NONLINEARIZABLE ACTIONS OF DIHEDRAL GROUPS
ON AFFINE SPACE

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Abstract. Let $G$ be a reductive, non-abelian, algebraic group defined over $\mathbb{C}$. We investigate algebraic $G$-actions on the total spaces of non-trivial algebraic $G$-vector bundles over $G$-modules with great interest in the case that $G$ is a dihedral group. We construct a map classifying such actions of a dihedral group in some cases and describe the spaces of those non-linearizable actions in some examples.

1. Introduction

Let $G$ be a reductive complex algebraic group. When $G$ is non-abelian, it is well-known that there exist non-linearizable actions of $G$ on complex affine space $\mathbb{A}^n$ for $n \geq 4$, i.e., algebraic actions of $G$ on $\mathbb{A}^n$ which are not conjugate to linear actions under polynomial automorphisms of $\mathbb{A}^n$. It is remarkable that non-linearizable actions on $\mathbb{A}^n$ known so far are all obtained from non-trivial algebraic $G$-vector bundles over $G$-modules. An algebraic $G$-vector bundle over a $G$-variety $X$ is defined to be an algebraic vector bundle $p : E \to X$, where $E$ is a $G$-variety, the projection $p$ is $G$-equivariant, and the morphism induced by $g \in G$ from $p^{-1}(x)$ to $p^{-1}(gx)$ is linear for all $g$ and $x \in X$. An algebraic $G$-vector bundle is called trivial if it is isomorphic to a product bundle $X \times Q \to X$ for some $G$-module $Q$. A total space of an algebraic $G$-vector bundle over a $G$-module is an affine space by the affirmative solution to the Serre conjecture by Quillen [19] and Suslin [21]. Thus, the $G$-action on a total space $E$ of a non-trivial $G$-vector bundle over a $G$-module is a candidate for a non-linearizable action on affine space. There are a couple of known conditions for such an action to be non-linearizable (Bass and Haboush [1], M. Masuda and Petrie [15]). Schwarz [20] (Kraft and Schwarz [7] for details) first showed that an algebraic $G$-vector bundle over a $G$-module can be non-trivial when the algebraic quotient of $P$ is of one dimension, and that there exist families of non-linearizable actions on affine space, by using the above conditions. After Schwarz, lots of examples of non-trivial algebraic $G$-vector bundles have been presented, and it turns out that many of the $G$-actions on their total spaces are non-linearizable (Knop [5], M. Masuda, Moser-Jauslin and Petrie [11], M. Masuda and Petrie [16]). For abelian groups, there are no known examples of non-linearizable actions on complex affine space. In fact, for an abelian group $G$, every algebraic $G$-vector bundle over
a $G$-module becomes trivial by the result of M. Masuda, Moser-Jauslin and Petrie [12], so, we cannot obtain non-linearizable actions from $G$-vector bundles. There are some affirmative results for the linearizability for torus actions (e.g. Bialynicki-Birula [2], Kaliman, Koras, Makar-Limanov and Russell [11]); however, it remains open whether or not every algebraic action of an abelian group on $\mathbb{A}^n$ ($n \geq 4$) is linearizable. Especially for a finite abelian group $G$, e.g. for a cyclic group $\mathbb{Z}/n\mathbb{Z}$, we never know even whether any $G$-action on $\mathbb{A}^3$ is linearizable or not.

For finite groups, M. Masuda and Petrie [11] showed that there exists a family of non-linearizable actions of a dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for $n$ even and $\geq 18$ on $\mathbb{A}^3$. They considered $D_n$-actions derived from algebraic $D_n$-vector bundles which become trivial by adding certain trivial bundles, and showed that those actions form a family in some cases. Later, Mederer [13] showed that non-trivial algebraic $D_n$-vector bundles form a huge family of infinite dimension for $n$ odd and $\geq 3$. In this article, we investigate $G$-actions derived from non-trivial algebraic $G$-vector bundles. We are most interested in the case that $G$ is a dihedral group. We present a new condition for such $D_n$-actions to be non-linearizable and construct a map which classifies such non-linearizable $D_n$-actions without imposing triviality on $D_n$-vector bundles under the addition of certain trivial bundles. We also describe the spaces of those non-linearizable $D_n$-actions in some examples.

2. Families of non-linearizable actions

Let $G$ be a reductive, non-abelian algebraic group and let $Z$ be an affine $G$-variety. We denote by $\mathbb{C}[Z]$ the coordinate ring of $Z$ and by $\mathbb{C}[Z]^G$ the ring of invariants. The algebraic quotient $Z//G$ is the affine variety defined by $Z//G = \text{Spec} \mathbb{C}[Z]^G$ and the quotient morphism $\pi_Z : Z \to Z//G$ is the morphism corresponding to the inclusion $\mathbb{C}[Z]^G \hookrightarrow \mathbb{C}[Z]$. Let $P$ and $Q$ be $G$-modules and let $X \subset P$ be a $G$-subvariety containing the origin of $P$. We denote by $\text{Vec}_G(X,Q)$ the set of algebraic $G$-vector bundles over $X$ whose fiber over the origin is isomorphic to $Q$, and by $\text{Vec}_G(X,Q)$ the set of $G$-isomorphism classes in $\text{Vec}_G(X,Q)$. An element $E \to X$ of $\text{Vec}_G(X,Q)$ is represented by the total space $E$, and the isomorphism class of $E \in \text{Vec}_G(X,Q)$ is denoted by $[E]$. The set $\text{Vec}_G(X,Q)$ is called trivial if $\text{Vec}_G(X,Q)$ consists of the unique class $[\Theta_Q]$, where $\Theta_Q$ denotes the product bundle with fiber $Q$. When $\dim P//G = 1$, Schwarz [20] showed that $\text{Vec}_G(P,Q)$ has an additive group structure and is isomorphic to a vector group $\mathbb{C}^q$ for a non-negative integer $q$. Mederer [13] (cf. [9]) extended the result of Schwarz to the case where the base space is a $G$-equivariant affine cone $X$ with $\dim X//G = 1$. When $\dim P//G \geq 2$, $\text{Vec}_G(P,Q)$ can be non-trivial and of countably or uncountably infinite dimension ([9], [11], [18]).

We assume that $\text{Vec}_G(P,Q)$ is non-trivial. Let $E \in \text{Vec}_G(P,Q)$. The following are the known conditions for the $G$-action on the total space $E$ to be non-linearizable.

