NONLINEARIZABLE ACTIONS OF DIHEDRAL GROUPS ON AFFINE SPACE

KAYO MASUDA

Abstract. Let $G$ be a reductive, non-abelian, algebraic group defined over $\mathbb{C}$. We investigate algebraic $G$-actions on the total spaces of non-trivial algebraic $G$-vector bundles over $G$-modules with great interest in the case that $G$ is a dihedral group. We construct a map classifying such actions of a dihedral group in some cases and describe the spaces of those non-linearizable actions in some examples.

1. Introduction

Let $G$ be a reductive complex algebraic group. When $G$ is non-abelian, it is well-known that there exist non-linearizable actions of $G$ on complex affine space $\mathbb{A}^n$ for $n \geq 4$, i.e., algebraic actions of $G$ on $\mathbb{A}^n$ which are not conjugate to linear actions under polynomial automorphisms of $\mathbb{A}^n$. It is remarkable that non-linearizable actions on $\mathbb{A}^n$ known so far are all obtained from non-trivial algebraic $G$-vector bundles over $G$-modules. An algebraic $G$-vector bundle over a $G$-variety $X$ is defined to be an algebraic vector bundle $p : E \to X$, where $E$ is a $G$-variety, the projection $p$ is $G$-equivariant, and the morphism induced by $g \in G$ from $p^{-1}(x)$ to $p^{-1}(gx)$ is linear for all $g$ and $x \in X$. An algebraic $G$-vector bundle is called trivial if it is isomorphic to a product bundle $X \times Q \to X$ for some $G$-module $Q$. A total space of an algebraic $G$-vector bundle over a $G$-module is an affine space by the affirmative solution to the Serre conjecture by Quillen [19] and Suslin [21]. Thus, the $G$-action on a total space $E$ of a non-trivial $G$-vector bundle over a $G$-module is a candidate for a non-linearizable action on affine space. There are a couple of known conditions for such an action to be non-linearizable (Bass and Haboush [1], M. Masuda and Petrie [15]). Schwarz [20] (Kraft and Schwarz [7] for details) first showed that an algebraic $G$-vector bundle over a $G$-module $P$ can be non-trivial when the algebraic quotient of $P$ is of one dimension, and that there exist families of non-linearizable actions on affine space, by using the above conditions. After Schwarz, lots of examples of non-trivial algebraic $G$-vector bundles have been presented, and it turns out that many of the $G$-actions on their total spaces are non-linearizable (Knop [5], M. Masuda, Moser-Jauslin and Petrie [11], M. Masuda and Petrie [16]). For abelian groups, there are no known examples of non-linearizable actions on complex affine space. In fact, for an abelian group $G$, every algebraic $G$-vector bundle over...
a $G$-module becomes trivial by the result of M. Masuda, Moser-Jauslin and Petrie [12], so, we cannot obtain non-linearizable actions from $G$-vector bundles. There are some affirmative results for the linearizability for torus actions (e.g. Bialynicki-Birula [2], Kaliman, Koras, Makar-Limanov and Russell [4]); however, it remains open whether or not every algebraic action of an abelian group on $\mathbb{A}^n \ (n \geq 4)$ is linearizable. Especially for a finite abelian group $G$, e.g. for a cyclic group $\mathbb{Z}/n\mathbb{Z}$, we never know even whether any $G$-action on $\mathbb{A}^3$ is linearizable or not.

For finite groups, M. Masuda and Petrie [16] showed that there exists a family of non-linearizable actions of a dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ for $n$ even and $\geq 18$ on $\mathbb{A}^3$. They considered $D_n$-actions derived from algebraic $D_n$-vector bundles which become trivial by adding certain trivial bundles, and showed that those actions form a family in some cases. Later, Mederer [18] showed that non-trivial algebraic $D_n$-vector bundles form a huge family of infinite dimension for $n$ odd and $\geq 3$. In this article, we investigate $G$-actions derived from non-trivial algebraic $G$-vector bundles. We are most interested in the case that $G$ is a dihedral group. We present a new condition for such $D_n$-actions to be non-linearizable and construct a map which classifies such non-linearizable $D_n$-actions without imposing triviality on $D_n$-vector bundles under the addition of certain trivial bundles. We also describe the spaces of those non-linearizable $D_n$-actions in some examples.

2. FAMILIES OF NON-LINEARIZABLE ACTIONS

Let $G$ be a reductive, non-abelian algebraic group and let $Z$ be an affine $G$-variety. We denote by $\mathbb{C}[Z]$ the coordinate ring of $Z$ and by $\mathbb{C}[Z]^G$ the ring of invariants. The algebraic quotient $Z//G$ is the affine variety defined by $Z//G = \text{Spec} \mathbb{C}[Z]^G$ and the quotient morphism $\pi_Z : Z \to Z//G$ is the morphism corresponding to the inclusion $\mathbb{C}[Z]^G \hookrightarrow \mathbb{C}[Z]$. Let $P$ and $Q$ be $G$-modules and let $X \subset P$ be a $G$-subvariety containing the origin of $P$. We denote by $\text{Vec}_G(X, Q)$ the set of algebraic $G$-vector bundles over $X$ whose fiber over the origin is isomorphic to $Q$, and by $\text{Vec}_G(X, Q)$ the set of $G$-isomorphism classes in $\text{Vec}_G(X, Q)$. An element $E \to X$ of $\text{Vec}_G(X, Q)$ is represented by the total space $E$, and the isomorphism class of $E \in \text{Vec}_G(X, Q)$ is denoted by $[E]$. The set $\text{Vec}_G(X, Q)$ is called trivial if $\text{Vec}_G(X, Q)$ consists of the unique class $[\Theta_Q]$, where $\Theta_Q$ denotes the product bundle with fiber $Q$. When dim $P//G = 1$, Schwarz [20] showed that $\text{Vec}_G(P, Q)$ has an additive group structure and is isomorphic to a vector group $\mathbb{C}^d$ for a non-negative integer $d$. Mederer [18] (cf. [3]) extended the result of Schwarz to the case where the base space is a $G$-equivariant affine cone $X$ with dim $X//G = 1$. When dim $P//G \geq 2$, $\text{Vec}_G(P, Q)$ can be non-trivial and of countably or uncountably infinite dimension [9], [19], [18].

We assume that $\text{Vec}_G(P, Q)$ is non-trivial. Let $E \in \text{Vec}_G(P, Q)$. The following are the known conditions for the $G$-action on the total space $E$ to be non-linearizable.

Proposition 2.1. Let $E, E' \in \text{Vec}_G(P, Q)$.

