

SURFACE SUPERCONDUCTIVITY IN 3 DIMENSIONS

XING-BIN PAN

ABSTRACT. We study the Ginzburg-Landau system for a superconductor occupying a 3-dimensional bounded domain, and improve the estimate of the upper critical field H_{C_3} obtained by K. Lu and X. Pan in *J. Diff. Eqns.*, **168** (2000), 386-452. We also analyze the behavior of the order parameters. We show that, under an applied magnetic field lying below and not far from H_{C_3} , order parameters concentrate in a vicinity of a sheath of the surface that is tangential to the applied field, and exponentially decay both in the normal and tangential directions away from the sheath in the L^2 sense. As the applied field decreases further but keeps in between and away from H_{C_2} and H_{C_3} , the superconducting sheath expands but does not cover the entire surface, and superconductivity at the surface portion orthogonal to the applied field is always very weak. This phenomenon is significantly different to the surface superconductivity on a cylinder of infinite height studied by X. Pan in *Comm. Math. Phys.*, **228** (2002), 327-370, where under an axial applied field lying in-between H_{C_2} and H_{C_3} the entire surface is in the superconducting state.

§1. INTRODUCTION

Surface nucleation of superconductivity in applied magnetic fields decreasing from the upper critical field H_{C_3} has been studied based on the Ginzburg-Landau theory by many authors; see, for instance, [SdG], [dG], [SST] and [T] for early references, and [C], [BPT], [GP], [BS], [DFS], [LP1], [LP2], [LP3], [LP4], [HM1], [HM2] and [HP] for recent mathematical research. It is the estimate of H_{C_3} and the location of nucleation that we are most interested in.

In [LP1], [LP2], [LP3] and [HP] we studied the surface nucleation phenomena for a cylindrical superconductor of infinite height placed in an axial applied magnetic field \mathcal{H} . Then the Ginzburg-Landau system is reduced to a two-dimensional problem on the cross section D of the cylinder, and hence this case is called a *2-dimensional superconductivity*. It has been proved in [HP] that, for large κ ,

$$(1.1) \quad H_{C_3}(\kappa) = \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}} \kappa_{\max} + O(\kappa^{-1/3}),$$

where κ is the Ginzburg-Landau parameter given by the ratio of the penetration depth and the coherence length of the superconductor, β_0 is the lowest eigenvalue of the Schrödinger operator with the unit magnetic field on the half plane, C_1 is a universal constant (see Section 2 for the definition of β_0 and C_1), and κ_{\max} is the maximum value of the curvature of the boundary curve ∂D of the cross section

Received by the editors October 12, 2001 and, in revised form, May 19, 2003.

2000 *Mathematics Subject Classification*. Primary 35Q55, 82D55.

Key words and phrases. Ginzburg-Landau system, superconductivity, nucleation, upper critical field, Schrödinger operator with a magnetic field.

D. Moreover, when the applied field is spatially homogeneous and decreases from H_{C_3} , a superconducting sheath nucleates on a subset of the lateral surface where the curvature of ∂D is maximal. In [P1] we further showed that, if the applied field lies in-between and keeps away from H_{C_2} and H_{C_3} , the sample is in the *surface superconducting state*, namely, superconductivity persists uniformly at a thin sheath surrounding the entire lateral surface of the sample, while the interior of the sample remains close to the normal state. For relative works see, for instance, [C], [BPT], [GP], [BS], [HM1], [SS], [A1].

Our understanding of surface nucleation on a 3-dimensional bounded superconductor is far from being complete. Before stating our main questions and results, we recall that, according to the Ginzburg-Landau theory, superconductivity is described by a complex-valued function ψ (order parameter) and a real vector field \mathbf{A} (magnetic potential), and (ψ, \mathbf{A}) is a minimizer of the Ginzburg-Landau functional. Let us consider a homogeneous applied field $\mathcal{H} = \sigma \mathbf{h}$ where \mathbf{h} is a constant unit vector and σ is a parameter, and set $\mathbf{A} = \sigma \mathbf{A}$. With proper scaling, we may rewrite the Ginzburg-Landau functional as follows ([GL], [dG], [CHO], [DGP]):

$$\mathcal{E}[\psi, \mathbf{A}] = \int_{\Omega} \{ |\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \} dx + \kappa^2\sigma^2 \int_{\mathbb{R}^3} |\operatorname{curl} \mathbf{A} - \mathbf{h}|^2 dx,$$

where $\nabla_{\kappa\sigma\mathbf{A}}\psi = \nabla\psi - i\kappa\sigma\mathbf{A}\psi$, $i = \sqrt{-1}$. Then we write the Ginzburg-Landau system for (ψ, \mathbf{A}) (the Euler equation of the functional \mathcal{E}) as follows:

$$(1.2) \quad \begin{cases} -\nabla_{\kappa\sigma\mathbf{A}}^2\psi = \kappa^2(1 - |\psi|^2)\psi & \text{in } \Omega, \\ \operatorname{curl}^2\mathbf{A} = -\frac{i}{2\kappa\sigma}(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}) - |\psi|^2\mathbf{A} & \text{in } \Omega, \\ \operatorname{curl}^2\mathbf{A} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \\ (\nabla_{\kappa\sigma\mathbf{A}}\psi) \cdot \nu = 0, \quad [\nu \times \mathbf{A}] = 0, \quad [\nu \times \operatorname{curl}\mathbf{A}] = \mathbf{0} & \text{on } \partial\Omega, \\ \operatorname{curl}\mathbf{A} \rightarrow \mathbf{h} \quad \text{as } |x| \rightarrow \infty, \end{cases}$$

where $\nabla_{\mathbf{A}}^2\psi = (\nabla - i\mathbf{A})^2\psi = \Delta\psi - i(2\mathbf{A} \cdot \nabla\psi + \psi\operatorname{div} \mathbf{A}) - |\mathbf{A}|^2\psi$, ν is the unit outer normal vector at the boundary of Ω , and $[\cdot]$ denotes the jump in the enclosed quantity across $\partial\Omega$. Note that in the above system the unit of length is the penetration depth. A (global) minimizer of the Ginzburg-Landau functional is called a *minimal solution* of (1.2). Throughout this paper we assume that Ω is a bounded, smooth (say, C^4) and simply-connected domain in \mathbb{R}^3 . Let $\mathbf{F}_{\mathbf{h}}$ be a smooth vector field on \mathbb{R}^3 such that

$$(1.3) \quad \operatorname{curl} \mathbf{F}_{\mathbf{h}} = \mathbf{h} \quad \text{and} \quad \operatorname{div} \mathbf{F}_{\mathbf{h}} = 0 \quad \text{in } \mathbb{R}^3.$$

For a given \mathbf{h} , $\mathbf{F}_{\mathbf{h}}$ is unique up to a harmonic gradient field (gradient of a harmonic function).¹ $(0, \mathbf{F}_{\mathbf{h}})$ is a trivial solution of (1.2) and corresponds to the normal state. A mathematical definition of the upper critical field H_{C_3} has been given in [LP3], [LP4]:

$$H_{C_3}(\mathbf{h}, \kappa) = \inf\{\sigma > 0 : (0, \mathbf{F}_{\mathbf{h}}) \text{ is a minimizer of } \mathcal{E}\}.$$

The nucleation of superconductivity can be described by the fact that, as an applied field decreases from H_{C_3} , (1.2) has a nontrivial minimal solution (ψ, \mathbf{A}) with $|\psi|$ being small; see [LP3], [LP4].

¹Hence the eigenvalues of the operators involving $\mathbf{F}_{\mathbf{h}}$ discussed in this paper are independent of the choice of $\mathbf{F}_{\mathbf{h}}$.

In [LP4] we proved that, for a 3-dimensional bounded, smooth and simply-connected domain,

$$(1.4) \quad \frac{\kappa}{\beta_0} - C\kappa^{1/3} \leq H_{C_3}(\mathbf{h}, \kappa) = \frac{\kappa}{\beta_0} + o(\kappa),$$

and showed that, as the applied field decreases from $H_{C_3}(\mathbf{h}, \kappa)$, a superconducting sheath nucleates at the *tangential set* $(\partial\Omega)_{\mathbf{h}}$, which is a subset of the surface that is tangential to the applied field²:

$$(\partial\Omega)_{\mathbf{h}} = \{x \in \partial\Omega : \mathbf{h} \cdot \nu(x) = 0\}.$$

Note that if the applied field is very close to H_{C_3} , the *nucleation set*, to which superconductivity is confined, is a proper subset of $(\partial\Omega)_{\mathbf{h}}$. As the applied field decreases further, the order parameters will grow up and the superconducting layer will expand; see [SST], p. 87. Thus we want to ask

Question 1. *Find the geometric characterization of the nucleation set, and find the profile of the superconducting layer when the applied field is below and close to H_{C_3} .*

Question 2. *Find the concentration behavior of order parameters when the applied field lies in-between and keeps away from H_{C_2} and H_{C_3} .*

In this paper we are able to give an answer to Question 2, and give some preparation for solving Question 1. In order to answer Question 1, we have to improve the estimate (1.4) by catching the next higher order term in the expansion, which will enable us to locate the nucleation set more precisely. For this purpose we need the following notation. For $x \in \partial\Omega$, let $\kappa_1(x)$ and $\kappa_2(x)$ be the principal curvatures of $\partial\Omega$ at x , and let $\theta_{\mathbf{h}} = \theta_{\mathbf{h}}(x)$ be the angle between \mathbf{h} and the principal direction corresponding to $\kappa_1(x)$. We define a function $P(x)$ on $\partial\Omega$ by

$$P(x) = [\kappa_1^2 \cos^2 \theta_{\mathbf{h}} + \kappa_2^2 \sin^2 \theta_{\mathbf{h}} - \alpha_0(\kappa_1 - \kappa_2)^2 \cos^2 \theta_{\mathbf{h}} \sin^2 \theta_{\mathbf{h}}]^{1/3},$$

where the constant α_0 will be given in (2.4) in Section 2, and $0 < \alpha_0 < 1$. Let

$$(1.5) \quad \begin{aligned} P_{\min} &= \min_{x \in (\partial\Omega)_{\mathbf{h}}} P(x), \\ \mathcal{P} &= \{x \in (\partial\Omega)_{\mathbf{h}} : P(x) = P_{\min}\}. \end{aligned}$$

Let λ_0 be the lowest eigenvalue of (2.14) in Section 2, and let

$$(1.6) \quad C_2 = \lambda_0(1 - \alpha_0)^{1/3}.$$

We have:

Theorem 1. *Let Ω be a bounded, smooth and simply-connected domain in \mathbb{R}^3 and \mathbf{h} a unit vector. We have, for large κ :*

$$(1.7) \quad \frac{\kappa}{\beta_0} - \frac{C_2}{\beta_0^{5/3}} P_{\min} \kappa^{1/3} + O(\kappa^{2/9}) \leq H_{C_3}(\mathbf{h}, \kappa) \leq \frac{\kappa}{\beta_0} + M\kappa^{1/2},$$

where M is a constant independent of \mathbf{h} and κ .

²The situation will be different for samples with nonsmooth surface; see [JRS], [J] and [P2]

In the proof of Theorem 1, the eigenvalue estimate and classification of the eigenfunctions of the Schrödinger operator with a nondegenerately vanishing magnetic field established by Montgomery [M], Helffer [H] and Pan-Kwek [PK] play an important role.

The right side estimate in (1.7) should be improved. In fact, we have the following

Conjecture. *Let Ω be a bounded, smooth and simply-connected domain in \mathbb{R}^3 and \mathbf{h} a unit vector. Then*

$$(1.8) \quad H_{C_3}(\mathbf{h}, \kappa) = \frac{\kappa}{\beta_0} - \frac{C_2}{\beta_0^{5/3}} P_{\min} \kappa^{1/3} + O(\kappa^{2/9}).$$

The estimate (1.8) will enable us to answer Question 1 by proving that, as the applied field decreases from H_{C_3} , superconductivity nucleates first at the subset \mathcal{P} of $\partial\Omega$, where the applied field is tangential to $\partial\Omega$, and the function $P(x)$ is minimal.³

With only (1.7) in hand, however, we are still able to investigate the behavior of the order parameters for a superconductor in an applied field which is below H_{C_3} by a gap of scale $\kappa^{1/2}$, and give a satisfactory answer to Question 2; see Theorem 2. We use the following notation: Given a unit vector \mathbf{h} and $x \in \partial\Omega$, let $\varphi_{\mathbf{h}}(x)$ denote the angle between \mathbf{h} and the outer normal vector $\nu(x)$, $0 \leq \varphi_{\mathbf{h}}(x) \leq \pi$. Note that $\varphi_{\mathbf{h}}(x) = \pi/2$ on $(\partial\Omega)_{\mathbf{h}}$. We can extend $\varphi_{\mathbf{h}}(x)$ to a continuous function on $\bar{\Omega}$; see Remark 2.4 in Section 2. Given $\delta > 0$, let

$$\Omega(\delta) = \{x \in \bar{\Omega} : \text{dist}(x, \partial\Omega) \leq \delta\}.$$

Using Theorem 1 and adapting the methods from [HP] involving Agmon’s idea [A] and the techniques developed by Helffer [H], we can prove the following.

Theorem 2. *Let Ω be a bounded, smooth and simply-connected domain in \mathbb{R}^3 and \mathbf{h} a unit vector. Under the applied field $\mathcal{H} = \sigma\mathbf{h}$, the order parameters ψ have the following concentration behavior for large κ :*

(1) *If $\sigma = H_{C_3}(\mathbf{h}, \kappa) - a_1\kappa^{1/2}$ where $a_1 > 0$, there exist positive constants η , γ_1 and M_1 , such that for large κ ,*

$$(1.9) \quad \int_{\Omega \setminus \mathcal{S}_{\kappa}(\mathbf{h})} |\psi|^2 dx \leq M_1 \exp(-\gamma_1 \kappa^{5/8}) \int_{\Omega(\kappa^{-1})} |\psi|^2 dx,$$

where $\mathcal{S}_{\kappa}(\mathbf{h})$ is a thin layer surrounding $(\partial\Omega)_{\mathbf{h}}$ and is defined by

$$\mathcal{S}_{\kappa}(\mathbf{h}) = \left\{ x \in \Omega(\kappa^{-1}) : \left| \varphi_{\mathbf{h}}(x) - \frac{\pi}{2} \right| \leq \eta \kappa^{-3/16} \right\}.$$

(2) *If $\sigma = \kappa/b$, where $\beta_0 < b < 1$, there exist positive constants $\vartheta(b)$, $\eta(\kappa)$, γ_2 and M_2 , all depending on b , where $0 < \vartheta(b) < \pi/2$ and $\eta(\kappa) \rightarrow 0$ as $\kappa \rightarrow \infty$, such that for large κ ,*

$$(1.10) \quad \int_{\Omega \setminus \mathcal{S}_{b,\kappa}(\mathbf{h})} |\psi|^2 dx \leq M_2 \exp(-\gamma_2 \kappa^{3/4}) \int_{\Omega(\kappa^{-1})} |\psi|^2 dx,$$

where

$$\mathcal{S}_{b,\kappa}(\mathbf{h}) = \left\{ x \in \Omega(\kappa^{-1}) : \vartheta(b) - \eta(\kappa) \leq \varphi_{\mathbf{h}}(x) \leq \pi - \vartheta(b) + \eta(\kappa) \right\}.$$

³Note that, if $P_{\min} = 0$, (1.8) is still not good enough, and we should carry out further analysis to catch a non-zero next higher order term in the expansion of H_{C_3} , and we expect a formula similar to (1.1) holds, with κ_{\max} replaced by the principal curvature corresponding to the direction orthogonal to \mathbf{h} .

Remark 1.1. From (1.9) we see that, if the applied field $\mathcal{H} = \sigma \mathbf{h}$ is not far from H_{C_3} in the sense that $\sigma = H_{C_3}(\mathbf{h}, \kappa) - a_1 \kappa^{1/2}$ with $a_1 > 0$, superconductivity is confined to a vicinity of $(\partial\Omega)_{\mathbf{h}}$, and in the L^2 sense the order parameters exponentially decay away from $(\partial\Omega)_{\mathbf{h}}$ both in the direction normal to the surface and in the directions tangential to the surface.

Remark 1.2. From (1.10), if the applied field lies in-between and keeps away from H_{C_2} and H_{C_3} in the sense that

$$(1.11) \quad \sigma = \frac{\kappa}{b}, \quad b \text{ is fixed, } \beta_0 < b < 1, \quad \kappa \text{ is large,}$$

then superconductivity is confined to the vicinity of a portion of the surface where the angle between the applied field and either the normal vector ν or $-\nu$ is smaller than $\vartheta(b)$. Note that if κ is fixed and b increases to 1, then $\mathcal{S}_{b,\kappa}(\mathbf{h})$ expands and gradually covers the entire surface, which means that when the applied field decreases to H_{C_2} , the whole surface will turn into the superconducting state; however, before the applied field reaches H_{C_2} , there always exists a subset of the surface with positive area which remains close to the normal state. In particular, the subset of the surface that is orthogonal to the applied field is always close to the normal state. This phenomenon illustrates a significant difference between 2-dimensional superconductivity (on a cylinder of infinite height) and 3-dimensional superconductivity (on a 3-dimensional bounded domain). As was proved in [P1] and mentioned above that, for a 2-dimensional superconductor, as the applied field decreases from H_{C_3} , a superconducting sheath forms quickly over the entire boundary, and the sample turns into the surface superconducting state much earlier than the interior nucleation occurs. However, for a 3-dimensional superconductor, superconducting sheath will not cover the entire surface when the applied field is between H_{C_2} and H_{C_3} . Thus the uniform surface superconducting state will not occur on a 3-dimensional bounded superconductor. Our Theorem 2 is a mathematical verification for the observations of Saint-James and Sarma [SST] mentioned above.

In order to estimate the value of H_{C_3} and prove Theorem 1, we must study the eigenvalue problem for the Schrödinger operator $-\nabla_{\varepsilon^{-2}\mathbf{F}_{\mathbf{h}}}^2$ in Ω :

$$(1.12) \quad -\nabla_{\varepsilon^{-2}\mathbf{F}_{\mathbf{h}}}^2 \phi = \mu \phi \quad \text{in } \Omega, \quad (\nabla_{\varepsilon^{-2}\mathbf{F}_{\mathbf{h}}} \phi) \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where $\varepsilon > 0$ is a small real number. It has been proved in [LP4] (Appendix) that, for a 3-dimensional bounded, smooth and simply-connected domain,

$$(1.13) \quad \frac{1}{\varepsilon^2}(\beta_0 + o(1)) \leq \mu(\varepsilon^{-2}\mathbf{F}_{\mathbf{h}}) \leq \frac{1}{\varepsilon^2}(\beta_0 + C\varepsilon^{2/3}).$$

In this paper we are able to improve the estimate (1.13) and prove a satisfactory upper bound

$$(1.14) \quad \mu(\varepsilon^{-2}\mathbf{F}_{\mathbf{h}}) \leq \frac{1}{\varepsilon^2} \{ \beta_0 + C_2 P_{\min} \varepsilon^{2/3} + O(\varepsilon^{5/6}) \}$$

for all small $\varepsilon > 0$, where C_2 is the number given in (1.6). We expect that the equality in (1.14) holds up to an error $O(\varepsilon^{7/9})$, namely

$$(1.15) \quad \mu(\varepsilon^{-2}\mathbf{F}_{\mathbf{h}}) = \frac{1}{\varepsilon^2} \{ \beta_0 + C_2 P_{\min} \varepsilon^{2/3} + O(\varepsilon^{7/9}) \}.$$

Remark 1.3. In this paper we consider the homogeneous boundary condition for order parameters ψ . As in [LP1], [LP2], [LP3] and [LP4], we can also consider a boundary condition

$$\nabla_{\kappa\sigma\mathbf{A}}\psi \cdot \nu + \gamma\psi = 0$$

with $\gamma > 0$, which was posed by de Gennes [dG] for superconductors adjacent to other material. It will produce more technical complexity.

This paper is organized as follows. In Section 2 we discuss several auxiliary linear differential equations, leaving some computations in Appendix A. An analysis on the local coordinates at boundary is given in Section 3 and some computational details are given in Appendix B. This analysis is needed in Section 4, where we prove (1.14) and obtain a lower bound estimate for H_{C_3} . The proof of (1.14) consists of careful analysis of functions and vector fields near a boundary point, carefully choosing test functions, and controlling error terms. In order to show clearly the intuition of choosing the test functions, we present a formal analysis in Appendix C. In Section 5 we establish the upper bound estimate for H_{C_3} based on a series of elliptic estimates. In Section 6 we discuss the location of nucleation of superconductivity as well as the behavior of order parameters for applied fields lying in-between and keeping away from H_{C_2} and H_{C_3} .

