

## VISCOSITY SOLUTIONS, ALMOST EVERYWHERE SOLUTIONS AND EXPLICIT FORMULAS

BERNARD DACOROGNA AND PAOLO MARCELLINI

ABSTRACT. Consider the differential inclusion  $Du \in E$  in  $\mathbb{R}^n$ . We exhibit an explicit solution that we call *fundamental*. It also turns out to be a *viscosity solution* when properly defining this notion. Finally, we consider a Dirichlet problem associated to the differential inclusion and we give an iterative procedure for finding a solution.

### 1. INTRODUCTION

Existence of *almost everywhere* solutions of the first order Dirichlet problem related to *implicit differential equations* of the type

$$(1) \quad \begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

has recently been extensively studied in the book [6] by the authors. Here  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function and we look for a Lipschitz-continuous solution  $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ . A wide literature on this subject can be found in [6], not only for scalar problems such as this one, but also for *vector-valued solutions of first order systems* related to maps  $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^N$ , for some  $m, N \geq 1$ .

Existence of *viscosity solutions* of the Dirichlet problem (1) is now well established. It has been studied by many authors starting with Hopf, Lax, Kruzkov and Crandall-Lions; see for example [1] or [6] for more historical comments. One of the earliest and still one of the most complete monographs on the subject is [10] by P.L. Lions. The research in this field remains very active; in particular H. Ishii and P. Loreti [8], motivated by an optimization problem, recently gave an existence result of viscosity solutions of the Dirichlet problem (1). See also [2] and [9].

In this paper we give some existence results, either in the case of *almost everywhere* solutions, or, when possible, of *viscosity solutions*. One of our aims is to give some constructive explicit formulas (cf. Theorems 1 and 6). Moreover, if the geometry of the set  $\Omega$  and the assumptions on the function  $F$  make it possible, following [3] we give (cf. Corollary 8) an explicit formula for a viscosity solution of the Dirichlet problem (1), simply in terms of sup and inf. Otherwise, with general  $F$  and  $\Omega$ , we propose, in Section 4, an iteration scheme for characterizing a solution.

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Received by the editors December 2, 2002 and, in revised form, August 21, 2003.

2000 *Mathematics Subject Classification*. Primary 34A60, 35F30, 49L25.

*Key words and phrases*. Almost everywhere solutions, viscosity solutions of nonlinear first order partial differential equations.

In Section 2 we introduce the notion of viscosity solution of a *differential inclusion*; namely: given a closed set  $E$ , we say that a function  $u$  is a viscosity solution of the differential inclusion

$$(2) \quad Du(x) \in E, \quad x \in \Omega,$$

if  $u$  is a viscosity solution of the equation

$$(3) \quad F(Du(x)) = 0, \quad x \in \Omega,$$

where  $F(\xi) = \text{dist}\{\xi, E\}$ . We will prove in Theorem 6 that the function  $L : \mathbb{R}^n \rightarrow \mathbb{R}$ , defined by

$$L(x) = \max\{\langle \xi, x \rangle : \xi \in E\},$$

is a viscosity solution of the differential inclusion (2), i.e. it is a *fundamental solution* of the equation (3).

## 2. FUNDAMENTAL SOLUTION AND VISCOSITY SOLUTIONS OF DIFFERENTIAL INCLUSIONS

We start by recalling some classical definitions and notations in convex analysis. We say that  $\xi \in \mathbb{R}^n$  is an *extreme point* for a convex set  $K \subset \mathbb{R}^n$  if the conditions

$$\begin{cases} \xi = t\xi_1 + (1-t)\xi_2, \\ \xi_1, \xi_2 \in K, \quad t \in (0, 1), \end{cases}$$

imply that  $\xi = \xi_1 = \xi_2$ .

If  $E$  is a set (not necessarily convex) of  $\mathbb{R}^n$ , we denote by  $E_{\text{ext}}$  the set of extreme points of the convex hull of  $E$  denoted by  $\text{co } E$  (note that  $E_{\text{ext}} \subset E$ ).

We also recall that the domain of a convex function  $L : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined as

$$\text{dom } L = \{x \in \mathbb{R}^n : L(x) < +\infty\}.$$

Theorem 1 below generalizes an analogous result obtained by Ischii and Loreti (see the proof of Theorem 2.2 in [8]) in the case that  $E$  is the level set of a continuous, positively homogeneous function of degree one, equal to zero only at the origin of  $\mathbb{R}^n$ .