1. ([15]) Suppose that there exists a subgroup $H$ of $G$ such that $(P \oplus Q)^H = P$. Then $E$ and $E'$ are isomorphic as $G$-varieties if and only if $E$ and the pullback $\varphi^* E'$ are isomorphic as $G$-vector bundles for some $G$-automorphism $\varphi$ of $P$.

2. ([1]) If the Whitney sum $E \oplus \Theta_P$ is non-trivial, then the $G$-action on $E$ is non-linearizable.
Let \( \text{VAR}_G(P, Q) \) be the set of \( G \)-isomorphism classes of affine \( G \)-spaces represented as the total spaces of elements of \( \text{Vec}_G(P, Q) \). The group \( \text{Aut}(P)^G \) of \( G \)-equivariant automorphisms of \( P \) acts on \( \text{Vec}_G(P, Q) \) by pull-backs. There exists a surjection \( \Psi \) from the orbit space of \( \text{Vec}_G(P, Q) \) under the action of \( \text{Aut}(P)^G \) to \( \text{VAR}_G(P, Q) \). Under the assumption in Proposition 2.1 (1), \( \Psi \) is an isomorphism.

**Example 2.1.** Let \( G = O(2) = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z} \) and let \( V_m \) \((m \geq 1)\) be a two-dimensional \( O(2) \)-module such that

\[
\lambda(x, y) = (\lambda^m x, \lambda^{-m} y) \quad \text{for } \lambda \in \mathbb{C}^*,
\]

\[
\tau(x, y) = (y, x) \quad \text{for the generator } \tau \in \mathbb{Z}/2\mathbb{Z}.
\]

Then \( V_m/O(2) = \text{Spec } \mathbb{C}[t] = \mathbb{A}^1 \), where \( t = xy \), and \( \text{Aut}(V_m)^G = \mathbb{C}^* \), namely, \( \text{Aut}(V_m)^G \) consists of scalar multiplications.

Let \( n \) be odd. Then \( \text{Vec}_G(V_2, V_n) \cong \mathbb{C}^{(n-1)/2} \) and the Whitney sum with \( \Theta_{V_2} \) induces an isomorphism between \( \text{Vec}_G(V_2, V_n) \) and \( \text{Vec}_G(V_2, V_n \oplus V_2) \) \((\mathbb{Z}^2)\). By Proposition 2.1 (1) or (2), if \( E \in \text{Vec}_{O(2)}(V_2, V_n) \) is non-trivial, then the \( O(2) \)-action on \( E \) is non-linearizable. We shall describe \( \text{VAR}_{O(2)}(V_2, V_n) \). Since \( (V_2 \oplus V_n)^{\mathbb{Z}/2\mathbb{Z}} = V_2 \), where \( \mathbb{Z}/2\mathbb{Z} \) is a subgroup of \( \mathbb{C}^* \subset O(2) \), it follows from Proposition 2.1 (1) that

\[
\text{VAR}_{O(2)}(V_2, V_n) \cong \text{Vec}_{O(2)}(V_2, V_n)/\mathbb{C}^*.
\]

In order to look at the action of \( \text{Aut}(V_2)^G = \mathbb{C}^* \) on \( \text{Vec}_G(V_2, V_n) \), recall the isomorphism \( \text{Vec}_G(V_2, V_n) \cong \mathbb{C}^{(n-1)/2} \). For the details, we refer to Kraft and Schwarz \([7]\). Let \( F = \pi^{-1}_V(1) \), which is the \( G \)-subvariety of \( V_2 \) defined by \( xy = 1 \). Then \( F \cong G/H \), where \( H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and \( V_n \) is multiplicity-free with respect to \( H \), namely, each irreducible \( H \)-module occurs in \( V_n \) with multiplicity at most one when \( V_n \) is viewed as an \( H \)-module. We set \( m = \text{Mor}(F, \text{End} V_n)^G \), the module of \( G \)-equivariant morphisms from \( F \) to \( \text{End} V_n \).

Let \( \mathbb{B} = \text{Spec } \mathbb{C}[t] \) be the double cover of \( \mathbb{A}^1 = \text{Spec } \mathbb{C}[t] \), where \( s^2 = t \). Then the group \( \Gamma := \{ \pm 1 \} \) acts on \( \mathbb{B} \) and on \( F \) by scalar multiplication. We denote by \( \mathbb{B} \times \Gamma F \) the quotient of \( \mathbb{B} \times F \) by \( \Gamma \) which acts by \( (b, f) \mapsto (b\gamma, -1 f) \) for \( \gamma \in \Gamma, b \in \mathbb{B}, \) and \( f \in F \). The group \( G \) acts on \( \mathbb{B} \times \Gamma F \) through \( F \). We define a \( G \)-equivariant morphism \( \varphi \) by

\[
\varphi : \mathbb{B} \times \Gamma F \to V_2,
\]

\[
[b, f] \mapsto bf,
\]

which is a \( G \)-isomorphism from \((\mathbb{B} - \{0\}) \times \Gamma F \) onto \( V_2 - \pi^{-1}_V(0) \). Note that \( \mathbb{C}[\mathbb{B} \times \Gamma F]^G \cong \mathbb{C}[\mathbb{B}]^\Gamma = \mathbb{C}[t] = \mathbb{C}[V_2]^G \). The morphism \( \varphi \) induces a homomorphism

\[
\varphi^\# : \text{Mor}(V_2, \text{End} V_n)^G \to \text{Mor}(\mathbb{B} \times \Gamma F, \text{End} V_n)^G
\]

\[
= \text{Mor}(\mathbb{B}, m)^\Gamma =: m(\mathbb{B})^\Gamma.
\]

The modules \( \text{Mor}(V_2, \text{End} V_n)^G \) and \( m(\mathbb{B})^\Gamma \) are finite free modules over \( \mathbb{C}[t] \). In fact, a basis of \( \text{Mor}(V_2, \text{End} V_n)^G \cong (\mathbb{C}[V_2] \otimes \text{End} V_n)^G \) over \( \mathbb{C}[t] \) is written in a matrix form as

\[
\begin{Bmatrix}
A_0 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
A_1 = \begin{pmatrix}
0 & x^n \\
y^n & 0
\end{pmatrix}
\end{Bmatrix}
\]

and a basis of \( m(\mathbb{B})^\Gamma \cong (\mathbb{C}[s] \otimes m)^\Gamma \) over \( \mathbb{C}[t] \) is

\[
\begin{Bmatrix}
C_0 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
C_1 = s(A_1|_F)
\end{Bmatrix}.
\]
The module \( \text{Mor}(V_2, \text{End} V_n)^G \) (resp. \( \mathfrak{m}(\mathbb{B})^\Gamma \)) inherits a grading from \( \mathbb{C}[V_2] \) (resp. \( \mathbb{C}[s] \)), and \( \varphi_\# \) is a homomorphism of degree 0. Let \( \text{Mor}(V_2, \text{End} V_n)^G \) (resp. \( \mathfrak{m}(\mathbb{B})^\Gamma \)) be the submodule of \( \text{Mor}(V_2, \text{End} V_n)^G \) (resp. \( \mathfrak{m}(\mathbb{B})^\Gamma \)) consisting of elements with positive degrees. Then \( \text{Vec}_G(V_2, V_n) \) is isomorphic to the quotient module \( \mathfrak{m}(\mathbb{B})^\Gamma_1 / \varphi_\# \text{Mor}(V_2, \text{End} V_n)^G_1 \). Since

\[
\varphi_\#(A_0) = C_0 \quad \text{and} \quad \varphi_\#(A_1) = i \frac{m-1}{2} C_1,
\]