We proved the upper bound estimate (1.14) and conjectured the formula (1.15) independently and simultaneously with B. Helffer and A. Morame. Just after we submitted the first version of this paper in October 2001, we received from Professor Helffer his preprint with A. Morame: “*Magnetic bottles for the Neumann problem: Curvature effects in the case of dimension 3*”, where they proved an asymptotic estimate for $\mu(\varepsilon^{-2}\mathbf{F}_h)$ under some geometric assumptions on the domain Ω . In March 2002 we received from Professor Helffer the preprint: “*Magnetic bottles for the Neumann problem: Curvature effects in the case of dimension 3 (general case)*”, which is an improved version of the first one, and in which they proved the asymptotic estimate for $\mu(\varepsilon^{-2}\mathbf{F}_h)$ under a very slight condition on the domain, and they also proved that their formula is equivalent to (1.15) up to the error term $O(\varepsilon^{2/3+\delta})$ for some $\delta > 0$. In February 2003 we received from Professor Y. Almog his preprint: “*Non-linear surface superconductivity in three dimensions in the large κ limit*”, where he considered superconductivity on a 3-dimensional bounded domain under an applied field $\mathcal{H} = \sigma\mathbf{h}$ with $\sigma - \kappa \gg \kappa^{1/2}$, and proved that order parameters exponentially decay in the normal direction away from surface.

§2. AUXILIARY EQUATIONS

In this section we study several auxiliary linear differential equations and collect some facts which will be used in this paper. For fixed z , let $\beta(z)$ denote the lowest eigenvalue of the following eigenvalue problem

$$(2.1) \quad \begin{cases} -u'' + (z+t)^2u = \beta(z)u & \text{for } t > 0, \\ u'(0) = 0, \quad u(+\infty) = 0. \end{cases}$$

The following results were proved by Dauge-Helffer in [DH]; also see [LP1], and [LP3] (Section 2 and Appendix).

Lemma 2.1. (1) *There is a unique z_0 , $z_0 < 0$, such that $\beta(z_0) = \inf_{z \in \mathbb{R}} \beta(z) = \beta_0$. Moreover, $\beta_0 = |z_0|^2$.*

(2) Let $u(t)$ be a positive eigenfunction of (2.1) for $z = z_0$ associated with the lowest eigenvalue β_0 . $u(t)$ is strictly decreasing in t and exponentially decays as $t \rightarrow +\infty$, and

$$(2.2) \quad \int_0^{+\infty} (z_0 + t)u^2(t)dt = 0, \quad \int_0^{+\infty} (t + z_0)^2 u(t)dt = \frac{\beta_0}{2} \int_0^{+\infty} u^2(t)dt.$$

It has been proved in [LP3] that $0.5 < \beta_0 < 0.76$. The number C_1 appearing in (1.1) for 2-dimensional superconductors is defined by an eigenfunction u of (2.1):

$$C_1 = \frac{u^2(0)}{3\|u\|_{L^2(\mathbb{R}_+)}^2}.$$

It was shown in [LP3] that

$$C_1 = \frac{1}{\|u\|_{L^2(\mathbb{R}_+)}^2} \left\{ \frac{u^2(0)}{2} - \int_0^{+\infty} (t + z_0)^3 u^2(t)dt \right\}.$$

Lemma 2.2. *Given a solution u of (2.1), the solutions of the equation*

$$(2.3) \quad \begin{cases} -u_1'' + (t + z_0)^2 u_1 - \beta_0 u_1 = -(t + z_0)u & \text{for } t > 0, \\ u_1'(0) = 0, \quad u_1(\infty) = 0 \end{cases}$$

are given by $u_1 = \frac{u'}{2} + cu$, where c is an arbitrary constant. For any solution u_1 of (2.3), let

$$(2.4) \quad \alpha_0 = -\frac{4}{\|u\|_{L^2(\mathbb{R}_+)}^2} \int_0^{+\infty} (t + z_0)uu_1 dt.$$

α_0 is independent of the choice of u_1 , and $0 < \alpha_0 < 1$.

Proof. $u''(0) = 0$ since $\beta_0 = z_0^2$. Hence $\frac{1}{2}u'(t)$ satisfies (2.3). Obviously, the solutions u_1 of (2.3) are given by $u'/2 + cu$. Integrating by parts we find that

$$\alpha_0 = -\frac{4}{\|u\|_{L^2(\mathbb{R}_+)}^2} \int_0^{+\infty} (t + z_0)uu_1 dt = 1 + \frac{z_0 u^2(0)}{\|u\|_{L^2(\mathbb{R}_+)}^2} = 1 - 3C_1|z_0| < 1.$$

On the other hand, multiplying (2.3) by u_1 and integrating, and using the simplicity of the eigenvalue β_0 , we find that

$$-\int_0^{+\infty} (t + z_0)uu_1 dt = \int_0^{+\infty} \{|u_1'|^2 + (t + z_0)^2 u_1^2 - \beta_0 u_1^2\} dt > 0,$$

so $\alpha_0 > 0$. □

In the following we shall always choose the positive eigenfunction u of (2.1) such that $\|u\|_{L^2(\mathbb{R}_+)} = 1$. Let $u_1 = \frac{u'}{2} + \frac{3C_1}{2}u$. Then u_1 is a solution of (2.3) and satisfies $\int_0^{+\infty} uu_1 dt = 0$. α_0 always denotes the number given in (2.4). u and u_1 will be used to construct test functions in Sections 3 and 4.

Next, given a unit vector \mathbf{h} , let $\mathbf{F}_\mathbf{h}$ be the vector field satisfying (1.3), and let $\lambda_\mathbf{h}$ be the lowest eigenvalue of the following problem in the half space $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) : x_3 > 0\}$:

$$-\nabla_{\mathbf{F}_\mathbf{h}}^2 \psi = \lambda \psi \quad \text{in } \mathbb{R}_+^3, \quad \nabla_{\mathbf{F}_\mathbf{h}} \psi \cdot \nu = 0 \quad \text{on } \partial\mathbb{R}_+^3.$$

Let $\vartheta(\partial\mathbb{R}_+^3, \mathbf{h})$ be the angle between \mathbf{h} and the unit outer normal of $\partial\mathbb{R}_+^3$. $0 \leq \vartheta(\partial\mathbb{R}_+^3, \mathbf{h}) \leq \pi$.

Lemma 2.3. *There exists a continuous function $\lambda(\vartheta)$ on $[0, \pi]$ such that $\lambda_{\mathbf{h}} = \lambda(\vartheta(\partial\mathbb{R}_+^3, \mathbf{h}))$ for all unit vectors \mathbf{h} .*

(1) $\lambda(\vartheta)$ is strictly decreasing for $0 \leq \vartheta \leq \pi/2$, $\lambda(0) = 1$, $\lambda(\frac{\pi}{2}) = \beta_0$, and $\lambda(\vartheta) = \lambda(\pi - \vartheta)$.

(2) For $0 < \vartheta < \pi/2$, it holds that

$$(2.5) \quad \beta_0 \sin^2 \vartheta + \cos^2 \vartheta < \lambda(\vartheta) < \beta_0 \sin \vartheta + \cos \vartheta.$$

(3) For small $\varepsilon > 0$,

$$\begin{aligned} \left| \vartheta - \frac{\pi}{2} \right| &\leq (1 - \beta_0)^{-1/2} \varepsilon^{1/2} + O(\varepsilon^{3/2}) && \text{if } \lambda(\vartheta) \leq \beta_0 + \varepsilon, \\ \left| \vartheta - \frac{\pi}{2} \right| &\geq \varepsilon + O(\varepsilon^2) && \text{if } \lambda(\vartheta) \geq \beta_0 + \varepsilon. \end{aligned}$$

Proof. The conclusions of Lemma 2.3, except (2) and (3), were established in [LP4] (Theorem 4.2). It was also proved there that, if $0 < \vartheta < \pi/2$, then $\lambda(\vartheta) = b(\vartheta)$, where

$$b(\vartheta) = \inf_{\phi \in W^{1,2}(\mathbb{R}_+^2)} \frac{\int_{\mathbb{R}_+^2} \{ |\nabla \phi|^2 + (x_1 \cos \vartheta - x_2 \sin \vartheta)^2 |\phi|^2 \} dx}{\|\phi\|_{L^2(\mathbb{R}_+^2)}^2}.$$

The right side inequality of (2.5) was proved in [LP4] (Lemma 3.8) by making a change of variables. To prove the left side inequality, we make a change of variables $y_1 = x_1 \cos \vartheta$, $y_2 = x_2 \sin \vartheta$, $w(y) = \phi(x)$, and find⁴

$$\begin{aligned} b(\vartheta) &= \inf_{w \in W^{1,2}(\mathbb{R}_+^2)} \left\{ \cos^2 \vartheta \frac{\int_{\mathbb{R}_+^2} [|\partial_{y_1} w|^2 + (y_1 - y_2)^2 |w|^2] dy}{\|w\|_{L^2(\mathbb{R}_+^2)}^2} \right. \\ &\quad \left. + \sin^2 \vartheta \frac{\int_{\mathbb{R}_+^2} [|\partial_{y_2} w|^2 + (y_2 - y_1)^2 |w|^2] dy}{\|w\|_{L^2(\mathbb{R}_+^2)}^2} \right\} \\ &\geq \cos^2 \vartheta \inf_{w \in W^{1,2}(\mathbb{R}_+^2)} \frac{\int_{\mathbb{R}_+^2} [|\partial_{y_1} w|^2 + (y_1 - y_2)^2 |w|^2] dy}{\|w\|_{L^2(\mathbb{R}_+^2)}^2} \\ &\quad + \sin^2 \vartheta \inf_{w \in W^{1,2}(\mathbb{R}_+^2)} \frac{\int_{\mathbb{R}_+^2} [|\partial_{y_2} w|^2 + (y_2 - y_1)^2 |w|^2] dy}{\|w\|_{L^2(\mathbb{R}_+^2)}^2} = \cos^2 \vartheta + \beta_0 \sin^2 \vartheta. \end{aligned}$$

Therefore, $b(\vartheta) \geq \cos^2 \vartheta + \beta_0 \sin^2 \vartheta$. If the equality holds for some $0 < \vartheta < \pi/2$, then $b(\vartheta)$ is achieved by a function $\phi(x) \in W^{1,2}(\mathbb{R}_+^2)$ (see [LP4], Theorem 3.1). Let

$$w(y) = \phi\left(\frac{y_1}{\cos \vartheta}, \frac{y_2}{\sin \vartheta}\right).$$

Then $w(y)$ must satisfy

$$\int_{\mathbb{R}_+^2} [|\partial_{y_2} w|^2 + (y_2 - y_1)^2 |w|^2] dy = \beta_0 \|w\|_{L^2(\mathbb{R}_+^2)}^2.$$

This contradicts conclusion (1) of Lemma 2.1.

To prove (3), we assume $0 < \vartheta < \pi/2$. If $\lambda(\vartheta) \leq \beta_0 + \varepsilon$, from (2.5) we have

$$\beta_0 \sin^2 \vartheta + \cos^2 \vartheta \leq \lambda(\vartheta) \leq \beta_0 + \varepsilon,$$

⁴Such techniques were used several times in [LP4] and [P2] in the estimates of various eigenvalue problems.

so $\cos \vartheta \leq \sqrt{\varepsilon/(1 - \beta_0)}$, and hence

$$\frac{\pi}{2} - \vartheta \leq \sqrt{\frac{\varepsilon}{1 - \beta_0}} + O(\varepsilon^{3/2}).$$

If $\lambda(\vartheta) \geq \beta_0 + \varepsilon$, using (2.5) again we have

$$\beta_0 \sin \vartheta + \cos \vartheta \geq \lambda(\vartheta) \geq \beta_0 + \varepsilon,$$

and hence $f(\vartheta) \equiv \beta_0(\sin \vartheta - 1) + \cos \vartheta \geq \varepsilon$. The function $f(\vartheta)$ is strictly decreasing for $\vartheta \in (\arctan(\beta_0), \pi/2)$. So $\vartheta \leq f^{-1}(\varepsilon) \equiv \vartheta_\varepsilon$. Let $t_\varepsilon = \pi/2 - \vartheta_\varepsilon$. Then

$$\varepsilon = f(\vartheta_\varepsilon) = f\left(\frac{\pi}{2} - t_\varepsilon\right) = \beta_0(\cos t_\varepsilon - 1) + \sin t_\varepsilon = t_\varepsilon - \frac{\beta_0}{2}t_\varepsilon^2 + O(t_\varepsilon^3).$$

Thus $t_\varepsilon = O(\varepsilon)$ and $\frac{\beta_0}{2}t_\varepsilon^2 - t_\varepsilon + \varepsilon + O(\varepsilon^3) = 0$. So

$$t_\varepsilon = \frac{1}{\beta_0}[1 - \sqrt{1 - 2\beta_0\varepsilon + O(\varepsilon^3)}] = \varepsilon + O(\varepsilon^2),$$

and

$$\frac{\pi}{2} - \vartheta \geq \frac{\pi}{2} - \vartheta_\varepsilon = t_\varepsilon = \varepsilon + O(\varepsilon^2).$$

□

We would like to mention that the inequality $\lambda(\vartheta) \geq \cos^2 \vartheta + \beta_0 \sin^2 \vartheta$ for $0 < \vartheta < \pi/2$ has been proved in [HM2]. Our argument yields the strict inequality.

Remark 2.4. Let $\varphi_{\mathbf{h}}(x)$ be the angle between \mathbf{h} and the outer normal vector $\nu(x)$ on $\partial\Omega$, as defined in Section 1. $\varphi_{\mathbf{h}}(x)$ is well defined on $\partial\Omega$ and is continuously differentiable in the set $\{x \in \partial\Omega : 0 < \varphi_{\mathbf{h}}(x) < \pi\}$. Let $\lambda(\vartheta)$ be the function given in Lemma 2.3. Then $\lambda(\varphi_{\mathbf{h}}(x))$ is a continuous function, and

$$(2.6) \quad \lambda(\varphi_{\mathbf{h}}(x)) = \beta_0 \quad \text{on } (\partial\Omega)_{\mathbf{h}}, \quad \beta_0 < \lambda(\varphi_{\mathbf{h}}(x)) \leq 1 \quad \text{on } \partial\Omega \setminus (\partial\Omega)_{\mathbf{h}}.$$

In Sections 5 and 6 we shall extend the function $\varphi_{\mathbf{h}}(x)$ onto $\bar{\Omega}$. The extension can be done as follows. There exists a number $\delta_0 > 0$ such that, for any $x \in \Omega(\delta_0)$ we can find a unique point $P(x) \in \partial\Omega$ such that $|P(x) - x| = \text{dist}(x, \partial\Omega)$. Then we define

$$\varphi_{\mathbf{h}}(x) = \varphi_{\mathbf{h}}(P(x)) \quad \text{for } x \in \Omega(\delta_0).$$

We can further extend it over $\bar{\Omega}$ to get a continuous function on $\bar{\Omega}$, such that $0 \leq \varphi_{\mathbf{h}}(x) \leq \pi$, and $\varphi_{\mathbf{h}}(x)$ is continuously differentiable in the set $\{x \in \Omega(\delta_0) : 0 < \varphi_{\mathbf{h}}(x) < \pi\}$. Now $\lambda(\varphi_{\mathbf{h}}(x))$ is also well defined and continuous on $\bar{\Omega}$, and $\beta_0 \leq \lambda(\varphi_{\mathbf{h}}(x)) \leq 1$.

In Section 3 we need bounded solutions of an Euler equation associated with the variational problem for the functional p :

$$p[\phi] = \int_{\mathbb{R}^2} \left\{ c_{11} |\partial_{y_1} \phi + ia_2^{(1)} y_2^2 \phi|^2 + 2c_{12} \Re[(\partial_{y_1} \phi + ia_2^{(1)} y_2^2 \phi)(\partial_{y_2} \bar{\phi} + ia_1^{(2)} y_1^2 \bar{\phi})] + c_{22} |\partial_{y_2} \phi - ia_1^{(2)} y_1^2 \phi|^2 \right\} dy.$$

Here $a_2^{(1)}, a_1^{(2)}$ are real numbers, $c_{11} > 0, c_{22} > 0$, and $c_{11}c_{22} - c_{12}^2 > 0$. Define

$$(2.7) \quad \mu_p = \inf_{\phi \in \mathcal{W}(\mathbb{R}^2)} \frac{p[\phi]}{\|\phi\|_{L^2(\mathbb{R}^2)}^2},$$

where

$$\mathcal{W}(\mathbb{R}^2) = \{\phi \in L^2(\mathbb{R}^2) \cap W_{\text{loc}}^{1,2}(\mathbb{R}^2) : p[\phi] < \infty\}.$$

We can show that the infimum μ_p is not achieved in $\mathcal{W}(\mathbb{R}^2)$, however, the associated Euler equation

$$\begin{aligned} c_{11}(\partial_{y_1} + ia_2^{(1)}y_2^2)^2\psi + c_{12}\{(\partial_{y_1} + ia_2^{(1)}y_2^2)(\partial_{y_2} - ia_1^{(2)}y_1^2)\psi \\ + (\partial_{y_2} - ia_1^{(2)}y_1^2)(\partial_{y_1} + ia_2^{(1)}y_2^2)\psi\} + c_{22}(\partial_{y_2} - ia_1^{(2)}y_1^2)^2\psi + \mu_p\psi = 0 \end{aligned}$$

in \mathbb{R}^2 has a nontrivial bounded solution $\psi(y)$. This equation can be written as follows:

$$(2.8) \quad c_{11}\partial_{y_1y_1}\psi + 2c_{12}\partial_{y_1y_2}\psi + c_{22}\partial_{y_2y_2}\psi - 2if_1\partial_{y_1}\psi - 2if_2\partial_{y_2}\psi - f_0\psi + \mu_p\psi = 0,$$

where

$$\begin{aligned} f_0 &= c_{11}(a_2^{(1)})^2y_2^4 - 2c_{12}a_2^{(1)}a_1^{(2)}y_1^2y_2^2 + c_{22}(a_1^{(2)})^2y_1^4 \\ &\quad + 2ic_{12}(a_1^{(2)}y_1 - a_2^{(1)}y_2), \\ f_1 &= c_{12}a_1^{(2)}y_1^2 - c_{11}a_2^{(1)}y_2^2, \\ f_2 &= c_{22}a_1^{(2)}y_1^2 - c_{12}a_2^{(1)}y_2^2. \end{aligned} \quad (2.9)$$

In order to simplify equation (2.8), we introduce new variables $\xi = (\xi_1, \xi_2)$ by

$$\begin{aligned} y_1 &= \frac{\sqrt{c_{11}}}{2}(\varepsilon_1\xi_1 + \varepsilon_2\xi_2), \\ y_2 &= \frac{\sqrt{c_{22}}}{2}(\varepsilon_1\xi_1 - \varepsilon_2\xi_2), \\ \psi(y) &= \psi_1(\xi), \end{aligned} \quad (2.10)$$

where

$$\varepsilon_1 = \left(2 + \frac{2c_{12}}{\sqrt{c_{11}c_{22}}}\right)^{1/2}, \quad \varepsilon_2 = \left(2 - \frac{2c_{12}}{\sqrt{c_{11}c_{22}}}\right)^{1/2}.$$

Then we can write the equation for $\psi_1(\xi)$ in the following form:

$$(2.11) \quad -\nabla_{\mathbf{Q}}^2\psi_1 = \mu_2\psi_1 \quad \text{in } \mathbb{R}^2,$$

where

$$\mathbf{Q}(\xi) = -\frac{|\xi|^2}{2}(\gamma_1, \gamma_2) + \nabla\eta(\xi).$$

Here $\eta(\xi)$ is a homogeneous polynomial of ξ_1, ξ_2 of degree 3, and

$$\begin{aligned} \gamma_1 &= 2\frac{c_{11}c_{22} - (c_{12})^2}{\varepsilon_1\sqrt{c_{11}c_{22}}} \left(\sqrt{c_{22}}a_2^{(1)} - \sqrt{c_{11}}a_1^{(2)}\right), \\ \gamma_2 &= 2\frac{c_{11}c_{22} - (c_{12})^2}{\varepsilon_2\sqrt{c_{11}c_{22}}} \left(\sqrt{c_{22}}a_2^{(1)} + \sqrt{c_{11}}a_1^{(2)}\right). \end{aligned} \quad (2.12)$$

For the details see Appendix A.