**Theorem 1.** *Let  $E$  be a compact set of  $\mathbb{R}^n$ . For every  $x \in \mathbb{R}^n$  let*

$$L(x) = \max\{\langle \xi, x \rangle : \xi \in E\}.$$

*Then*

$$DL(x) \in E \quad \text{a.e. } x \in \mathbb{R}^n.$$

*Remark 2.* (i) It should be noted that in fact the theorem is more precise, namely

$$DL(x) \in \overline{E_{\text{ext}}} \subset E \cap \partial \text{co } E \quad \text{a.e. } x \in \mathbb{R}^n.$$

(ii) If  $E$  is any set, not necessarily closed or bounded, then the proof gives (replacing max by sup) that

$$DL(x) \in \overline{E} \quad \text{a.e. } x \in \text{dom } L.$$

(iii) In terms of convex analysis and anticipating on (4) we can say that  $L$  is the support function of  $\text{co } E$ .

Before proceeding with the proof it might be interesting to rewrite the theorem in terms of equations.

**Corollary 3.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function such that*

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$$

*is a bounded set. Let  $L(x) = \max \{\langle \xi, x \rangle : F(\xi) = 0\}$ ; then*

$$F(DL(x)) = 0 \text{ a.e. } x \in \mathbb{R}^n.$$

*Proof.* The following representation formula for  $L$  holds (see Rockafellar [12], Theorem 32.2):

$$(4) \quad L(x) = \max \{\langle \xi, x \rangle : \xi \in \text{co } E\} = \max \{\langle \xi, x \rangle : \xi \in E\}, \quad \forall x \in \mathbb{R}^n.$$

In fact one has the more precise result (see Rockafellar [12], Corollary 32.3.2)

$$(5) \quad L(x) = \max \{\langle \xi, x \rangle : \xi \in \text{co } E\} = \max \{\langle \xi, x \rangle : \xi \in E_{\text{ext}}\}, \quad \forall x \in \mathbb{R}^n.$$

Let  $\{\xi_h\}_{h \in \mathbb{N}}$  be a (finite or) countable dense subset of  $E_{\text{ext}} \subset E$  and, analogously to (4), for every  $h \in \mathbb{N}$  and for every  $x \in \mathbb{R}^n$  let us define

$$L_h(x) = \max \{\langle \xi_1, x \rangle, \langle \xi_2, x \rangle, \dots, \langle \xi_h, x \rangle\}.$$

Clearly the gradient  $DL_h$  exists almost everywhere in  $\mathbb{R}^n$  and

$$(6) \quad DL_h(x) \in \{\xi_1, \xi_2, \dots, \xi_h\} \subset E_{\text{ext}}, \quad \text{a.e. } x \in \mathbb{R}^n.$$

For every  $x \in \mathbb{R}^n$  the sequence  $L_h(x)$  is increasing with respect to  $h \in \mathbb{N}$  and we have

$$L(x) = \sup \{L_h(x) : h \in \mathbb{N}\} = \lim_{h \rightarrow +\infty} L_h(x).$$

For every  $h \in \mathbb{N}$  the sequence  $L_h(x)$  is convex with respect to  $x \in \mathbb{R}^n$  and

$$\text{dom } L_h = \text{dom } L = \mathbb{R}^n.$$

Thus we can apply Lemma 4 and we obtain that, at every point where  $L_h$  and  $L$  are differentiable (i.e., almost everywhere in  $\mathbb{R}^n$ ),

$$DL_h(x) \rightarrow DL(x).$$

Therefore, by (6), we get the conclusion

$$DL(x) \in \overline{E_{\text{ext}}} \subset E, \quad \text{a.e. } x \in \mathbb{R}^n.$$

□

In the proof of Theorem 1 we used a result given in [11] (Lemma 5.9), that we recall here in a form more appropriate to the applications given in this paper.

**Lemma 4.** *Let  $\{L_h\}_{h \in \mathbb{N}}$  be a sequence of convex functions, defined on  $\mathbb{R}^n$  with values on  $\mathbb{R} \cup \{+\infty\}$ , with pointwise limit  $L : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ . At every point  $x \in \text{int} [(\bigcap_{h \in \mathbb{N}} \text{dom } L_h) \cap \text{dom } L]$ , where  $L_h$  and  $L$  are differentiable, the gradient  $DL_h(x)$  converges in  $\mathbb{R}^n$  to the gradient  $DL(x)$ .*

*Proof.* For every  $h \in \mathbb{N}$  let  $\text{dom } L_h$  and  $\text{dom } L$  be the domains of  $L_h$  and  $L$ . Then each  $L_h$  is locally Lipschitz-continuous in  $\text{int } \text{dom } L_h$  and  $L$  is locally Lipschitz-continuous in  $\text{int } \text{dom } L$ . Therefore, for every  $h \in \mathbb{N}$ , there exists a set  $N_h \subset \text{dom } L_h \subset \mathbb{R}^n$  of zero measure such that  $L_h$  is differentiable at every point of  $\text{dom } L_h \setminus N_h$ . Analogously, there exists a set  $N \subset \text{dom } L$  of zero measure such that

