TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 12, Pages 5025–5043
S 0002-9947(04)03469-5
Article electronically published on June 29, 2004

BLAKERS-MASSEY ELEMENTS AND EXOTIC DIFFEOMORPHISMS OF S^6 AND S^{14} VIA GEODESICS

C. E. DURÁN, A. MENDOZA, AND A. RIGAS

Abstract. We use the geometry of the geodesics of a certain left-invariant metric on the Lie group $Sp(2)$ to find explicit related formulas for two topological objects: the Blakers-Massey element (a generator of $\pi_6(S^3)$) and an exotic (i.e. not isotopic to the identity) diffeomorphism of S^6 (C. E. Durán, 2001). These formulas depend on two quaternions and their conjugates and we produce their extensions to the octonions through formulas for a generator of $\pi_{14}(S^7)$ and exotic diffeomorphisms of S^{14}, thus giving explicit gluing maps for half of the 15-dimensional exotic spheres expressed as the union of two 15-disks.

1. Introduction

1.1. **Explicit topological phenomena via algebra.** In this section we illustrate how algebraic phenomena can be used to *explicitly* model (differential) topological constructions, via a sequence of examples which converges to the topics treated in this paper; we denote the quaternions by $\mathbb{H}$ and the Cayley octonions by $\mathbb{O}$:

- Hopf fibrations: The Hopf fibration $S^1 \to S^3 \to S^2$ can be written as $H(q) = q\mathbf{i}\bar{q}$, where $q \in S^3$ is regarded as a unit quaternion, which suggests that the Hopf fibration $S^1 \to S^3 \to S^2$ is non-trivial because the quaternions do not commute. The Hopf fibration $S^3 \to S^7 \to S^4$ can be written as $H(q) = (e_1 q)(\bar{q} e_2)$, where $q \in S^7$ is regarded as a unit octonion ([3]). Analogously, this suggests that its non-triviality is due to the non-associativity of the octonions.
- The Blakers-Massey element, which generates $\pi_6(S^3)$ and therefore classifies S^3 bundles over S^7, can be written as follows: let $M : S^3 \times S^3 \to S^3$ be the commutator of unit quaternions by
$$M(x, y) = xyx^{-1}y^{-1}.$$
Since $M(x, 1) = M(1, y) = 1$, M descends to the smash product, and we have a map $m : S^3 \wedge S^3 \cong S^6$ onto S^3, which can be shown to generate $\pi_6(S^3)$ ([10, 17]). Therefore the existence of non-trivial S^3 bundles over S^7 is due to the non-commutativity of the quaternions. This result extends to the octonions: if in the definition of M we substitute quaternions by octonions everywhere, we get a generator of $\pi_{14}(S^7)$ ([10, 17]).

Received by the editors March 21, 2003 and, in revised form, July 29, 2003.
1991 *Mathematics Subject Classification.* Primary 53C22; Secondary 58F07, 57R15.
Key words and phrases. Geometry of geodesics, Blakers-Massey element, exotic spheres.

- An explicit covering of $T : S^3 \times S^3 \to SO(4)$ is given by $T(u,v)(\xi) = u\xi\bar{v}$, where u and v are unit quaternions acting on an arbitrary quaternion ξ. The classification of $SO(4)$ bundles over S^4 is governed by $\pi_3(SO(4) \cong \mathbb{Z} \times \mathbb{Z}$, and a explicit isomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to \pi_3(SO(4))$ can readily be given as $\phi(i,j) = [T(u^i, u^j)]$, that is, $\phi(i,j)$ is the class of $\xi \mapsto u^i \xi u^{-j}$. Note how the non-commutativity of the quaternions appears again. Again, this classification remains valid for $SO(8)$ bundles over S^8 substituting quaternions by octonions everywhere [4]. Associated S^3 bundles corresponding to some of the bundles above produce 7-dimensional exotic spheres, as shown by Milnor ([4, 13]), and we also have the corresponding 15-dimensional result for octonions ([6, 16]).
- One of these bundles, corresponding to $i = 2, j = 1$ in the Milnor construction, was given a geometric realization as a quotient of the Lie group $Sp(2)$ by Gromoll and Meyer ([7]), thus producing a metric of non-negative curvature on the exotic sphere $M^7_{2,-1}$.
- Exotic spheres can alternately be described as two disks joined at their boundaries by a diffeomorphism that is not isotopic to the identity. Such a diffeomorphism was explicitly given in [5], and it goes as follows: let

  $$S^6 = \{(p,w) \in \mathbb{H} \times \mathbb{H} : |p|^2 + |w|^2 = 1, \operatorname{Re}(p) = 0\},$$

  and let $\sigma : S^6 \to S^6$ be given by

  $$\sigma(p,w) = \begin{cases} \left(\frac{1}{(1+p^2)^2}\bar{w}e^{-\pi p}wp\bar{w}e^{\pi p}w, \frac{1}{(1+p^2)}\bar{w}e^{-\pi p}we^{\pi p}w\right), & w \neq 0, \\ (p,0), & w = 0. \end{cases}$$

  Then σ is a diffeomorphism, $M^7_{2,-1} = D^7 \cup_\sigma D^7$, and σ is not isotopic to the identity. Note that σ acts as the identity when p and w commute.