Let us introduce ρ and ϑ by

$$\rho = (\gamma_1^2 + \gamma_2^2)^{1/2}, \quad \gamma_1 = \rho \cos \vartheta, \quad \gamma_2 = \rho \sin \vartheta,$$

and introduce new variables $x = (x_1, x_2)$ by

$$(2.13) \quad \begin{aligned} \xi_1 &= \frac{1}{\rho^{1/3}}(x_1 \cos \vartheta - x_2 \sin \vartheta), \\ \xi_2 &= \frac{1}{\rho^{1/3}}(x_1 \sin \vartheta + x_2 \cos \vartheta), \\ \varphi(x) &= \psi_1(\xi) \exp\left(\frac{i}{6}x_1^3 - i\eta(\xi)\right). \end{aligned}$$

Let

$$\lambda_0 = \mu_p \rho^{-2/3}.$$

Then we can write the equation for $\varphi(x)$ as follows:

$$(2.14) \quad -\Delta\varphi - ix_2^2\partial_{x_1}\varphi + \frac{1}{4}x_2^4\varphi = \lambda_0\varphi \quad \text{in } \mathbb{R}^2.$$

Equation (2.14) has been studied by Montgomery [M], Helffer [H] and Pan-Kwek [PK]. Montgomery [M] proved that the lowest eigenvalue λ_0 of (2.14) is given by $\lambda_0 = \inf_{\tau \in \mathbb{R}} \lambda(\tau)$, where $\lambda(\tau)$ is the lowest eigenvalue of the following ordinary differential equation

$$(2.15) \quad -v'' + \frac{1}{4}(t^2 + 2\tau)^2v = \lambda(\tau)v, \quad -\infty < t < \infty.$$

Pan-Kwek [PK] further showed that the function $\lambda(\tau)$ has a unique minimum point τ_0 , $\tau_0 < 0$,

$$(2.16) \quad \lambda(\tau_0) = \lambda_0 = \inf_{\tau \in \mathbb{R}} \lambda(\tau),$$

and the eigenfunctions of (2.14) corresponding to λ_0 are given by

$$\varphi(x) = ce^{i\tau_0 x_1} v(x_2),$$

where $v(t)$ is the eigenfunction of (2.15) with $\tau = \tau_0$. Hence, the lowest eigenvalue of (2.8) is $\mu_p = \lambda_0 \rho^{2/3}$, and the bounded eigenfunctions do not lie in $\mathcal{W}(\mathbb{R}^2)$. Thus we have obtained the following conclusion.

Proposition 2.5. *The lowest eigenvalue of (2.8) is*

$$(2.17) \quad \mu_p = \lambda_0(\gamma_1^2 + \gamma_2^2)^{1/3},$$

where γ_1, γ_2 are given in (2.12), and λ_0 is the lowest eigenvalue of (2.14). μ_p is not achieved in $\mathcal{W}(\mathbb{R}^2)$. After the transformations of coordinates (2.10) and (2.13), the bounded solutions of (2.8) associated with the lowest eigenvalue μ_p are $\psi(y) = c\psi_p(y)$, where

$$(2.18) \quad \psi_p(y) = v(x_2) \exp\left(i\tau_0 x_1 - \frac{i}{6}x_1^3 + i\eta(\xi)\right);$$

here τ_0 is given in (2.16), v is the eigenfunction of (2.15) with $\tau = \tau_0$, and $\eta(\xi)$ is a homogeneous polynomial of degree 3.

Remark 2.6. Let $\psi_p(y)$ be the function given in (2.18). For any $R_1, R_2 > 0$, we have

$$\int_{\{|y_1| < R_1, |y_2| < R_2\}} y_j |\psi_p(y)|^2 dy = 0, \quad j = 1, 2.$$

§3. LOCAL COORDINATES NEAR BOUNDARY

In this section we give some analysis in the local coordinates determined by a diffeomorphism that straightens a piece of surface. This analysis is needed in Section 4 to obtain estimates of the eigenvalues. Let us fix a point $X_0 \in \partial\Omega$, and introduce new variables y_1, y_2 such that $\partial\Omega$ be represented (at least near X_0) by $\mathbf{r} = \mathbf{r}(y_1, y_2)$, and $\mathbf{r}(0, 0) = X_0$. Here and henceforth we let

$$y = (y_1, y_2)$$

and use the notation $\mathbf{r}_j(y) = \partial_{y_j}\mathbf{r}(y)$, $\mathbf{r}_{ij} = \partial_{y_i y_j}\mathbf{r}(y)$, etc. Let

$$\mathbf{n}(y) = \frac{\mathbf{r}_1(y) \times \mathbf{r}_2(y)}{|\mathbf{r}_1(y) \times \mathbf{r}_2(y)|}.$$

We choose (y_1, y_2) in such a way that $\mathbf{n}(y)$ is the inward normal of $\partial\Omega$; and that the y_1 - and y_2 -curves on $\partial\Omega$ are the lines of curvatures; thus, $\mathbf{r}_1(y)$ and $\mathbf{r}_2(y)$ are orthogonal to each other everywhere on $\partial\Omega$. Let

$$g_{ij}(y) = \mathbf{r}_i(y) \cdot \mathbf{r}_j(y), \quad g(y) = \det(g_{ij}(y)) = g_{11}(y)g_{22}(y).$$

Let $\Omega_{ij}(y)$ denote the coefficients of the second fundamental form of $\partial\Omega$. According to the choice of y_1 and y_2 , we have

$$\Omega_{11} = g_{11}\kappa_1, \quad \Omega_{12} = 0, \quad \Omega_{22} = g_{22}\kappa_2;$$

here κ_1 and κ_2 are the principal curvatures of $\partial\Omega$. Let ∂_j denote ∂_{y_j} for $j = 1, 2$, and ∂_3 denote ∂_z . For scalar functions f , let $f_{,j}$ denote the partial derivative in y_j . Let us define a map \mathcal{F} by

$$X = \mathcal{F}(y, z) = \mathbf{r}(y_1, y_2) + z\mathbf{n}(y_1, y_2).$$

\mathcal{F} is a diffeomorphism in a small neighborhood of O . As in [LP4] (Appendix) we calculate $\partial_j\mathcal{F}$ and $G_{ij}(y, z) = \partial_i\mathcal{F} \cdot \partial_j\mathcal{F}$:

$$\begin{aligned} \partial_1\mathcal{F} &= [1 - \kappa_1(y)z]\mathbf{r}_1(y), & \partial_2\mathcal{F} &= [1 - \kappa_2(y)z]\mathbf{r}_2(y), & \partial_z\mathcal{F} &= \mathbf{n}(y); \\ G_{11}(y, z) &= g_{11}(y)[1 - \kappa_1(y)z]^2, \\ G_{22}(y, z) &= g_{22}(y)[1 - \kappa_2(y)z]^2, \\ G_{33}(y, z) &= 1, \\ G_{12}(y, z) &= G_{13}(y, z) = G_{23}(y, z) = 0, \\ G(y, z) &= \det(G_{ij}(y, z)) = G_{11}(y, z)G_{22}(y, z). \end{aligned}$$

Let $G^{ij}(y, z)$ denote the elements of the inverse of the matrix $(G_{ij}(y, z))$. Then

$$G^{11} = \frac{1}{G_{11}}, \quad G^{22} = \frac{1}{G_{22}}, \quad G^{33} = 1, \quad G^{12} = G^{13} = G^{23} = 0.$$

Given a function ψ and a vector field \mathbf{A} , in the set $\Omega(\delta)$ (see Section 1 for the definition) we can write them in the new variables (y, z) as follows:

$$\begin{aligned} \tilde{\psi}(y, z) &= \psi(\mathcal{F}(y, z)), \\ (3.1) \quad \tilde{\mathbf{A}}(y, z) &= \mathbf{A}(\mathcal{F}(y, z)) = \sum_{j=1}^3 G^{jj}(y, z)a_j(y, z)\partial_j\mathcal{F}(y, z), \\ &\text{where } a_j(y, z) = \mathbf{A}(\mathcal{F}(y, z)) \cdot \partial_j\mathcal{F}(y, z). \end{aligned}$$

Later we need the Taylor expansions near X_0 of various functions and vector fields. Let

$$(3.2) \quad \mathbf{e}_1 = \mathbf{r}_1(0, 0), \quad \mathbf{e}_2 = \mathbf{r}_2(0, 0), \quad \mathbf{e}_3 = \mathbf{n}(0, 0).$$

Note that \mathbf{e}_1 and \mathbf{e}_2 may not be unit vectors. In the following computations, $g, g_{,j}, g_{jj}, \kappa_j, \Omega_{jj}$, etc. denote the values of these quantities at X_0 . The indices i, j, k, l, m , etc. run from 1 to 2. We also take the summation convention, that is, when the summation is taken over repeated indices, the summation symbol is omitted. We have

$$(3.3) \quad \begin{aligned} \mathbf{r}_1(y) &= (1 + \Gamma_{1j}^1 y_j) \mathbf{e}_1 + \Gamma_{1j}^2 y_j \mathbf{e}_2 + \Omega_{11} y_1 \mathbf{e}_3 + O(|y|^2), \\ \mathbf{r}_2(y) &= \Gamma_{2j}^1 y_j \mathbf{e}_1 + (1 + \Gamma_{2j}^2 y_j) \mathbf{e}_2 + \Omega_{22} y_2 \mathbf{e}_3 + O(|y|^2), \\ \mathbf{n}(y) &= \mathbf{e}_3 - g^{jj} \Omega_{jj} y_j \mathbf{e}_j + O(|y|^2), \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} g_{11}(y) &= g_{11} + g_{11,j} y_j + O(|y|^2), \\ g_{22}(y) &= g_{22} + g_{22,j} y_j + O(|y|^2), \\ \sqrt{g(y)} &= \sqrt{g} \left[1 + \frac{1}{2} \left(\frac{g_{11,j}}{g_{11}} + \frac{g_{22,j}}{g_{22}} \right) y_j \right] + O(|y|^2), \\ \sqrt{\frac{g_{11}(y)}{g_{22}(y)}} &= \sqrt{\frac{g_{11}}{g_{22}}} \left[1 + \frac{1}{2} \left(\frac{g_{11,j}}{g_{11}} - \frac{g_{22,j}}{g_{22}} \right) y_j \right] + O(|y|^2), \\ \sqrt{\frac{g_{22}(y)}{g_{11}(y)}} &= \sqrt{\frac{g_{22}}{g_{11}}} \left[1 + \frac{1}{2} \left(\frac{g_{22,j}}{g_{22}} - \frac{g_{11,j}}{g_{11}} \right) y_j \right] + O(|y|^2), \end{aligned}$$

where Γ_{ij}^k denote the value of the Christoffel symbols at X_0 , and $|y| = \sqrt{y_1^2 + y_2^2}$. Let O_k denote $O(|y|^k + |z|^k)$. We have

$$(3.5) \quad \begin{aligned} G_{11}(y, z) &= g_{11} + g_{11,j} y_j - 2\Omega_{11} z + O_2, \\ G_{22}(y, z) &= g_{22} + g_{22,j} y_j - 2\Omega_{22} z + O_2, \\ G(y, z) &= g + g_{,j} y_j - 2g(\kappa_1 + \kappa_2) z + O_2, \\ \sqrt{G(y, z)} &= \sqrt{g} \left[1 + \frac{g_{,j} y_j}{2g} - (\kappa_1 + \kappa_2) z \right] + O_2, \\ \sqrt{\frac{G_{22}(y, z)}{G_{11}(y, z)}} &= \sqrt{\frac{g_{22}}{g_{11}}} \left[1 + \frac{1}{2} \left(\frac{g_{22,j}}{g_{22}} - \frac{g_{11,j}}{g_{11}} \right) y_j + (\kappa_1 - \kappa_2) z \right] + O_2, \\ \sqrt{\frac{G_{11}(y, z)}{G_{22}(y, z)}} &= \sqrt{\frac{g_{11}}{g_{22}}} \left[1 + \frac{1}{2} \left(\frac{g_{11,j}}{g_{11}} - \frac{g_{22,j}}{g_{22}} \right) y_j + (\kappa_2 - \kappa_1) z \right] + O_2. \end{aligned}$$

Now we fix a unit vector \mathbf{h} . Let

$$(3.6) \quad h_1 = \mathbf{h} \cdot \mathbf{e}_1, \quad h_2 = \mathbf{h} \cdot \mathbf{e}_2, \quad h_3 = \mathbf{h} \cdot \mathbf{e}_3.$$

We have

$$(3.7) \quad \frac{h_1^2}{g_{11}} + \frac{h_2^2}{g_{22}} + h_3^2 = 1.$$

Note that if $X_0 \in (\partial\Omega)_{\mathbf{h}}$, then $h_3 = 0$.

Lemma 3.1. *Assume $X_0 \in (\partial\Omega)_{\mathbf{h}}$. Given any constant ξ_0 , we can define a vector field $\mathbf{F}_{\mathbf{h}}$ in a neighborhood of X_0 such that $\text{curl } \mathbf{F}_{\mathbf{h}} = \mathbf{h}$, and in the coordinates (y, z) introduced above with $\mathbf{r}(0, 0) = X_0$, $\mathbf{F}_{\mathbf{h}}$ has the expression*

$$(3.8) \quad \mathbf{F}_{\mathbf{h}}(y, z) = \sum_{j=1}^3 G^{jj}(y, z) \hat{a}_j(y, z) \partial_j \mathcal{F}(y, z),$$

where

$$(3.9) \quad \begin{aligned} \hat{a}_1(y, z) &= h_2 \sqrt{\frac{g_{11}}{g_{22}}}(z + \xi_0) - a_2^{(1)} y_2^2 + a_{13}^{(1)} y_1(z + \xi_0) \\ &\quad + a_{23}^{(1)} y_2(z + \xi_0) + a_{33}^{(1)} z^2 + O_3, \\ \hat{a}_2(y, z) &= -h_1 \sqrt{\frac{g_{22}}{g_{11}}}(z + \xi_0) + a_1^{(2)} y_1^2 + a_{13}^{(2)} y_1(z + \xi_0) \\ &\quad + a_{23}^{(2)} y_2(z + \xi_0) + a_{33}^{(2)} z^2 + O_3, \\ \hat{a}_3(y, z) &= 0, \end{aligned}$$

where

$$(3.10) \quad \begin{aligned} a_2^{(1)} &= -\frac{h_2}{2} \sqrt{g} \kappa_2, \\ a_{13}^{(1)} &= \frac{1}{2\sqrt{g}} (h_1 g_{11,2} + h_2 g_{11,1}), \\ a_{23}^{(1)} &= \frac{1}{2\sqrt{g}} (-h_1 g_{22,1} + h_2 g_{11,2}), \\ a_{33}^{(1)} &= -\frac{h_2}{2} \sqrt{\frac{g_{11}}{g_{22}}} \kappa_1, \\ a_1^{(2)} &= -\frac{h_1}{2} \sqrt{g} \kappa_1, \\ a_{13}^{(2)} &= a_{23}^{(1)}, \\ a_{23}^{(2)} &= -\frac{1}{2\sqrt{g}} (h_1 g_{22,2} + h_2 g_{22,1}), \\ a_{33}^{(2)} &= \frac{h_1}{2} \sqrt{\frac{g_{22}}{g_{11}}} \kappa_2. \end{aligned}$$

The conclusion is obtained by direct computations. The details are given in Appendix B.

Under the conditions of Lemma 3.1, the eigenvalue equation for the operator $-\nabla_{\varepsilon}^{-2} \mathbf{F}_{\mathbf{h}}$ can be written as follows:

$$(3.11) \quad \begin{aligned} &\sum_{j=1}^2 \frac{1}{G_{jj}(y, z)} \{ \partial_{jj} \psi - 2i\varepsilon^{-2} \hat{a}_j(y, z) \partial_1 \psi - \varepsilon^{-4} |\hat{a}_j(y, z)|^2 \psi \} + \partial_{33} \psi \\ &+ \sum_{j=1}^3 \beta_j(y, z) \partial_j \psi - i\varepsilon^{-2} \sum_{j=1}^2 \left\{ \frac{\alpha_j(y, z)}{\sqrt{G(y, z)}} \partial_j \hat{a}_j(y, z) \right. \\ &\quad \left. + \beta_j(y, z) \hat{a}_j(y, z) \right\} \psi + \mu \psi = 0, \end{aligned}$$

where

$$\alpha_1(y, z) = \sqrt{\frac{G_{22}(y, z)}{G_{11}(y, z)}}, \quad \alpha_2(y, z) = \sqrt{\frac{G_{11}(y, z)}{G_{22}(y, z)}}, \quad \alpha_3(y, z) = \sqrt{G(y, z)},$$

$$\beta_j(y, z) = \frac{\partial_j \alpha_j(y, z)}{\sqrt{G(y, z)}}, \quad j = 1, 2, 3.$$

The boundary condition is

$$\partial_z \psi(y, 0) = 0 \quad \text{if } z = 0.$$

For later use we choose the constant ξ_0 in Lemma 3.1 to be εz_0 , where z_0 is the number given in Lemma 2.1. Consider the following rescaling for small $\varepsilon > 0$:

$$\begin{aligned} \phi(y, z) &= \psi(\varepsilon^{2/3}y, \varepsilon z), \\ G_{jj,\varepsilon}(y, z) &= G_{jj}(\varepsilon^{2/3}, \varepsilon z), \\ G_\varepsilon(y, z) &= G(\varepsilon^{2/3}y, \varepsilon z), \\ b_j(y, z) &= \varepsilon^{-1} \hat{a}_j(\varepsilon^{2/3}y, \varepsilon z), \\ \alpha_{j,\varepsilon}(y, z) &= \alpha_j(\varepsilon^{2/3}y, \varepsilon z), \\ \beta_{j,\varepsilon}(y, z) &= \beta_j(\varepsilon^{2/3}y, \varepsilon z), \quad j = 1, 2, 3. \end{aligned}$$

From (3.9) we find that, when $X_0 \in (\partial\Omega)_{\mathbf{h}}$,

$$\begin{aligned} b_1(y, z) &= h_2 \sqrt{\frac{g_{11}}{g_{22}}}(z + z_0) - a_2^{(1)} \varepsilon^{1/3} y_2^2 \\ &\quad + \varepsilon^{2/3} [a_{13}^{(1)} y_1 + a_{23}^{(1)} y_2](z + z_0) + a_{33}^{(1)} \varepsilon z^2 \\ &\quad + O(\varepsilon) |y|^3 + O(\varepsilon^{4/3}), \\ (3.12) \quad b_2(y, z) &= -h_1 \sqrt{\frac{g_{22}}{g_{11}}}(z + z_0) + a_1^{(2)} \varepsilon^{1/3} y_1^2 \\ &\quad + \varepsilon^{2/3} [a_{13}^{(2)} y_1 + a_{23}^{(2)} y_2](z + z_0) + a_{33}^{(2)} \varepsilon z^2 \\ &\quad + O(\varepsilon) |y|^3 + O(\varepsilon^{4/3}), \\ b_3(y, z) &= 0. \end{aligned}$$

Equation (3.11) is changed to

$$\begin{aligned} (3.13) \quad &\sum_{j=1}^2 \frac{1}{G_{jj,\varepsilon}(y, z)} \{ \varepsilon^{2/3} \partial_{jj} \phi - 2i \varepsilon^{1/3} b_j(y, z) \partial_j \phi - |b_j(y, z)|^2 \phi \} + \partial_{33} \phi \\ &+ \varepsilon \left\{ \beta_{3,\varepsilon} \partial_3 \phi - i \sum_{j=1}^2 \left[\frac{\alpha_{j,\varepsilon}}{\sqrt{G_\varepsilon}} \frac{\partial_j b_j}{\varepsilon^{2/3}} + \beta_{j,\varepsilon} b_j \right] \phi \right\} \\ &+ \varepsilon^{4/3} \sum_{j=1}^2 \beta_{j,\varepsilon} \partial_j \phi + \varepsilon^2 \mu \phi = 0. \end{aligned}$$

Note that

$$(3.14) \quad \partial_j b_j(y, z) = O(\varepsilon^{2/3}), \quad j = 1, 2.$$

These computational results will be used in Section 4.

§4. UPPER BOUND OF $\mu(\varepsilon^{-2}\mathbf{F}_h)$ AND LOWER BOUND OF $H_{C_3}(\mathbf{h}, \mathbf{k})$

Let the vector field \mathbf{F}_h satisfy (1.3). In this section we give an upper bound estimate for the lowest eigenvalue $\mu(\varepsilon^{-2}\mathbf{F}_h)$, which is optimal up to the second term.

Theorem 4.1. *Assume that Ω is a bounded 3-dimensional domain of class C^4 . We have*

$$(4.1) \quad \mu(\varepsilon^{-2}\mathbf{F}_h) \leq \frac{1}{\varepsilon^2} \{ \beta_0 + \varepsilon^{2/3} C_2 P_{\min} + O(\varepsilon^{5/6}) \},$$

where P_{\min} is given in (1.5) and C_2 is given in (1.6).