$L$  is differentiable at every point of  $\text{dom } L \setminus N$ . Then the set of points  $x \in \mathbb{R}^n$  where  $L_h$  and  $L$  are differentiable is (possibly empty and) given by

$$\left( \bigcap_{h \in \mathbb{N}} (\text{dom } L_h \setminus N_h) \right) \cap (\text{dom } L \setminus N)$$

and differs from the intersection of their domains  $(\bigcap_{h \in \mathbb{N}} \text{dom } L_h) \cap \text{dom } L$  by (at most) a set  $(\bigcup_{h \in \mathbb{N}} N_h) \cup N$  of zero measure. Let  $x \in \text{int} [(\bigcap_{h \in \mathbb{N}} \text{dom } L_h) \cap \text{dom } L]$  be a point of  $\mathbb{R}^n$  where  $L_h$  and  $L$  are differentiable. Let  $i \in \{1, 2, \dots, n\}$  and  $h \in \mathbb{N}$  be fixed. Then at  $x = (x_1, \dots, x_i, \dots, x_n)$  the partial derivatives  $\partial L_h / \partial x_i$  and  $\partial L / \partial x_i$  are well defined. An elementary application of the convex inequality for the function  $L_h$  gives the monotonicity of the difference quotient; precisely, if  $t > 0$  is sufficiently small and if, as usual, we denote by  $x \pm te_i$  the two points of  $\mathbb{R}^n$  with coordinates respectively  $(x_1, \dots, x_{i-1}, x_i \pm t, x_{i+1}, \dots, x_n)$ , we have

$$\frac{L_h(x - te_i) - L_h(x)}{-t} \leq \frac{\partial L_h}{\partial x_i}(x) \leq \frac{L_h(x + te_i) - L_h(x)}{t}$$

and, in the limit as  $h \rightarrow +\infty$ ,

$$\frac{L(x - te_i) - L(x)}{-t} \leq \liminf_{h \rightarrow +\infty} \frac{\partial L_h}{\partial x_i}(x) \leq \limsup_{h \rightarrow +\infty} \frac{\partial L_h}{\partial x_i}(x) \leq \frac{L(x + te_i) - L(x)}{t}.$$

Since  $L$  is differentiable at  $x$ , as  $t \rightarrow 0^+$  we obtain that  $\partial L_h / \partial x_i(x)$  converges to  $\partial L / \partial x_i$ . The property being such that for every  $i \in \{1, 2, \dots, n\}$ , we have the conclusion, i.e., that the gradient  $DL_h(x)$  converges in  $\mathbb{R}^n$  to the gradient  $DL(x)$ . □

*Remark 5.* With a slightly different proof, as in Lemma 5.9 in [11], we can give a compactness result. Precisely, we can show that from every locally bounded sequence  $\{L_h\}_{h \in \mathbb{N}}$  of convex functions ( $\{L_h\}_{h \in \mathbb{N}}$  uniformly bounded in  $L^\infty_{\text{loc}}(\Omega)$ , with  $\Omega$  open set in  $\mathbb{R}^n$ ) it is possible to select a subsequence  $\{L_{h_k}\}_{k \in \mathbb{N}}$  whose gradients  $\{DL_{h_k}\}_{k \in \mathbb{N}}$  converge almost everywhere in  $\Omega$ , and at the same time  $\{L_{h_k}\}_{k \in \mathbb{N}}$  converges in the strong topology of  $W^{1,q}_{\text{loc}}(\Omega)$ , for every  $q \in [1, +\infty)$ .

With the help of the above construction we can give a definition of what we mean by viscosity solutions of differential inclusions. Given a closed set  $E$ , we say that a function  $u$  is a *viscosity solution of the differential inclusion*

$$Du(x) \in E, \quad x \in \mathbb{R}^n,$$

if  $u$  is a viscosity solution of the equation

$$F(Du(x)) = 0, \quad x \in \mathbb{R}^n,$$

where  $F(\xi) = \text{dist}\{\xi, E\}$ . We therefore have the following result.

**Theorem 6.** *Let  $E$  be a compact set of  $\mathbb{R}^n$ . For every  $x \in \mathbb{R}^n$  let*

$$L(x) = \max\{\langle \xi, x \rangle : \xi \in E\}.$$

*Then  $L$  is a viscosity solution of*

$$DL(x) \in E, \quad x \in \mathbb{R}^n.$$

*Proof.* The function  $L$  being convex we have that  $D^+L(x)$  (the superdifferential of  $L$  at  $x$ ; see [1] and [6] for the precise definition of this set) is either empty or reduced to  $\{DL(x)\}$ , i.e.  $x$  is a point of differentiability of  $L$  and we know by Theorem 1 that at such points  $DL(x) \in E$ . We therefore have that

$$F(p) = 0, \forall p \in D^+L(x),$$

which means that  $L$  is a viscosity subsolution (see Proposition 4.7 of [6]) of  $F(Du) = 0$ .