The last item naturally leads us to the next section. Also, note that the Gromoll-Meyer construction (which is involved in the last two items) does *not* have an octonionic analogue ([11]).

1.2. **Explicit topological phenomena via geometry of geodesics.** The exotic diffeomorphism σ is constructed in [5] by using the geometry of the geodesics of a peculiar left-invariant (but not bi-invariant!) metric g on $Sp(2)$. This metric descends to a metric on $M^7_{2,-1}$ which has the *Wiedersehen property*, i.e. there are "north" and "south" poles and every geodesic emanating from the north pole reaches the south pole at length π. The map σ is then the change of coordinates of exponential charts from each pole.

From σ we noticed:

(1) The same geometry of the geodesics of the metric g allows the writing of the boundary map of the exact homotopy sequence of the fibration $S^3 \to Sp(2) \to S^7$, and thus an explicit differentiable formula $b : S^6 \to S^3$ for the Blakers-Massey element.
(2) The exotic diffeomorphisms σ "factors" through b, and b is σ-invariant.
(3) Both maps, σ and b, are expressed in an algebra generated by two quaternions. This lead us to suspect that substituting quaternions by octonions everywhere, we might get analogous results for the octonionic case.

We prove:

Theorem I. *Let* $S^6 = \{(p,w) \in \mathbb{H} \times \mathbb{H} \ : \ \mathrm{Re}(p) = 0, |p|^2 + |w|^2 = 1\}$. *The map* $b : S^6 \to S^3$ *given by*

$$b(p,w) = \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, & w \neq 0, \\ -1, & w = 0, \end{cases}$$

is a real analytic map that generates $\pi_6(S^3) \cong Z_{12}$, *and therefore represents the Blakers-Massey element.*

We then extend the result to the octonionic case:

Theorem II. *Let* $S^{14} = \{(p,w) \in \mathbb{O} \times \mathbb{O} \ : \ \mathrm{Re}(p) = 0, |p|^2 + |w|^2 = 1\}$. *The map* $b : S^{14} \to S^7$ *given by*

$$b(p,w) = \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, & w \neq 0, \\ -1, & w = 0, \end{cases}$$

is a real analytic map that generates $\pi_{14}(S^7) \cong \mathbb{Z}_{120}$, *and therefore represents the Blakers-Massey element in the octonion case.*

We also take advantage of the formulas involving only the algebra generated by two octonions to show that σ also works in this case, and in fact it is related to the Blakers-Massey element:

Theorem III. *Let* $S^{14} = \{(p,w) \in \mathbb{O} \times \mathbb{O} \ : \ |p|^2 + |w|^2 = 1, \mathrm{Re}(p) = 0\}$, *and let* $\sigma : S^{14} \to S^{14}$ *be given by*

$$\sigma(p,w) = \left(\overline{b(p,w)}\, p\, b(p,w)\, ,\, \overline{b(p,w)}\, w\, b(p,w)\right),$$

where $b : S^{14} \to S^7$ *is the map representing the Blakers-Massey element of Theorem* II. *Then* σ *is a diffeomorphism, and the manifold* $M^{15} = D^{15} \cup_\sigma D^{15}$ *is an exotic sphere. Therefore* σ *is not isotopic to the identity.*

Let us remark that Theorems II and III involve a lot more than just substituting quaternions by octonions everywhere in the respective quaternionic theorems; the Gromoll-Meyer construction has no obvious generalization to the 15-dimensional case, and in fact it has been proven that the Gromoll-Meyer sphere is the only one thast can be expressed as a biquotient ([11]).

When we want to substitute octonions everywhere, we lose the geometry of $Sp(2)$. Thus the gist of the constructions is to use the geometry of geodesics in the quaternionic case to *inspire* formulas for the octonionic analogues, and the moral of this story is that the geodesics—being one-dimensional objects—give us a good chance that everything will go through due to the alternating property of the octonions.

2. The Blakers-Massey element

In this section we use the geometric methods of [5] in order to find a smooth map $b : S^6 \to S^3$ which generates $\pi_6(S^3)$ in terms of quaternions. We then extend the formula to octonions and produce a smooth map $b : S^{14} \to S^7$ which generates $\pi_{14}(S^7)$.

2.1. **Topology.** Let $Sp(2)$ be the group of 2×2 matrices with quaternionic entries satisfying

$$\begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The group S^3 acts freely in $Sp(2)$ as follows:

$$q \bullet \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \bar q \end{pmatrix},$$

producing a principal fibration $S^3 \to Sp(2) \xrightarrow{\rho} S^7$, where S^7 is the standard 7-sphere. In fact, the projection ρ of $Sp(2)$ onto S^7 is just the $\rho(A) = 1^{\text{st}}$ column of A, and in what follows we will regard S^7 as the unit sphere of $\mathbb{H}^2$.