Proof. Let us recall the definition of $(\partial\Omega)_h$, the function $P(x)$ and \mathcal{P} given in Section 1. In the proof we write

$$(4.2) \quad \mu_p = C_2 P_{\min} = \lambda_0(1 - \alpha_0)^{1/3} P_{\min}.$$

We fix a point $X_0 \in \mathcal{P}$, and in a neighborhood of X_0 we use the coordinates (y, z) introduced in Section 3. To save notation, we will not distinguish a function in the original coordinates or in new coordinates. As in Section 3, we choose $\mathbf{F}_h = \sum_{j=1}^3 G^{jj} \hat{a}_j \partial_j \mathcal{F}$; here \hat{a}_j 's are given in (3.9). For a function $\psi(x)$ with compact support, we have

$$\begin{aligned} \int_{\Omega} |\nabla_{\varepsilon^{-2}\mathbf{F}_h} \psi|^2 dx &= \int_{\mathbb{R}_+^3} \sum_{j=1}^3 \sqrt{G(y, z)} G^{jj}(y, z) |\partial_{y_j} \psi - i\varepsilon^{-2} \hat{a}_j \psi|^2 dy dz, \\ \int_{\Omega} |\psi|^2 dx &= \int_{\mathbb{R}_+^3} \sqrt{G(y, z)} |\psi|^2 dy dz. \end{aligned}$$

Here $dy = dy_1 dy_2$. Let us define $G_{jj, \varepsilon}$, G_{ε} and $b_j(y, z)$ as in Section 3 (see (3.12)), and choose a test function in the form

$$\psi(x) = \phi\left(\frac{y}{\varepsilon^{2/3}}, \frac{z}{\varepsilon}\right).$$

We compute

$$\mu(\varepsilon^{-2}\mathbf{F}_h) \leq \frac{\int_{\Omega} |\nabla_{\varepsilon^{-2}\mathbf{F}_h} \psi|^2 dx}{\int_{\Omega} |\psi|^2 dx} = \varepsilon^{-2} \frac{I[\phi]}{II[\phi]},$$

where

$$\begin{aligned} I[\psi] &= \int_{\mathbb{R}_+^3} \left\{ \sqrt{\frac{G_{22, \varepsilon}}{G_{11, \varepsilon}}} |\varepsilon^{1/3} \partial_{y_1} \phi - ib_1 \phi|^2 + \sqrt{\frac{G_{11, \varepsilon}}{G_{22, \varepsilon}}} |\varepsilon^{1/3} \partial_{y_2} \phi - ib_2 \phi|^2 \right. \\ &\quad \left. + \sqrt{G_{\varepsilon}} |\partial_z \phi|^2 \right\} dy dz, \\ II[\psi] &= \int_{\mathbb{R}_+^3} \sqrt{G_{\varepsilon}} |\phi|^2 dy dz. \end{aligned}$$

Let us choose

$$\begin{aligned} \phi(y, z) &= \phi_0(y, z) + \varepsilon^{1/3} \phi_1(y, z), \\ \text{where } \phi_0(y, z) &= w(y)u(z), \quad \phi_1(y, z) = w_1(y)u_1(z), \end{aligned}$$

where u is a positive eigenfunction of (2.1) with $z = z_0$ and $\beta = \beta_0$, and u_1 is a solution of (2.3) for this u (see Section 2); w is to be determined later and w_1 is

determined by w through the equality

$$(4.3) \quad w_1(y) = \frac{2}{\sqrt{g}}[ih_2\partial_1 w - ih_1\partial_2 w - (a_1^{(2)}h_1y_1^2 + a_2^{(1)}h_2y_2^2)w],$$

where $a_2^{(1)}$ and $a_1^{(2)}$ are given in (3.10). We also assume that u, u_1, w and w_1 satisfy

$$(4.4) \quad \begin{aligned} \|u\|_{L^2(\mathbb{R}_+)} &= 1, & \|u_1\|_{L^2(\mathbb{R}_+)} &= 1, \\ \|w\|_{L^2(\mathbb{R}^2)} &= 1, & \|w_1\|_{L^2(\mathbb{R}^2)} &= 1, \\ \int_0^{+\infty} u(z)u_1(z)dz &= 0, & \int_{\mathbb{R}^2} y_j|w(y)|^2dy &= 0, \quad j = 1, 2. \end{aligned}$$

Recall that u and u_1 exponentially decay. So, in the following computations we can drop various small terms. In the following, g_{jj} , etc. denote the value of the involved functions at X_0 . Using (3.5) and equation (3.13), we find

$$\begin{aligned} \sqrt{\frac{G_{22,\varepsilon}}{G_{11,\varepsilon}}|\varepsilon^{1/3}\partial_{y_1}\phi - ib_1\phi|^2} &= h_2^2\sqrt{\frac{g_{11}}{g_{22}}}(z+z_0)^2|\phi_0|^2 + 2\varepsilon^{1/3}h_2(z+z_0)\Re(i\bar{\phi}_0R_1) \\ &+ \varepsilon^{2/3}\left\{\sqrt{\frac{g_{22}}{g_{11}}}|R_1|^2 + c_jh_2^2\sqrt{\frac{g_{11}}{g_{22}}}(z+z_0)^2y_j|\phi_0|^2 + 2h_2(z+z_0)\Re(i\bar{\phi}_0R_2)\right\} + O(\varepsilon), \\ \sqrt{\frac{G_{11,\varepsilon}}{G_{22,\varepsilon}}|\varepsilon^{1/3}\partial_{y_2}\phi - ib_2\phi|^2} &= h_1^2\sqrt{\frac{g_{22}}{g_{11}}}(z+z_0)^2|\phi_0|^2 - 2\varepsilon^{1/3}h_1(z+z_0)\Re(i\bar{\phi}_0Z_1) \\ &+ \varepsilon^{2/3}\left\{\sqrt{\frac{g_{11}}{g_{22}}}|Z_1|^2 - c_jh_1^2\sqrt{\frac{g_{22}}{g_{11}}}(z+z_0)^2y_j|\phi_0|^2 - 2h_1(z+z_0)\Re(i\bar{\phi}_0Z_2)\right\} + O(\varepsilon), \\ \sqrt{G_\varepsilon}|\partial_z\phi|^2 &= \sqrt{g}|\partial_3\phi_0|^2 + 2\varepsilon^{1/3}\sqrt{g}\Re(\partial_z\phi_0\partial_z\bar{\phi}_1) + \varepsilon^{2/3}\sqrt{g}(d_jy_j|\partial_z\phi_0|^2 + |\partial_z\phi_1|^2) \\ &+ O(\varepsilon), \end{aligned}$$

where

$$(4.5) \quad \begin{aligned} c_j &= \frac{1}{2}\left(\frac{g_{22,j}}{g_{22}} - \frac{g_{11,j}}{g_{11}}\right), \\ d_j &= \frac{1}{2}\left(\frac{g_{11,j}}{g_{11}} + \frac{g_{22,j}}{g_{22}}\right), \quad j = 1, 2, \\ R_1 &= \partial_1\phi_0 + ia_2^{(1)}y_2^2\phi_0 - ih_2\sqrt{\frac{g_{11}}{g_{22}}}(z+z_0)\phi_1, \\ R_2 &= \partial_1\phi_1 + ia_2^{(1)}y_2^2\phi_1 - i(a_{13}^{(1)}y_1 + a_{23}^{(1)}y_2)(z+z_0)\phi_0, \\ Z_1 &= \partial_2\phi_0 - ia_1^{(2)}y_1^2\phi_0 + ih_1\sqrt{\frac{g_{22}}{g_{11}}}(z+z_0)\phi_1, \\ Z_2 &= \partial_2\phi_1 - ia_1^{(2)}y_1^2\phi_1 - i(a_{13}^{(2)}y_1 + a_{23}^{(2)}y_2)(z+z_0)\phi_0. \end{aligned}$$

We compute

$$I[\phi] = \sqrt{g}\{I_0 + 2\varepsilon^{1/3}I_1 + \varepsilon^{2/3}I_2\} + O(\varepsilon),$$

where

$$\begin{aligned} I_0 &= \int_{\mathbb{R}_+^3} \left\{ |\partial_z \phi_0|^2 + \left(\frac{h_1^2}{g_{11}} + \frac{h_2^2}{g_{22}} \right) (z + z_0)^2 |\phi_0|^2 \right\} dy dz, \\ I_1 &= \int_{\mathbb{R}_+^3} \left\{ \Re(\partial_z \bar{\phi}_0 \partial_z \phi_1) + \frac{z + z_0}{\sqrt{g}} \Re[i \bar{\phi}_0 (h_2 R_1 - h_1 Z_1)] \right\} dy dz, \\ I_2 &= \int_{\mathbb{R}_+^3} \left\{ |\partial_z \phi_1|^2 + d_j y_j |\partial_z \phi_0|^2 + \frac{1}{g_{11}} |R_1|^2 + \frac{1}{g_{22}} |Z_1|^2 \right. \\ &\quad \left. + c_j \left(\frac{h_2^2}{g_{22}} - \frac{h_1^2}{g_{11}} \right) y_j (z + z_0)^2 |\phi_0|^2 + \frac{2(z + z_0)}{\sqrt{g}} \Re[i \bar{\phi}_0 (h_2 R_2 - h_1 Z_2)] \right\} dy dz. \end{aligned}$$

Recall that $h_1^2/g_{11} + h_2^2/g_{22} = 1$, and

$$\begin{aligned} \int_0^{+\infty} \{u' u_1' + (z + z_0)^2 u u_1\} dz &= \beta_0 \int_0^{+\infty} u u_1 dz = 0, \\ \int_0^{+\infty} \{|u_1'|^2 + (z + z_0)^2 u_1^2\} dz &= \beta_0 \|u_1\|_{L^2(\mathbb{R}_+)}^2 + \frac{\alpha_0}{4}; \end{aligned}$$

see Section 2. Using (4.4) and (4.5) we find $I_0 = \beta_0$, $I_1 = 0$, and

$$\begin{aligned} I_2 &= \int_0^{+\infty} \{|u_1'|^2 + (z + z_0)^2 u_1^2\} dz \int_{\mathbb{R}^2} |w_1|^2 dy \\ &\quad + \|u\|_{L^2(\mathbb{R}_+)}^2 \int_{\mathbb{R}^2} \left\{ \frac{1}{g_{11}} |\partial_{y_1} w + ia_2^{(1)} y_2^2 w|^2 + \frac{1}{g_{22}} |\partial_{y_2} w - ia_1^{(2)} y_1^2 w|^2 \right\} dy \\ &\quad + \left\{ d_j \|u'\|_{L^2(\mathbb{R}_+)}^2 + c_j \left(\frac{h_2^2}{g_{22}} - \frac{h_1^2}{g_{11}} \right) \int_0^{+\infty} (z + z_0)^2 u^2 dz \right\} \int_{\mathbb{R}^2} y_j |w|^2 dy \\ &\quad + \frac{2}{\sqrt{g}} \int_0^{+\infty} (z + z_0)^2 u^2 dz \int_{\mathbb{R}^2} [h_2 (a_{13}^{(1)} y_1 + a_{23}^{(1)} y_2) - h_1 (a_{13}^{(2)} y_1 + a_{23}^{(2)} y_2)] |w|^2 dy \\ &\quad + \frac{4}{\sqrt{g}} \int_0^{+\infty} (z + z_0) u u_1 dz \int_{\mathbb{R}^2} \Re \left\{ i \bar{w}_1 [h_2 (\partial_{y_1} w + ia_2^{(1)} y_2^2 w) \right. \\ &\quad \left. - h_1 (\partial_{y_2} w - ia_1^{(2)} y_1^2 w)] \right\} dy \\ &= \beta_0 \|u_1\|_{L^2(\mathbb{R}_+)}^2 \|w_1\|_{L^2(\mathbb{R}^2)}^2 + \frac{\alpha_0}{4} \|w_1\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad + \int_{\mathbb{R}^2} \left\{ \frac{1}{g_{11}} |\partial_{y_1} w + ia_2^{(1)} y_2^2 w|^2 + \frac{1}{g_{22}} |\partial_{y_2} w - ia_1^{(2)} y_1^2 w|^2 \right\} dy \\ &\quad + \frac{4}{\sqrt{g}} \int_0^{+\infty} (z + z_0) u u_1 dz \int_{\mathbb{R}^2} \Re \left\{ i \bar{w}_1 [h_2 (\partial_{y_1} w + ia_2^{(1)} y_2^2 w) \right. \\ &\quad \left. - h_1 (\partial_{y_2} w - ia_1^{(2)} y_1^2 w)] \right\} dy \\ &= \beta_0 \|u_1\|_{L^2(\mathbb{R}_+)}^2 \|w_1\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \left\{ c_{11} |\partial_{y_1} w + ia_2^{(1)} y_2^2 w|^2 + c_{22} |\partial_{y_2} w - ia_1^{(2)} y_1^2 w|^2 \right. \\ &\quad \left. + 2c_{12} \Re[(\partial_{y_1} w + ia_2^{(1)} y_2^2 w)(\partial_{y_2} \bar{w} + ia_1^{(2)} y_1^2 \bar{w})] \right\} dy \\ &= \beta_0 \|u_1\|_{L^2(\mathbb{R}_+)}^2 \|w_1\|_{L^2(\mathbb{R}^2)}^2 + p[w], \end{aligned}$$

where $p[w]$ is the functional defined in Section 2, and

$$\begin{aligned}
 c_{11} &= \frac{1}{g_{11}} \left(1 - \alpha_0 \frac{h_2^2}{g_{22}}\right), \\
 c_{12} &= \frac{\alpha_0 h_1 h_2}{g_{11} g_{22}}, \\
 c_{22} &= \frac{1}{g_{22}} \left(1 - \alpha_0 \frac{h_1^2}{g_{11}}\right), \\
 f_0 &= c_{11} (a_2^{(1)})^2 y_2^4 - 2c_{12} a_2^{(1)} a_1^{(2)} y_1^2 y_2^2 + c_{22} (a_1^{(2)})^2 y_1^4 \\
 &\quad + 2ic_{12} (a_1^{(2)} y_1 - a_2^{(1)} y_2), \\
 f_1 &= c_{12} a_1^{(2)} y_1^2 - c_{11} a_2^{(1)} y_2^2, \\
 f_2 &= c_{22} a_1^{(2)} y_1^2 - c_{12} a_2^{(1)} y_2^2.
 \end{aligned}
 \tag{4.6}$$

On the other hand, we have

$$II[\phi] = \sqrt{g} \{1 + \varepsilon^{2/3} \|u_1\|_{L^2(\mathbb{R}_+)}^2 \|w_1\|_{L^2(\mathbb{R}^2)}^2\} + O(\varepsilon).$$

Hence

$$\frac{I[\phi]}{II[\phi]} = I_0 + \varepsilon^{2/3} \{I_2 - I_0 \|u_1\|_{L^2(\mathbb{R}_+)}^2 \|w_1\|_{L^2(\mathbb{R}^2)}^2\} + O(\varepsilon) = I_0 + \varepsilon^{2/3} p[w] + O(\varepsilon).$$

Now we look for a proper function w which satisfies all the requirements needed in the above computations, and such that all the error terms of order $O(\varepsilon)$ in the estimates can be controlled. By careful analysis on each term of order $O(\varepsilon)$, we see that they can be controlled by the integral of the function $\varepsilon(|y|^2 + |z|^2)|\phi|^2$. From Proposition 2.5 we know that (2.8) has a bounded solution $\psi_p(y)$ given in (2.18). In the new coordinates $x' = (x'_1, x'_2)$ determined by the transformations (2.10) and (2.13), we have

$$|\psi_p(y)| = cv(x'_1),$$

which depends only on x'_1 (here, in order to avoid confusion, we write the new coordinates given by (2.13) by $x' = (x'_1, x'_2)$, instead of x). Moreover, $\psi_p(y)$ and its partial derivatives exponentially decay as x'_1 goes to infinity. However, $\psi_p(y)$ does not decay in x'_2 direction. Hence we choose w in the following form:

$$w(y) = c\psi_p(y)\eta(x'_2),$$

where $\eta(x'_2) \in L^2(\mathbb{R})$ and is an even function. Then, using Remark 2.6 we have

$$\int_{\mathbb{R}^2} y_j |w(y)|^2 dy = 0, \quad j = 1, 2.$$

Since $u(z)$ and $u_1(z)$ exponentially decay in z , and $|\psi_p(y)| = cv(x'_1)$ exponentially decays in x'_1 , we find that

$$\frac{I[\phi]}{II[\phi]} \leq \beta_0 + \varepsilon^{2/3} p[w] + M_1 \varepsilon \int_{\mathbb{R}^2} |y|^2 |w|^2 dy + M_2 \varepsilon.$$

To determine the function η , we consider the following variational problem:

$$\delta(\varepsilon) = \inf_{\phi \in W^{1,2}(\mathbb{R})} \frac{\int_{-\infty}^{+\infty} \{|\phi'(t)|^2 + M_1 \varepsilon^{1/3} t^2 |\phi(t)|^2\} dt}{\|\phi\|_{L^2(\mathbb{R})}^2}.
 \tag{4.7}$$

By rescaling $s = \varepsilon^{1/12}t$, we see that $\delta(\varepsilon) = \varepsilon^{1/6}\delta_0$, where

$$(4.8) \quad \delta_0 = \inf_{\varphi \in W^{1,2}(\mathbb{R})} \frac{\int_{-\infty}^{+\infty} \{|\varphi'(s)|^2 + M_1 s^2 |\varphi(s)|^2\} ds}{\|\varphi\|_{L^2(\mathbb{R})}^2}.$$

The minimizers $\varphi_0(s)$ of the latter problem (4.8) exist. They are even functions and exponentially decay in s . So the minimizers $\phi_0(t)$ of the former problem (4.7) exist, they are even functions and exponentially decay in t , and

$$\phi_0(t) = \varphi_0(s) = \varphi_0(\varepsilon^{1/12}t).$$

Now we choose a cut-off function $\eta_n(t)$ such that, $\eta_n(t)$ is an even function, $\eta_n(t) = 1$ if $|t| < n$, $\eta_n(t) = 0$ if $|t| > 2n$, and $|\eta'_n(t)| \leq 2/n$. Then we choose a suitable integer n_ε , and let

$$(4.9) \quad \begin{aligned} \eta(x'_2) &= c(\varepsilon)\eta_{n_\varepsilon}(x'_2)\varphi_0(\varepsilon^{1/12}x'_2), \\ w(y) &= c(\varepsilon)\psi_p(y)\eta(x'_2), \end{aligned}$$

where $c(\varepsilon)$ is chosen so that the requirement $\|w\|_{L^2(\mathbb{R}^2)} = 1$ is satisfied. Using this w , we defined the function w_1 by (4.3). Then we let

$$(4.10) \quad \begin{aligned} \phi(y, z) &= c(\varepsilon)\psi_p(y)\eta(x'_2)u(z) + \varepsilon^{1/3}w_1(y)u_1(z), \\ \psi(x) &= \phi\left(\frac{y}{\varepsilon^{2/3}}, \frac{z}{\varepsilon}\right). \end{aligned}$$

We have

$$\begin{aligned} & p[w] + M_1\varepsilon^{1/3} \int_{\mathbb{R}^2} |y|^2 |w|^2 dy \\ &= [\mu_p + \delta(\varepsilon) + O(\varepsilon)] \|w\|_{L^2(\mathbb{R}^2)}^2 = [\mu_p + O(\varepsilon^{1/6})] \|w\|_{L^2(\mathbb{R}^2)}^2. \end{aligned}$$

Hence

$$\varepsilon^2 \mu(\varepsilon^{-2}\mathbf{F}_h) \leq \frac{I[\phi]}{II[\phi]} \leq \beta_0 + \varepsilon^{2/3}[\mu_p + O(\varepsilon^{1/6})] \leq \beta_0 + \varepsilon^{2/3}\mu_p + O(\varepsilon^{5/6}).$$

□

Theorem 4.2 (Lower bound estimate for H_{C_3}). *Assume that Ω is a bounded 3-dimensional domain of class C^4 . Then for any unit vector \mathbf{h} and large κ , we have*

$$(4.11) \quad H_{C_3}(\mathbf{h}, \kappa) \geq \frac{\kappa}{\beta_0} - \frac{C_2}{\beta_0^{5/3}} P_{\min} \kappa^{1/3} + O(\kappa^{2/9}).$$

Proof. Note that, if $\mu(\sigma\kappa\mathbf{F}_h) < \kappa^2$, then the Ginzburg-Landau functional \mathcal{E} has a nontrivial minimizer; see [LP3] (Lemma 2.1). Thus, the lower bound of H_{C_3} can be obtained by using the upper bound of the lowest eigenvalue $\mu(\sigma\kappa\mathbf{F}_h)$ given in Theorem 4.1, and using the argument in [LP3]. □

§5. ELLIPTIC ESTIMATES

In this section we establish some estimates for the minimizers of the Ginzburg-Landau functional \mathcal{E} . Throughout this section we assume that Ω is a bounded and simply-connected 3-dimensional domain of class C^4 . For convenience, we introduce

$$\varepsilon = \frac{1}{\sqrt{\sigma\kappa}}, \quad \lambda_\varepsilon = \frac{\kappa}{\sigma}.$$

Note that λ_ε depends actually on both κ and σ . From (1.4), $H_{C_3}(\mathbf{h}, \kappa) = \frac{\kappa}{\beta_0} + o(\kappa)$. We also believe that $H_{C_2}(\kappa, \mathbf{h}) \sim \kappa$ for large value of κ . Since we are considering applied fields in the range between H_{C_2} and H_{C_3} , we assume in the following that

$$(5.1) \quad \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = \lambda, \quad \beta_0 \leq \lambda \leq 1.$$