Proposition 2.1. Let $E, E' \in \text{Vec}_G(P,Q)$.

1. ([15]) Suppose that there exists a subgroup $H$ of $G$ such that $(P \oplus Q)^H = P$. Then $E$ and $E'$ are isomorphic as $G$-varieties if and only if $E$ and the pull-back $\varphi^*E'$ are isomorphic as $G$-vector bundles for some $G$-automorphism $\varphi$ of $P$.

2. ([11]) If the Whitney sum $E \oplus \Theta_P$ is non-trivial, then the $G$-action on $E$ is non-linearizable.
Let $\text{VAR}_G(P, Q)$ be the set of $G$-isomorphism classes of affine $G$-spaces represented as the total spaces of elements of $\text{Vec}_G(P, Q)$. The group $\text{Aut}(P)^G$ of $G$-equivariant automorphisms of $P$ acts on $\text{Vec}_G(P, Q)$ by pull-backs. There exists a surjection $\Psi$ from the orbit space of $\text{Vec}_G(P, Q)$ under the action of $\text{Aut}(P)^G$ to $\text{VAR}_G(P, Q)$. Under the assumption in Proposition 2.1 (1), $\Psi$ is an isomorphism.

**Example 2.1.** Let $G = O(2) = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ and let $V_m$ ($m \geq 1$) be a two-dimensional $O(2)$-module such that

$$\lambda(x, y) = (\lambda^m x, \lambda^{-m} y) \quad \text{for} \quad \lambda \in \mathbb{C}^*,$$

$$\tau(x, y) = (y, x) \quad \text{for the generator} \quad \tau \in \mathbb{Z}/2\mathbb{Z}.$$  

Then $V_m/O(2) = \text{Spec} \mathbb{C}[t] = \mathbb{A}^1$, where $t = xy$, and $\text{Aut}(V_m)^G = \mathbb{C}^*$, namely, $\text{Aut}(V_m)^G$ consists of scalar multiplications.

Let $n$ be odd. Then $\text{Vec}_G(V_2, V_n) \cong \mathbb{C}^{(n-1)/2}$ and the Whitney sum with $\Theta_{V_2}$ induces an isomorphism between $\text{Vec}_G(V_2, V_n)$ and $\text{Vec}_G(V_2, V_n \oplus V_2)$ ([20]). By Proposition 2.1 (1) or (2), if $E \in \text{Vec}_{O(2)}(V_2, V_n)$ is non-trivial, then the $O(2)$-action on $E$ is non-linearizable. We shall describe $\text{VAR}_{O(2)}(V_2, V_n)$. Since $(V_2 \oplus V_n)^{\mathbb{Z}/2\mathbb{Z}} = V_2$, where $\mathbb{Z}/2\mathbb{Z}$ is a subgroup of $\mathbb{C}^* \subset O(2)$, it follows from Proposition 2.1 (1) that

$$\text{VAR}_{O(2)}(V_2, V_n) \cong \text{Vec}_{O(2)}(V_2, V_n)/\mathbb{C}^*.$$  

In order to look at the action of $\text{Aut}(V_2)^G = \mathbb{C}^*$ on $\text{Vec}_G(V_2, V_n)$, recall the isomorphism $\text{Vec}_G(V_2, V_n) \cong \mathbb{C}^{(n-1)/2}$ . For the details, we refer to Kraft and Schwarz [7]. Let $F = \pi_{V_2}^{-1}(1)$, which is the $G$-subvariety of $V_2$ defined by $xy = 1$. Then $F \cong G/H$, where $H = \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, and $V_n$ is multiplicity-free with respect to $H$, namely, each irreducible $H$-module occurs in $V_n$ with multiplicity at most one when $V_n$ is viewed as an $H$-module. We set $m = \text{Mor}(F, \text{End} V_n)^G$, the module of $G$-equivariant morphisms from $F$ to $\text{End} V_n$. Let $B = \text{Spec} \mathbb{C}[t]$ be the double cover of $\mathbb{A}^1 = \text{Spec} \mathbb{C}[t]$, where $s^2 = t$. Then the group $\Gamma := \{\pm 1\}$ acts on $B$ and on $F$ by scalar multiplication. We denote by $B \times_\Gamma F$ the quotient of $B \times F$ by $\Gamma$ which acts by $(b, f) \mapsto (b\gamma, \gamma^{-1}f)$ for $\gamma \in \Gamma$, $b \in B$, and $f \in F$. The group $G$ acts on $B \times_\Gamma F$ through $F$. We define a $G$-equivariant morphism $\varphi$ by

$$\varphi : B \times_\Gamma F \rightarrow V_2, \quad [b, f] \mapsto bf.$$  

which is a $G$-isomorphism from $(B - \{0\}) \times_\Gamma F$ onto $V_2 - \pi_{V_2}^{-1}(0)$. Note that $\mathbb{C}[B \times_\Gamma F]^G \cong \mathbb{C}[B]^G = \mathbb{C}[t] = \mathbb{C}[V_2]^G$. The morphism $\varphi$ induces a homomorphism

$$\varphi_# : \text{Mor}(V_2, \text{End} V_n)^G \rightarrow \text{Mor}(B \times_\Gamma F, \text{End} V_n)^G = \text{Mor}(B, \mathfrak{m})^F =: \mathfrak{m}(B)^F.$$  