Since  $F \geq 0$  we deduce trivially that

$$F(p) \geq 0, \forall p \in D^-L(x),$$

where  $D^-L(x)$  is the subdifferential of  $L$  at  $x$ . This means that  $L$  is a viscosity supersolution of  $F(Du) = 0$ .

Combining these two results we have indeed that  $L$  is a viscosity solution of  $F(Du) = 0$  and hence of  $Du \in E$ .  $\square$

### 3. FUNDAMENTAL SOLUTION AND THE BOUNDARY CONDITION

We now want to discuss a Dirichlet problem in a bounded domain. We first fix the notations.

We let  $\Omega \subset \mathbb{R}^n$  be a bounded open convex set and denote by  $\nu(y)$  the outward unit normal at  $y \in \partial\Omega$  (that exists at almost all points  $y \in \partial\Omega$ , since  $\Omega$  is convex).

We next let  $E \subset \mathbb{R}^n$  be a compact set with  $0 \in \text{intco } E$ . We then associate to  $\text{co } E$  its *gauge*  $\rho$ , which is a convex and positively homogeneous of degree one function, such that

$$\text{co } E = \{\xi \in \mathbb{R}^n : \rho(\xi) \leq 1\}.$$

Recall also that

$$L(x) = \max \{\langle \xi, x \rangle : \xi \in E\}.$$

We should immediately note that, with our hypotheses on  $E$  (and invoking (4)), the function  $L$  is in fact the polar of  $\rho$ , denoted also sometimes by  $\rho^0$ .

We finally consider the Dirichlet problem

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

We could also consider the case of a more general boundary datum of class  $C^1$  but the analysis can then be carried in a straightforward manner.

We have the following theorem that is inspired by Cardaliaguet-Dacorogna-Gangbo-Georgy [3] (see also [6]).

**Theorem 7.** *Let  $\Omega, \nu, E, \rho$  and  $L$  be as above and satisfy in addition*

$$(7) \quad \frac{-\nu(y)}{\rho(-\nu(y))} \in E, \text{ a.e. } y \in \partial\Omega;$$

*then the function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ , defined by*

$$(8) \quad u(x) = \min \{L(x - y) : y \in \partial\Omega\},$$

*solves the Dirichlet problem*

$$(9) \quad \begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

As before we rewrite this theorem in terms of functions.

**Corollary 8.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous with  $F(\xi) \rightarrow \infty$  as  $|\xi| \rightarrow \infty$  and  $F(0) < 0$ . Set*

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}.$$

*Let  $\Omega, \nu, \rho$  and  $L$  be as above. If*

$$(10) \quad F\left(\frac{-\nu(y)}{\rho(-\nu(y))}\right) = 0 \text{ a.e. } y \in \partial\Omega,$$

*then*

$$(11) \quad u(x) = \min \{L(x - y) : y \in \partial\Omega\}$$

*solves*

$$(12) \quad \begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

*Furthermore if  $E \subset \partial \text{co } E$ , then  $u$  is a viscosity solution.*

*Remark 9.* (i) The first part of the corollary follows immediately from the theorem. The fact that  $u$  is a viscosity solution (when  $E \subset \partial \text{co } E$ ) was established in [3].

(ii) Note that if, in addition,  $\partial \text{co } E \subset E$  (which happens if, for instance,  $F$  is convex or more generally if the set  $\{\xi : F(\xi) \leq 0\}$  is convex), then (10) is always satisfied. In fact, since  $\rho$  is positively homogeneous of degree one,

$$\rho\left(\frac{-\nu(y)}{\rho(-\nu(y))}\right) = 1 \Rightarrow \frac{-\nu(y)}{\rho(-\nu(y))} \in \partial \text{co } E \subset E.$$

Moreover, if  $E = \partial \text{co } E$ ,  $u$  defined in (11) is the unique viscosity solution of (12).

(iii) According to Theorem 4.1 of Lions [10], the Dirichlet problem (12) always has a viscosity solution. However the solution given by (11) is not necessarily a viscosity solution; it is so when  $E \subset \partial \text{co } E$ .

We can now proceed with the proof of the theorem.

*Proof of Theorem 7.* We recall the following two facts (the first one is just the Hopf-Lax formula and the second one is Lemma 2.9 in [3] or Lemma 4.17 in [6]). We also use the standard notation  $D^+u(x)$ , respectively  $D^-u(x)$ , for the superdifferential, respectively the subdifferential, of  $u$  at  $x$  (see [6] for more details).