The functional \mathcal{E} can be written as follows:

$$\mathcal{E}[\psi, \mathbf{A}] = \int_{\Omega} \left\{ |\nabla_{\varepsilon^{-2}\mathbf{A}}\psi|^2 - \frac{\lambda_\varepsilon}{\varepsilon^2}|\psi|^2 + \frac{\lambda_\varepsilon}{2\varepsilon^2}|\psi|^4 \right\} dx + \frac{1}{\varepsilon^4} \int_{\mathbb{R}^3} |\operatorname{curl}\mathbf{A} - \mathbf{h}|^2 dx.$$

Let $W^{1,2}(\Omega, \mathbb{C})$ be the Sobolev space of all complex-valued functions defined on $\bar{\Omega}$. Let

$$\begin{aligned} \mathbf{D}^{1,2}(\mathbb{R}^3) &= \{ \mathbf{A} \in L^1_{\text{loc}}(\mathbb{R}^3) : |\nabla\mathbf{A}| \in L^2(\mathbb{R}^3) \}, \\ \mathbf{D}^{1,2}(\mathbb{R}^3, \operatorname{div}) &= \{ \mathbf{A} \in \mathbf{D}^{1,2}(\mathbb{R}^3) : \operatorname{div}\mathbf{A} = 0 \text{ in } \mathbb{R}^3 \}. \end{aligned}$$

Given a unit vector \mathbf{h} , let $\mathbf{F}_\mathbf{h}$ be the smooth vector field satisfying (1.3) and let

$$\mathcal{W}(\Omega) = \{ (\psi, \mathbf{A}) : \psi \in W^{1,2}(\Omega, \mathbb{C}), \mathbf{A} - \mathbf{F}_\mathbf{h} \in \mathbf{D}^{1,2}(\mathbb{R}^3) \}.$$

We consider the variational problem for \mathcal{E} on $\mathcal{W}(\Omega)$, and define

$$C(\varepsilon) = \inf_{(\psi, \mathbf{A}) \in \mathcal{W}(\Omega)} \mathcal{E}[\psi, \mathbf{A}].$$

It is easy to show that the minimizers exist. By the gauge invariance of \mathcal{E} , we can restrict the functional \mathcal{E} on a subspace

$$\mathcal{W}(\Omega, \operatorname{div}) = \{ (\psi, \mathbf{A}) \in \mathcal{W}(\Omega) : \operatorname{div}\mathbf{A} = 0 \text{ in } \mathbb{R}^3 \}.$$

Then, the minimizers satisfy the Euler equation

$$(5.2) \quad \begin{cases} -\nabla_{\varepsilon^{-2}\mathbf{A}}^2 \psi = \frac{\lambda_\varepsilon}{\varepsilon^2}(1 - |\psi|^2)\psi & \text{in } \Omega, \\ \operatorname{curl}^2(\mathbf{A} - \mathbf{F}_\mathbf{h}) = \varepsilon^2 \Im(\bar{\psi} \nabla_{\varepsilon^{-2}\mathbf{A}} \psi) \chi_\Omega & \text{in } \mathbb{R}^3, \\ (\nabla_{\varepsilon^{-2}\mathbf{A}} \psi) \cdot \nu = 0, & \text{on } \partial\Omega, \\ \mathbf{A} - \mathbf{F}_\mathbf{h} \in \mathbf{D}^{1,2}(\mathbb{R}^3, \operatorname{div}), \end{cases}$$

where $\chi_\Omega(x) = 1$ for $x \in \Omega$ and $\chi_\Omega(x) = 0$ for $x \notin \Omega$. (5.2) is a weak form of (1.2); see [P4], also see [L] (Chapter 5, Section 4). It has been proved by many authors that the minimal solutions of (5.2) satisfy

$$\|\psi\|_{L^\infty(\Omega)} \leq 1.$$

The regularity of the minimal solutions has been discussed in [P4]. In the following we shall always write a minimizer of \mathcal{E} by $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$ and assume $\operatorname{div}\mathbf{A}^\varepsilon = 0$ in Ω . We always let C denote a generic positive constant that is independent of ε , but may vary from line to line. Let B_R denote the open ball centered at the origin and with radius R , $B_R^+ = \{y \in B_R : y_3 > 0\}$ and $\Gamma_R = \{(y_1, y_2, 0) : y_1^2 + y_2^2 < R\}$. Let

$$t(x) = \operatorname{dist}(x, \partial\Omega).$$

Lemma 5.1. *Under the condition (5.1), $\psi^\varepsilon \in C^{2+\alpha}(\bar{\Omega})$ and $\mathbf{A}^\varepsilon \in C^{1+\alpha}(\bar{\Omega})$ for any $0 < \alpha < 1$. Moreover, we have the following estimates:*

(1) *There exists $C > 0$ independent of ε such that*

$$\begin{aligned} \|\mathbf{A}^\varepsilon - \mathbf{F}_\mathbf{h}\|_{L^6(\mathbb{R}^3)} &\leq C\varepsilon \|\psi^\varepsilon\|_{L^4(\Omega)}^2, \\ \|\operatorname{curl}\mathbf{A}^\varepsilon - \mathbf{h}\|_{L^2(\mathbb{R}^3)} &\leq C\varepsilon \|\psi^\varepsilon\|_{L^4(\Omega)}^2. \end{aligned}$$

(2) For any $0 < \alpha < 1$, and for any $R > 0$ such that $\bar{\Omega} \subset B_R$, there exist $C(\alpha, R) > 0$ independent of ε and $q = 3/(1 - \alpha)$ such that, for all small $\varepsilon > 0$,

$$\|\nabla(\mathbf{A}^\varepsilon - \mathbf{F}_h)\|_{C^\alpha(R_R)} \leq C(\alpha, R)\{\varepsilon\|\psi^\varepsilon\|_{L^4(\Omega)}^2 + \varepsilon^2\|\tilde{\psi}^\varepsilon\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon}\psi^\varepsilon\|_{L^q(\Omega)}\}.$$

The proof is similar to the proof of Lemma 3.3 in [P4] and hence is omitted here. In the next lemma we need the estimate of the solutions near boundary. In a neighborhood of a point $X_0 \in \partial\Omega$ we introduce the local coordinates (y, z) by the diffeomorphism \mathcal{F} introduced in Section 3 such that $\mathcal{F}(0, 0) = X_0$. We define $\tilde{\psi}^\varepsilon$ and $\tilde{\mathbf{A}}^\varepsilon$ as in (3.1):

$$(5.3) \quad \begin{aligned} \tilde{\psi}^\varepsilon(y, z) &= \psi^\varepsilon(\mathcal{F}(y, z)), \\ \tilde{\mathbf{A}}^\varepsilon(y, z) &= \mathbf{A}^\varepsilon(\mathcal{F}(y, z)) = \sum_{j=1}^3 G^{jj}(y, z)a_j^\varepsilon(y, z)\partial_j\mathcal{F}(y, z), \end{aligned}$$

where $y = (y_1, y_2)$, $\partial_1 = \partial_{y_1}$, $\partial_2 = \partial_{y_2}$, $\partial_3 = \partial_z$, $a_j^\varepsilon(y, z) = \mathbf{A}^\varepsilon(\mathcal{F}(y, z)) \cdot \partial_j\mathcal{F}(y, z)$. Then, since $\nu = -\partial_3\mathcal{F}$ on $\partial\Omega$ and $G_{12} = G_{13} = G_{23} = 0$, $G_{33} = 1$, we have

$$(5.4) \quad \begin{cases} -\frac{1}{\sqrt{G}}\sum_{j=1}^3\left\{\partial_j(\sqrt{G}G^{jj}\partial_j\tilde{\psi}^\varepsilon) - 2\varepsilon^{-2}i\sqrt{G}G^{jj}a_j^\varepsilon\partial_j\tilde{\psi}^\varepsilon - \varepsilon^{-4}\sqrt{G}G^{jj}|a_j^\varepsilon|^2\tilde{\psi}^\varepsilon\right\} = \frac{\lambda_\varepsilon}{\varepsilon^2}(1 - |\tilde{\psi}^\varepsilon|^2)\tilde{\psi}^\varepsilon & \text{in } B_{R_0}^+, \\ \partial_3\tilde{\psi}^\varepsilon - ia_3^\varepsilon\tilde{\psi}^\varepsilon = 0 & \text{on } \Gamma_{R_0}. \end{cases}$$

Lemma 5.2. Assume that (5.1) holds.

(1) There exist positive constants C and ε_0 , such that for any $0 < \varepsilon < \varepsilon_0$ we have

$$\|\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon}\psi^\varepsilon\|_{C(\bar{\Omega})} \leq \frac{C}{\varepsilon}.$$

(2) Given $0 < \alpha < 1$, there exist positive constants $C_1(\alpha)$ and $\varepsilon_1(\alpha)$, and $q = 3/(1 - \alpha)$, such that for any $0 < \varepsilon < \varepsilon_1(\alpha)$,

$$\|\nabla(\mathbf{A}^\varepsilon - \mathbf{F}_h)\|_{C^\alpha(\bar{\Omega})} \leq \varepsilon C_1(\alpha)\{\|\psi^\varepsilon\|_{L^4(\Omega)}^2 + \|\psi^\varepsilon\|_{L^q(\Omega)}\|\psi^\varepsilon\|_{L^\infty(\Omega)}\}.$$

(3) Given $0 < \alpha < 2\sqrt{1 - \beta_0}$, there exist positive constants $C_2(\alpha)$ and $\varepsilon_2(\alpha)$, such that for all $0 < \varepsilon < \varepsilon_2(\alpha)$,

$$(5.5) \quad \int_\Omega \exp\left(\frac{\alpha}{\varepsilon}t(x)\right)\{|\psi^\varepsilon|^2 + \varepsilon^2|\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon}\psi^\varepsilon|^2\}dx \leq C_2(\alpha)\varepsilon.$$

Proof. Note that, in the 2-dimensional case we have $C^{2+\alpha}$ estimates on the closed domain for the magnetic potential, while in the 3-dimensional case, we only have $\mathbf{A}^\varepsilon \in C_{loc}^3(\Omega) \cap C^{1+\alpha}(\bar{\Omega})$ for any $0 < \alpha < 1$.

To prove conclusion (1), we modify the blow-up arguments that were used to study the 2-dimensional case in [LP3] (Proposition 3.1) and [HP] (Proposition 4.2). Let X^ε be the maximum point of the function $|\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon}\psi^\varepsilon|$. In the interior blow-up case, namely, if $\text{dist}(X^\varepsilon, \partial\Omega) \gg \varepsilon$, then the argument of [HP] remains valid in the 3-dimensional case and we find

$$|\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon}\psi^\varepsilon(X^\varepsilon)| \leq C\varepsilon^{-1}.$$

Now let us consider the boundary blow-up case, namely, $\text{dist}(X^\varepsilon, \partial\Omega) \leq C\varepsilon$ for all small ε . Choose $Q^\varepsilon \in \partial\Omega$ such that $|X^\varepsilon - Q^\varepsilon| = \text{dist}(X^\varepsilon, \partial\Omega)$. Let $\mathcal{F}_\varepsilon(y, z)$ be a diffeomorphism which straightens a portion of $\partial\Omega$ surrounding Q^ε as described

in Section 3 such that $\mathcal{F}_\varepsilon(0, 0) = Q^\varepsilon$. Similar to (5.3) we define $\tilde{\psi}^\varepsilon(y, z)$ and $\tilde{\mathbf{A}}^\varepsilon(y, z)$, with \mathcal{F} replaced by \mathcal{F}_ε . Then $\tilde{\psi}^\varepsilon$ satisfies an equation in the form of (5.4), with coefficients depending on \mathcal{F}_ε instead of depending on \mathcal{F} . We make gauge transformation and rescaling

$$\begin{aligned} \tilde{\phi}_\varepsilon(y, z) &= e^{-i\chi_\varepsilon} \tilde{\psi}^\varepsilon(\varepsilon y, \varepsilon z), \\ \tilde{\mathbf{A}}_\varepsilon(y, z) &= \frac{1}{\varepsilon} [\tilde{\mathbf{A}}^\varepsilon(\varepsilon y, \varepsilon z) - \tilde{\mathbf{A}}^\varepsilon(0, 0)], \end{aligned}$$

where χ_ε is chosen properly; see [HP] (Proposition 4.2). Then $\tilde{\phi}_\varepsilon$ satisfies an elliptic equation in the form

$$\begin{cases} -\Delta_\varepsilon \tilde{\phi}_\varepsilon = \lambda_\varepsilon(1 - |\tilde{\phi}_\varepsilon|^2) \tilde{\phi}_\varepsilon & \text{in } B_{R_0/\varepsilon}^+, \\ \partial_{y_3} \tilde{\phi}_\varepsilon = i\varepsilon a_{\varepsilon,3} \tilde{\phi}_\varepsilon & \text{on } \Gamma_{R_0/\varepsilon}, \end{cases}$$

where Δ_ε is the representation of $\nabla_{\mathbf{A}_\varepsilon}^2$ in the coordinates (y, z) , and $\mathbf{A}_\varepsilon(x)$ is a vector field that corresponds in the coordinates x to $\tilde{\mathbf{A}}_\varepsilon$. Repeating the proof of Lemma 5.1 (see [P4], Lemma 3.3) we find that $\{\tilde{\mathbf{A}}_\varepsilon\}$ is bounded in $C_{\text{loc}}^{1+\alpha}(\mathbb{R}_+^3)$. Applying the Hölder estimate to the above equation we find that $\{\tilde{\phi}_\varepsilon\}$ is bounded in $C_{\text{loc}}^{2+\alpha}(\mathbb{R}_+^3)$. So $|\nabla_{\tilde{\mathbf{A}}_\varepsilon} \tilde{\psi}^\varepsilon(0, 0)| \leq C$, and hence $|\nabla_{\varepsilon^{-2}\mathbf{A}_\varepsilon} \psi^\varepsilon(X^\varepsilon)| \leq C\varepsilon^{-1}$.

Conclusion (2) follows from conclusion (1) and Lemma 5.1. Conclusion (3) can be proved by using (1) and (2). We omit the details but refer to [HP] (Lemma 3.2) and [P1] (Lemma 7.2) for similar discussions in the 2-dimensional case. \square

Conclusion (3) of Lemma 5.2 describes in the L^2 sense that, under condition (5.1), the order parameters concentrate in a thin layer at boundary, and decay exponentially in the normal direction away from the boundary. It means that, under the applied field above H_{C_2} , superconductivity persists within a thin sheath near surface with thickness $O(\kappa^{-1})$. More precise information of the concentration behavior of the order parameters will be given in Section 6. From conclusion (3) we also find that, for any nonnegative integer k ,

$$(5.6) \quad \int_{\Omega} t(x)^k \{|\psi^\varepsilon|^2 + \varepsilon^2 |\nabla_{\varepsilon^{-2}\mathbf{A}_\varepsilon} \psi^\varepsilon|^2\} dx \leq C_k \varepsilon^{k+1}.$$

In particular, we have

$$(5.7) \quad \int_{\Omega} |\psi^\varepsilon|^2 dx \leq C\varepsilon.$$

Let, for $q > 3$,

$$(5.8) \quad d(\varepsilon) = \|\psi^\varepsilon\|_{L^4(\Omega)}^2 + \|\psi^\varepsilon\|_{L^q(\Omega)} \|\psi^\varepsilon\|_{L^\infty(\Omega)}.$$

From (5.7) we find

$$(5.9) \quad d(\varepsilon) \leq C \{ \varepsilon^{1/2} \|\psi^\varepsilon\|_{L^\infty(\Omega)} + \varepsilon^{1/q} \|\psi^\varepsilon\|_{L^\infty(\Omega)}^{2(q-1)/q} \}.$$

Now we recall the function $\lambda(\vartheta)$ given in Lemma 2.3, and the function $\varphi_{\mathbf{h}}(x)$ given in Remark 2.4.

Lemma 5.3. *Assume condition (5.1). There exists a constant $C > 0$ such that, for any smooth function χ and small $\varepsilon > 0$, we have*

$$(5.10) \quad \int_{\Omega} |\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon}(\chi\psi^\varepsilon)|^2 dx \geq \varepsilon^{-2} \int_{\Omega} W_{\mathbf{h}}(x) |\chi\psi^\varepsilon|^2 dx - C\varepsilon^{-4/3} d(\varepsilon)^{2/3} \int_{\Omega} |\chi\psi^\varepsilon|^2 dx,$$

where $d(\varepsilon)$ is given in (5.8) and

$$(5.11) \quad W_{\mathbf{h}}(x) = \begin{cases} 1 - C\varepsilon^{1/2} & \text{if } \text{dist}(x, \partial\Omega) > 2\varepsilon^{1/4}, \\ \lambda(\varphi_{\mathbf{h}}(x)) - C\varepsilon^{1/2} & \text{if } \text{dist}(x, \partial\Omega) \leq 2\varepsilon^{1/4}. \end{cases}$$

Proof. The function $W_{\mathbf{h}}(x)$ was given by Helffer-Morame in [HM2],⁵ and it was given in [HM2] (Theorem 4.3) that, for any small $\varepsilon > 0$ and for any $\phi \in W^{1,2}(\Omega)$, the following inequality holds:

$$(5.12) \quad \int_{\Omega} |\nabla_{\varepsilon^{-2}\mathbf{F}_{\mathbf{h}}} \phi|^2 dx \geq \frac{1}{\varepsilon^2} \int_{\Omega} W_{\mathbf{h}}(x) |\phi|^2 dx.$$

To prove (5.12) one applies the local analysis techniques developed in [HM1], and uses the upper bound of $\mu(\varepsilon^{-2}\mathbf{F}_{\mathbf{h}})$ given in (1.13).

Lemma 5.3 can be proved exactly as in [P3] (proof of Theorem 6.2, Step 2), and the details are omitted here. Note that, a result in the 2-dimensional case similar to (5.10) has been proved in [HP], and in the proof the fact that we can choose a gauge such that the magnetic potential vanishes at the boundary makes it easy to control the error term; see (3.3) in [HP]. In the 3-dimensional case, however, the magnetic potentials do not vanish at boundary, and harder estimates including the decomposition of vector fields near boundary are needed. Then we can apply (5.12) and the local analysis techniques in [HM1] to prove (5.10). For more details see [P3]. \square

Lemma 5.4. *Let ϕ^ε be the eigenfunction of the operator $-\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon}^2$ associated with the lowest eigenvalue $\mu(\varepsilon^{-2}\mathbf{A}^\varepsilon)$. Then (5.5) and (5.10) hold for ϕ^ε .*

Proof. The conclusion that the eigenfunctions ϕ^ε satisfy (5.5) was given in [HM2], and it can be proved exactly as for ψ^ε . Note that (5.10) holds for any function satisfying (5.5), and hence is true for ϕ^ε . \square

Theorem 5.5. *Assume that Ω is a bounded and simply-connected 3-dimensional domain of class C^4 . Given a unit vector \mathbf{h} we have, for large κ ,*

$$(5.13) \quad H_{C_3}(\mathbf{h}, \kappa) \leq \frac{\kappa}{\beta_0} + M\kappa^{1/2},$$

where M is a constant independent of \mathbf{h} and κ .

Proof. We consider an applied field $\mathcal{H} = \sigma\mathbf{h}$ that is close to $H_{C_3}(\mathbf{h}, \kappa)$. From (1.4), we can choose $\sigma \sim \kappa/\beta_0$ as $\kappa \rightarrow \infty$. Hence $\varepsilon = (\sigma\kappa)^{-1/2} \sim \sqrt{\beta_0\kappa^{-1}}$. Let $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$ be a nontrivial minimizer of the functional \mathcal{E} . We show that

$$(5.14) \quad \mu(\varepsilon^{-2}\mathbf{A}^\varepsilon) \geq \frac{1}{\varepsilon^2}(\beta_0 - M_1\varepsilon^{1/2}),$$

⁵The notation ϑ and $\sigma(\vartheta)$ used in [HM2] are related to our notation $\varphi_{\mathbf{h}}$ and $\lambda(\varphi_{\mathbf{h}})$ through $\vartheta = \frac{\pi}{2} - \varphi_{\mathbf{h}}$, $\sigma(\vartheta) = \lambda(\frac{\pi}{2} - \varphi_{\mathbf{h}})$.

where M_1 is a constant independent of ε and \mathbf{h} . To prove (5.14), we let ϕ^ε be an eigenfunction of $-\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon}^2$ associated with $\mu(\varepsilon^{-2}\mathbf{A}^\varepsilon)$. From Lemma 5.4 we have

$$\int_{\Omega} |\nabla_{\varepsilon^{-2}\mathbf{A}^\varepsilon} \phi^\varepsilon|^2 dx \geq \varepsilon^{-2} \int_{\Omega} W_{\mathbf{h}}(x) |\phi^\varepsilon|^2 dx - C\varepsilon^{-4/3} d(\varepsilon)^{2/3} \int_{\Omega} |\phi^\varepsilon|^2 dx.$$

For the applied field close to H_{C_3} we have $\|\psi^\varepsilon\|_{L^\infty(\Omega)} = o(1)$. From (5.9) we find $d(\varepsilon) = o(\varepsilon^{1/q})$ for $q > 3$. From (5.11), we have

$$\mu(\varepsilon^{-2}\mathbf{A}^\varepsilon) \geq \frac{1}{\varepsilon^2} \{ \beta_0 - C\varepsilon^{1/2} - o(1)\varepsilon^{2(1+q)/(3q)} \}.$$

By choosing $q < 6$ in the above inequality we get (5.14).