The modules $\text{Mor}(V_2, \text{End} V_n)^G$ and $\mathfrak{m}(B)^F$ are finite free modules over $\mathbb{C}[t]$. In fact, a basis of $\text{Mor}(V_2, \text{End} V_n)^G \cong (\mathbb{C}[V_2] \otimes \text{End} V_n)^G$ over $\mathbb{C}[t]$ is written in a matrix form as

$$\begin{cases} 
A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 
A_1 = \begin{pmatrix} 0 & x^n \\ y^n & 0 \end{pmatrix} 
\end{cases}$$

and a basis of $\mathfrak{m}(B)^F \cong (\mathbb{C}[s] \otimes \mathfrak{m})^F$ over $\mathbb{C}[t]$ is

$$\begin{cases} 
C_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 
C_1 = s(A_1|_F) 
\end{cases}.$$
The module $\text{Mor}(V_2, \text{End} V_n)^G$ (resp. $\text{m}(\mathbb{B})^\Gamma$) inherits a grading from $\mathbb{C}[V_2]$ (resp. $\mathbb{C}[s]$), and $\varphi_*$ is a homomorphism of degree 0. Let $\text{Mor}(V_2, \text{End} V_n)^G$ (resp. $\text{m}(\mathbb{B})^\Gamma$) be the submodule of $\text{Mor}(V_2, \text{End} V_n)$ (resp. $\text{m}(\mathbb{B})$) consisting of elements with positive degrees. Then $\text{Vec}_G(V_2, V_n)$ is isomorphic to the quotient module $\text{m}(\mathbb{B})^\Gamma/\varphi_* \text{Mor}(V_2, \text{End} V_n)^G$. Since 

$$\varphi_*(A_0) = C_0 \quad \text{and} \quad \varphi_*(A_1) = t^{\frac{n-1}{2}} C_1,$$

the set $\{t^{i-1}C_1; 1 \leq i \leq \frac{n-1}{2}\}$ forms a $\mathbb{C}$-basis of $\text{m}(\mathbb{B})^\Gamma/\varphi_* \text{Mor}(V_2, \text{End} V_n)^G$, and hence

$$\text{Vec}_G(V_2, V_n) \cong \text{m}(\mathbb{B})^\Gamma/\varphi_* \text{Mor}(V_2, \text{End} V_n)^G \cong \mathbb{C}^{\frac{n-1}{2}}.$$

Note that $\text{deg}(t^{i-1}C_1) = 2i - 1$. The scalar multiplication on $V_2$ corresponds to a scalar multiplication on $\mathbb{B}$ via $\varphi$. Hence $\text{Vec}_G(V_2, V_n) \cong \bigoplus_{i=1}^{(n-1)/2} W(2i - 1)$ as a module of $\text{Aut}(V_2)^G = \mathbb{C}^*$, where $W(i)$ denotes the representation space of $\mathbb{C}^*$ with weight $i$. Thus we obtain by Proposition 2.1 (1) that

$$\text{VAR}_{O(2)}(V_2, V_n) \cong \left( \bigoplus_{i=1}^{(n-1)/2} W(2i - 1) \right) / \mathbb{C}^* \cong \mathbb{P}_s(2i - 1; 1 \leq i \leq \frac{n-1}{2}).$$

Here $\mathbb{P}_s(2i - 1; 1 \leq i \leq (n - 1)/2)$ consists of the “vertex” $\star$, and the weighted projective space $\mathbb{P}(2i - 1; 1 \leq i \leq (n - 1)/2)$ of dimension $(n - 3)/2$ with weight $2i - 1$ for $1 \leq i \leq (n - 1)/2$. The “vertex” corresponds to the linearizable action and the weighted projective space to non-linearizable actions (cf. [16]).

**Example 2.2.** Let $G = \text{SL}_2$ and let $R_n$ be the $\text{SL}_2$-module of binary forms of degree $n \geq 1$. Then $\text{Vec}_G(R_2, R_{n}) \cong \mathbb{C}^{[(n-1)^2/4]}$ and $\text{Aut}(R_2)^G = \mathbb{C}^*$ ([20], [7]). As a module of $\mathbb{C}^* = \text{Aut}(R_2)^G$, $\text{Vec}_G(R_2, R_n)$ is isomorphic to $\bigoplus_{i=1}^{n-2} m_i W(i)$ with multiplicity $m_i = \lfloor \frac{mi}{2} \rfloor$. Suppose $n$ is odd. Then $(R_2 \oplus R_n)^{2n/2n} = R_2$. Hence by Proposition 2.1 (1),

$$\text{VAR}_{\text{SL}_2}(R_2, R_n) \cong \left( \bigoplus_{i=1}^{n-2} m_i W(i) \right) / \mathbb{C}^* \cong \mathbb{P}_s(i, m_i; 1 \leq i \leq n - 2).$$

In this case, the space of non-linearizable $\text{SL}_2$-actions is isomorphic to the weighted projective space of dimension $[(n - 1)^2/4] - 1$ with weight $i$ of multiplicity $m_i$ for $1 \leq i \leq n - 2$.

**Example 2.3.** Let $G$ be semisimple and let $\mathfrak{g}$ be the adjoint representation of $G$. Let $\Sigma$ be a system of simple roots of $G$ and $F$ an irreducible $G$-module with the highest weight $\chi$. Knop [5] constructed a map associated with $\alpha \in \Sigma$,

$$\Phi_\alpha : \text{Vec}_G(\mathfrak{g}, F) \to \text{Vec}_{\text{SL}_2}(R_2, R_m),$$

where $m = \langle \chi, \alpha \rangle$. The map $\Phi_\alpha$ is surjective if the $\alpha$-string of $\chi$ is regular ([5], [4]). We recall the construction of $\Phi_\alpha$. Let $T \subset G$ be a maximal torus with the Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let $L$ be the subgroup of $G$ generated by $T$ and the root subgroups $U_\alpha$ and $U_{-\alpha}$. We denote by $L'$ the commutator subgroup of $L$ and by $Z$ the center of $L$. Then $L = L/Z$, and $L'$ is isomorphic to $\text{SL}_2$ or $SO_3$. Let $\mathfrak{l}$ be the Lie algebra of $L$. Then $\mathfrak{l}$ is isomorphic to $\mathfrak{sl}_2 \oplus \mathbb{C}^{n-1}$ as an $L'$-module,
where \( n = \text{rank } \mathfrak{t} \). For \( E \in \text{Vec}_G(\mathfrak{g}, F) \), the restricted bundle \( E|_i \) is an \( L \)-vector bundle with fiber \( F' \) which is \( F \) viewed as an \( L \)-module. Take a \( \xi_0 \in \mathfrak{t} \) so that the centralizer of \( \xi_0 \) is exactly \( L \), and fix it. Then \( \mathfrak{a} := \xi_0 + \text{Lie} L' \subseteq \mathfrak{g} \) is \( L \)-stable and isomorphic to \( \mathfrak{sl}_2 \cong R_2 \) as an \( L' \)-variety. Since \( Z \) acts trivially on \( \mathfrak{t} \), hence on \( \mathfrak{a} \), \( E|_a \) decomposes to a Whitney sum of eigenbundles of \( Z \). Let \( (E|_a)_\chi \) be the eigenbundle corresponding to the restricted weight of \( \chi \) onto \( Z \). Then the \( L' \)-vector bundle \( (E|_a)_\chi \) is considered as an element of \( \text{Vec}_{SL_2}(R_2, R_m) \). The map \( \Phi_\alpha \) is defined by \( \Phi_\alpha(E) = (E|_a)_\chi \). By the construction of \( \Phi_\alpha \), \( \Phi_\alpha \) decomposes to the maps

\[
\phi_\alpha : \text{Vec}_G(\mathfrak{g}, F) \to \text{Vec}_L(1, F') \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)
\]

and

\[
\phi_{\xi_0} : \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m) \to \text{Vec}_{SL_2}(R_2, R_m).
\]