It is well known ([9], [14]) that this fibration generates the principal S^3 bundles over S^7.

Let us look at the relevant piece of the exact sequence of this fibering:

$$\ldots \to \pi_7(S^7) \xrightarrow{\partial} \pi_6(S^3) \to \pi_6(Sp(2)) \to \ldots.$$

Since $\pi_6(Sp(2)) = 0$ ([9]), it follows that the image of a generator of $\pi_7(S^7) \backsimeq \mathbb{Z}$ under the boundary map generates $\pi_6(S^3)$.

Denote by $s = (-1, 0) \in \mathbb{H}\times\mathbb{H}$ the "south pole" of S^7. By construction of the boundary map of the homotopy sequence of a fibration, if we find a map $e : (D^7, S^6) \to (S^7, s)$ representing a generator of $\pi_7(S^7)$ and a lift $E : (D^7, S^6) \to (Sp(2), S^3)$ of e, then the restriction $b = E|_{S^6}$ will represent $\partial(e)$ and therefore generate $\pi_6(S^3)$. We will explicitly give the maps e and E via a geometrical construction in the next section.

2.2. **Geometry.** The typical element ξ of the Lie algebra $sp(2)$ of $Sp(2)$ looks like

$$\xi = \begin{pmatrix} p & -\bar w \\ w & s \end{pmatrix},$$

where p and s are pure quaternions. We will denote such an element by $\xi = (p, w, s)$.

Let us define a left invariant metric g on $Sp(2)$. Its associated norm in the Lie algebra is given by

$$\|(p, w, s)\|_g^2 = |p|^2 + |r|^2 + |s|^2,$$

and we extend the definition of g by left invariance.

This metric is thoroughly studied in ([5]), with a rather different objective. It is not too difficult to show that g descends to the canonical metric g_{can} of constant curvature 1 in S^7, so that $S^3 \to (Sp(2), g) \to (S^7, g_{can})$ is a Riemannian submersion, and that the horizontal space at the identity is given by vectors of the form $(p, r, 0)$ (see section 3 of [5]).

Remark. It is a somewhat surprising fact that the bi-invariant metric of $Sp(2)$ (whose norm in the Lie algebra has the expression $\|(p, r, s)\|^2_{\text{biinv}} = |p|^2 + 2|w|^2 + |s|^2$) does *not* descend to the round metric of S^7.

Now let us explain how we use this metric to find the maps e and E alluded to in the previous section. Let us establish some notation:

$$\begin{aligned} S^7 &= \{(a,b) \in \mathbb{H} \times \mathbb{H} \ : \ |a|^2 + |b|^2 = 1\}, \\ T_{(1,0)}S^7 &= \{(p,w) \in \mathbb{H} \times \mathbb{H} \ : \ \mathrm{Re}(p) = 0\}, \\ D^7 &= \{(p,w) \in T_{(1,0)}S^7 \ : \ |p|^2 + |w|^2 \leq 1\}, \\ S^6 &= \{(p,w) \in T_{(1,0)}S^7 \ : \ |p|^2 + |w|^2 = 1\}. \end{aligned}$$

Thus whenever we refer to S^6 or D^7 we are thinking of it as contained in the tangent space at $(1,0)$ of $S^7 \subset \mathbb{H} \times \mathbb{H}$.

The map $e : (D^7, S^6) \to (S^7, s)$ is simply the exponential map from $T_{(1,0)}S^7$, $e(p,w) = exp(\pi(p,w))$, i.e.

$$e(p,w) = \cos(\pi|(p,w)|) + \sin(\pi|(p,w)|)\frac{(p,w)}{|(p,w)|}.$$

Now we arrive at the main construction, which is the lift of the map e. We will lift e by the exponential map in $(Sp(2), g)$ of horizontal geodesics through the identity. Let $\iota : D^7 \to sp(2)$ be given by

$$\iota(p,w) = \begin{pmatrix} p & -\bar{w} \\ w & 0 \end{pmatrix},$$

that is, ι is the horizontal lift $\iota : T_{(1,0)}S^7 \to T_{Id}Sp(2) = sp(2)$, and $\iota \circ \rho_* = Id_{T_nS^7}$.

By the general theory of Riemannian submersions, we have that

$$\rho \exp_{(Sp(2),g)}(\pi\iota(p,w)) = \exp_{(S^7,can)}(\pi(p,w)).$$

Therefore if we define $E(p,w) = \exp_{(Sp(2),g)}(\pi\iota(p,w))$, it follows that E lifts e. See Figure 1 for a picture of what is happening.