Since $\psi^\varepsilon \not\equiv 0$, from [LP3] (Lemma 2.1) we know that $\mu(\varepsilon^{-2}\mathbf{A}^\varepsilon) < \kappa^2$. Now we have $\varepsilon^{-2}(\beta_0 - M_1\varepsilon^{1/2}) < \kappa^2$. So

$$\sigma < \frac{\kappa}{\beta_0 - M_1\varepsilon^{1/2}} = \frac{\kappa}{\beta_0} \left[1 + \frac{M_1}{\beta_0} \varepsilon^{1/2} + O(\varepsilon) \right] \leq \frac{\kappa}{\beta_0} + M_2\sqrt{\kappa}.$$

Thus (5.13) is true. To show that M can be chosen independent of \mathbf{h} , we replace \mathbf{h} by a unit vector \mathbf{h}_ε which depends on ε , and repeat the above discussion. \square

Proof of Theorem 1. Theorem 4.2 + Theorem 5.5. \square

§6. SURFACE NUCLEATION

In this section we study the nucleation of superconductivity of a bounded 3-dimensional superconductor with large κ and subjected to an applied field lying in-between H_{C_2} and H_{C_3} . We shall use Agmon’s idea [A] and the techniques developed in [H], and adapt the arguments from [HP]. For this purpose we shall modify $\lambda(\varphi_{\mathbf{h}}(x)) - \beta_0$ to get a smooth function which measures the tangential distance from $(\partial\Omega)_{\mathbf{h}}$, where $\lambda(\vartheta)$ and $\varphi_{\mathbf{h}}(x)$ are given in Lemma 2.3 and Remark 2.4.⁶

Lemma 6.1. *Fix a small $\delta > 0$. There exists a function $\zeta_{\mathbf{h}}$ with the following properties:*

- (1) $\zeta_{\mathbf{h}} \in C^1(\bar{\Omega})$; $\zeta_{\mathbf{h}}(x) \geq 0$ on $\bar{\Omega}$, $\zeta_{\mathbf{h}}(x) = 0$ on $(\partial\Omega)_{\mathbf{h}}$, and $\zeta_{\mathbf{h}}(x) > 0$ on $(\partial\Omega) \setminus (\partial\Omega)_{\mathbf{h}}$.
- (2) If $x \in \Omega(\delta)$ and $\delta \leq \varphi_{\mathbf{h}}(x) \leq \pi - \delta$, then $\zeta_{\mathbf{h}}(x) = \zeta_0(\varphi_{\mathbf{h}}(x))$, where

$$\zeta_0(\varphi) = \left| \int_{\varphi}^{\pi/2} [\lambda(\vartheta) - \beta_0]^{1/2} d\vartheta \right|.$$

- (3) There exists a positive constant C_δ depending only on Ω , \mathbf{h} and δ , such that

$$(6.1) \quad |\nabla \zeta_{\mathbf{h}}(x)| \leq C_\delta [\lambda(\varphi_{\mathbf{h}}(x)) - \beta_0]^{1/2} \quad \text{on } \bar{\Omega}.$$

- (4) For φ close to $\pi/2$ we have the following estimate for the function $\zeta_0(\varphi)$:

$$\frac{1}{2} \sqrt{1 - \beta_0} \left(\varphi - \frac{\pi}{2} \right)^2 + O\left(\left(\varphi - \frac{\pi}{2} \right)^4 \right) \leq \zeta_0(\varphi) \leq \frac{2}{3} \left| \varphi - \frac{\pi}{2} \right|^{3/2} + O\left(\left| \varphi - \frac{\pi}{2} \right|^{5/2} \right).$$

⁶We cannot simply take $\lambda(\varphi_{\mathbf{h}}(x)) - \beta_0$ to measure the tangential distance from $(\partial\Omega)_{\mathbf{h}}$, because $\varphi_{\mathbf{h}}(x)$ is not differentiable on $(\partial\Omega)_{\mathbf{h}}$, and we only know that $\lambda(\vartheta)$ is differentiable almost everywhere but we do not know yet whether $\lambda'(\vartheta) \in L^\infty[0, \pi]$.

Proof. Recall that $\lambda(\vartheta)$ is continuous for $\vartheta \in [0, \pi]$ and $\beta_0 \leq \lambda(\vartheta) \leq 1$. Thus ζ_0 is a C^1 function on $[0, \pi]$. We first define $\zeta_{\mathbf{h}}(x) = \zeta_0(\varphi_{\mathbf{h}}(x))$ on $\{x \in \Omega(\delta) : \delta \leq \varphi_{\mathbf{h}}(x) \leq \pi - \delta\}$, and then extend it over $\bar{\Omega}$ to get a C^1 function. It is easy to check that $\zeta_{\mathbf{h}}$ satisfies (1), (2), (3). To verify (4), we use (2.5). For $0 < \varphi < \pi/2$ we have

$$\begin{aligned} \zeta_0(\varphi) &\geq \int_{\varphi}^{\pi/2} \sqrt{1 - \beta_0} \cos \vartheta \, d\vartheta = \sqrt{1 - \beta_0} [1 - \cos(\frac{\pi}{2} - \varphi)] \\ &= \frac{1}{2} \sqrt{1 - \beta_0} (\frac{\pi}{2} - \varphi)^2 + O((\frac{\pi}{2} - \varphi)^4). \end{aligned}$$

Now if $\pi/2 - \varphi > 0$ is small, we have

$$\begin{aligned} \zeta_0(\varphi) &\leq \int_{\varphi}^{\pi/2} [\beta_0(\sin \vartheta - 1) + \cos \vartheta]^{1/2} \, d\vartheta = \int_0^{\pi/2 - \varphi} [\beta_0(\cos t - 1) + \sin t]^{1/2} \, dt \\ &= \int_0^{\pi/2 - \varphi} [t - \frac{\beta_0}{2} t^2 + O(t^3)]^{1/2} \, dt = \frac{2}{3} (\frac{\pi}{2} - \varphi)^{3/2} + O((\frac{\pi}{2} - \varphi)^{5/2}). \end{aligned}$$

□

Similarly, given $\beta_0 < b < 1$, we introduce $\zeta_{\mathbf{h},b}(x)$ to measure the tangential distance from the set $\{x \in \partial\Omega : \lambda(\varphi_{\mathbf{h}}(x)) \leq b\}$. Recall that $\lambda(\vartheta)$ is strictly decreasing on $[0, \pi/2]$. So the inverse function $\lambda^{-1}(b)$ exists and is continuous for $b \in [\beta_0, 1]$, and $\lambda^{-1}(\beta_0) = \pi/2$. In the following we use the notation $a_+ = \max\{a, 0\}$.

Lemma 6.2. *Given a unit vector \mathbf{h} and a number $b \in (\beta_0, 1)$, for small $\delta > 0$, there exists a function $\zeta_{\mathbf{h},b} \in C(\bar{\Omega})$ with the following properties:*

(1) $\zeta_{\mathbf{h},b}(x) \geq 0$ on $\bar{\Omega}$, $\zeta_{\mathbf{h},b}(x) = 0$ on $\{x \in \partial\Omega : \lambda(\varphi_{\mathbf{h}}(x)) \leq b\}$, and $\zeta_{\mathbf{h},b}(x) > 0$ on $\partial\Omega \setminus \{x \in \partial\Omega : \lambda(\varphi_{\mathbf{h}}(x)) \leq b\}$; $\nabla \zeta_{\mathbf{h},b}(x)$ exists almost everywhere and $|\nabla \zeta_{\mathbf{h},b}(x)| \in L^\infty(\Omega)$.

(2) If $x \in \Omega(\delta)$ and $\delta \leq \varphi_{\mathbf{h}}(x) \leq \pi - \delta$, then $\zeta_{\mathbf{h},b}(x) = \zeta_b(\varphi_{\mathbf{h}}(x))$, where

$$\zeta_b(\varphi) = \int_{\varphi}^{\lambda^{-1}(b)} [\lambda(\vartheta) - b]^{1/2} \, d\vartheta \quad \text{if } 0 \leq \varphi \leq \lambda^{-1}(b),$$

$\zeta_b(\varphi) = 0$ if $\lambda^{-1}(b) < \varphi \leq \pi/2$, and $\zeta_b(\varphi) = \zeta_b(\pi - \varphi)$ if $\pi/2 < \varphi \leq \pi$.

(3) There exists $C_{\delta,b} > 0$ depending only on Ω , \mathbf{h} , δ and b , such that

$$|\nabla \zeta_{\mathbf{h},b}(x)| \leq C_{\delta,b} [(\lambda(\varphi_{\mathbf{h}}(x)) - b)_+]^{1/2}.$$

(4) We have the following estimate for the function $\zeta_b(\varphi)$: For any fixed $l > 0$, there exists $c_l(\varepsilon)$ satisfying $c_l(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, such that, if $\lambda(\varphi) \leq b + l\varepsilon^{1/2}$, then

$$0 \leq \zeta_b(\varphi) \leq c_l(\varepsilon) \varepsilon^{1/4}.$$

Proof. We only need to prove (4), and we only consider the case where $0 < \varphi \leq \pi/2$. Assume $\lambda(\varphi) \leq b + l\varepsilon^{1/2}$. Then $\varphi \geq \lambda^{-1}(b + l\varepsilon^{1/2})$. Since $\lambda(\vartheta)$ is strictly decreasing in $[0, \pi/2]$, we have

$$\begin{aligned} \zeta_b(\varphi) &= \int_{\varphi}^{\lambda^{-1}(b)} [\lambda(\vartheta) - b]^{1/2} \, d\vartheta \leq [\lambda^{-1}(b) - \varphi] [\lambda(\varphi) - b]^{1/2} \\ &\leq [\lambda^{-1}(b) - \varphi] l^{1/2} \varepsilon^{1/4} \leq [\lambda^{-1}(b) - \lambda^{-1}(b + l\varepsilon^{1/2})] l^{1/2} \varepsilon^{1/4} = c_l(\varepsilon) \varepsilon^{1/4}, \end{aligned}$$

where

$$c_l(\varepsilon) = l^{1/2} \left[\lambda^{-1}(b) - \lambda^{-1}(b + l\varepsilon^{1/2}) \right]$$

and $c_l(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. □

Theorem 6.3. *Let Ω be a bounded, smooth and simply-connected domain in \mathbb{R}^3 and \mathbf{h} a unit vector. Under the applied field $\mathcal{H} = \sigma\mathbf{h}$, the order parameters ψ have the following concentration behavior for large κ :*

(1) *If $\sigma = H_{C_3}(\mathbf{h}, \kappa) - a_1\kappa^{1/2}$ where $a_1 > 0$, there exist positive constants α_1, c_1, M_1 such that*

$$(6.2) \quad \int_{\Omega} \exp(\alpha_1\kappa[\zeta_{\mathbf{h}}(x) - c_1\kappa^{-3/8}])|\psi|^2 dx \leq M_1\kappa^{1/2} \int_{\Omega(\kappa^{-1})} |\psi|^2 dx.$$

In particular, there exist positive constants η and γ_1 such that

$$(6.3) \quad \int_{\Omega \setminus \mathcal{S}_{\kappa}(\mathbf{h})} |\psi|^2 dx \leq M_1 \exp(-\gamma_1\kappa^{5/8}) \int_{\Omega(\kappa^{-1})} |\psi|^2 dx,$$

where

$$\mathcal{S}_{\kappa}(\mathbf{h}) = \Omega(\kappa^{-1}) \cap \{|\varphi_{\mathbf{h}}(x) - \pi/2| \leq \eta\kappa^{-3/16}\}.$$

(2) *If $\sigma = \kappa/b$, where $\beta_0 < b < 1$, there exist positive constants α_2, c_2 and M_2 , all depending on b , such that*

$$(6.4) \quad \int_{\Omega} \exp(\alpha_2\kappa[\zeta_{\mathbf{h},b}(x) - c_2\kappa^{-1/4}])|\psi|^2 dx \leq M_2\kappa^{1/2} \int_{\Omega(\kappa^{-1})} |\psi|^2 dx.$$

In particular, there exist positive constants γ_2 and $\eta(\kappa)$ depending on b , and $\vartheta(b) = \lambda^{-1}(b)$, where $\eta(\kappa) \rightarrow 0$ as $\kappa \rightarrow \infty$, such that

$$(6.5) \quad \int_{\Omega \setminus \mathcal{S}_{b,\kappa}(\mathbf{h})} |\psi|^2 dx \leq M_2 \exp(-\gamma_2\kappa^{3/4}) \int_{\Omega(\kappa^{-1})} |\psi|^2 dx,$$

where

$$\mathcal{S}_{b,\kappa}(\mathbf{h}) = \Omega(\kappa^{-1}) \cap \{\vartheta(b) - \eta(\kappa) \leq \varphi_{\mathbf{h}}(x) \leq \pi - \vartheta(b) + \eta(\kappa)\}.$$

Proof. As in Section 5, we let $\varepsilon = 1/\sqrt{\sigma\kappa}$ and $\lambda_{\varepsilon} = \kappa/\sigma$, and let $(\psi^{\varepsilon}, \mathbf{A}^{\varepsilon})$ denote the minimal solutions of (5.2). Let

$$\eta = \exp(\alpha\varepsilon^{-1}\zeta),$$

where α is a positive constant and ζ is a real-valued C^1 function, both to be determined later. We multiply the first equation in (5.2) by $\eta^2\bar{\psi}^{\varepsilon}$ and integrate, and use (5.10) to get

$$(6.6) \quad \begin{aligned} 0 &= \int_{\Omega} \{|\nabla_{\varepsilon^{-2}\mathbf{A}^{\varepsilon}}(\eta\psi^{\varepsilon})|^2 - |\nabla\eta|^2|\psi^{\varepsilon}|^2 - \frac{\lambda_{\varepsilon}}{\varepsilon^2}(1 - |\psi^{\varepsilon}|^2)|\eta\psi^{\varepsilon}|^2\} dx \\ &\geq \varepsilon^{-2} \int_{\Omega} \{W_{\mathbf{h}}(x) - \alpha^2|\nabla\zeta|^2 - \lambda_{\varepsilon} + o(\varepsilon^{2/3})\} |\eta\psi^{\varepsilon}|^2 dx. \end{aligned}$$

Let

$$\rho = H_{C_3}(\mathbf{h}, \kappa) - \sigma, \quad C_3 = C_2\beta_0^{-5/3}P_{\min}.$$

From Theorem 1 we have

$$\sigma = H_{C_3}(\mathbf{h}, \kappa) - \rho \geq \frac{\kappa}{\beta_0} \{1 - \beta_0\kappa^{-1}\rho - C_3\beta_0\kappa^{-2/3} + O(\kappa^{-7/9})\}.$$

If $\kappa \gg \rho \geq c_0 \kappa^{1/3}$, we have

$$(6.7) \quad \lambda_\varepsilon = \frac{\kappa}{\sigma} \leq \beta_0 + \beta_0^2 \kappa^{-1} \rho + C_3 \beta_0^2 \kappa^{-2/3} + O(\kappa^{-7/9}) + O(\rho^2 \kappa^{-2}).$$

Step 1. In case (1) we have

$$\varepsilon = \sqrt{\beta_0} \kappa^{-1} + O(\kappa^{-3/2}), \quad \beta_0^2 \kappa^{-1} \rho = d\varepsilon^{1/2}$$

for some $d > 0$. Then from (6.7),

$$\lambda_\varepsilon \leq \beta_0 + d\varepsilon^{1/2} + O(\varepsilon^{2/3}).$$

Let $\zeta = \zeta_{\mathbf{h}}$ and choose $\alpha > 0$ such that $\alpha^2 C_\delta^2 < 1/2$, where C_δ is given in Lemma 6.1. From (6.1) and (6.6), we can find a constant $c > 0$ such that

$$\begin{aligned} 0 &\geq \int_{\Omega} \{W_{\mathbf{h}}(x) - \alpha^2 |\nabla \zeta_{\mathbf{h}}|^2 - \lambda_\varepsilon + o(\varepsilon^{2/3})\} |\eta \psi^\varepsilon|^2 dx \\ &\geq \int_{\Omega(2\varepsilon^{1/4})} \{\lambda(\varphi_{\mathbf{h}}(x)) - \alpha^2 C_\delta^2 [\lambda(\varphi_{\mathbf{h}}(x)) - \beta_0] - \beta_0 - (c+d)\varepsilon^{1/2}\} |\eta \psi^\varepsilon|^2 dx \\ &\geq \frac{1}{2} \int_{\Omega(2\varepsilon^{1/4})} \{\lambda(\varphi_{\mathbf{h}}(x)) - \beta_0 - 2(c+d)\varepsilon^{1/2}\} |\eta \psi^\varepsilon|^2 dx \\ &\geq \left(\frac{l}{2} - c - d\right) \varepsilon^{1/2} \int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) \geq \beta_0 + l\varepsilon^{1/2}\}} |\eta \psi^\varepsilon|^2 dx \\ &\quad - \left[\frac{\beta_0}{2} + (c+d)\varepsilon^{1/2}\right] \int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) < \beta_0 + l\varepsilon^{1/2}\}} |\eta \psi^\varepsilon|^2 dx, \end{aligned}$$

where l is chosen such that $l > 2(c+d)$. So, for some $L_1 > 0$,

$$\begin{aligned} &\int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) \geq \beta_0 + l\varepsilon^{1/2}\}} \exp(2\alpha\varepsilon^{-1}\zeta_{\mathbf{h}}(x)) |\psi^\varepsilon|^2 dx \\ &\leq L_1 \varepsilon^{-1/2} \int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) < \beta_0 + l\varepsilon^{1/2}\}} \exp(2\alpha\varepsilon^{-1}\zeta_{\mathbf{h}}(x)) |\psi^\varepsilon|^2 dx. \end{aligned}$$

In both sides we add

$$\int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) < \beta_0 + l\varepsilon^{1/2}\}} \exp(2\alpha\varepsilon^{-1}\zeta_{\mathbf{h}}(x)) |\psi^\varepsilon|^2 dx$$

to get

$$\begin{aligned} &\int_{\Omega(2\varepsilon^{1/4})} \exp(2\alpha\varepsilon^{-1}\zeta_{\mathbf{h}}(x)) |\psi^\varepsilon|^2 dx \\ &\leq (L_1 \varepsilon^{-1/2} + 1) \int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) < \beta_0 + l\varepsilon^{1/2}\}} \exp(2\alpha\varepsilon^{-1}\zeta_{\mathbf{h}}(x)) |\psi^\varepsilon|^2 dx. \end{aligned}$$

From Lemma 2.3 (conclusion (3)), on the set

$$\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) < \beta_0 + l\varepsilon^{1/2}\}$$

we have

$$\left|\varphi_{\mathbf{h}}(x) - \frac{\pi}{2}\right| \leq l^{1/2} (1 - \beta_0)^{-1/2} \varepsilon^{1/4} + O(\varepsilon^{3/4}).$$

Then from Lemma 6.1 (conclusion (4)),

$$\begin{aligned} \zeta_{\mathbf{h}}(x) &= \zeta_0(\varphi_{\mathbf{h}}(x)) \leq \frac{2}{3} [l^{1/2} (1 - \beta_0)^{-1/2} \varepsilon^{1/4} + O(\varepsilon^{3/4})]^{3/2} + O(\varepsilon^{5/4}) \\ &= \frac{2}{3} l^{3/4} (1 - \beta_0)^{-3/4} \varepsilon^{3/8} + O(\varepsilon^{7/8}) \leq c(l) \varepsilon^{3/8}, \end{aligned}$$

where $c(l)$ is independent of ε . So, for some constant L_2 ,

$$\begin{aligned} & \int_{\Omega(2\varepsilon^{1/4})} \exp(2\alpha\varepsilon^{-1}\zeta_{\mathbf{h}}(x))|\psi^\varepsilon|^2 dx \\ & \leq L_2\varepsilon^{-1/2} \int_{\Omega(2\varepsilon^{1/4})\cap\{\lambda(\varphi_{\mathbf{h}}(x))<\beta_0+l\varepsilon^{1/2}\}} \exp(2\alpha\varepsilon^{-1}\zeta_{\mathbf{h}}(x))|\psi^\varepsilon|^2 dx \\ & \leq L_2\varepsilon^{-1/2} \int_{\Omega(2\varepsilon^{1/4})\cap\{\lambda(\varphi_{\mathbf{h}}(x))<\beta_0+l\varepsilon^{1/2}\}} \exp(2\alpha\varepsilon^{-1}c(l)\varepsilon^{3/8})|\psi^\varepsilon|^2 dx. \end{aligned}$$

Thus

$$\begin{aligned} & \int_{\Omega(2\varepsilon^{1/4})\cap\{\lambda(\varphi_{\mathbf{h}}(x))\geq\beta_0+l\varepsilon^{1/2}\}} \exp(2\alpha\varepsilon^{-1}[\zeta_{\mathbf{h}}(x) - c(l)\varepsilon^{3/8}])|\psi^\varepsilon|^2 dx \\ & \leq L_2\varepsilon^{-1/2} \int_{\Omega(2\varepsilon^{1/4})\cap\{\lambda(\varphi_{\mathbf{h}}(x))<\beta_0+l\varepsilon^{1/2}\}} |\psi^\varepsilon|^2 dx. \end{aligned}$$

Then we use (5.5) to control the integrals on $\Omega \setminus \Omega(\varepsilon)$ and find

$$\int_{\Omega} \exp(2\alpha\varepsilon^{-1}[\zeta_{\mathbf{h}}(x) - c(l)\varepsilon^{3/8}])|\psi^\varepsilon|^2 dx \leq L_3\varepsilon^{-1/2} \int_{\Omega(\varepsilon)} |\psi^\varepsilon|^2 dx,$$

where $L_3 > 0$. Now (6.2) is proved.