\( \{ t^{i-1}C_1; 1 \leq i \leq \frac{m-1}{2} \} \) forms a \( \mathbb{C} \)-basis of \( \mathfrak{m}(\mathbb{B})^\Gamma_1 / \varphi_\# \text{Mor}(V_2, \text{End} V_n)^G_1 \), and hence

\[
\text{Vec}_G(V_2, V_n) \cong \mathfrak{m}(\mathbb{B})^\Gamma_1 / \varphi_\# \text{Mor}(V_2, \text{End} V_n)^G_1 \cong \mathbb{C}^{n-1}.
\]

Note that \( \deg(t^{i-1}C_1) = 2i - 1 \). The scalar multiplication on \( V_2 \) corresponds to a scalar multiplication on \( \mathbb{B} \) via \( \varphi \). Hence \( \text{Vec}_G(V_2, V_n) \cong \bigoplus_{i=1}^{(n-1)/2} W(2i-1) \) as a module of \( \text{Aut}(V_2)^G = \mathbb{C}^* \), where \( W(i) \) denotes the representation space of \( \mathbb{C}^* \) with weight \( i \). Thus we obtain by Proposition 2.1 (1) that

\[
\text{VAR}_{O(2)}(V_2, V_n) \cong \left( \bigoplus_{i=1}^{(n-1)/2} W(2i-1) \right) / \mathbb{C}^* \cong \mathbb{P}_s(2i-1; 1 \leq i \leq \frac{n-1}{2}).
\]

Here \( \mathbb{P}_s(2i-1; 1 \leq i \leq (n-1)/2) \) consists of the “vertex” \( \bullet \), and the weighted projective space \( \mathbb{P}(2i-1; 1 \leq i \leq (n-1)/2) \) of dimension \( (n-3)/2 \) with weight \( 2i-1 \) for \( 1 \leq i \leq (n-1)/2 \). The “vertex” corresponds to the linearizable action and the weighted projective space to non-linearizable actions (cf. [16]).

**Example 2.2.** Let \( G = SL_2 \) and let \( R_n \) be the \( SL_2 \)-module of binary forms of degree \( n \geq 1 \). Then \( \text{Vec}_G(R_2, R_n) \cong \mathbb{C}^{([n-1]^2/4)} \) and \( \text{Aut}(R_2)^G = \mathbb{C}^* \) ([20], [7]). As a module of \( \mathbb{C}^* = \text{Aut}(R_2)^G \), \( \text{Vec}_G(R_2, R_n) \) is isomorphic to \( \bigoplus_{i=1}^{n-2} m_i W(i) \) with multiplicity \( m_i = \left[ \frac{[n-1]^2}{4} \right] \). Suppose \( n \) is odd. Then \( (R_2 \oplus R_n)^{Z/2Z} = R_2 \). Hence by Proposition 2.1 (1),

\[
\text{VAR}_{SL_2}(R_2, R_n) \cong \left( \bigoplus_{i=1}^{n-2} m_i W(i) \right) / \mathbb{C}^* \cong \mathbb{P}_s(i, m_i; 1 \leq i \leq n-2).
\]

In this case, the space of non-linearizable \( SL_2 \)-actions is isomorphic to the weighted projective space of dimension \( [([n-1]^2/4)] \) with weight \( i \) of multiplicity \( m_i \) for \( 1 \leq i \leq n-2 \).

**Example 2.3.** Let \( G \) be semisimple and let \( g \) be the adjoint representation of \( G \). Let \( \Sigma \) be a system of simple roots of \( G \) and \( F \) an irreducible \( G \)-module with the highest weight \( \chi \). Knop [3] constructed a map associated with \( \alpha \in \Sigma \),

\[
\Phi_\alpha : \text{Vec}_G(g, F) \to \text{Vec}_{SL_2}(R_2, R_m),
\]

where \( m = \langle \chi, \alpha \rangle \). The map \( \Phi_\alpha \) is surjective if the \( \alpha \)-string of \( \chi \) is regular ([5], [14]). We recall the construction of \( \Phi_\alpha \). Let \( T \subseteq G \) be a maximal torus with the Lie algebra \( t \subseteq g \). Let \( L \) be the subgroup of \( G \) generated by \( T \) and the root subgroups \( U_\alpha \) and \( U_{-\alpha} \). We denote by \( L' \) the commutator subgroup of \( L \) and by \( Z \) the center of \( L \). Then \( L = L'Z \), and \( L' \) is isomorphic to \( SL_2 \) or \( SO_3 \). Let \( I \) be the Lie algebra of \( L \). Then \( I \) is isomorphic to \( \mathfrak{sl}_2 \oplus \mathbb{C}^{n-1} \) as an \( L' \)-module,
where \( n = \text{rank } \mathfrak{t} \). For \( E \in \text{Vec}_G(\mathfrak{g}, F) \), the restricted bundle \( E|_l \) is an \( L \)-vector bundle with fiber \( F' \) which is \( F \) viewed as an \( L \)-module. Take a \( \xi_0 \in \mathfrak{t} \) so that the centralizer of \( \xi_0 \) is exactly \( L \), and fix it. Then \( \mathfrak{a} := \xi_0 + \text{Lie}L' \subseteq \mathfrak{g} \) is \( L \)-stable and isomorphic to \( \mathfrak{sl}_2 \cong R_2 \) as an \( L' \)-variety. Since \( Z \) acts trivially on \( \mathfrak{t} \), hence on \( \mathfrak{a} \), \( E|_{\mathfrak{a}} \) decomposes to a Whitney sum of eigenbundles of \( Z \). Let \(( E|_{\mathfrak{a}} )_\chi \) be the eigenbundle corresponding to the restricted weight \( \chi \) onto \( Z \). Then the \( L' \)-vector bundle \(( E|_{\mathfrak{a}} )_\chi \) is considered as an element of \( \text{Vec}_{SL_2}(R_2, R_m) \). The map \( \Phi_\alpha \) is defined by \( \Phi_\alpha(E) = (E|_{\mathfrak{a}})_\chi \). By the construction of \( \Phi_\alpha \), \( \Phi_\alpha \) decomposes to the maps

\[
\phi_\alpha : \text{Vec}_G(\mathfrak{g}, F) \to \text{Vec}_L(l, F') \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)
\]

and

\[
\phi_{\xi_0} : \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m) \to \text{Vec}_{SL_2}(R_2, R_m).
\]