The fundamental calculation is to express the horizontal geodesics through the identity of $(Sp(2), g)$. This is done in section 4 of [5], from where the following formula is taken:

Theorem 1. *The horizontal geodesics through the identity of $(Sp(2), g)$ have the following expression: let $\gamma : \mathbb{R} \to Sp(2)$ be a unit horizontal geodesic,*

$$\gamma(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma'(0) = \begin{pmatrix} p & -\bar{w} \\ w & 0 \end{pmatrix},$$

where p is a pure quaternion and $|p|^2 + |w|^2 = 1$. Then

$$\gamma(t) = \begin{pmatrix} \cos(t) + \sin(t)p & -\sin(t)e^{tp}\bar{w} \\ \sin(t)w & \frac{w}{|w|}(\cos(t) - \sin(t)p)e^{tp}\frac{\bar{w}}{|w|} \end{pmatrix}, \tag{1}$$

in the "generic" case $w \neq 0$. In the case $w = 0$,

$$\gamma(t) = \begin{pmatrix} \cos(t) + \sin(t)p & 0 \\ 0 & (\cos(t) - \sin(t)p)e^{tp} \end{pmatrix}.$$

Note that the expression of the geodesics is broken into the two cases $w = 0$ and $w \neq 0$, but the differentiability of this expression is guaranteed since it is a formula for an *a priori* smooth map, the exponential map of $(Sp(2), g)$.

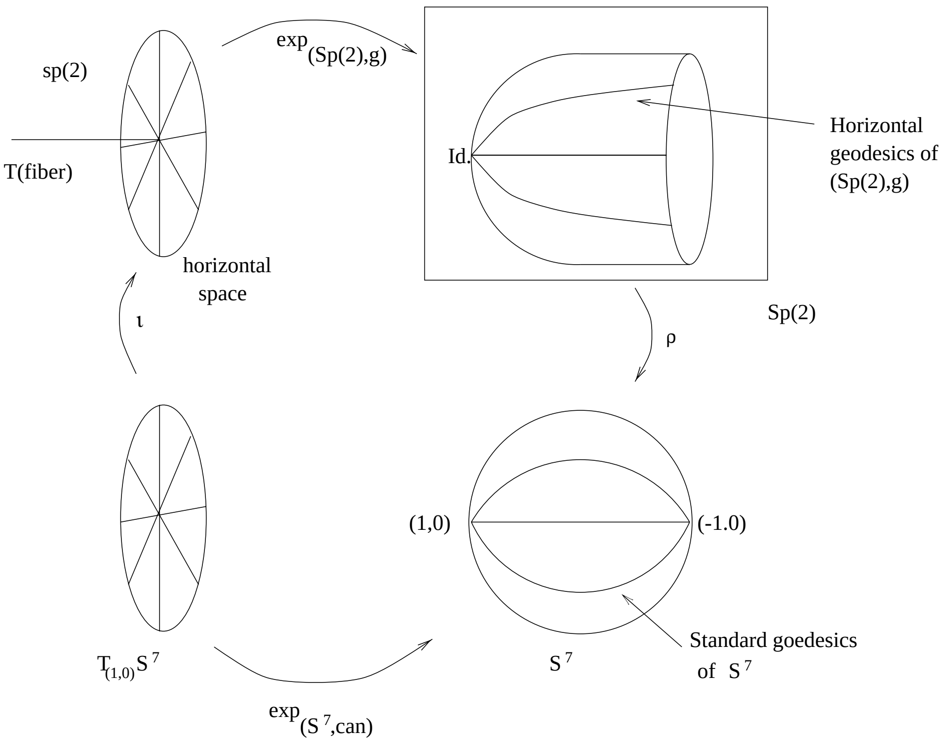

FIGURE 1.

We are interested in the values of E restricted to S^6. Setting $t = \pi$ in the previous formula gives us, when $|(p,w)| = 1$,

$$E(p,w) = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & -\frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|} \end{pmatrix}, & w \neq 0, \\ \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & w = 0. \end{cases}$$

Recall that since $|(p,w)| = 1$, $w = 0$ implies $|p| = 1$ and therefore $e^{\pi p} = -1$.

Note that the $(2,2)$ component of this matrix spans the fiber over the south pole $s = (-1,0)$ of S^7. The minus sign in front of the $(2,2)$ entry is irrelevant for our present intentions since multiplication by -1 in S^3 induces an isomorphism in homotopy. Again, recall that we know *a priori* that b is smooth. Then we have arrived at our first main theorem:

Theorem I. *Let* $S^6 = \{(p,w) \in \mathbb{H} \times \mathbb{H} \ : \ \mathrm{Re}(p) = 0, |p|^2 + |w|^2 = 1\}$. *The map* $b : S^6 \to S^3$ *given by*

$$b(p,w) = \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, & w \neq 0, \\ -1, & w = 0, \end{cases}$$

is a real analytic map that generates $\pi_6(S^3) \cong \mathbb{Z}_{12}$, *and therefore represents the Blakers-Massey element.*

2.3. **Analysis.** Once the geometry provides us with the map, one can also proceed to prove directly that b is smooth and represents the Blakers-Massey element without going through the geometry of the geodesics in $Sp(2)$: this will be done in this section. This direct approach has two advantages: first, it immediately generalizes to a generator of $\pi_{14}(S^7)$ by replacing "quaternions" with "octonions" everywhere. Also the analysis of the formula helps to clarify the geometry of the map itself (as opposed to the geometry *leading to* the map).