If $x \in \Omega(\delta)$ and $|\varphi_{\mathbf{h}}(x) - \frac{\pi}{2}| \geq c\kappa^{-3/16}$, from Lemma 6.1 (conclusion (4)) we have

$$\begin{aligned} \zeta_{\mathbf{h}}(x) &= \zeta_0(\varphi_{\mathbf{h}}(x)) \geq \frac{1}{2}\sqrt{1-\beta_0}(\varphi_{\mathbf{h}}(x) - \frac{\pi}{2})^2 + O((\varphi_{\mathbf{h}}(x) - \frac{\pi}{2})^4) \\ &\geq \frac{c^2}{2}\sqrt{1-\beta_0}\kappa^{-3/8} + O(\kappa^{3/4}). \end{aligned}$$

Let the constant c be chosen sufficiently large. Then we can find a positive constant m , such that on the set

$$\Omega(\delta) \cap \left\{ \left| \varphi_{\mathbf{h}}(x) - \frac{\pi}{2} \right| \geq c\kappa^{-3/16} \right\}$$

we have

$$\alpha_1\kappa[\zeta_{\mathbf{h}}(x) - c_1\kappa^{-5/8}] \geq m\kappa^{5/8}$$

for all large κ , where α_1 and c_1 are the constants in (6.2). From (6.2) we have

$$\int_{\Omega(\delta)\cap\{|\varphi_{\mathbf{h}}(x)-\frac{\pi}{2}|\geq c\kappa^{-3/16}\}} |\psi|^2 dx \leq M_1\kappa^{1/2} \exp(-m\kappa^{-5/8}) \int_{\Omega(\kappa^{-1})} |\psi|^2 dx.$$

Now we choose $0 < \gamma_1 < m$ such that

$$\kappa^{1/2} \exp(-m\kappa^{-5/8}) \leq \exp(-\gamma_1\kappa^{-5/8})$$

holds for all large κ . Then (6.3) follows.

Step 2. In case (2) we have $\varepsilon = \sqrt{b}\kappa^{-1}$ and $\lambda_\varepsilon = b$. We choose $\zeta = \zeta_{\mathbf{h},b}$ and choose α such that $\alpha^2 C_{\delta,b}^2 \leq 1/2$, where $C_{\delta,b}$ is given in Lemma 6.2. From (6.6)

and Lemma 6.2 (conclusion (3)),

$$\begin{aligned} 0 &\geq \int_{\Omega} \{W_{\mathbf{h}}(x) - \alpha^2 |\nabla \zeta_{\mathbf{h},b}|^2 - b + o(\varepsilon^{2/3})\} |\eta \psi^\varepsilon|^2 dx \\ &\geq \int_{\Omega(2\varepsilon^{1/4})} \{\lambda(\varphi_{\mathbf{h}}(x)) - \alpha^2 C_{\delta,b}^2 [\lambda(\varphi_{\mathbf{h}}(x)) - b]_+ - b - c\varepsilon^{1/2}\} |\eta \psi^\varepsilon|^2 dx \\ &\geq \int_{\Omega(2\varepsilon^{1/4})} \{\lambda(\varphi_{\mathbf{h}}(x)) - b - \frac{1}{2} [\lambda(\varphi_{\mathbf{h}}(x)) - b]_+ - c\varepsilon^{1/2}\} |\eta \psi^\varepsilon|^2 dx \\ &\geq \left(\frac{l}{2} - c\right) \varepsilon^{1/2} \int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) \geq b + l\varepsilon^{1/2}\}} |\eta \psi^\varepsilon|^2 dx \\ &\quad - (b + c\varepsilon^{1/2}) \int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) < b + l\varepsilon^{1/2}\}} |\eta \psi^\varepsilon|^2 dx, \end{aligned}$$

where l is chosen such that $l > 2c$. Then we argue as in Step 1 and use Lemma 6.2 (conclusion (4)) to find

$$\begin{aligned} &\int_{\Omega(2\varepsilon^{1/4})} \exp(2\alpha\varepsilon^{-1}[\zeta_{\mathbf{h},b}(x) - c_l(\varepsilon)\varepsilon^{1/4}]) |\psi^\varepsilon|^2 dx \\ &\leq L_4 \varepsilon^{-1/2} \int_{\Omega(2\varepsilon^{1/4}) \cap \{\lambda(\varphi_{\mathbf{h}}(x)) < b + l\varepsilon^{1/2}\}} |\psi^\varepsilon|^2 dx, \end{aligned}$$

and hence

$$\int_{\Omega} \exp(2\alpha\varepsilon^{-1}[\zeta_{\mathbf{h},b}(x) - c_l(\varepsilon)\varepsilon^{1/4}]) |\psi^\varepsilon|^2 dx \leq L_5 \varepsilon^{-1/2} \int_{\Omega(\varepsilon)} |\psi^\varepsilon|^2 dx,$$

where L_4 and L_5 are positive constants. So (6.4) is true.

Let c_2 be the constant in (6.4). For large κ we define

$$\eta(\kappa) = \sup \left\{ \eta > 0 : \int_{\lambda^{-1}(b)-\eta}^{\lambda^{-1}(b)} [\lambda(\vartheta) - b]^{1/2} d\vartheta \leq 2c_2 \kappa^{-1/4} \right\}.$$

Obviously, $\lim_{\kappa \rightarrow \infty} \eta(\kappa) = 0$. If $0 \leq \varphi \leq \lambda^{-1}(b) - \eta(\kappa)$, then

$$\zeta_b(\varphi) \geq \int_{\lambda^{-1}(b)-\eta(\kappa)}^{\lambda^{-1}(b)} [\lambda(\vartheta) - b]^{1/2} d\vartheta \geq 2c_2 \kappa^{-1/4}.$$

If $\pi - \lambda^{-1}(b) + \eta(\kappa) \leq \varphi \leq \pi$, then $0 \leq \pi - \varphi \leq \lambda^{-1}(b) - \eta(\kappa)$, and by the definition of ζ_b we have

$$\zeta_b(\varphi) = \zeta_b(\pi - \varphi) \geq 2c_2 \kappa^{-1/4}.$$

Thus we can use (6.4) and argue as in the last part of Step 1 to derive (6.5). □

Proof of Theorem 2. It is a corollary of Theorem 6.3. □

APPENDIX A. COMPUTATIONAL DETAILS FOR EQUATION (2.8)

Consider equation (2.8). Using the new variables $\xi = (\xi_1, \xi_2)$ defined in (2.10), we write equation (2.8) as follows:

$$(A.1) \quad -\nabla_{\mathbf{Q}}^2 \psi_1 = \mu_p \psi_1 + f(\xi) \psi_1, \quad \xi \in \mathbb{R}^2,$$

where the vector field $\mathbf{Q}(\xi) = (Q_1(\xi), Q_2(\xi))$ and the function $f(\xi)$ are given by

$$\begin{aligned} Q_1(\xi) &= \frac{1}{\varepsilon_1} \left[\frac{1}{\sqrt{c_{11}}} f_1(y) + \frac{1}{\sqrt{c_{22}}} f_2(y) \right], \\ Q_2(\xi) &= \frac{1}{\varepsilon_2} \left[\frac{1}{\sqrt{c_{11}}} f_1(y) - \frac{1}{\sqrt{c_{22}}} f_2(y) \right], \\ f(\xi) &= f_1^2(y) + f_2^2(y) - f_0(y) + i \operatorname{div} \mathbf{Q}, \end{aligned}$$

where f_0, f_1 and f_2 are given in (2.9). We compute $Q_1(\xi)$ and $Q_2(\xi)$ to get

$$\begin{aligned} Q_1(\xi) &= \alpha_{11} \xi_1^2 + 2\alpha_{12} \xi_1 \xi_2 + \alpha_{22} \xi_2^2, \\ Q_2(\xi) &= \beta_{11} \xi_1^2 + 2\beta_{12} \xi_1 \xi_2 + \beta_{22} \xi_2^2, \end{aligned}$$

where

$$\begin{aligned} \alpha_{11} &= \frac{\varepsilon_1}{4} [-(\sqrt{c_{11}}c_{22} + c_{12}\sqrt{c_{22}})a_2^{(1)} + (c_{11}\sqrt{c_{22}} + \sqrt{c_{11}}c_{12})a_1^{(2)}], \\ \alpha_{12} &= \frac{\varepsilon_2}{4} [(\sqrt{c_{11}}c_{22} + c_{12}\sqrt{c_{22}})a_2^{(1)} + (c_{11}\sqrt{c_{22}} + \sqrt{c_{11}}c_{12})a_1^{(2)}], \\ \alpha_{22} &= \frac{\varepsilon_2^2}{4\varepsilon_1} [-(\sqrt{c_{11}}c_{22} + c_{12}\sqrt{c_{22}})a_2^{(1)} + (c_{11}\sqrt{c_{22}} + \sqrt{c_{11}}c_{12})a_1^{(2)}], \\ \beta_{11} &= \frac{\varepsilon_1^2}{4\varepsilon_2} [(-\sqrt{c_{11}}c_{22} + c_{12}\sqrt{c_{22}})a_2^{(1)} + (-c_{11}\sqrt{c_{22}} + \sqrt{c_{11}}c_{12})a_1^{(2)}], \\ \beta_{12} &= \frac{\varepsilon_1}{4} [(\sqrt{c_{11}}c_{22} - c_{12}\sqrt{c_{22}})a_2^{(1)} + (-c_{11}\sqrt{c_{22}} + \sqrt{c_{11}}c_{12})a_1^{(2)}], \\ \beta_{22} &= \frac{\varepsilon_2}{4} [(-\sqrt{c_{11}}c_{22} + c_{12}\sqrt{c_{22}})a_2^{(1)} + (-c_{11}\sqrt{c_{22}} + \sqrt{c_{11}}c_{12})a_1^{(2)}]. \end{aligned}$$

Computations show that

$$f(\xi) \equiv 0.$$

Hence we can write the equation for $\psi_1(\xi)$ in the following form:

$$(A.2) \quad -\nabla_{\mathbf{Q}}^2 \psi_1 = \mu_2 \psi_1 \quad \text{in } \mathbb{R}^2,$$

which is exactly (2.11). We can make a gauge transformation to rewrite (2.11) in a form more convenient to us. Let us choose

$$(A.3) \quad \eta(\xi) = \frac{1}{3}(\alpha_{11} - \alpha_{22} + \beta_{12})\xi_1^2 + \alpha_{12}\xi_1^2\xi_2 + \beta_{12}\xi_1\xi_2^2 + \frac{1}{3}(\alpha_{12} - \beta_{11} + \beta_{22})\xi_2^3,$$

and let

$$\begin{aligned} \gamma_1 &= \beta_{12} - \alpha_{22} = 2 \frac{c_{11}c_{22} - (c_{12})^2}{\varepsilon_1 \sqrt{c_{11}c_{22}}} (\sqrt{c_{22}}a_2^{(1)} - \sqrt{c_{11}}a_1^{(2)}), \\ \gamma_2 &= \alpha_{12} - \beta_{11} = 2 \frac{c_{11}c_{22} - (c_{12})^2}{\varepsilon_2 \sqrt{c_{11}c_{22}}} (\sqrt{c_{22}}a_2^{(1)} + \sqrt{c_{11}}a_1^{(2)}); \end{aligned}$$

see (2.12). Then

$$\mathbf{Q}(\xi) = -\frac{|\xi|^2}{2}(\gamma_1, \gamma_2) + \nabla \eta(\xi).$$

Let

$$\begin{aligned} \rho &= (\gamma_1^2 + \gamma_2^2)^{1/2}, \quad \gamma_1 = \rho \cos \vartheta, \quad \gamma_2 = \rho \sin \vartheta, \\ \mathbf{A}_{\vartheta}(\xi) &= -\frac{|\xi|^2}{2}(\cos \vartheta, \sin \vartheta). \end{aligned}$$

Then we have

$$\mathbf{Q} = \rho \mathbf{A}_\vartheta(\xi) + \nabla \eta.$$

We introduce $\psi_2(\xi) = \psi_1(\xi)e^{-i\eta}$, and derive an equation for ψ_2 from (2.11):

$$(A.4) \quad -\nabla_{\rho \mathbf{A}_\vartheta}^2 \psi_2 = \mu_p \psi_2 \quad \text{in } \mathbb{R}^2.$$

We make further change of variables as in (2.13), and let

$$\varphi(x) = \psi_2(\xi) \exp\left(\frac{i}{6} x_1^3\right), \quad \lambda_0 = \mu_p \rho^{-2/3}.$$

Then $\varphi(x)$ satisfies (2.14).

APPENDIX B. PROOF OF LEMMA 3.1

We keep the notation in Section 3. In particular, let $X_0 \in \partial\Omega$ be fixed and (y, z) be the new variables introduced there. We first consider the following problem: Given a vector field \mathbf{R} satisfying $\operatorname{div} \mathbf{R} = 0$, find a vector field \mathbf{A} such that

$$(B.1) \quad \operatorname{curl} \mathbf{A} = \mathbf{R} \quad \text{in } \Omega(\delta).$$

To solve this question, we write

$$\mathbf{R} = \sum_{j=1}^3 G^{jj} R_j \partial_j \mathcal{F}, \quad \text{where } R_j = \mathbf{R} \cdot \partial_j \mathcal{F}.$$

Since $\operatorname{div} \mathbf{R} = 0$, we have

$$(B.2) \quad \sum_{j=1}^3 \partial_j (\sqrt{G} G^{jj} R_j) = 0.$$

(B.1) has a solution

$$\mathbf{A} = \sum_{j=1}^3 G^{jj} a_j \partial_j \mathcal{F},$$

where a_j are given by

$$(B.3) \quad \begin{aligned} a_1(y, z) &= \int_0^z \sqrt{G(y, t)} G^{22}(y, t) R_2(y, t) dt, \\ a_2(y, z) &= - \int_0^z \sqrt{G(y, t)} G^{11}(y, t) R_1(y, t) dt + p_2(y), \\ a_3(y, z) &= 0, \end{aligned}$$

where

$$p_2(y) = \int_0^{y_1} \sqrt{G(s, y_2, 0)} R_3(s, y_2, 0) ds.$$

Now, given a unit vector \mathbf{h} , we can use (B.3) to find a vector field $\mathbf{A}_\mathbf{h}$ that satisfies

$$(B.4) \quad \operatorname{curl} \mathbf{A} = \mathbf{h} \quad \text{in } \Omega(\delta).$$

Write

$$\mathbf{h} = \sum_{j=1}^3 G^{jj} R_j \partial_j \mathcal{F},$$

where

$$\begin{aligned} R_1 &= [1 - \kappa_1(y)z] \mathbf{h} \cdot \mathbf{r}_1(y), \\ R_2 &= [1 - \kappa_2(y)z] \mathbf{h} \cdot \mathbf{r}_2(y), \\ R_3 &= \mathbf{h} \cdot \mathbf{n}(y). \end{aligned}$$

Then the solution of (B.4) is

$$\mathbf{A} = \mathbf{A}_{\mathbf{h}} + \nabla \chi,$$

where χ is an arbitrary smooth function, and

$$\mathbf{A}_{\mathbf{h}} = \sum_{j=1}^3 G^{jj} a_j \partial_j \mathcal{F},$$

where

$$\begin{aligned} (B.5) \quad a_1(y, z) &= \sqrt{\frac{g_{11}(y)}{g_{22}(y)}} \left[z - \frac{1}{2} \kappa_2(y) z^2 \right] \mathbf{h} \cdot \mathbf{r}_2(y), \\ a_2(y, z) &= -\sqrt{\frac{g_{22}(y)}{g_{11}(y)}} \left[z - \frac{1}{2} \kappa_1(y) z^2 \right] \mathbf{h} \cdot \mathbf{r}_1(y) \\ &\quad + \int_0^{y_1} \sqrt{g(s, y_2)} \mathbf{h} \cdot \mathbf{n}(s, y_2) ds, \\ a_3(y, z) &= 0. \end{aligned}$$

In the following we look for a solution to (B.4) which has a simple expansion in a neighborhood of X_0 . Choose \mathbf{e}_j as in (3.2). Then (3.3) holds. For the unit vector \mathbf{h} , we have

$$\begin{aligned} (B.6) \quad \mathbf{h} \cdot \mathbf{r}_1(y) &= h_1 + (h_1 \Gamma_{1j}^1 + h_2 \Gamma_{1j}^2) y_j + h_3 \Omega_{11} y_1 + O(|y|^2), \\ \mathbf{h} \cdot \mathbf{r}_2(y) &= h_2 + (h_1 \Gamma_{2j}^1 + h_2 \Gamma_{2j}^2) y_j + h_3 \Omega_{22} y_2 + O(|y|^2), \\ \mathbf{h} \cdot \mathbf{n}(y) &= h_3 - h_j g^{jj} \Omega_{jj} y_j + O(|y|^2). \end{aligned}$$

From (3.4), (3.6), (B.5) and (B.6) we get

$$\begin{aligned} (B.7) \quad a_1(y, z) &= h_2 \sqrt{\frac{g_{11}}{g_{22}}} z + a_{13}^{(1)} y_1 z + a_{23}^{(1)} y_2 z + a_{33}^{(1)} z^2 + O(|y|^3 + |z|^3) \\ a_2(y, z) &= h_3 \sqrt{g} y_1 - h_1 \sqrt{\frac{g_{22}}{g_{11}}} z + a_1^{(2)} y_1^2 + 2a_2^{(1)} y_1 y_2 + a_{13}^{(2)} y_1 z \\ &\quad + a_{23}^{(2)} y_2 z + a_{33}^{(2)} z^2 + O(|y|^3 + |z|^3), \\ a_3(y, z) &= 0, \end{aligned}$$

where

$$\begin{aligned}
 a_2^{(1)} &= \sqrt{g} \left[\frac{h_3}{4} \left(\frac{g_{11,2}}{g_{11}} + \frac{g_{22,2}}{g_{22}} \right) - \frac{h_2}{2} \kappa_2 \right], \\
 a_{13}^{(1)} &= \sqrt{\frac{g_{11}}{g_{22}}} \left[\frac{h_2}{2} \left(\frac{g_{11,1}}{g_{11}} - \frac{g_{22,1}}{g_{22}} \right) + h_1 \Gamma_{21}^1 + h_2 \Gamma_{21}^2 \right], \\
 a_{23}^{(1)} &= \sqrt{\frac{g_{11}}{g_{22}}} \left[\frac{h_2}{2} \left(\frac{g_{11,2}}{g_{11}} - \frac{g_{22,2}}{g_{22}} \right) + h_1 \Gamma_{22}^1 + h_2 \Gamma_{22}^2 + h_3 \Omega_{22} \right], \\
 a_{33}^{(1)} &= -\frac{h_2}{2} \sqrt{\frac{g_{11}}{g_{22}}} \kappa_1, \\
 a_1^{(2)} &= \sqrt{g} \left[\frac{h_3}{4} \left(\frac{g_{11,1}}{g_{11}} + \frac{g_{22,1}}{g_{22}} \right) - \frac{h_1}{2} \kappa_1 \right], \\
 a_{13}^{(2)} &= \sqrt{\frac{g_{22}}{g_{11}}} \left[\frac{h_1}{2} \left(\frac{g_{11,1}}{g_{11}} - \frac{g_{22,1}}{g_{22}} \right) - h_1 \Gamma_{11}^1 - h_2 \Gamma_{11}^2 - h_3 \Omega_{11} \right], \\
 a_{23}^{(2)} &= \sqrt{\frac{g_{22}}{g_{11}}} \left[\frac{h_1}{2} \left(\frac{g_{11,2}}{g_{11}} - \frac{g_{22,2}}{g_{22}} \right) - h_1 \Gamma_{12}^1 - h_2 \Gamma_{12}^2 \right], \\
 a_{33}^{(2)} &= \frac{h_1}{2} \sqrt{\frac{g_{22}}{g_{11}}} \kappa_2.
 \end{aligned}
 \tag{B.8}$$

From (B.2) we have

$$\begin{aligned}
 &\partial_1 \left\{ \sqrt{\frac{g_{22}(y)}{g_{11}(y)}} [1 - \kappa_2(y)z] \mathbf{h} \cdot \mathbf{r}_1(y) \right\} + \partial_2 \left\{ \sqrt{\frac{g_{11}(y)}{g_{22}(y)}} [1 - \kappa_1(y)z] \mathbf{h} \cdot \mathbf{r}_2(y) \right\} \\
 &+ \partial_z \left\{ \sqrt{g(y)} [1 - \kappa_1(y)z] [1 - \kappa_2(y)z] \mathbf{h} \cdot \mathbf{n}(y) \right\} = 0.
 \end{aligned}$$

Checking the Taylor expansion of this equality we find that

$$a_{23}^{(1)} = a_{13}^{(2)} + h_3(\kappa_1 + \kappa_2)\sqrt{g}.
 \tag{B.9}$$

Now we fix the point $X_0 \in (\partial\Omega)_{\mathbf{h}}$, namely, \mathbf{h} is orthogonal to \mathbf{n} at X_0 . Thus

$$\begin{aligned}
 a_2^{(1)} &= -\frac{h_2}{2} \sqrt{g} \kappa_2, & a_1^{(2)} &= -\frac{h_1}{2} \sqrt{g} \kappa_1, & a_{23}^{(1)} &= a_{13}^{(2)}, \\
 h_3 &= 0, & \frac{h_1^2}{g_{11}} + \frac{h_2^2}{g_{22}} &= 1.
 \end{aligned}
 \tag{B.10}$$

Recall that y_1, y_2 were chosen such that the y_1 - and y_2 -curves are the lines of curvatures on $\partial\Omega$. Using (B.10) and the relation between the Christoffel symbols and $g_{ii,j}$ we find

$$\begin{aligned}
 a_{13}^{(1)} &= \frac{1}{2\sqrt{g}} (h_1 g_{11,2} + h_2 g_{11,1}), \\
 a_{23}^{(1)} &= \frac{1}{2\sqrt{g}} (-h_1 g_{22,1} + h_2 g_{11,2}), \\
 a_{23}^{(2)} &= -\frac{1}{2\sqrt{g}} (h_1 g_{22,2} + h_2 g_{22,1}).
 \end{aligned}
 \tag{B.11}$$

From (B.8), (B.10), (B.11) we get (3.10).