From the choice of \( \xi_0 \), \( \phi_{\xi_0} \) is surjective. In fact, \( \phi_{\xi_0} \circ \text{pr}^* = \text{id} \), where

\[
\text{pr}^* : \text{Vec}_{SL_2}(R_2, R_m) \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)
\]

is the induced map from the projection \( R_2 \oplus \mathbb{C}^{n-1} \to R_2 \). When \( \Phi_\alpha \) is surjective, \( \phi_\alpha \) is also surjective since \( \phi_{\xi_0} \) is surjective. By \([8]\),

\[
\text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m) \cong \text{Vec}_{SL_2}(R_2, R_m) \otimes_\mathbb{C} \mathbb{C}[\mathbb{C}^{n-1}].
\]

Hence we obtain the following.

**Theorem 2.2.** Under the notation above, if the \( \alpha \)-string of \( \chi \) is regular, then

\[
\phi_\alpha : \text{Vec}_G(\mathfrak{g}, F) \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)
\]

\[
\cong \mathbb{C}^{(m-1)^2/4} \otimes_\mathbb{C} \mathbb{C}[y_1, \ldots, y_{n-1}]
\]

is surjective. Furthermore, if there is a subgroup \( H \) such that \(( \mathfrak{g} \oplus F)^H = \mathfrak{g} \), then \( \phi_\alpha \) induces a surjection

\[
\text{VAR}_G(\mathfrak{g}, F) \to \mathbb{C}^{(m-1)^2/4} \otimes_\mathbb{C} \mathbb{C}[y_1, \ldots, y_{n-1}]/\mathbb{C}^*,
\]

where \( \mathbb{C}^* \) acts on \( \mathbb{C}^{(m-1)^2/4} \) with weight \( i \) of multiplicity \( m_i = [(m-i)/2] \) and on \( y_i \) with weight 1.

**Proof.** The first assertion follows from the above observation. For the second assertion, note that \( \text{Aut}(\mathfrak{g})^G = \mathbb{C}^* \) \([7]\). From Proposition 2.1 (1), there is an isomorphism \( \text{VAR}_G(\mathfrak{g}, F) \cong \text{Vec}_G(\mathfrak{g}, F)/\mathbb{C}^* \). Hence \( \phi_\alpha \) induces a surjection

\[
\text{VAR}_G(\mathfrak{g}, F) \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)/\mathbb{C}^*.
\]

The assertion follows from the statement in Example 2.2. \( \square \)

**Remark.** When the \( \alpha \)-string of \( \chi \) is singular, the image of \( \Phi_\alpha \) contains a subspace of dimension \( [m/2][(m/2) - 1]/2 \) \([13]\).

By Theorem 2.2 and its remark, \( \text{Vec}_G(\mathfrak{g}, F) \) is of infinite dimension if \( m \geq 4 \) and \( n \geq 2 \). Furthermore, if \(( \mathfrak{g} \oplus F)^H = \mathfrak{g} \) for a subgroup \( H \), then \( \text{VAR}_G(\mathfrak{g}, F) \) is of infinite dimension.

Now, we give a new condition for the \( G \)-action on \( E \in \text{Vec}_G(P, Q) \) to be non-linearizable, which is used as a basic fact in the next section.

**Proposition 2.3.** Let \( E, E' \in \text{Vec}_G(P, Q) \). Suppose that there exist reductive subgroups \( H \) and \( K \) such that \( H \subset K \) and satisfying the following conditions;

1. \( \mathbb{Q}^K = \mathbb{Q}^H \).
2. \( \dim \mathbb{P}^H = 1 \) and \( \dim \mathbb{P}^K = 0 \).
If $E \cong E'$ as $G$-varieties, then the restricted bundles $E|_X$ and $(c^* E')|_X$ are isomorphic as $G$-vector bundles, where $X = G \cdot P^H$ and $c$ is a scalar multiplication on $P$. In particular, if $E|_X$ is a non-trivial $G$-vector bundle, then the $G$-action on $E$ is non-linearizable.

**Proof.** Let $\phi : E \cong E'$ be an isomorphism of $G$-varieties. Then $\phi$ restricts to an isomorphism $\phi_H : E^H \cong E'^{H'}$. Since $E^H$ and $E'^{H'}$ are trivial $(H)$-vector bundles over $P^H$ with fiber $Q^H$ (cf. [5]), it follows that $E^H \cong E'^{H'} \cong P^H \times Q^H$. Similarly, $E^K \cong E'^K \cong Q^K = Q^H$ since $\dim P^K = 0$. Since $E^K$ (resp. $E'^K$) is a subbundle of $E^H$ (resp. $E'^{H'}$), we get $E^H = E^K \times P^H$ and $E'^{H'} = E'^K \times P^{H'}$. Let $x$ be a coordinate variable of $P^H \times Q^H$ such that $P^H = \text{Spec } \mathbb{C}[x]$. Then the ideal corresponding to $E^K$ is $(x)$, and the ideal for $E'^K$ is the same. Since $\phi$, hence $\phi_H$, maps $E^K$ to $E'^K$ isomorphically, the ideal $(x)$ must be fixed by the algebra isomorphism corresponding to $\phi_H$. This implies that $\phi_H$, hence $\phi$, induces an isomorphism $\tilde{c}$ on $P^H$ such that $p_H' \circ \phi_H = \tilde{c} \circ p_H$, where $p_H : E^H \to P^H$ and $p_H' : E'^{H'} \to P^{H'}$ are projections. Note that $\tilde{c}$ is a scalar multiplication on $P^H$. Hence $\tilde{c}$ extends to a scalar multiplication $c$ on $P$. Since the following diagram commutes, $\phi$ restricts to a variety isomorphism $E|_{P^H} \cong E'|_{P^H}$:

$$
\begin{array}{ccc}
E & \xrightarrow{\phi} & E' \\
\downarrow^{E^H} & & \downarrow^{E'^{H'}} \\
P^H & \xrightarrow{c} & P^H
\end{array}
$$

where the diagonal arrows are inclusions. Furthermore, since $\phi$ is a $G$-isomorphism, $\phi$ in fact restricts to a $G$-isomorphism $\phi_X : E|_X \to E'|_{X}$ such that $p'_X \circ \phi_X = (c|_X) \circ p_X$, where $p_X : E|_X \to X$ and $p'_X : E'|_{X} \to X$ are projections. Thus $E|_X \cong (c^* E')|_X$ as $G$-vector bundles, and the assertion follows. \qed

Proposition 2.3 enables us to classify elements of $\text{VAR}_G(P, Q)$.

