \ell, m \). Thus, for any residue class \( 0 \leq r \leq |G| - 1 \) and any sequence of positive integers \( k_i \gg 0 \), by Corollary [1.2] we have a descending chain of base loci:

\[
\text{Bs}(\{H^0(M, A^{\otimes r})\}) \supseteq \text{Bs}(\{H^0(M, A^{\otimes (r+k_1)})\}) \supseteq \text{Bs}(\{H^0(M, A^{\otimes (r+k_1+k_2)})\}) \supseteq \cdots
\]

This implies the first statement. The rest is obvious.

**Proof of Corollary [1.4]** If \( G_x = G \) and \( k \equiv r \pmod{|G|} \), then

\[
\Pi_{i,k}(x, x) = \frac{\dim(V_i)}{|G|} \cdot \gamma_{i,r}(x) \cdot \Pi_k(p, p).
\]

Thus, if \( \gamma_{i,r}(x) = 0 \), then \( s(x) = 0 \) for every \( s \in H(M, A^{\otimes k}) \).

**Proof of Corollary [1.5]** The first statement follows from Theorem [1.1] while the second is a consequence of the proof of Proposition [1.1].
Proof of Theorem 1.3. Suppose $B_{i,N} = \emptyset$ so that, perhaps after replacing $N$ by $N+k|G|$ for $k \gg 0$, $|H(S)_{1,N}|$ is base-point-free. The claim is that if $U' \subset U$ is open with compact closure in $U$ and $N \gg 0$, then $\Phi_{i,N}$ is an immersion on $U'$. We shall be done by proving that $N^{-1}\Phi_{i,N}^*(\omega^{(N)}_{FS}) - \omega = O(1/N)$ on connected compact subsets of $U$, where $\omega^{(N)}_{FS}$ is the Fubini-Study symplectic form on $\mathbb{P}(H(M,A^{\otimes_k})^*)$. In turn, this will follow if we prove that $N^{-1}\Phi_{i,N}^*(\tilde{\omega}_N) - \pi^*(\omega) = O(1/N)$ on horizontal vectors, over compact subsets of $S$; here $\tilde{\omega}_N = \frac{4}{\pi}\partial \bar{\partial} \log |\xi|^2$ on $H(M,A^{\otimes_k}) \setminus \{0\}$ (with its natural hermitian structure), and $\pi : S \to M$ is the projection.

Now, if $d^1$ and $d^2$ denote exterior differentiation on the first and second component of $S \times S$, respectively, then $N^{-1}\Phi_{i,N}^*\tilde{\omega}_N = \text{diag}^*(d^1d^2\log \tilde{\Pi}_{i,N})$, where $\text{diag} : S \to S \times S$ is the diagonal map ($\text{SZ2}$, proof of Theorem 3.1 (b)). If $x, y \in M$ lie in the same connected component $V$ of $U$, $G_y = G_x$. Thus $b_x := (\alpha^N_x, \chi_i)_Gx$ is constant on $V$, say equal to $b_V$. Hence, if $p, q \in \pi^{-1}(V)$ and $x = \pi(p)$,

$$\tilde{\Pi}_{i,N}(p, q) = \frac{\dim(V)}{|G|} \cdot \left\{ b_V \cdot \tilde{\Pi}_{i,N}(p, q) + \sum_{g \notin G_x} \chi_i(g) \tilde{\Pi}_{i,N}(gp, q) \right\}.$$

By the proof of Theorem 3.1 (b) of $\text{SZ2}$, $(i/2N) \text{diag}^*(d^1d^2\log \Pi_N) \to \pi^*\omega$ in $C^k$-norm for any $k$ on $M$. Therefore, we shall be done by proving that

$$N^{-1}d_1d_2(\tilde{\Pi}_{i,N}(gp, q)/\tilde{\Pi}_{i,N}(p, q)) \to 0$$

and

$$N^{-1}d_1(\tilde{\Pi}_{i,N}(gp, q)/\tilde{\Pi}_{i,N}(p, q)) \wedge d_2(\tilde{\Pi}_{i,N}(g'p, q)/\tilde{\Pi}_{i,N}(p, q)) \to 0$$

for $g, g' \notin G_x$ near compact subsets of $\text{diag}(V)$.

Then let $K \subset V$ be a compact subset, and suppose $x \in K$, $g \notin G_x$, and $u, v \in T_xM$ have unit length. Let $U, V$ be horizontal vector fields of unit length on $S$, of unit length near $S_x$ and extending the horizontal lifts of $u$ and $v$. We want to estimate $N^{-1}U_1 \circ V_2(\tilde{\Pi}_{i,N}(gp, q)/\tilde{\Pi}_{i,N}(p, q))$ over $K$, where $U_1 = (U, 0)$ and $V_2 = (0, V)$ are horizontal vector fields on $S \times S$.