Fact 1: The function  $u$  is the viscosity solution of

$$(13) \quad \begin{cases} \rho(Du(x)) = 1, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Fact 2: Let  $y(x) \in \partial\Omega$  be such that

$$u(x) = L(x - y(x)).$$

Then, if  $p \in D^-u(x)$  (i.e.  $D^-u(x)$  is non empty), the outward unit normal  $\nu(y(x))$  is well defined and there exists  $\lambda(y(x)) > 0$  such that

$$(14) \quad p = -\lambda(y(x))\nu(y(x)).$$

Since we are interested in almost everywhere solutions we need only to consider points  $x \in \Omega$  where  $D^+u(x) = D^-u(x) = \{Du(x)\}$ . Combining (13) and (14) with  $p = Du(x)$  and the homogeneity of  $\rho$ , we get that  $\lambda(y) = 1/\rho(-\nu(y))$  and hence

$$Du(x) = \frac{-\nu(y)}{\rho(-\nu(y))}.$$

The hypothesis (7) leads to the result  $Du \in E$ . □

4. THE ITERATION SCHEME

As above we let  $\Omega \subset \mathbb{R}^n$  be a nonempty bounded open set. We want to find, with the help of the previous construction, a solution  $u \in W_0^{1,\infty}(\Omega)$  of the differential inclusion

$$Du(x) \in E, \text{ a.e. } x \in \Omega,$$

where  $E \subset \mathbb{R}^n$  is a compact set with  $0 \in \text{intco } E$ . We let  $\rho$  be the gauge associated to  $\text{co } E$ .

We will find a sequence of disjoint convex open sets  $\Omega_i \subset \Omega$  so that

$$\text{meas} \left[ \Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i \right] = 0$$

and the function  $u$  will be defined as

$$u(x) = \begin{cases} \inf \{L(x - y) : y \in \partial\Omega_i\}, & x \in \Omega_i, \\ 0, & x \in \Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i. \end{cases}$$

Observe that  $u$  is a viscosity solution of the Dirichlet problem  $Du \in E$  in  $\Omega_i$ ,  $u = 0$ , on  $\partial\Omega_i$  for every  $i$  (but not globally in  $\Omega$ ).

Any Vitali covering by level sets of the function  $L$  has all the above requirements. However we will choose, among them, one with some maximality properties. In particular we want that  $\Omega_1 = \Omega$  if  $\Omega$  is convex and  $\frac{-\nu}{\rho(-\nu)} \in E$ , a.e. on  $\partial\Omega$ , where  $\nu$  is the outward unit normal to  $\Omega$  (recall that this always happens if  $E = \partial \text{co } E$  or if  $\Omega$  is the level set of the function  $L$ ).

Before describing this construction we need to introduce some notations.

*Notation 10.* Let  $x_0 \in \mathbb{R}^n$ . We let  $G_{x_0}$  be the set of all gauges centered at  $x_0$ . In other words this is the set of all convex functions  $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying

$$\begin{aligned} \gamma(x_0) = 0, \gamma(x) > 0, \forall x \in \mathbb{R}^n \setminus \{x_0\}, \\ \gamma(t(x - x_0) + x_0) = t\gamma(x), \forall x \in \mathbb{R}^n, \forall t > 0. \end{aligned}$$

**Proposition 11.** (i) *If  $\gamma \in G_{x_0}$  is differentiable at  $x \in \mathbb{R}^n$  (this happens at almost all points), then it is differentiable at any  $x_t \in \mathbb{R}^n$  of the form  $x_t = t(x - x_0) + x_0$ ,  $t > 0$  and*

$$D\gamma(x_t) = D\gamma(x).$$

*In particular  $\gamma$  is differentiable at almost all points of  $\{x \in \mathbb{R}^n : \gamma(x) = 1\}$ .*

(ii) *Let  $C \subset \mathbb{R}^n$  be a nonempty bounded open convex set and  $x_0 \in \text{int } C$ . The gauge of  $C$  centered at  $x_0$  is defined as*

$$\gamma_{C,x_0}(x) = \inf \left\{ \lambda \geq 0 : x_0 + \frac{x - x_0}{\lambda} \in C \right\}.$$

*Then  $\gamma_{C,x_0} \in G_{x_0}$  and*

$$\begin{aligned} C &= \{x \in \mathbb{R}^n : \gamma_{C,x_0}(x) < 1\}, \\ \partial C &= \{x \in \mathbb{R}^n : \gamma_{C,x_0}(x) = 1\}. \end{aligned}$$

*Remark 12.* At almost every point  $x \in \partial C$ ,  $\gamma_{C,x_0}$  is differentiable and  $D\gamma_{C,x_0}(x)$  is then an outward normal to  $C$ .