**Corollary 2.4.** Under the assumption and notation in Proposition 2.3, there exists a map

$$
\Phi : \text{VAR}_G(P, Q) \to \text{Vec}_G(X, Q)/\mathbb{C}^*,
$$

where the target space is the orbit space of $\text{Vec}_G(X, Q)$ under the action of $\mathbb{C}^*$, which is a subgroup of $\text{Aut}(X)^G$ consisting of scalar multiplications.

When $H$ is an isotropy group of a point $x \in P$ whose orbit is closed, then $X = G \cdot P^H$ is a $G$-equivariant affine cone in $P$ with $\dim X//G = 1$. In this case, $\text{Vec}_G(X, Q)$ is isomorphic to a finite-dimensional module of $\mathbb{C}^* \subset \text{Aut}(X)^G$ ([18]). Hence $\text{Vec}_G(X, Q)/\mathbb{C}^*$ is isomorphic to a weighted projective space with a “vertex”.

**Example 2.4.** Let $G = O(2)$ and consider $\text{Vec}_{O(2)}(V_1, V_m)$. Then applying Proposition 2.3 for $H = \mathbb{Z}/2\mathbb{Z}$ (the reflection subgroup) and $K = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we obtain $X = V_1$, and hence, a map $\text{VAR}_{O(2)}(V_1, V_m) \to \text{Vec}_{O(2)}(V_1, V_m)/\mathbb{C}^*$, which is an isomorphism. Since $\text{Vec}_{O(2)}(V_1, V_m) \cong \bigoplus_{i=1}^m W(2i) \oplus [7]$, we have
(cf. \cite{16}, \cite{17})

\[ \text{VAR}_{O(2)}(V_1, V_m) \cong \left( \bigoplus_{i=1}^{m-1} W(2i) \right)/\mathbb{C}^* \]
\[ \cong \mathbb{P}_4(2i; 1 \leq i \leq m-1). \]

We apply Proposition 2.3 and its corollary for dihedral groups and classify non-linearizable actions of dihedral groups in the next section.

3. Non-linearizable actions of dihedral groups

In this section, we investigate non-linearizable actions of dihedral groups. Let \( G \) be a dihedral group \( D_n = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) for \( n > 2 \). By considering \( D_n \) as a finite subgroup of \( O(2) = \mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z} \), an \( O(2) \)-module \( V_m \) is naturally considered as a \( D_n \)-module. Since \( V_m \cong V_{[m-n]} \) as a \( D_n \)-module, we may assume \( m \leq n/2 \); otherwise \( m = n \). Let \( k \) be a positive integer such that \( (k, n) = 1 \) and \( k \leq n/2 \). Let \( \{x, y\} \) be a coordinate system of \( V_k \) as in Example 2.1. Then \( V_k/D_n = \text{Spec} \mathbb{C}[t, u] \)

where \( t = xy \) and \( u = x^n + y^n \). Let \( X_k = D_n \cdot V_k^{Z/2\mathbb{Z}} \), where \( Z/2\mathbb{Z} \) is the reflection subgroup. Then \( X_k \) is the \( D_n \)-subvariety of \( V_k \) defined by \( x^n - y^n = 0 \) for \( n \) odd, and \( x^{n/2} - y^{n/2} = 0 \) for \( n \) even. The algebraic quotient of \( X_k \) is

\[ X_k/D_n = \left\{ \begin{array}{ll}
\text{Spec} \mathbb{C}[t, u]/(u^2 - 4t^n) & \text{for } n \text{ odd}, \\
\text{Spec} \mathbb{C}[t] & \text{for } n \text{ even}.
\end{array} \right. \]

The variety \( X_k \) is the \( D_n \)-equivariant affine cone in \( V_k \) with one-dimensional quotient. Hence \( \text{Vec}_{D_n} (X_k, V_m) \cong \mathbb{C}^q \) for some \( q \) (\cite{18}, \cite{8}).

We shall classify elements of \( \text{VAR}_{D_n}(V_k, V_m) \) under a certain condition.

**Proposition 3.1.** Let \( E, E' \in \text{Vec}_{D_n}(V_k, V_m) \) and let \( X_k \) be as above. Suppose that \( (m, n) > 1 \). Then, if \( E \cong E' \) as \( D_n \)-varieties, then the restricted bundles \( E|_{X_k} \) and \( (e^*E')|_{X_k} \) are isomorphic as \( D_n \)-vector bundles, where \( e \) is a scalar multiplication on \( V_k \).

**Proof.** By taking \( H = \mathbb{Z}/2\mathbb{Z} \) (the reflection subgroup) and \( K = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), where \( p = (m, n) \) in Proposition 2.3, the assertion follows. \( \square \)

Under the assumption in Proposition 3.1, there exists a map

\[ \Phi_{k,m} : \text{VAR}_{D_n}(V_k, V_m) \rightarrow \text{Vec}_{D_n}(X_k, V_m)/\mathbb{C}^*. \]

Let \( i^*_k : \text{Vec}_{D_n}(V_k, V_m) \rightarrow \text{Vec}_{D_n}(X_k, V_m) \) be the restriction induced by the inclusion \( i_k : X_k \hookrightarrow V_k \). There exists a sequence

\[ \text{Vec}_{O(2)}(V_k, V_m) \xrightarrow{d_n} \text{Vec}_{D_n}(V_k, V_m) \xrightarrow{i^*_k} \text{Vec}_{D_n}(X_k, V_m), \]

where \( d_n \) is the group restriction.

**Theorem 3.2** (cf. \cite{16}). Let \( n \) be odd, and let \( k = 2 \) and \( m = n \) in the notation above.

1. The composite map \( i^*_2 \circ d_n : \text{Vec}_{O(2)}(V_2, V_n) \rightarrow \text{Vec}_{D_n}(X_2, V_n) \) is injective and

\[ \text{Im}(i^*_2 \circ d_n) \cong \mathbb{C}^{n-1}. \]

2. The image of \( \Phi_{2,n} \) is isomorphic to \( \mathbb{P}_4(2i; 1 \leq i \leq (n-1)/2) \).
(3) The map $\text{VAR}_{O(2)}(V_2, V_n) \to \text{VAR}_{D_n}(V_2, V_n)$ is injective. Hence, if $E \in \text{Vec}_{O(2)}(V_2, V_n)$ is a non-trivial $O(2)$-vector bundle, then the $D_n$-action on $E$ is non-linearizable.