Let us consider again the distribution $\tilde{\Pi}_g = \alpha^*_g \tilde{\Pi} \in J^{1/2}(S \times S, g^*\Sigma)$, discussed before the proof of Proposition 4.1. If $P$ is a horizontal differential operator of degree $\ell$ on $S \times S$, its principal symbol vanishes on $g^*\Sigma$ and therefore $P(\tilde{\Pi}_g) \in J^{(\ell+1)/2}(S \times S, \alpha^*_g \Sigma)$. As in $\text{BU2}$, Lemma 4.5, for $k \in \mathbb{N}$ we can find $\nu_{g, p, k} \in C^\infty(S)$, having an asymptotic expansion $\nu_{g, p, k}(p) = \sum_{j=0}^{\infty} k^{n+(\ell-j)/2} f_{g, p, k}^{(j)}(p)$, and real phase functions $\alpha_{g, p, k} \in C^\infty(S \times S)$ such that

$$G(p, q) = \sum_k \nu_{g, p, k}(p) e^{i\alpha_{g, p, k}(p, q)} e^{-kd(gp, q)^2/2} \in J^{(\ell+1)/2}(S \times S, \alpha^*_g \Sigma)$$

and $P(\tilde{\Pi}_g) - G \in J^{\ell/2}(S \times S, \alpha^*_g \Sigma)$. Since $P(\tilde{\Pi}_g)$ has definite (even) parity, we may assume without loss of generality that so does $G$. Therefore, the above asymptotic expansions may be assumed to go down by integer steps: $\nu_{g, p, k}(p) = \sum_{j=0}^{\infty} k^{n+\ell/2-j} f_{g, p, k}^{(j)}(p)$, and

$$(4) \quad |P(\tilde{\Pi}_{i,N}(gp, q))| = \nu_{g, p, a}(p) \cdot e^{-N(d(gp, q)^2/2} + O(N^{n+\ell/2-1})$$

Because $K \subset U$ is compact and $g \notin G_x$ for $x \in K$, there is $\epsilon > 0$ such that $d(gp, p) > \epsilon$ for all $p \in \pi^{-1}(K)$. Thus, $P(\tilde{\Pi}_{i,N}(gp, q)) = O(N^{n+\ell/2-1})$
on $\pi^{-1}(K)$. Developing $N^{-1}U_1 \circ V_2(\tilde{\Pi}_N(gp,q)/\tilde{\Pi}_N(p,q))$, we see that $N^{-1}U_1 \circ V_2(\tilde{\Pi}_N(gp,q)/\tilde{\Pi}_N(p,q)) = O(1/N)$ over $K$, uniformly in $U$ and $V$. This proves (2); the proof of the other estimate is similar.

**Proof of Proposition 1.2**. Notation being as above, we may assume that $K$ is $G$-invariant. Suppose then, by contradiction, that for a sequence $k_j \to +\infty$ we can find $x_{k_j}, y_{k_j} \in K$ with $d_G(x_{k_j}, y_{k_j}) > 0$ and $\Phi_{i,N+k_j|G}(x_{k_j}) = \Phi_{i,N+k_j|G}(y_{k_j})$. Set $N_j = N + k_j|G|$. We conclude that $d_G(x_{k_j}, y_{k_j}) \leq C/\sqrt{n_j}$. Following [BU2], proof of Corollary 4.6, pick $p_{k_j} \in \pi^{-1}(x_{k_j}), q_{k_j} \in \pi^{-1}(y_{k_j})$. Then $\tilde{\Phi}_{i,N_j}(x_{k_j}) = \lambda_j \tilde{\Phi}_{i,N_j}(y_{k_j})$ for some $\lambda_j \in \mathbb{C}$; it follows that $||\tilde{\Phi}_{i,N_j}(p_{k_j})||^2 = ||\lambda_j||^2 \cdot ||\tilde{\Phi}_{i,N_j}(q_{k_j})||^2$. However, $||\tilde{\Phi}_{i,N_j}(p)||^2 = \Pi_{i,N_j}(p,p) (p \in \Sigma)$, and therefore by (1) above $||\lambda_j|| = 1 + O(N_j^{-1/2})$. We also have $|\lambda_j||\Pi_{i,N_j}(p_{k_j}, q_{k_j})|| = ||\Pi_{i,N_j}(p_{k_j}, q_{k_j})||$, and on the other hand, again by (1),

$$||\Pi_{i,N_j}(p_{k_j}, q_{k_j})|| \leq C/\sqrt{n_j} \leq C/\sqrt{N_j},$$

as claimed. Hence, after replacing $x_{k_j}$ by $g_j \cdot x_{k_j}$ for a suitable $g_j \in G$, we may assume $d(x_{k_j}, y_{k_j}) \leq C/\sqrt{N_j}$ and $d(x_{k_j}, y_{k_j}) = d_G(x_{k_j}, y_{k_j})$ for every $j$.

Since $d(gx, x) > \epsilon$ for some fixed $\epsilon > 0$ and all $x \in K$ and $g \not\in G$, $x_{k_j}$ is the only point in $G \cdot x_{k_j}$ minimizing the distance from $y_{k_j}$, for every $j$.

We may now apply the argument of the proof of Theorem 3.2 (b) of [SZ2], with minor changes.

**References**


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