2.3.1. *Smoothness of b.*

Proposition 1. *The map b is smooth.*

Proof. The only problem is at $w = 0$. The proof is completely elementary, but we include it because we want to remark on a general heuristic principle which appears repeatedly in this work and might be useful in similar cases: "the commutator is faster than division by zero". In the expression

$$\frac{w}{|w|}e^{\pi p}\frac{\bar{w}}{|w|},$$

if w commutes with $e^{\pi p}$, then $b(p,w) = e^{\pi p}$ and the dependence on w disappears. As $w \to 0$ the quotient $w/|w|$ loses meaning. But since $|p|^2 + |w|^2 = 1$, as $w \to 0$ $|p| \to 1$, and therefore $e^{\pi p} \to -1$. Then $e^{\pi p}$ "almost commutes" with $w/|w|$, and the indeterminacy is avoided. This kind of behavior seems to be behind the smooth expressions of twisted phenomena.

Let us give the computational proof:

Spelling out $e^{\pi p} = \cos(\pi|p|) + \sin(\pi|p|)\frac{p}{|p|}$, we have

$$\begin{aligned} b(p,w) &= \cos(\pi|p|) + \sin(\pi|p|)\frac{wp\bar{w}}{|p||w|^2} \\ &= \cos(\pi|p|) + \sin(\pi(1-|p|))\frac{wp\bar{w}}{|p|(1-|p|)(1+|p|)} \\ &= \cos(\pi|p|) + \left[\frac{\sin(\pi(1-|p|))}{1-|p|}\right]\frac{wp\bar{w}}{|p|(1+|p|)}. \end{aligned}$$

The term between square brackets is analytic in a neighborhood of $w = 0$, which takes care of the smoothness issue at points where $w = 0$. □

Remark. Note that if p and w are assumed to be octonions instead of quaternions, the proof works without change. There is no problem with associativity since the algebra generated by two octonions is associative.

2.3.2. *The map b generates $\pi_6(S^3)$.* We now prove that the element b generates $\pi_6(S^3)$. The strategy is to show that b is homotopic to a map which factors through the "classical" Blakers-Massey element.

Let $B(p,w) = b(p,w)e^{-\pi p} = [w/|w|, e^{\pi p}]$. Then B is homotopic to b, the homotopy being given by $H(t,(p,w)) = b(p,w)e^{t\pi p}$.

Let us consider the commutative diagram

$$\begin{array}{ccccc} S^6_0 & \xrightarrow{\alpha} & S^3 \times S^3 & \xrightarrow{\rho} & S^3 \wedge S^3 \cong S^6 \\ & & \mathcal{M} \searrow & & \swarrow m \\ & & & S^3 & \end{array},$$

where $\alpha(p,w) = (w/|w|, -e^{\pi p})$ and ρ is the projection to the quotient. Then $B = \mathcal{M}\alpha = m\rho\alpha$. The map α is, of course, *not* continuous as a map $S^6 \to S^3 \times S^3$, not being defined at points where $w = 0$. However, we claim

Proposition 2. *The composition* $\rho\alpha : S^6 - \{w = 0\}$ *can be extended to a continuous map* $\phi : S^6 \to S^6$. *Moreover,* ϕ *induces an isomorphism in the homotopy group* $\pi_6(S^6)$.

Proof. Extend ϕ to all of S^6 by defining $\phi(p, 0)$ to be the distinguished point of $S^3 \wedge S^3$. We will show continuity via an explicit identification of the quotient $S^n \wedge S^n$ with S^{2n} (see also section 23 of [15]):

Denote by $\mathcal{S}_n$ the stereographic projection of $S^n - \{N\}$ into $\mathbb{R}^n$, where N is the north pole $(1, 0, \ldots, 0)$ (note that in the case of $n = 0, 1, 3, 7$, the unit spheres of division algebras N is the unit element). Consider the map $F : S^n \times S^n \to S^{2n}$ given by

$$F(x,y) = \begin{cases} \mathcal{S}_{2n}^{-1}(\mathcal{S}_n(x), \mathcal{S}_n(y)), & x \neq 1, y \neq 1, \\ N, & x = 1 \text{ or } y = 1\,. \end{cases}$$

If we denote an arbitrary point in $x \in S^n$ by (x_0, x_1), where $x_0 \in \mathbb{R}$, $x_1 \in \mathbb{R}^n$, the expresion for F is given by

$$F(x,y) = \left(1 - 2\left(1 + \frac{|x_1|^2}{(1-x_0)^2} + \frac{|y_1|^2}{(1-y_0)^2}\right)^{-1}, \right.$$
$$\frac{2\,x_1}{1-x_0}\left(1 + \frac{|x_1|^2}{(1-x_0)^2} + \frac{|y_1|^2}{(1-y_0)^2}\right)^{-1},$$
$$\left.\frac{2\,y_1}{1-y_0}\left(1 + \frac{|x_1|^2}{(1-x_0)^2} + \frac{|y_1|^2}{(1-y_0)^2}\right)^{-1}\right).$$

Clearly F factors through a homeomorphism $f : S^n \wedge S^n \to S^{2n}$, and the projection map $\rho : S^n \times S^n \to S^n \wedge S^n \cong S^{2n}$ can be substituted everywhere by $F : S^n \times S^n \to S^{2n}$.