From the choice of \( \xi_0 \), \( \phi_{\xi_0} \) is surjective. In fact, \( \phi_{\xi_0} \circ pr^* = id \), where

\[
pr^* : \text{Vec}_{SL_2}(R_2, R_m) \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)
\]

is the induced map from the projection \( R_2 \oplus \mathbb{C}^{n-1} \to R_2 \). When \( \Phi_\alpha \) is surjective, \( \phi_\alpha \) is also surjective since \( \phi_{\xi_0} \) is surjective. By \([9]\),

\[
\text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m) \cong \text{Vec}_{SL_2}(R_2, R_m) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{C}^{n-1}].
\]

Hence we obtain the following.

**Theorem 2.2.** Under the notionation above, if the \( \alpha \)-string of \( \chi \) is regular, then

\[
\phi_\alpha : \text{Vec}_G(\mathfrak{g}, F) \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)
\]

\[
\cong \mathbb{C}^{(m-1)^2/4} \otimes_{\mathbb{C}} \mathbb{C}[y_1, \ldots, y_{n-1}]
\]

is surjective. Furthermore, if there is a subgroup \( H \) such that \( (\mathfrak{g} \oplus F)^H = \mathfrak{g} \), then \( \phi_\alpha \) induces a surjection

\[
\text{VAR}_G(\mathfrak{g}, F) \to (\mathbb{C}^{(m-1)^2/4} \otimes_{\mathbb{C}} \mathbb{C}[y_1, \ldots, y_{n-1}]) / \mathbb{C}^*
\]

where \( \mathbb{C}^* \) acts on \( \mathbb{C}^{(m-1)^2/4} \) with weight \( i \) of multiplicity \( m_i = [(m - i)/2] \) and on \( y_i \) with weight 1.

**Proof.** The first assertion follows from the above observation. For the second assertion, note that \( \text{Aut}(\mathfrak{g})^G = \mathbb{C}^* \) \([7]\). From Proposition 2.1 (1), there is an isomorphism \( \text{VAR}_G(\mathfrak{g}, F) \cong \text{Vec}_G(\mathfrak{g}, F)/\mathbb{C}^* \). Hence \( \phi_\alpha \) induces a surjection

\[
\text{VAR}_G(\mathfrak{g}, F) \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)/\mathbb{C}^*.
\]

The assertion follows from the statement in Example 2.2. \( \square \)

**Remark.** When the \( \alpha \)-string of \( \chi \) is singular, the image of \( \Phi_\alpha \) contains a subspace of dimension \( [m/2][(m/2) - 1]/2 \) \([11]\).

By Theorem 2.2 and its remark, \( \text{Vec}_G(\mathfrak{g}, F) \) is of infinite dimension if \( m \geq 4 \) and \( n \geq 2 \). Furthermore, if \( (\mathfrak{g} \oplus F)^H = \mathfrak{g} \) for a subgroup \( H \), then \( \text{VAR}_G(\mathfrak{g}, F) \) is of infinite dimension.

Now, we give a new condition for the \( G \)-action on \( E \in \text{Vec}_G(P, Q) \) to be non-linearizable, which is used as a basic fact in the next section.

**Proposition 2.3.** Let \( E, E' \in \text{Vec}_G(P, Q) \). Suppose that there exist reductive subgroups \( H \) and \( K \) such that \( H \subset K \) and satisfying the following conditions;

1. \( Q^K = Q^H \).
2. \( \dim P^H = 1 \) and \( \dim P^K = 0 \).
If \( E \cong E' \) as \( G \)-varieties, then the restricted bundles \( E|_X \) and \( (c^* E')|_X \) are isomorphic as \( G \)-vector bundles, where \( X = G \cdot P^H \) and \( c \) is a scalar multiplication on \( P \). In particular, if \( E|_X \) is a non-trivial \( G \)-vector bundle, then the \( G \)-action on \( E \) is non-linearizable.

**Proof.** Let \( \phi : E \cong E' \) be an isomorphism of \( G \)-varieties. Then \( \phi \) restricts to an isomorphism \( \phi_H : E^H \cong E'^H \). Since \( E^H \) and \( E'^H \) are trivial \( (H-) \)vector bundles over \( P^H \) with fiber \( Q^H \) (cf. [8]), it follows that \( E^H \cong E'^H \cong P^H \times Q^H \). Similarly, \( E^K \cong E'^K \cong Q^K = Q^H \) since \( \dim Q^K = 0 \). Since \( E^K \) (resp. \( E'^K \)) is a subbundle of \( E^H \) (resp. \( E'^H \)), we get \( E^H = E^K \times P^H \) and \( E'^H = E'^K \times P^H \). Let \( x \) be a coordinate variable of \( P^H \times Q^H \) such that \( P^H = \text{Spec} \mathbb{C}[x] \). Then the ideal corresponding to \( E^K \) is \((x)\), and the ideal for \( E'^K \) is the same. Since \( \phi \), hence \( \phi_H \), maps \( E^K \) to \( E'^K \) isomorphically, the ideal \((x)\) must be fixed by the algebra isomorphism corresponding to \( \phi_H \). This implies that \( \phi_H \), hence \( \phi \), induces an isomorphism \( \bar{c} \) on \( P^H \) such that \( p_H^t \circ \phi_H = \bar{c} \circ p_H \), where \( p_H : E^H \to P^H \) and \( p_H^t : E'^H \to P^H \) are projections. Note that \( \bar{c} \) is a scalar multiplication on \( P^H \). Hence \( \bar{c} \) extends to a scalar multiplication \( c \) on \( P \). Since the following diagram commutes, \( \phi \) restricts to a variety isomorphism \( E|_{P^H} \cong E'|_{P^H} \):