Proof. (1) By applying the method of Mederer, we can show that $\text{Vec}_{D_n}(X_2, V_n)$ is isomorphic to a vector group $\mathbb{C}^{n-1}$. For the detailed argument, we refer to Mederer [15]. We shall give a basis of $\text{Vec}_{D_n}(X_2, V_n) \cong \mathbb{C}^{n-1}$. We use the notation in Example 2.1 and denote $X_2$ simply by $X$. Let $\nu : \mathcal{B} = \text{Spec} \mathbb{C}[s] \to X/\text{D}_n = \text{Spec} \mathbb{C}[t, u]/(u^2 - 4t^n)$ be the normalization, where $t = s^2$ and $u = 2s^n$, and let $F_X = \pi_X^{-1}(\nu(1))$. Then $F_X \cong \text{D}_n/H'$, where $H' = \mathbb{Z}/2\mathbb{Z}$ (the reflection subgroup) and $V_n$ is multiplicity free with respect to $H'$. There is a $D_n$-equivariant morphism

$$\varphi^X : \mathcal{B} \times F_X \to X,$$

$$(b, f) \mapsto bf,$$

which is an isomorphism from $(\mathcal{B} - \{0\}) \times F_X$ onto $X - \pi_X^{-1}(\nu(0))$. Note that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{B} \times F_X & \xrightarrow{\varphi^X} & X \\
\downarrow & & \downarrow \\
\mathbb{B} \times 1 & \xrightarrow{\varphi} & V_2
\end{array}$$

where the vertical maps are inclusions. Let $\mathfrak{m}_X = \text{Mor}(F_X, \text{End} V_n)^{D_n}$. Then $\mathfrak{m}_X(\mathcal{B}) := \text{Mor}(\mathcal{B}, \mathfrak{m}_X)$ is a free $\mathbb{C}[s]$-module with a grading induced from $\mathbb{C}[s]$. The $\mathbb{C}[X]^{D_n}$-module $\text{Mor}(X, \text{End} V_n)^{D_n} \cong (\mathbb{C}[X] \otimes \text{End} V_n)^{D_n}$ inherits a grading from $\mathbb{C}[X] \subset \mathbb{C}[V_2]$. Note that $\mathfrak{m}_X(\mathcal{B})$ is considered as a $\mathbb{C}[X]^{D_n}$-module via $\nu$. The morphism $\varphi^X$ induces

$$\varphi^X : \text{Mor}(X, \text{End} V_n)^{D_n} \to \text{Mor}(\mathcal{B} \times F_X, \text{End} V_n)^{D_n} = \mathfrak{m}_X(\mathcal{B}),$$

which is a $\mathbb{C}[X]^{D_n}$-homomorphism of degree 0. Note that $\text{Mor}(X, \text{End} V_n)^{D_n}$ is a finite free module over $\mathbb{C}[X]^{D_n}$. In fact, $\text{Mor}(V_2, \text{End} V_n)^{D_n}$ is free over $\mathbb{C}[t, u]$ with a basis

$$\begin{align*}
\bar{A}_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\bar{A}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\bar{A}_2 &= \begin{pmatrix} x^n - y^n & 0 \\ 0 & -(x^n - y^n) \end{pmatrix}, \\
\bar{A}_3 &= \begin{pmatrix} 0 & x^n - y^n \\ -(x^n - y^n) & 0 \end{pmatrix}.
\end{align*}$$

Hence $\text{Mor}(X, \text{End} V_n)^{D_n}$ is a free module over $\mathbb{C}[t, u]/(u^2 - 4t^n)$ with a basis $
\{\bar{A}_0, \bar{A}_1\}$. Let $\text{Mor}(X, \text{End} V_n)^{D_n}_{1}$ (respectively $\mathfrak{m}_X(\mathcal{B})_1$) be the submodule of $\text{Mor}(X, \text{End} V_n)^{D_n}$ (respectively $\mathfrak{m}_X(\mathcal{B})$) of elements with positive degrees. Then $\text{Vec}_{D_n}(X, V_n)$ is isomorphic to the quotient module of $\mathfrak{m}_X(\mathcal{B})_1$ by

$$\varphi^X_{\#} \text{Mor}(X, \text{End} V_n)^{D_n}_{1}.$$ 

The module $\mathfrak{m}_X(\mathcal{B})_1$ is free over $\mathbb{C}[s]$ with a basis

$$\left\{ \begin{pmatrix} \bar{C}_0 = s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \bar{C}_1 = s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}. $$

Since $\varphi^X_{\#}(t\bar{A}_i) = s\bar{C}_i$ and $\varphi^X_{\#}(u\bar{A}_i) = 2s^{n-1}\bar{C}_i$ for $i = 0, 1$,

$$\text{Vec}_{D_n}(X, V_n) \cong \mathfrak{m}_X(\mathcal{B})_1/\varphi^X_{\#} \text{Mor}(X, \text{End} V_n)^{D_n}_{1} \cong \mathbb{C}^{n-1}.$$
with a basis \( \{ s^{2(j-1)}C_i ; i = 0, 1, 1 \leq j \leq (n-1)/2 \} \). The inclusions \( \mathbb{B} \times F \hookrightarrow \mathbb{B} \times F \) and \( X \hookrightarrow V_2 \) give rise to a homomorphism
\[
i \mapsto \mathfrak{m}(\mathbb{B})^I_1 \mapsto \mathfrak{m} X (\mathbb{B})^I_1 \mapsto \mathfrak{m} X (X, \text{End} V_n)_1^D_n,
\]
which corresponds to \( \iota_n \circ d_n \). Since \( \iota(t^{-1}C_1) = s^{2(t^{-1})C_1} \), it follows that \( \iota_n \circ d_n \) is injective and \( \text{Im} \circ d_n \cong \mathbb{C}^{(n-1)/2} \) with a basis \( \{ s^{2(j-1)}C_i ; 1 \leq j \leq (n-1)/2 \} \).

(2) From Proposition 3.1, there is a map
\[
\Phi_{2,n} : \mathrm{VAR}_{D_n}(V_2, V_n) \rightarrow \mathrm{VAR}_{D_n}(X_2, V_n)/C^*.
\]
From (1), \( \text{Im} \iota_n^2 \) contains a subspace
\[
\bigoplus_{i=1}^{(n-1)/2} W(2i-1).
\]
In fact, \( \text{Im} \iota_n^2 \quireq \bigoplus_{i=1}^{(n-1)/2} W(2i-1) \) (cf. [38 III 3,4]). Hence the assertion follows.