We now specialize in the case at hand, where $n = 3$. Showing that $\rho\alpha$ is continuous is the same thing, by the comments above, as showing that $F\alpha$ is continuous, and the only problem is at the points where $w = 0$. Computing, we have

$$F \circ \alpha(p,w) = \left(1 - \frac{2\,(|w| - w_0)^2}{w_1^2 + (|w| - w_0)^2 \sec(\pi\,|p|/2)^2}, \right.$$
$$\left. \frac{2\,(|w| - w_0)\,w_1}{w_1^2 + (|w| - w_0)^2 \sec(\pi\,|p|/2)^2}, \frac{-2\,(|w| - w_0)^2\, \tan(\pi\,|p|/2)\,p}{\left(w_1^2 + (|w| - w_0)^2 \sec(\pi\,|p|/2)^2\right)\,|p|}\right).$$

The first ("height") component of $F\alpha$ can be written as

$$(F \circ \alpha)_0(p,w) = 1 - 2\left(\frac{w_1^2}{(|w| - w_0)^2} + \sec^2(\pi|p|/2)\right)^{-1}.$$

Clearly, as $w \to 0$, $|p| \to 1$ and $(F\alpha)_0(p,w) \to 1$. Since $|F \circ \alpha(p,w)| \equiv 1$, this means that the other components tend to zero as $w \to 0$, which proves by continuity that $\lim_{w\to 0} F \circ \alpha(p,w) = (1, 0, \ldots, 0) = F \circ \alpha(p, 0)$.

Let us now prove the second part of the proposition. A moment's reflection shows that ϕ is onto. Also, ϕ is almost one-to-one: ϕ is injective in the complement of the set $\mathcal{N} = \{w \in \mathbb{R},\ w \geq 0\}$; the set $\mathcal{N}$ is the inverse image of the north pole of S^6 (which corresponds to the distinguished point of $S^3 \wedge S^3$). Let $X = S^6/\mathcal{N}$ be the space obtained from S^6 by identifying $\mathcal{N}$ to a point, and let $q : S^6 \to X$ be the projection onto the quotient. Then ϕ factors through q, that is, there is $\tilde{\phi} : X \to S^6$ such that $\phi = \tilde{\phi} q$. By definition, $\tilde{\phi}$ is continuous, one-to-one, and onto, and therefore a homeomorphism.

Since $\mathcal{N}$ is a disk and a neighbourhood deformation retract, the homology exact sequence of the pair $(S^6, \mathcal{N})$ shows that $q_* : H_6(S^6) \to H_6(S^6, \mathcal{N}) \cong H_6(X)$ is an isomorphism. Thus $\phi_* = \tilde{\phi}_* q_*$ is also an isomorphism in, and ϕ has degree one. Therefore ϕ induces an isomorphism in $\pi_6(S^6)$. □

Since m is known to generate $\pi_6(S^3)$, then it follows that b is also a generator of $\pi_6(S^3)$.

Again, this argument goes through *verbatim* if we substitute quaternions everywhere by octonions. The commutator map M and its factoring m are known to generate $\pi_{14}(S^7)$ ([17] and §9 of [10]). Therefore we have

Theorem II. *Let* $S^{14} = \{(p, w) \in \mathbb{O} \times \mathbb{O} \ :\ \mathrm{Re}(p) = 0, |p|^2 + |w|^2 = 1\}$. *The map* $b : S^{14} \to S^7$ *given by*

$$b(p, w) = \begin{cases} \frac{\bar{w}}{|w|} e^{\pi p} \frac{w}{|w|}, & w \neq 0, \\ -1, & w = 0, \end{cases}$$

is a real analytic map that generates $\pi_{14}(S^7) \cong \mathbb{Z}_{120}$.

3. Exotic spheres

In this section we study exotic spheres in dimensions 7 and 15. In the 7-dimensional case, there are three manifolds involved:

- $M^7_{2,-1}$, the original exotic sphere of Milnor expressed as an S^3 bundle over S^4 with structure group $SO(4)$. This is the join $\mathbb{R}^4 \times S^3 \cup_T \mathbb{R}^4 \times S^3$, where $T : \mathbb{R}^4 - \{0\} \times S^3 \to \mathbb{R}^4 - \{0\} \times S^3$ is given by $T(\xi, k) = (\xi/|\xi|^2, \frac{\xi^2}{|\xi|^2} \cdot k \cdot \frac{\bar{\xi}}{|\xi|})$. The $\mathbb{R}^4$ factors are stereographic coordinates for the sphere S^4; let us remark that in section 3.3 we shall do the relevant constructions directly on S^4.
- The Gromoll-Meyer sphere M^7_{G-M}, the quotient of $Sp(2)$ by the action

$$q \bigstar \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} qa\bar{q} & qc \\ qb\bar{q} & qd \end{pmatrix}.$$

- M^7_σ, the join of two disks by the diffeomorphism σ defined in Theorem III with quaternions everywhere.