\[
\begin{array}{ccc}
E & \xrightarrow{\phi} & E' \\
\downarrow & & \downarrow \\
E^H & \xrightarrow{\phi_H} & E'^H \\
\downarrow & & \downarrow \\
P^H & \xrightarrow{c} & P^H
\end{array}
\]

where the diagonal arrows are inclusions. Furthermore, since \( \phi \) is a \( G \)-isomorphism, \( \phi \) in fact restricts to a \( G \)-isomorphism \( \phi_X : E|_X \to E'|_X \) such that \( p_X \circ \phi_X = (c|_X) \circ p_X \), where \( p_X : E|_X \to X \) and \( p_X^t : E'|_X \to X \) are projections. Thus \( E|_X \cong (c^* E')|_X \) as \( G \)-vector bundles, and the assertion follows. \( \square \)

Proposition 2.3 enables us to classify elements of \( \text{VAR}_G(P, Q) \).

**Corollary 2.4.** Under the assumption and notation in Proposition 2.3, there exists a map

\[
\Phi : \text{VAR}_G(P, Q) \to \text{Vec}_G(X, Q)/\mathbb{C}^*,
\]

where the target space is the orbit space of \( \text{Vec}_G(X, Q) \) under the action of \( \mathbb{C}^* \), which is a subgroup of \( \text{Aut}(X)^G \) consisting of scalar multiplications.

When \( H \) is an isotropy group of a point \( x \in P \) whose orbit is closed, then \( X = \overline{G \cdot P^H} \) is a \( G \)-equivariant affine cone in \( P \) with \( \dim X/\overline{G} = 1 \). In this case, \( \text{Vec}_G(X, Q) \) is isomorphic to a finite-dimensional module of \( \mathbb{C}^* \subset \text{Aut}(X)^G \) ([13]). Hence \( \text{Vec}_G(X, Q)/\mathbb{C}^* \) is isomorphic to a weighted projective space with a “vertex”.

**Example 2.4.** Let \( G = O(2) \) and consider \( \text{Vec}_{O(2)}(V_1, V_m) \). Then applying Proposition 2.3 for \( H = \mathbb{Z}/2\mathbb{Z} \) (the reflection subgroup) and \( K = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), we obtain \( X = V_1 \), and hence, a map \( \text{VAR}_{O(2)}(V_1, V_m) \to \text{Vec}_{O(2)}(V_1, V_m)/\mathbb{C}^* \), which is an isomorphism. Since \( \text{Vec}_{O(2)}(V_1, V_m) \cong \bigoplus_{i=1}^{m-1} W(2i) \) ([7]), we have
nonlinearizable actions of dihedral groups

Let $G$ be a dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ for $n > 2$. By considering $D_n$ as a finite subgroup of $O(2) = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$, an $O(2)$-module $V_m$ is naturally considered as a $D_n$-module. Since $V_m \cong V_{n-m}$ as a $D_n$-module, we may assume $m \leq n/2$; otherwise $m = n$. Let $k$ be a positive integer such that $(k, n) = 1$ and $k \leq n/2$. Let \{x, y\} be a coordinate system of $V_k$ as in Example 2.1. Then $V_k / D_n = \text{Spec } \mathbb{C}[t, u]$, where $t = xy$ and $u = x^2 + y^n$. Let $X_k = D_n \cdot V_k^{\mathbb{Z}/2\mathbb{Z}}$, where $\mathbb{Z}/2\mathbb{Z}$ is the reflection subgroup. Then $X_k$ is the $D_n$-subvariety of $V_k$ defined by $x^n - y^n = 0$ for $n$ odd, and $x^{n/2} - y^{n/2} = 0$ for $n$ even. The algebraic quotient of $X_k$ is

$$X_k / D_n = \begin{cases} \text{Spec } \mathbb{C}[t, u]/(u^2 - 4t^n) & \text{for } n \text{ odd}, \\ \text{Spec } \mathbb{C}[t] & \text{for } n \text{ even.} \end{cases}$$

The variety $X_k$ is the $D_n$-equivariant affine cone in $V_k$ with one-dimensional quotient. Hence $\text{Vec}_{D_n}(X_k, V_m) \cong \mathbb{C}^q$ for some $q$ (cf. [18], [8]).

We shall classify elements of $\text{VAR}_{D_n}(V_k, V_m)$ under a certain condition.

**Proposition 3.1.** Let $E, E' \in \text{Vec}_{D_n}(V_k, V_m)$ and let $X_k$ be as above. Suppose that $(m, n) > 1$. Then, if $E \cong E'$ as $D_n$-varieties, then the restricted bundles $E|_{X_k}$ and $(c^*E')|_{X_k}$ are isomorphic as $D_n$-vector bundles, where $c$ is a scalar multiplication on $V_k$.

**Proof.** By taking $H = \mathbb{Z}/2\mathbb{Z}$ (the reflection subgroup) and $K = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $p = (m, n)$ in Proposition 2.3, the assertion follows. \qed

Under the assumption in Proposition 3.1, there exists a map

$$\Phi_{k,m} : \text{VAR}_{D_n}(V_k, V_m) \to \text{Vec}_{D_n}(X_k, V_m)/\mathbb{C}^*.$$

Let $i_k^* : \text{Vec}_{D_n}(V_k, V_m) \to \text{Vec}_{D_n}(X_k, V_m)$ be the restriction induced by the inclusion $i_k : X_k \hookrightarrow V_k$. There exists a sequence

$$\text{Vec}_{O(2)}(V_k, V_m) \xrightarrow{d_n} \text{Vec}_{D_n}(V_k, V_m) \xrightarrow{i_k^*} \text{Vec}_{D_n}(X_k, V_m),$$

where $d_n$ is the group restriction.

**Theorem 3.2** (cf. [16]). Let $n$ be odd, and let $k = 2$ and $m = n$ in the notation above.

1. The composite map $i_2^* \circ d_n : \text{Vec}_{O(2)}(V_2, V_n) \to \text{Vec}_{D_n}(X_2, V_n)$ is injective and

$$\text{Im}(i_2^* \circ d_n) \cong \mathbb{C}^{n-1}.$$

2. The image of $\Phi_{2,n}$ is isomorphic to $\mathbb{P}_*(2i-1; 1 \leq i \leq (n-1)/2)$. 

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(3) The map $\text{VAR}_{O(2)}(V_2, V_n) \to \text{VAR}_{D_n}(V_2, V_n)$ is injective. Hence, if $E \in \text{Vec}_{O(2)}(V_2, V_n)$ is a non-trivial $O(2)$-vector bundle, then the $D_n$-action on $E$ is non-linearizable.