We will proceed inductively to define  $\Omega_i$ . We start by choosing a sequence of points in  $\Omega$ ,  $\{x^N\}_{N=1}^\infty$ , dense in  $\Omega$ . We set  $\Omega_0 = \emptyset$  and assume that  $\Omega_i$  has already been defined. If  $\Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k = \emptyset$ , then the procedure is already over. We then define,  $N = N(i + 1)$ ,

$$N(i + 1) = \min \left\{ N : x^N \in \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k \right\}$$

and we label  $x_{i+1} = x^{N(i+1)}$  (so that  $x_1 = x^1$ ). We then choose  $r_{i+1} > 0$  sufficiently small so that

$$\left\{ x \in \mathbb{R}^n : l_{r_{i+1}}(x) \equiv \frac{L(x_{i+1} - x)}{r_{i+1}} < 1 \right\} \subset \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k,$$

where

$$L(x) = \max \{ \langle \xi, x \rangle : \xi \in E \}.$$

This is always possible since  $\Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k$  is an open set,  $x_{i+1} \in \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k$ ,  $L(0) = 0$  and  $L$  is locally Lipschitz.

We next define

$$\begin{aligned} & \Gamma \left( x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k \right) \\ &= \left\{ \begin{array}{l} \gamma \in G_{x_{i+1}} : \frac{-D\gamma(x)}{\rho(-D\gamma(x))} \in E, \text{ a.e. } x \in \mathbb{R}^n \\ \{ x \in \mathbb{R}^n : l_{r_{i+1}}(x) < 1 \} \subset \{ x \in \mathbb{R}^n : \gamma(x) < 1 \} \subset \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k \end{array} \right\}. \end{aligned}$$

Note that  $l_{r_{i+1}} \in \Gamma \left( x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k \right)$ , since, by Theorem 1,  $DL \in E$  and  $\rho(DL) =$

1. Observe also that if  $\gamma \in \Gamma \left( x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k \right)$ , then

$$(15) \quad \gamma \leq l_{r_{i+1}}.$$

We now claim that there exists  $\gamma_{i+1} \in \Gamma \left( x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k \right)$  such that if

$$\Omega_{i+1} = \{ x \in \mathbb{R}^n : \gamma_{i+1}(x) < 1 \},$$

then

$$\text{meas}(\Omega_{i+1}) = \sup_{\gamma \in \Gamma \left( x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k \right)} [\text{meas} \{ x \in \mathbb{R}^n : \gamma(x) < 1 \}].$$

Indeed let  $\{\gamma^s\}$  be a maximizing sequence. From (15), we deduce that up to a subsequence, that we still label  $\{\gamma^s\}$ , the sequence converges to an element  $\gamma_{i+1} \in \Gamma \left( x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \bar{\Omega}_k \right)$ . In fact all the conditions are easily checked. By Remark 5 we have

$$\frac{-D\gamma_{i+1}(x)}{\rho(-D\gamma_{i+1}(x))} \in E.$$

Let us prove, for example, that  $\gamma_{i+1}(x) \neq 0$  if  $x \neq x_{i+1}$ . Assume for the sake of contradiction that there exists  $y \neq x_{i+1}$  with  $\gamma_{i+1}(y) = 0$ . We would deduce that  $\gamma_{i+1} \equiv 0$  on the half line  $x_{i+1} + t(y - x_{i+1})$ ,  $t \geq 0$ , which contradicts,  $\Omega$  being bounded, the inclusion  $\{x \in \mathbb{R}^n : \gamma_{i+1}(x) < 1\} \subset \Omega$ .

Since the measure is upper semicontinuous (in fact even continuous), cf. Proposition 14, with respect to the type of convergence under consideration we have the result.

Observe that, as wished,  $\Omega_1 = \Omega$  if  $\Omega$  is convex and  $\frac{-\nu}{\rho(-\nu)} \in E$ , a.e. on  $\partial\Omega$  (because choosing  $\omega$  the gauge of  $\Omega$  centered at  $x_1$ , we would have  $\omega \in \Gamma(x_1, \Omega)$ ).