The fact that $M^7_{2,-1} \cong M^7_{G-M}$ is Theorem I of [7], and that $M^7_{G-M} \cong M^7_\sigma$ is the subject of [5] and is summarized in section 3.1. Of course, $M^7_{2,-1} \cong M^7_\sigma$ follows by transitivity.

3.1. **Geometry of geodesics and exotic spheres.** Exotic spheres M^n are constructed by gluing two disks D^n by their boundaries, using a diffeomorphism $\sigma : S^{n-1} \to S^{n-1}$ that is not differentiably isotopic to the identity ([12, 13]).

In [5], such an exotic diffeomorphism $\sigma : S^6 \to S^6$ is constructed and a formula for σ is given; after substituting the Blakers-Massey element

$$b(p, w) = \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, & w \neq 0\,, \\ -1, & w = 0\,, \end{cases}$$

in Theorem II of [5], we get

Theorem. *Let* $S^6 = \{(p, w) \in \mathbb{H} \times \mathbb{H} \, : \, |p|^2 + |w|^2 = 1, \mathrm{Re}(p) = 0\}$*, and let* $\sigma : S^6 \to S^6$ *be given by*

$$\sigma(p, w) = (\overline{b(p, w)}\, p\, b(p, w)\,,\, \overline{b(p, w)}\, w\, b(p, w))\,.$$

Then σ *is a diffeomorphism,* $M^7_{2,-1} = D^7 \cup_\sigma D^7$*, and* σ *is not isotopic to the identity.*

Let us summarize the method employed in [5] to construct σ, which also uses the geometry of $Sp(2)$ provided with the left-invariant metric g. This metric is invariant under the Gromoll-Meyer action of [7], and therefore descends to a metric on $M^7_{2,-1}$. This metric has the *Wiedersehen property*: there is a "north pole" and a "south pole", and each geodesic emanating from the north pole reaches the south pole at time π. The Wiedersehen property implies that the exponential map from each pole is a diffeomorphism up to length π, and the exponential balls from each pole cover $M^7_{2,1}$ diffeomorphically. Then $M^7_{2,-1}$ is expressed as the union of the (embedded) images of the exponential maps from the unit disks of two antipodal points; see Figure 2, where ρ_+ and ρ_- are fixed identifications of $\mathbb{R}^7$ with the tangent spaces of $M^7_{2,-1}$ at the north and south poles, respectively.

The explicit formulas for the relevant geodesics turn into the formula for the gluing map σ. A chart that covers M^7_σ minus a point is given by

$$(t, p, w) \mapsto \left[\begin{pmatrix} \cos(t) + \sin(t) p & -\sin(t) e^{tp} \bar{w} \\ \sin(t) w & \frac{w}{|w|}(\cos(t) - \sin(t) p) e^{tp} \frac{\bar{w}}{|w|} \end{pmatrix} \right]\,,$$

since this is the formula for geodesics from the north pole $=[Id]$. The change of coordinates is just describing how the same point looks from the south pole by following the geodesics; it is given by

$$(t_1, p_1, w_1) = (\pi - t, \sigma(-p, -w)) = (\pi - t, -\overline{b(p, w)}\, p\, b(p, w), -\overline{b(p, w)}\, w\, b(p, w))\,.$$

See [5] for details.

3.2. **The Blakers-Massey element and the exotic diffeomorphism σ.** It is remarkable how the Blakers-Massey element appears in the formula above and interacts with the exotic diffeomorphism σ. This relationship between the Blakers-Massey element and the exotic diffeomorphism σ greatly simplifies some formulas.

Proposition 3. *The Blakers-Massey element* $b : S^6 \to S^3$ *is* σ*-invariant.*

Proof. If $w = 0$, we have that $\sigma(p, 0) = (p, 0)$, and there is nothing to prove.