Proof. (1) By applying the method of Mederer, we can show that $\text{Vec}_{D_n}(X_2, V_n)$ is isomorphic to a vector group $\mathbb{C}^{n-1}$. For the detailed argument, we refer to Mederer [15]. We shall give a basis of $\text{Vec}_{D_n}(X_2, V_n) \cong \mathbb{C}^{n-1}$. We use the notation in Example 2.1 and denote $X_2$ simply by $X$. Let $\nu : B = \text{Spec} \mathbb{C}[s] \to X//D_n = \text{Spec} \mathbb{C}[t, u]/(u^2 - 4t^n)$ be the normalization, where $t = s^2$ and $u = 2s^n$, and let $F_X = \pi_X^{-1}(\nu(1))$. Then $F_X \cong D_n/H'$, where $H' = \mathbb{Z}/2\mathbb{Z}$ (the reflection subgroup) and $V_n$ is multiplicity free with respect to $H'$. There is a $D_n$-equivariant morphism

$$\varphi^X : B \times F_X \to X,$$

$$(b, f) \mapsto bf,$$

which is an isomorphism from $(B - \{0\}) \times F_X$ onto $X - \pi_X^{-1}(\nu(0))$. Note that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{B} \times F_X & \xrightarrow{\varphi^X} & X \\
\downarrow & & \downarrow \\
\mathbb{B} \times F & \xrightarrow{\varphi} & V_2
\end{array}
$$

where the vertical maps are inclusions. Let $m_X = \text{Mor}(F_X, \text{End}_n V_n)$. Then $m_X(\mathbb{B}) := \text{Mor}(\mathbb{B}, m_X)$ is a free $\mathbb{C}[s]$-module with a grading induced from $\mathbb{C}[s]$. The $\mathbb{C}[X]^{D_n}$-module $\text{Mor}(X, \text{End}_n V_n)^{D_n}$, which inherits a grading from $\mathbb{C}[X] \subset \mathbb{C}[V_2]$. Note that $m_X(\mathbb{B})$ is considered as a $\mathbb{C}[X]^{D_n}$-module via $\nu$. The morphism $\varphi^X$ induces

$$\varphi^X : \text{Mor}(X, \text{End}_n V_n)^{D_n} \to \text{Mor}(\mathbb{B} \times F_X, \text{End}_n V_n)^{D_n} = m_X(\mathbb{B}),$$

which is a $\mathbb{C}[X]^{D_n}$-homomorphism of degree $0$. Note that $\text{Mor}(X, \text{End}_n V_n)^{D_n}$ is a finite free module over $\mathbb{C}[X]^{D_n}$. In fact, $\text{Mor}(V_2, \text{End}_n V_n)^{D_n}$ is free over $\mathbb{C}[t, u]$ with a basis

$$\{ \tilde{A}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tilde{A}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\begin{pmatrix} x^n - y^n \\ 0 \\ -(x^n - y^n) \end{pmatrix}, \begin{pmatrix} 0 \\ x^n - y^n \\ 0 \end{pmatrix} \}.$$

Hence $\text{Mor}(X, \text{End}_n V_n)^{D_n}$ is a free module over $\mathbb{C}[t, u]/(u^2 - 4t^n)$ with a basis $\{ \tilde{A}_0, \tilde{A}_1 \}$. Let $\text{Mor}(X, \text{End}_n V_n)^{D_n}(\text{respectively } m_X(\mathbb{B}^1))$ be the submodule of $\text{Mor}(X, \text{End}_n V_n)^{D_n}$ (respectively $m_X(\mathbb{B})$) of elements with positive degrees. Then $\text{Vec}_{D_n}(X, V_n)$ is isomorphic to the quotient module of $m_X(\mathbb{B}^1)$ by $\varphi^X : \text{Mor}(X, \text{End}_n V_n)^{D_n}$.

The module $m_X(\mathbb{B}^1)$ is free over $\mathbb{C}[s]$ with a basis

$$\{ \tilde{C}_0 = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tilde{C}_1 = s\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}.$$

Since $\varphi^X(\tilde{s}\tilde{A}_i) = s\tilde{C}_i$ and $\varphi^X(\tilde{s}\tilde{A}_i) = 2s^{n-1}\tilde{C}_i$ for $i = 0, 1$,

$$\text{Vec}_{D_n}(X, V_n) \cong m_X(\mathbb{B}^1)/\varphi^X : \text{Mor}(X, \text{End}_n V_n)^{D_n} \cong \mathbb{C}^{n-1}.$$
with a basis \( \{ s^{2(j-1)} C_i : i = 0, 1, 1 \leq j \leq (n-1)/2 \} \). The inclusions \( \mathcal{B} \times F_X \hookrightarrow \mathcal{B} \times_F F \) and \( X \hookrightarrow V_2 \) give rise to a homomorphism
\[
\iota : \mathfrak{m}(\mathcal{B})_1^F / \Phi_\# \operatorname{Mor}(V_2, \operatorname{End} V_n)^{(2)} \to \mathfrak{m}_X(\mathcal{B})_1^X / \Phi_\# \operatorname{Mor}(X, \operatorname{End} V_n)^{(2)},
\]
which corresponds to \( i_2^* \circ d_n \). Since \( \iota (t^{-1} C_1) = s^{2((n-1)/2)} C_1 \), it follows that \( i_2^* \circ d_n \) is injective and \( \operatorname{Im}(i_2^* \circ d_n) \cong \mathbb{C}^{(n-1)/2} \) with a basis \( \{ s^{2(j-1)} C_i : 1 \leq j \leq (n-1)/2 \} \).