Since we have, with this procedure, exhausted all elements of the sequence  $\{x^N\}$ , we have indeed

$$\text{meas} \left[ \Omega \setminus \bigcup_{i=1}^{\infty} \overline{\Omega}_i \right] = 0.$$

**Example 13.** Consider the case  $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$ ,  $u = u(x_1, x_2)$  and

$$\begin{cases} \left( \left( \frac{\partial u}{\partial x_1} \right)^2 - 1 \right)^2 + \left( \left( \frac{\partial u}{\partial x_2} \right)^2 - 1 \right)^2 = 0 & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Choosing the grid sequence  $\{x^N\}_{N=1}^{\infty}$  in a suitable way, starting with  $x^1 = (0, 0)$ , we find with our procedure

$$\Omega^1 = \{x \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\} \text{ and } u(x_1, x_2) = 1 - |x_1| - |x_2| \text{ in } \Omega_1.$$

Similarly for  $\Omega_i$ . Our construction is compatible with the numerical computations of [4].

We end up with an elementary convergence result that we used above.

**Proposition 14.** Let  $\{\gamma^s\}_{s \in \mathbb{N}}$  and  $\gamma^\infty$  be measurable functions defined on a bounded measurable set  $\Omega \subset \mathbb{R}^n$ . Let

$$\begin{aligned} \Omega^s &= \{x \in \Omega : \gamma^s(x) \leq 1\}, \\ \Omega^\infty &= \{x \in \Omega : \gamma^\infty(x) \leq 1\}. \end{aligned}$$

If  $\gamma^s \rightarrow \gamma^\infty$  a.e. in  $\Omega$ , then

$$\text{meas}(\Omega^\infty) \geq \limsup_{s \rightarrow \infty} \text{meas}(\Omega^s).$$

If, in addition,  $\gamma^s$  and  $\gamma^\infty$  are gauges centered at  $x_0 \in \Omega$  and  $\Omega$  is open, then

$$\text{meas}(\Omega^\infty) = \lim_{s \rightarrow \infty} \text{meas}(\Omega^s).$$

*Remark 15.* Note that if  $\gamma^s$  and  $\gamma^\infty$  are merely convex, then continuity does not hold, as the following example shows. Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set containing the unit ball  $B_1$ . If

$$\gamma^s(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ \frac{1}{s}|x| + \frac{s-1}{s} & \text{if } |x| > 1, \end{cases}$$

then  $\Omega^s = B_1$  for every  $s \in \mathbb{N}$ , while  $\Omega^\infty = \Omega$ .

*Proof.* 1) Define

$$\chi^s(x) = \begin{cases} 0 & \text{if } x \in \Omega^s, \\ 1 & \text{if } x \notin \Omega^s \end{cases}$$

and similarly for  $\chi^\infty$ . Note that because of the convergence of  $\gamma^s$  to  $\gamma^\infty$ , we have that, at almost all points where  $\chi^\infty(x) = 1$  (i.e.  $\gamma^\infty(x) > 1$ ) and for large enough  $s$ ,  $\chi^s(x) = 1$  and thus

$$\lim_{s \rightarrow \infty} \chi^s(x) = \chi^\infty(x), \text{ a.e. } x \notin \Omega^\infty.$$

Moreover, trivially,  $\liminf_{s \rightarrow \infty} \chi^s(x) \geq \chi^\infty(x) = 0$ , a.e.  $x \in \Omega^\infty$  and therefore

$$\liminf_{s \rightarrow \infty} \chi^s(x) \geq \chi^\infty(x), \text{ a.e. } x \in \Omega.$$

Therefore by Fatou's lemma

$$\begin{aligned} \liminf_{s \rightarrow \infty} [\text{meas}(\Omega) - \text{meas}(\Omega^s)] &= \liminf_{s \rightarrow \infty} \int_{\Omega} \chi^s(x) dx \\ &\geq \int_{\Omega} \chi^\infty(x) dx = \text{meas}(\Omega) - \text{meas}(\Omega^\infty) \end{aligned}$$

which gives the upper semicontinuity.

2) Let  $B_\varepsilon = \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}$  and for  $A \subset \mathbb{R}^n$  define

$$A + B_\varepsilon = \{x \in \mathbb{R}^n : x = y + z \text{ with } y \in A \text{ and } |z| \leq \varepsilon\}.$$

The Hausdorff distance between two sets is then defined as

$$d(A, B) = \inf \{\varepsilon \geq 0 : A \subset B + B_\varepsilon, B \subset A + B_\varepsilon\}.$$

Observe (see below) that since  $\gamma^s$  and  $\gamma^\infty$  are gauges then

$$(16) \quad d(\Omega^s, \Omega^\infty) \rightarrow 0, \text{ as } s \rightarrow \infty,$$

and therefore (see Theorem 6.2.17 in [13])

$$\text{meas}(\Omega^\infty) = \lim_{s \rightarrow \infty} \text{meas}(\Omega^s).$$

We now establish (16). We will prove that for every  $\varepsilon > 0$  we can find  $s$  sufficiently large so that

$$(17) \quad \Omega^\infty \subset \Omega^s + B_\varepsilon, \quad \Omega^s \subset \Omega^\infty + B_\varepsilon.$$