(3) follows from (1) and Proposition 3.1. \( \square \)

Remark. From Theorem 3.2 (1), \( d_n : \mathrm{Vec}_{O(2)}(V_2, V_n) \rightarrow \mathrm{Vec}_{D_n}(V_2, V_n) \) is an injection.

Let \( \varepsilon \) be the 1-dimensional sign representation and let \( \varepsilon^m \) be the direct sum of \( m \) copies of \( \varepsilon \). One can show by direct calculation that the composite map \( \iota_n^2 \circ d_n \) given by
\[
\begin{align*}
\text{Vec}_{O(2)}(V_2, V_n \oplus \mathbb{C}^m \oplus \varepsilon^{m_2}) & \xrightarrow{\iota_n^2} \text{Vec}_{D_n}(V_2, V_n \oplus \mathbb{C}^m \oplus \varepsilon^{m_2}) \quad \overset{d_n}{\longrightarrow} \quad \text{Vec}_{D_n}(X_2, V_n \oplus \mathbb{C}^m \oplus \varepsilon^{m_2})
\end{align*}
\]
is an injection. In fact, since the dimensions of \( V_2/O(2) \) and \( X_2/D_n \) are both equal to 1, the map \( \iota_n^2 \circ d_n \) is a homomorphism of \( \mathbb{C} \)-vector groups. Since the generators of the \( \mathbb{C} \)-vector group \( \text{Vec}_{O(2)}(V_2, V_n \oplus \mathbb{C}^m \oplus \varepsilon^{m_2}) \), which is isomorphic to \( \text{Vec}_{O(2)}(V_2, V_n) \), do not vanish by the homomorphism \( \iota_n^2 \circ d_n \) (cf. [7 VII 4], [38 III 5]), so \( \iota_n^2 \circ d_n \) is injective. The map
\[
\theta_2 : \text{Vec}_{D_n}(V_2, V_n) \rightarrow \text{Vec}_{D_n}(V_2, V_n \oplus \mathbb{C}^m \oplus \varepsilon^{m_2})
\]
sending \( [E] \) to \( [E \oplus \Theta_{\mathbb{C}^m \oplus \varepsilon^{m_2}}] \) induces a map
\[
\text{VAR}_{D_n}(V_2, V_n) \rightarrow \text{VAR}_{D_n}(V_2, V_n \oplus \mathbb{C}^m \oplus \varepsilon^{m_2})
\]
which is the product map with \( \mathbb{C}^m \times \varepsilon^{m_2} \).

**Theorem 3.3.** Let \( n \) be odd and let \( m_1 \) and \( m_2 \) be non-negative integers. Then the map
\[
\text{VAR}_{O(2)}(V_2, V_n) \rightarrow \text{VAR}_{D_n}(V_2, V_n)
\]
induced by \( \theta_2 \circ d_n \) is an injection.

**Proof.** Let \( E, E' \in \text{Vec}_{O(2)}(V_2, V_n) \) be such that \( E \times \mathbb{C}^{m_1} \times \varepsilon^{m_2} \cong E' \times \mathbb{C}^{m_1} \times \varepsilon^{m_2} \) as \( D_n \)-varieties. Then applying Proposition 2.3 to \( E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}} \) and \( E' \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}} \) with \( H = \mathbb{Z}/2\mathbb{Z} \) (the reflection subgroup) and \( K = D_n \), we have
\[
(E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}})|_{X_2} \cong (c^*E' \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}})|_{X_2}
\]
as $D_n$-vector bundles, where $c$ is a scalar multiplication of $V_2$. Since $i_k^* \circ d_n$ is injective, $E \oplus \Theta_{C^{m_1} \oplus C^{m_2}} \cong e^* E' \oplus \Theta_{C^{m_1} \oplus C^{m_2}}$ as $O(2)$-vector bundles. Since the Whitney sum with $\Theta_{C^{m_1} \oplus C^{m_2}}$ induces an isomorphism

$$\text{Vec}_{O(2)}(V_2, V_n) \cong \text{Vec}_{O(2)}(V_2, V_n \oplus C^{m_1} \oplus \varepsilon^{m_2}),$$

it follows that $E \cong e^* E'$ as $O(2)$-vector bundles, and the assertion follows. \hfill \Box

Remark. One of the first examples of non-linearizable actions by Schwarz is the $O(2)$-action on the total space of the non-trivial $E \in \text{Vec}_{O(2)}(V_2, V_3)$. By Theorem 3.2 (3), the action of $D_3$ on $E$ is non-linearizable. Furthermore, by Theorem 3.3, the $D_3$-action on $E \times C^{m_1} \times \varepsilon^{m_2}$ remains non-linearizable (cf. [3]). Since the map $\text{Vec}_{O(2)}(V_2, V_n) \to \text{Vec}_{O(2)}(V_2, V_n \oplus V_1)$ sending $[E]$ to $[E \oplus V_1]$ is trivial [20], the $D_3$-action on $E \times V_1$ is linearizable.

By a method similar to the proof of Theorem 3.2, we can show the following.

**Theorem 3.4 (cf. [16]).** Let $m$ and $n$ be even and $m \leq n/4$.

1. The composite map $i_1^* \circ d_n : \text{Vec}_{O(2)}(V_1, V_m) \to \text{Vec}_{D_n}(X_1, V_m)$ is an isomorphism. Hence, $d_n : \text{Vec}_{O(2)}(V_1, V_m) \to \text{Vec}_{D_n}(V_1, V_m)$ is injective and $i_1^* : \text{Vec}_{D_n}(V_1, V_m) \to \text{Vec}_{D_n}(X_1, V_m)$ is surjective.

2. The map

$$\Phi_{1,m} : \text{VAR}_{D_n}(V_1, V_m) \to \mathbb{P}_s(2i; 1 \leq i \leq m - 1)$$

is surjective.

3. The map $\text{VAR}_{O(2)}(V_1, V_m) \to \text{VAR}_{D_n}(V_1, V_m)$ is injective. Hence, if $E \in \text{Vec}_{O(2)}(V_1, V_m)$ is a non-trivial $O(2)$-vector bundle, then the $D_n$-action on $E$ is non-linearizable.

**Proof.** (1) By [20], $\text{Vec}_{O(2)}(V_1, V_m) \cong C^{m-1}$ and by [3], $\text{Vec}_{D_n}(X_1, V_m) \cong C^{m-1}$. We can show that $i_1^* \circ d_n$ is an isomorphism directly as in the proof of Theorem 3.2 (1).