To simplify our computations we modify \mathbf{A}_h by a gauge transformation. Let ξ_0 be a real number, the value of which is to be determined later. Set

$$\chi(y) = a_2^{(1)} y_1 y_2^2 - \xi_0 \left\{ h_2 \sqrt{\frac{g_{11}}{g_{22}}} y_1 - h_1 \sqrt{\frac{g_{22}}{g_{11}}} y_2 + \frac{1}{2} a_{13}^{(1)} y_1^2 + a_{23}^{(1)} y_1 y_2 + \frac{1}{2} a_{23}^{(2)} y_2^2 \right\},$$

and let

$$\mathbf{F}_h = \mathbf{A}_h - \nabla \chi = \sum_{j=1}^3 G^{jj} \hat{a}_j \partial_j \mathcal{F},$$

$$\text{where } \hat{a}_j(y, z) = a_j(y, z) - \partial_j \chi(y), \quad j = 1, 2, 3.$$

Using (B.10) we find that $\hat{a}_j(y, z)$ satisfy (3.9). Now Lemma 3.1 is proved.

APPENDIX C. FORMAL EXPANSION OF THE LOWEST EIGENVALUE

In this section we look for a formal solution of (3.13). We shall keep the notation in Section 3 and Appendix B. Recall that we fixed a point $X_0 \in (\partial\Omega)_h$. Write

$$\mu = \mu_0 \varepsilon^{-2} + \mu_1 \varepsilon^{-5/3} + \mu_2 \varepsilon^{-4/3} + O(\varepsilon^{-1}).$$

Using (3.5), (3.14) and (B.10), we can expand (3.13) as follows:

$$\begin{aligned} (C.1) \quad & \partial_{zz} \phi - (z + z_0)^2 \phi + \mu_0 \phi \\ & + \varepsilon^{1/3} \left\{ \frac{2(z + z_0)}{\sqrt{g}} [-ih_2 \partial_1 \phi + ih_1 \partial_2 \phi + (a_1^{(2)} h_1 y_1^2 + a_2^{(1)} h_2 y_2^2) \phi] + \mu_1 \phi \right\} \\ & + \varepsilon^{2/3} \left\{ \frac{\partial_{11} \phi}{g_{11}} + \frac{\partial_{22} \phi}{g_{22}} + 2i \frac{a_2^{(1)}}{g_{11}} y_2^2 \partial_1 \phi - 2i \frac{a_1^{(2)}}{g_{22}} y_1^2 \partial_2 \phi - \left[\frac{(a_1^{(2)})^2}{g_{22}} y_1^4 + \frac{(a_2^{(1)})^2}{g_{11}} y_2^4 \right] \phi \right. \\ & + (z + z_0)^2 \left[\frac{2h_1}{\sqrt{g}} (a_{13}^{(2)} y_1 + a_{23}^{(2)} y_2) - \frac{2h_2}{\sqrt{g}} (a_{13}^{(1)} y_1 + a_{23}^{(1)} y_2) \right. \\ & \quad \left. \left. + \frac{1}{g} (h_2^2 g_{11,j} + h_1^2 g_{22,j}) y_j \right] \phi + \mu_2 \phi \right\} \\ & + O(\varepsilon) = 0, \end{aligned}$$

where $\partial_j = \partial_{y_j}$, $j = 1, 2$. Using (B.8), (B.10) and (B.11) we find that

$$\begin{aligned} (C.2) \quad & h_1^2 \frac{g_{22,1}}{\sqrt{g}} + h_2^2 \frac{g_{11,1}}{\sqrt{g}} + 2h_1 a_{13}^{(2)} - 2h_2 a_{13}^{(1)} = 0, \\ & h_1^2 \frac{g_{22,2}}{\sqrt{g}} + h_2^2 \frac{g_{11,2}}{\sqrt{g}} + 2h_1 a_{23}^{(2)} - 2h_2 a_{23}^{(1)} = 0. \end{aligned}$$

Thus we can write the equation as follows:

$$\begin{aligned} (C.3) \quad & \partial_{zz} \phi - (z + z_0)^2 \phi + \mu_0 \phi \\ & + \varepsilon^{1/3} \left\{ \frac{2(z + z_0)}{\sqrt{g}} [-ih_2 \partial_1 \phi + ih_1 \partial_2 \phi + (a_1^{(2)} h_1 y_1^2 + a_2^{(1)} h_2 y_2^2) \phi] + \mu_1 \phi \right\} \\ & + \varepsilon^{2/3} \left\{ \frac{\partial_{11} \phi}{g_{11}} + \frac{\partial_{22} \phi}{g_{22}} + 2i \frac{a_2^{(1)}}{g_{11}} y_2^2 \partial_1 \phi - 2i \frac{a_1^{(2)}}{g_{22}} y_1^2 \partial_2 \phi \right. \\ & \quad \left. - \left[\frac{(a_1^{(2)})^2}{g_{22}} y_1^4 + \frac{(a_2^{(1)})^2}{g_{11}} y_2^4 \right] \phi + \mu_2 \phi \right\} \\ & + O(\varepsilon) = 0. \end{aligned}$$

We look for a solution in the form

$$\phi = \phi_0 + \varepsilon^{1/3}\phi_1 + \varepsilon^{2/3}\phi_2 + O(\varepsilon),$$

where ϕ_j are functions defined in \mathbb{R}_+^3 . Plugging it into (C.3) and checking the order 1 terms we find an equation for ϕ_0 :

$$\partial_{zz}\phi_0 - (z + z_0)^2\phi_0 + \mu_0\phi_0 = 0 \quad \text{in } \mathbb{R}_+^3, \quad \partial_z\phi_0 = 0 \quad \text{on } \partial\mathbb{R}_+^3.$$

Using Lemma 2.1 we find that

$$\mu_0 = \beta_0, \quad \phi_0(y, z) = w(y)u(z),$$

z_0 is the minimum point of $\beta(z)$ given in Lemma 2.1, and u is the eigenfunction of (2.1) for z_0 and β_0 .

Next, by checking the $O(\varepsilon^{1/3})$ terms in (C.3), we find an equation for ϕ_1 :

$$(C.4) \quad \begin{aligned} &\partial_{zz}\phi_1 - (z + z_0)^2\phi_1 + \beta_0\phi_1 \\ &+ \frac{2(z + z_0)u}{\sqrt{g}}[-ih_2\partial_1w + ih_1\partial_2w + (a_1^{(2)}h_1y_1^2 + a_2^{(1)}h_2y_2^2)w] + \mu_1uw = 0. \end{aligned}$$

We fix y , multiply (C.4) by $u(z)$, integrate the equality in z over $[0, +\infty)$, and use the first equality in (2.2) to find that

$$\mu_1 = 0.$$

Let us look for a solution ϕ_1 of (C.4) in the form

$$\phi_1 = w_1(y)u_1(z).$$

Plugging it into (C.4) we find that u_1 satisfies (2.3), and

$$(C.5) \quad w_1(y) = \frac{2}{\sqrt{g}}[ih_2\partial_1w - ih_1\partial_2w - (a_1^{(2)}h_1y_1^2 + a_2^{(1)}h_2y_2^2)w].$$

Now we check the $O(\varepsilon^{2/3})$ terms in (C.3) and find an equation for ϕ_2 :

$$\begin{aligned} &\partial_{zz}\phi_2 - (z + z_0)^2\phi_2 + \beta_0\phi_2 \\ &+ \frac{2(z + z_0)u_1}{\sqrt{g}}[-ih_2\partial_1w_1 + ih_1\partial_2w_1 + (a_1^{(2)}h_1y_1^2 + a_2^{(1)}h_2y_2^2)w_1] \\ &+ u \left\{ \frac{\partial_{11}w}{g_{11}} + \frac{\partial_{22}w}{g_{22}} + 2i\frac{a_2^{(1)}}{g_{11}}y_2^2\partial_1w - 2i\frac{a_1^{(2)}}{g_{22}}y_1^2\partial_2w \right. \\ &\quad \left. - \left[\frac{(a_1^{(2)})^2}{g_{22}}y_1^4 + \frac{(a_2^{(1)})^2}{g_{11}}y_2^4 \right]w + \mu_2w \right\} = 0. \end{aligned}$$

Multiplying the equation by u and integrating in z , and using (C.5) and (2.4), we find that, w satisfies the following equality:

$$(C.6) \quad c_{11}\partial_{11}w + 2c_{12}\partial_{12}w + c_{22}\partial_{22}w - 2if_1\partial_1w - 2if_2\partial_2w - f_0w + \mu_2w = 0,$$

where c_{ij} and f_j are given in (4.6). Computations show that

$$c_{11}c_{22} - c_{12}^2 = \frac{1 - \alpha_0}{g} > 0,$$

where α_0 is the number given in (2.4). Comparing (4.6) with (2.9), we see that the equation (C.6) is in the form of (2.8). From Proposition 2.5 we know that, the lowest eigenvalue of (C.6) is $\lambda_0(\gamma_1^2 + \gamma_2^2)^{1/3}$, where λ_0 is the lowest eigenvalue of

(2.14), and γ_1 and γ_2 are determined by (2.12). From the last equality in (B.10), we can write

$$(C.7) \quad \begin{aligned} \cos \theta(X_0) &= \frac{h_1}{\sqrt{g_{11}}} = \mathbf{h} \cdot \frac{\mathbf{r}_1(0, 0)}{\sqrt{g_{11}(0, 0)}}, \\ \sin \theta(X_0) &= \frac{h_2}{\sqrt{g_{22}}} = \mathbf{h} \cdot \frac{\mathbf{r}_2(0, 0)}{\sqrt{g_{22}(0, 0)}}. \end{aligned}$$

$\theta(X_0)$ is actually the angle between \mathbf{h} and the principal direction $\mathbf{e}_1(0, 0)$ at X_0 . Then we compute

$$\begin{aligned} \gamma_1^2 + \gamma_2^2 &= 4(c_{11}c_{22} - c_{12}^2)[c_{11}(a_1^{(2)})^2 + 2c_{12}a_2^{(1)}a_1^{(2)} + c_{22}(a_2^{(1)})^2] \\ &= (1 - \alpha_0) \left[\frac{h_1^2}{g_{11}} \kappa_1^2 + \frac{h_2^2}{g_{22}} \kappa_2^2 - \alpha_0 \frac{h_1^2 h_2^2}{g_{11} g_{22}} (\kappa_1 - \kappa_2)^2 \right] \\ &= (1 - \alpha_0) [\kappa_1^2 \cos^2 \theta(X_0) + \kappa_2^2 \sin^2 \theta(X_0) - \alpha_0 (\kappa_1 - \kappa_2)^2 \cos^2 \theta(X_0) \sin^2 \theta(X_0)] \\ &= (1 - \alpha_0) P(X_0)^3. \end{aligned}$$

So we obtain

$$\mu_2 = \lambda_0(1 - \alpha_0)^{1/3} P(X_0)$$

and

$$\mu = \beta_0 \varepsilon^{-2} + \lambda_0(1 - \alpha_0)^{1/3} P(X_0) \varepsilon^{-4/3} + O(\varepsilon^{-1}).$$

To get the lowest eigenvalue, we choose $X_0 \in \mathcal{P}$, i.e., $P(X_0) = P_{\min}$. Then we get

$$(C.8) \quad \begin{aligned} \mu_2 &= \lambda_0(1 - \alpha_0)^{1/3} P_{\min}, \\ \mu &= \beta_0 \varepsilon^{-2} + \lambda_0(1 - \alpha_0)^{1/3} P_{\min} \varepsilon^{-4/3} + O(\varepsilon^{-1}). \end{aligned}$$

This ends our formal analysis.

ACKNOWLEDGMENTS

The author would like to thank Professor B. Helffer for communicating his preprints [HM1], [HM2] before publication. This work was partially supported by the National Natural Science Foundation of China, the Science Foundation of the Ministry of Education of China, and NUS Academic Research Grant R-146-000-033-112. The main results of this paper were reported in the *International Conference on Nonlinear PDE and Applications*, held in Huangshan, China, June 2001; and the *Concentration Phenomena and Vortex Dynamics Workshop*, held in Pacific Institute for Mathematical Science, University of British Columbia, Canada, July 2001. The author would like to thank the organizers for the invitation.

REFERENCES

- [A] S. Agmon, *Lectures on exponential decay of solutions of second order elliptic equations: bounds on eigenfunctions of N-body Schrodinger operators*, Princeton Univ. Press, NJ, 1982. MR **85f**:35019
- [A1] Y. Almog, *Non-linear surface superconductivity for type II superconductors in the large domain limit*, Arch. Ration. Mech. Anal., **165** (2002), 271-293. MR **2003k**:82108
- [BPT] P. Bauman, D. Phillips and Q. Tang, *Stable nucleation for the Ginzburg-Landau system with an applied magnetic field*, Arch. Rat. Mech. Anal., **142** (1998), 1-43. MR **99g**:58040
- [BS] A. Bernoff and P. Sternberg, *Onset of superconductivity in decreasing fields for general domains*, J. Math. Phys. **39** (1998), 1272-1284. MR **99a**:82099

- [C] S. J. Chapman, *Nucleation of superconductivity in decreasing fields*, European J. Appl. Math., **5** (1994), part 1, 449-468; part 2, 469-494. MR **95m**:82119
- [CHO] S. J. Chapman, S. D. Howison and J. R. Ockendon, *Macroscopic models for superconductivity*, SIAM Review, **34** (1992), 529-560. MR **94b**:82037
- [DFS] M. del Pino, P. Felmer and P. Sternberg, *Boundary concentration for eigenvalue problems related to the onset of superconductivity*, Commun. Math. Phys., **210** (2000), 413-446. MR **2001k**:35231
- [dG] P. G. De Gennes, *Superconductivity of Metals and Alloys*, W. A. Benjamin, New York, 1966.
- [DGP] Q. Du, M. Gunzburger and J. Peterson, *Analysis and approximation of the Ginzburg-Landau model of superconductivity*, SIAM Review, **34** (1992), 45-81. MR **93g**:82109
- [DH] M. Dauge and B. Helffer, *Eigenvalues variation, I, Neumann problem for Sturm-Liouville operators*, J. Differential Equations, **104** (1993), 243-262. MR **94j**:47097
- [G] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, vol. 1, Springer-Verlag, New York, 1994. MR **95i**:35216a
- [GL] V. Ginzburg and L. Landau, *On the theory of superconductivity*, in Collected Papers, Gordon and Breach, New York, 1967, 546-568. MR **38**:5577
- [GP] T. Giorgi and D. Phillips, *The breakdown of superconductivity due to strong fields for the Ginzburg-Landau model*, SIAM J. Math. Anal., **30** (1999), 341-359. MR **2000b**:35235
- [H] B. Helffer, *Semi-Classical Analysis for the Schrödinger Operator and Applications*, Lecture Notes in Mathematics, vol. 1336, Springer-Verlag, Berlin, 1988. MR **90c**:81043
- [HM1] B. Helffer and A. Morame, *Magnetic bottles in connection with superconductivity*, J. Functional Analysis, **185** (2001), 604-680. MR **2002m**:81051
- [HM2] B. Helffer and A. Morame, *Magnetic bottles for the Neumann problem: the case of dimension 3*, Proceedings of the Indian Academy of Sciences-Mathematical Sciences, **112** : (1) (2002), 71-84. MR **2003e**:81048
- [HP] B. Helffer and X. B. Pan, *Upper critical field and location of surface nucleation of superconductivity*, Ann. L'I.H.P. Analyse non Linéaire, **20** (2003), 145-181.
- [J] H. Jadallah, *The onset of superconductivity in a domain with a corner*, J. Math. Phys., **42** (2001), 4101-4121. MR **2002k**:82086
- [JRS] H. T. Jadallah, J. Rubinstein and P. Sternberg, *Phase transition curves for mesoscopic superconducting samples*, Phys. Rev. Lett., **82** (1999), 2935-2938.
- [L] O. A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, New York, 1985. MR **87f**:35001
- [LP1] K. Lu and X. B. Pan, *Gauge invariant eigenvalue problems in \mathbb{R}^2 and in \mathbb{R}_+^2* , Trans. Amer. Math. Soc., **352** (2000), 1247-1276. MR **2000j**:35248
- [LP2] K. Lu and X. B. Pan, *Eigenvalue problems of Ginzburg-Landau operator in bounded domains*, J. Math. Phys., **40** (1999), 2647-2670. MR **2001c**:35167
- [LP3] K. Lu and X. B. Pan, *Estimates of the upper critical field for the Ginzburg-Landau equations of superconductivity*, Physica D, **127** : (1-2) (1999), 73-104. MR **2000a**:82075
- [LP4] K. Lu and X. B. Pan, *Surface nucleation of superconductivity in 3-dimension*, J. Differential Equations, **168** (2000), 386-452. MR **2002b**:82069
- [M] R. Montgomery, *Hearing the zero locus of a magnetic field*, Comm. Math. Phys., **168** (1995), 651-675. MR **96e**:81044
- [P1] X. B. Pan, *Surface superconductivity in applied magnetic fields above H_{C_2}* , Comm. Math. Phys., **228** (2002), 327-370. MR **2003i**:82094
- [P2] X. B. Pan, *Upper critical field for superconductors with edges and corners*, Calculus of Variations and PDE's, **14** (2002), 447-482. MR **2003f**:82105
- [P3] X. B. Pan, *Superconducting thin films and the effect of de Gennes parameter*, SIAM J. Math. Anal., **34** (2003), 957-991.
- [P4] X. B. Pan, *Superconductivity near critical temperature*, J. Math. Phys., **44** (2003), 2639-2678.
- [PK] X. B. Pan and K. H. Kwak, *Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains*, Trans. Amer. Math. Soc., **354** (2002), 4201-4227. MR **2003j**:81052
- [R] J. Rubinstein, *Six lectures on superconductivity*, in: *Boundaries, Interfaces, and Transitions*, M. Delfour ed., CRM Proceedings and Lecture Notes 13, Amer. Math. Soc., Providence, RI, 1998, pp. 163-184. MR **99e**:82101

- [SdG] D. Saint-James and P. G. De Gennes, *Onset of superconductivity in decreasing fields*, Physics Letters, **6** : (5) (1963), 306-308.
- [SS] E. Sandier and S. Serfaty, *The decrease of bulk-superconductivity close to the second critical field in the Ginzburg-Landau model*, SIAM J. Math. Anal., **34** (2003), 939-956.
- [SST] D. Saint-James and G. Sarma and E. J. Thomas, *Type II Superconductivity*, Pergamon Press, Oxford, 1969.
- [T] M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill, New York, 1975.

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, CHINA – AND –
DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE 119260

E-mail address: `amaxbpan@dial.zju.edu.cn`

E-mail address: `matpanxb@nus.edu.sg`