Assume without loss of generality that  $x_0 = 0$ . Since  $\Omega$  is bounded and  $\gamma^s$  are gauges that converge almost everywhere to a gauge  $\gamma^\infty$ , the convergence is, in fact, uniform. Furthermore there exist  $m, M > 0$  so that

$$m|x| \leq \gamma^s(x), \quad \gamma^\infty(x) \leq M|x|, \quad \forall x \in \Omega,$$

and, for  $s$  sufficiently large,

$$|\gamma^s(x) - \gamma^\infty(x)| \leq \varepsilon^2, \quad \forall x \in \Omega.$$

Let  $x \in \Omega^\infty$ , i.e.  $\gamma^\infty(x) \leq 1$ , and choose  $\delta > 0$  such that

$$\frac{\varepsilon^2}{1 + \varepsilon^2} \leq \delta \leq m\varepsilon$$

and observe that

$$\gamma^s((1 - \delta)x) = (1 - \delta)\gamma^s(x) \leq (1 - \delta)(\gamma^\infty(x) + \varepsilon^2) \leq (1 - \delta)(1 + \varepsilon^2) \leq 1,$$

$$|\delta x| \leq \delta \frac{\gamma^\infty(x)}{m} \leq \frac{\delta}{m} \leq \varepsilon.$$

Therefore  $x = (1 - \delta)x + \delta x \in \Omega^s + B_\varepsilon$  which is the first inclusion in (17). The second one being proved in a similar manner, we have the claim.  $\square$

## ACKNOWLEDGMENTS

The research of Bernard Dacorogna was partially supported by the *Fonds National Suisse* (21-61390-00). The research of Paolo Marcellini was partially supported by the Italian *Ministero dell'Istruzione, dell'Università e della Ricerca* (MIUR) and through the contribution of the Italian *Ministero degli Affari Esteri, Direzione Generale per la Promozione e la Cooperazione Culturale* (MAE).

## REFERENCES

- [1] M. Bardi and I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, 1997. MR **99e**:49001
- [2] F. Camilli and A. Siconolfi, Maximal subsolutions for a class of degenerate Hamilton-Jacobi problems, *Indiana Univ. Math. J.*, **48** (1999), 1111–1131. MR **2001a**:49028
- [3] P. Cardaliaguet, B. Dacorogna, W. Gangbo and N. Georgy, Geometric restrictions for the existence of viscosity solutions, *Annales Institut Henri Poincaré, Analyse Non Linéaire*, **16** (1999), 189–220. MR **99k**:35023
- [4] B. Dacorogna, R. Glowinski and T.W. Pan, Numerical methods for the solution of a system of eikonal equations with Dirichlet boundary conditions, *Comptes Rendus Acad. Sci. Paris*, **336** (2003), 511–518.
- [5] B. Dacorogna and P. Marcellini, General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial case, *Acta Mathematica*, **178** (1997), 1–37. MR **98d**:35029
- [6] B. Dacorogna and P. Marcellini, *Implicit partial differential equations, Progress in Non-linear Differential Equations and their Applications*, 37, Birkhäuser, Boston, 1999. MR **2000f**:35005
- [7] L. Hörmander, *Notions of convexity*, Birkhäuser, 1994. MR **95k**:00002
- [8] H. Ishii and P. Loreti, Relaxation in an  $L^\infty$ -optimization problem, *Proc. Royal Soc. Edinburgh*, **133** (2003), 599–615.
- [9] H. Ishii and M. Ramaswamy, Uniqueness results for a class of Hamilton-Jacobi equations with singular coefficients, *Comm. Par. Diff. Eq.*, **20** (1995), 2187–2213. MR **96k**:35026
- [10] P. L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Math. 69, Pitman, London, 1982. MR **84a**:49038
- [11] P. Marcellini, Non convex integrals of the calculus of variations, in: *Methods of nonconvex analysis*, ed. Cellina A., Lecture Notes in Math. 1446, Springer-Verlag, Berlin, 1990, 16–57. MR **91j**:49002
- [12] R.T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970. MR **43**:445
- [13] R. Webster, *Convexity*, Oxford University Press, Oxford, 1994. MR **98h**:52001

DÉPARTEMENT DE MATHÉMATIQUES, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, 1015 LAUSANNE, SWITZERLAND

*E-mail address*: [bernard.dacorogna@epfl.ch](mailto:bernard.dacorogna@epfl.ch)

DIPARTIMENTO DI MATEMATICA U. DINI, UNIVERSITÀ DI FIRENZE, FIRENZE, ITALY

*E-mail address*: [marcell@math.unifi.it](mailto:marcell@math.unifi.it)