(2) By [3], $\text{Vec}_{D_n}(X_1, V_m) \cong \bigoplus_{i=1}^{m-1} W(2i)$. From this together with (1), the assertion follows.

(3) follows from (1) and Proposition 3.1. \hfill \Box

Remarks. (1) When $m$ and $n$ are even and $n/4 < m < n/2$, one can show that $\text{Vec}_{D_n}(X_1, V_m) \cong \bigoplus_{i=1}^{n/2-m-1} W(2i)$ [3], and $i_1^* \circ d_n$ is a surjection. Hence $\Phi_{1,m}$ is a surjection from $\text{VAR}_{D_n}(V_1, V_m)$ onto $\mathbb{P}_s(2i; 1 \leq i \leq n/2 - m - 1)$.

(2) When $n$ is even, the Whitney sum maps

$$\text{Vec}_{O(2)}(V_1, V_m) \to \text{Vec}_{O(2)}(V_1, V_m \oplus C^{m_1} \oplus \varepsilon^{m_2})$$

and

$$\text{Vec}_{D_n}(X_1, V_m) \to \text{Vec}_{D_n}(X_1, V_m \oplus C^{m_1} \oplus \varepsilon^{m_2})$$

are trivial (cf. [20], [3]).

(3) Suppose $n$ is odd. Then the map

$$i_1^* \circ d_n : \text{Vec}_{O(2)}(V_1, V_m) \to \text{Vec}_{D_n}(X_1, V_m)$$

is injective and

$$\text{Im}(i_1^* \circ d_n) \cong \bigoplus_{i=1}^{m-1} W(2i)$$
(cf. [18]). Hence, when \((m,n) > 1\), \(\text{VAR}_{O(2)}(V_1, V_m) \rightarrow \text{VAR}_{D_n}(V_1, V_m)\) is injective.

Consider the commutative diagram for \(n\) odd:

\[
\begin{array}{ccc}
\text{Vec}_{D_n}(V_1, V_m) & \xrightarrow{i_1'} & \text{Vec}_{D_n}(X_1, V_m) \\
\theta_1 & & \\
\text{Vec}_{D_n}(V_1, V_m \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) & \xrightarrow{i_1''} & \text{Vec}_{D_n}(X_1, V_m \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})
\end{array}
\]

where the vertical maps are the Whitney sum maps with \(\Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}\). By [18],

\[
\text{Im} i_1'' \cong (\bigoplus_{i=1}^{2m-1} W(i)) \oplus (\bigoplus_{i=1}^{(n-1)/2-2m} W(2m - 1 + 2i))
\]

for \(m < n/4\),

\[
\text{Im} i_1' \cong (\bigoplus_{i=1}^{n-m-2} W(i)) \oplus (\bigoplus_{i=1}^{2m-(n+1)/2} W(n - 2m - 1 + 2i))
\]

for \(n/4 < m < n/2\), and

\[
\text{Im}(i_1'' \circ \theta_1) \cong (\bigoplus_{i=1}^{(n-2m-1)/2} W(2i-1)).
\]

Hence we obtain the following by applying Proposition 2.3.

**Theorem 3.5.** Suppose that \(n\) is odd and \((m,n) > 1\). Then the image of \(\Phi_{1,m}\) is isomorphic to the weighted projective space \(\mathbb{P}_*(((n-5)/2)\) with a vertex. The space \(\mathbb{P}_*((n-5)/2)\) is of dimension \((n-5)/2\) and contains the weighted projective space \(\mathbb{P}(2i-1; 1 \leq i \leq (n-2m-1)/2)\) whose inverse image under \(\Phi_{1,m}\) consists of elements \(E\) such that the \(D_n\)-action on \(E \times \mathbb{C}^{m_1} \times \varepsilon^{m_2}\) is non-linearizable.

**Remark.** Mederer [18] showed that \(\text{Vec}_{D_n}(V_1, V_1) \cong \Omega_C\), the module of Kähler differentials of \(C\) over \(Q\), and furthermore, there is a surjection from \(\text{Ker} i_1'\) in the above diagram for \(n \geq 5\) to \(\text{Vec}_{D_n}(V_1, V_1)\). Hence \(\text{Vec}_{D_n}(V_1, V_m)\) (\(n\) odd; \(n \geq 5\)) contains a space of uncountably-infinite dimension. Proposition 2.3 is, to our regret, not useful for classifying the \(D_n\)-actions derived from \(\text{Ker} i_k\) or \(\text{Vec}_{D_1}(V_1, V_1)\).

Suppose \(n\) is odd, and classify the \(D_n\)-actions derived from \(\text{Vec}_{D_n}(V_2 \oplus \varepsilon^m, V_n)\). By applying Proposition 2.3 for \(H = \mathbb{Z}/2\mathbb{Z}\) and \(K = D_n\), we obtain a surjection from \(\text{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n)\) to the orbit space of \(\text{Im} i_{2,m}''\) under the action of \(\mathbb{C}^*\), where \(i_{2,m}'' : \text{Vec}_{D_n}(V_2 \oplus \varepsilon^m, V_n) \rightarrow \text{Vec}_{D_n}(X_2, V_n)\) is the restriction induced by \(i_{2,m} : X_2 \rightarrow V_2 \oplus \varepsilon^m\). Let \(i_m : V_2 \rightarrow V_2 \oplus \varepsilon^m\) be the inclusion. Then \(i_{2,m}'' = i_{2,m}'\). Since \(i_{2,m}'\) is a surjection, \(\text{Im} i_{2,m}'' = \text{Im} i_{2,m}'\). Since \(\text{Im} i_{2,m}' \cong \bigoplus_{i=1}^{(n-1)/2} W(2i-1)\) (cf. the proof of Theorem 3.2 (2)), we have a surjection

\[
\text{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n) \rightarrow \mathbb{P}_i((2i-1; 1 \leq i \leq (n-1)/2).
\]

**Theorem 3.6.** Let \(m\) be a non-negative integer and let \(n\) be odd. Then there is a surjection from \(\text{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n)\) onto \(\mathbb{P}_i((2i-1; 1 \leq i \leq (n-1)/2).
\]

**Remark.** Let \(l\) be a non-negative integer and let \((m,n) > 1\). Then one obtains a similar result for \(\text{VAR}_{D_n}(V_1 \oplus \varepsilon^l, V_m)\).
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Mathematical Science II, Himeji Institute of Technology, 2167 Shosha, Himeji 671-2201, Japan
E-mail address: kayo@sci.himeji-tech.ac.jp