(2) From Proposition 3.1, there is a map
\[
\Phi_{2,n} : \operatorname{VAR}_{D_n}(V_2, V_n) \to \operatorname{Vec}_{D_n}(X_2, V_n)/\mathbb{C}^*.
\]
From (1), \( \operatorname{Im} i_2^* \) contains a subspace
\[
\bigoplus_{i=1}^{(n-1)/2} W(2i-1).
\]
In fact, \( \operatorname{Im} i_2^* \cong \bigoplus_{i=1}^{(n-1)/2} W(2i-1) \) (cf. [18, III 3,4]). Hence the assertion follows.

(3) follows from (1) and Proposition 3.1.

Remark. From Theorem 3.2 (1), \( d_n : \operatorname{Vec}_{O(2)}(V_2, V_n) \to \operatorname{Vec}_{D_n}(V_2, V_n) \) is an injection.

Let \( \varepsilon \) be the 1-dimensional sign representation and let \( \varepsilon^m \) be the direct sum of \( m \) copies of \( \varepsilon \). One can show by direct calculation that the composite map \( i_2^* \circ d_n \) given by
\[
\operatorname{Vec}_{O(2)}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) \xrightarrow{d_n} \operatorname{Vec}_{D_n}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) \xrightarrow{i_2^*} \operatorname{Vec}_{D_n}(X_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})
\]
is an injection. In fact, since the dimensions of \( V_2//O(2) \) and \( X_2//D_n \) are both equal to 1, the map \( i_2^* \circ d_n \) is a homomorphism of \( \mathbb{C} \)-vector groups. Since the generators of the \( \mathbb{C} \)-vector group \( \operatorname{Vec}_{O(2)}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) \), which is isomorphic to \( \operatorname{Vec}_{O(2)}(V_2, V_n) \), do not vanish by the homomorphism \( i_2^* \circ d_n \) (cf. [17, VII 4], [18, III 5]), so \( i_2^* \circ d_n \) is injective. The map
\[
\theta_2 : \operatorname{Vec}_{D_n}(V_2, V_n) \to \operatorname{Vec}_{D_n}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})
\]
sending \( [E] \) to \( [E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}] \) induces a map
\[
\operatorname{VAR}_{D_n}(V_2, V_n) \to \operatorname{VAR}_{D_n}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})
\]
which is the product map with \( \mathbb{C}^{m_1} \times \varepsilon^{m_2} \).

**Theorem 3.3.** Let \( n \) be odd and let \( m_1 \) and \( m_2 \) be non-negative integers. Then the map
\[
\operatorname{VAR}_{O(2)}(V_2, V_n) \to \operatorname{VAR}_{D_n}(V_2, V_n)
\]
\[
\to \operatorname{VAR}_{D_n}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})
\]
induced by \( \theta_2 \circ d_n \) is an injection.

**Proof.** Let \( E, E' \in \operatorname{Vec}_{O(2)}(V_2, V_n) \) be such that \( E \times \mathbb{C}^{m_1} \times \varepsilon^{m_2} \cong E' \times \mathbb{C}^{m_1} \times \varepsilon^{m_2} \) as \( D_n \)-varieties. Then applying Proposition 2.3 to \( E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}} \) and \( E' \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}} \) with \( H = \mathbb{Z}/2\mathbb{Z} \) (the reflection subgroup) and \( K = D_n \), we have
\[
(E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}})|_{X_2} \cong (c^* E' \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}})|_{X_2}
\]
as $D_n$-vector bundles, where $c$ is a scalar multiplication of $V_2$. Since $i^*_n \circ d_n$ is injective, $E \oplus \Theta_{C^{m_1} \oplus \varepsilon^{m_2}} \cong c^*E' \oplus \Theta_{C^{m_1} \oplus \varepsilon^{m_2}}$ as $O(2)$-vector bundles. Since the Whitney sum with $\Theta_{C^{m_1} \oplus \varepsilon^{m_2}}$ induces an isomorphism

$$\text{Vec}_{O(2)}(V_2, V_n) \cong \text{Vec}_{O(2)}(V_2, V_n \oplus C^{m_1} \oplus \varepsilon^{m_2}),$$

it follows that $E \cong c^*E'$ as $O(2)$-vector bundles, and the assertion follows. \hfill \Box

**Remark.** One of the first examples of non-linearizable actions by Schwarz is the $O(2)$-action on the total space of the non-trivial $E \in \text{Vec}_{O(2)}(V_2, V_3)$. By Theorem 3.2 (3), the action of $D_3$ on $E$ is non-linearizable. Furthermore, by Theorem 3.3, the $D_3$-action on $E \times \mathbb{C}^{m_1} \times \varepsilon^{m_2}$ remains non-linearizable (cf. [3]). Since the map $\text{Vec}_{O(2)}(V_2, V_n) \to \text{Vec}_{O(2)}(V_2, V_n \oplus V_1)$ sending $[E]$ to $[E \oplus \Theta_{V_1}]$ is trivial [20], the $D_3$-action on $E \times V_1$ is linearizable.

By a method similar to the proof of Theorem 3.2, we can show the following.

**Theorem 3.4** (cf. [16]). Let $m$ and $n$ be even and $m \leq n/4$.

1. The composite map $i^*_n \circ d_n : \text{Vec}_{O(2)}(V_1, V_m) \to \text{Vec}_{D_n}(X_1, V_m)$ is an isomorphism. Hence, $d_n : \text{Vec}_{O(2)}(V_1, V_m) \to \text{Vec}_{D_n}(V_1, V_m)$ is injective and $i^*_n : \text{Vec}_{D_n}(V_1, V_m) \to \text{Vec}_{D_n}(X_1, V_m)$ is surjective.

2. The map

$$\Phi_{1,m} : \text{VAR}_{D_n}(V_1, V_m) \to \mathbb{P}_* (2i; 1 \leq i \leq m - 1)$$

is surjective.

3. The map $\text{VAR}_{O(2)}(V_1, V_m) \to \text{VAR}_{D_n}(V_1, V_m)$ is injective. Hence, if $E \in \text{Vec}_{O(2)}(V_1, V_m)$ is a non-trivial $O(2)$-vector bundle, then the $D_n$-action on $E$ is non-linearizable.

**Proof.** (1) By [20], $\text{Vec}_{O(2)}(V_1, V_m) \cong C^{m-1}$ and by [8], $\text{Vec}_{D_n}(X_1, V_m) \cong C^{m-1}$.

We can show that $i^*_n \circ d_n$ is an isomorphism directly as in the proof of Theorem 3.2 (1).

(2) By [8], $\text{Vec}_{D_n}(X_1, V_m) \cong \bigoplus_{i=1}^{m-1} W(2i)$. From this together with (1), the assertion follows.

(3) follows from (1) and Proposition 3.1. \hfill \Box

**Remarks.** (1) When $m$ and $n$ are even and $n/4 < m < n/2$, one can show that $\text{Vec}_{D_n}(X_1, V_m) \cong \bigoplus_{i=1}^{n/2-m-1} W(2i)$ ([8]), and $i^*_n \circ d_n$ is a surjection. Hence $\Phi_{1,m}$ is a surjection from $\text{VAR}_{D_n}(V_1, V_m)$ onto $\mathbb{P}_*(2i; 1 \leq i \leq n/2 - m - 1)$.

(2) When $n$ is even, the Whitney sum maps

$$\text{Vec}_{O(2)}(V_1, V_m) \to \text{Vec}_{O(2)}(V_1, V_m \oplus C^{m_1} \oplus \varepsilon^{m_2})$$

and

$$\text{Vec}_{D_n}(X_1, V_m) \to \text{Vec}_{D_n}(X_1, V_m \oplus C^{m_1} \oplus \varepsilon^{m_2})$$

are trivial (cf. [20], [8]).

(3) Suppose $n$ is odd. Then the map

$$i^*_n \circ d_n : \text{Vec}_{O(2)}(V_1, V_m) \to \text{Vec}_{D_n}(X_1, V_m)$$

is injective and

$$\text{Im}(i^*_n \circ d_n) \cong \bigoplus_{i=1}^{m-1} W(2i)$$
(cf. [18]). Hence, when $(m, n) > 1$, $\text{VAR}_{O(2)}(V_1, V_m) \rightarrow \text{VAR}_{D_n}(V_1, V_m)$ is injective.

Consider the commutative diagram for $n$ odd:

$$
\begin{array}{ccc}
\text{Vec}_{D_n}(V_1, V_m) & \xrightarrow{i^*_1} & \text{Vec}_{D_n}(X_1, V_m) \\
\theta_1 \downarrow & & \downarrow \\
\text{Vec}_{D_n}(V_1, V_m \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) & \xrightarrow{i^*_1} & \text{Vec}_{D_n}(X_1, V_m \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})
\end{array}
$$

where the vertical maps are the Whitney sum maps with $\Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}$. By [18],

$$
\text{Im} i^*_1 \cong (\bigoplus_{i=1}^{2m-1} W(i)) \oplus (\bigoplus_{i=1}^{(n-1)/2-2m} W(2m - 1 + 2i))
$$

for $m < n/4$,

$$
\text{Im} i^*_1 \cong (\bigoplus_{i=1}^{n-2m-1} W(i)) \oplus (\bigoplus_{i=1}^{2m-(n+1)/2} W(n - 2m - 1 + 2i))
$$

for $n/4 < m < n/2$, and

$$
\text{Im}(\tilde{i}^*_1 \circ \theta_1) \cong (\bigoplus_{i=1}^{n-2m-1/2} W(2i - 1)).
$$

Hence we obtain the following by applying Proposition 2.3.

**Theorem 3.5.** Suppose that $n$ is odd and $(m, n) > 1$. Then the image of $\Phi_{1,m}$ is isomorphic to the weighted projective space $\mathbb{P}_s((n - 5)/2)$ with a vertex. The space $\mathbb{P}_s((n - 5)/2)$ is of dimension $(n - 5)/2$ and contains the weighted projective space $\mathbb{P}(2i - 1; 1 \leq i \leq (n - 2m - 1)/2)$ whose inverse image under $\Phi_{1,m}$ consists of elements $E$ such that the $D_n$-action on $E \times \mathbb{C}^{m_1} \times \varepsilon^{m_2}$ is non-linearizable.

**Remark.** Mederer [18] showed that $\text{Vec}_{D_k}(V_1, V_1) \cong \Omega_\mathbb{C}$, the module of Kähler differentials of $\mathbb{C}$ over $\mathbb{Q}$, and furthermore, there is a surjection from $\text{Ker} i^*_1$ in the above diagram for $n \geq 5$ to $\text{Vec}_{D_n}(V_1, V_1)$. Hence $\text{Vec}_{D_n}(V_1, V_m)$ ($n$ odd; $n \geq 5$) contains a space of uncountably-infinite dimension. Proposition 2.3 is, to our regret, not useful for classifying the $D_n$-actions derived from $\text{Ker} i^*_k$ or $\text{Vec}_{D_n}(V_1, V_1)$.

Suppose $n$ is odd, and classify the $D_n$-actions derived from $\text{Vec}_{D_n}(V_2 \oplus \varepsilon^m, V_n)$. By applying Proposition 2.3 for $H = \mathbb{Z}/2\mathbb{Z}$ and $K = D_n$, we obtain a surjection from $\text{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n)$ to the orbit space of $\text{Im} i^*_{2,m}$ under the action of $\mathbb{C}^*$, where $i^*_{2,m} : \text{Vec}_{D_n}(V_2 \oplus \varepsilon^m, V_n) \rightarrow \text{Vec}_{D_n}(X_2, V_n)$ is the restriction induced by $i^*_{2,m} : X_2 \rightarrow V_2 \oplus \varepsilon^m$. Let $i_m : V_2 \rightarrow V_2 \oplus \varepsilon^m$ be the inclusion. Then $i^*_{2,m} = i^*_m \circ i^*_{2,m}$. Since $i^*_m$ is a surjection, $\text{Im} i^*_{2,m} = \text{Im} i^*_m$. Since $\text{Im} i^*_m \cong \bigoplus_{i=1}^{(n-1)/2} W(2i - 1)$ (cf. the proof of Theorem 3.2 (2)), we have a surjection

$$
\text{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n) \rightarrow \mathbb{P}_s(2i - 1; 1 \leq i \leq (n - 1)/2).
$$

**Theorem 3.6.** Let $m$ be a non-negative integer and let $n$ be odd. Then there is a surjection from $\text{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n)$ onto $\mathbb{P}_s(2i - 1; 1 \leq i \leq (n - 1)/2)$.

**Remark.** Let $l$ be a non-negative integer and let $(m, n) > 1$. Then one obtains a similar result for $\text{VAR}_{D_n}(V_1 \oplus \varepsilon^l, V_m)$. 

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