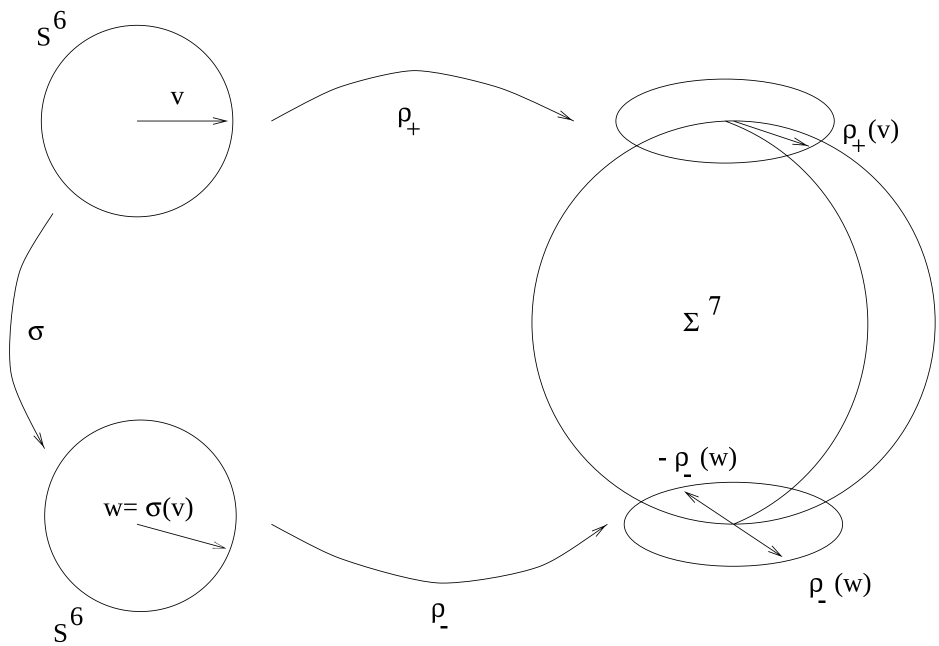

FIGURE 2.

If $w \neq 0$, set $(p_1, w_1) = \sigma(p, w)$. Note that $|p_1| = |p|$, $|w_1| = |w|$. Then

$$\begin{aligned} b(p_1, w_1) &= \frac{w_1}{|w_1|} e^{\pi p_1} \frac{\bar{w}_1}{|w_1|} \\ &= \frac{\overline{b(p,w)}\, w\, b(p,w)}{|w|} e^{\pi \overline{b(p,w)}\, p\, b(p,w)} \frac{\overline{b(p,w)}\, \bar{w}\, b(p,w)}{|w|} \\ &= \overline{b(p,w)} b(p,w)^2 = b(p,w)\,. \end{aligned}$$

□

Let us compute the iterates of σ. Again, the case $w = 0$ is trivial, and set $(p_1, w_1) = \sigma(p, w)$.

$$\begin{aligned} \sigma^2(p,w) &= \sigma(p_1, w_1) = (\overline{b(p_1,w_1)}\, p_1\, b(p_1, w_1), \overline{b(p_1,w_1)}\, w_1\, b(p_1, w_1)) \\ &= (\overline{b(p,w)}\, p_1\, b(p,w), \overline{b(p,w)}\, w_1\, b(p, w1)) \text{ since } b \text{ is } \sigma\text{-invariant} \\ &= (\overline{b(p,w)^2}\, p\ b(p,w)^2, \overline{b(p,w)^2}\, w\, b(p,w_1)^2). \end{aligned}$$

Therefore, by induction we have

Proposition 4. *The iterates of σ are given by powers of the Blakers-Massey element b, that is,*

$$\sigma^n(p,w) = (\overline{b(p,w)^n}\, p\, b(p,w)^n, \overline{b(p,w)^n}\, w\, b(p,w)^n)\,.$$

Remarks. (1) Note that the same argument shows that the inverse of σ is given by the group inverse of b, that is,

$$\sigma^{-1}(p,w) = (b(p,w)\, p\, \overline{b(p,w)}, b(p,w)\, w\, \overline{b(p,w)})\,.$$

(2) Also, we have

$$b(p,w)^n = \frac{w}{|w|} e^{n\pi p} \frac{\bar{w}}{|w|}\,.$$

Using the "little cubes" method we can observe that the map

$$\pi_0 Diff(S^6) \ni \sigma \longmapsto \Sigma_\sigma := D^7 \cup_\sigma D^7 \in \Gamma_7$$

is a group isomorphism, where Γ_n denotes the group of homotopy n-spheres with the connected sum operation. So, the powers of b define formulas for exotic diffeomorphisms of S^6 in each π_0-class, i.e. in each element of the group Z_{28}. The homotopy classes of b^{12} and b^{24} are trivial, since $\pi_6 S^3 = Z_{12}$, but the corresponding σ^{12} and σ^{24} are not trivial in $\pi_0 Diff(S^6)$. An immediate question is: Does there exist a generator τ of $\pi_0 Diff(S^6)$ resulting from a homotopically trivial element of $\pi_6 S^3$? If not, one has essentially two elements of $\pi_6 S^3$ to consider, b and b^5, and there may be a relation with the non-cancellation phenomenon of P. Hilton, J. Roitberg, et al. ([1, 8]).

Let us also note that σ is a composition of a diagonal inclusion of $SO(3)$ in $SO(7)$, i.e. let $\eta : S^6 \to SO(3)$ be given by $\eta = \mathrm{proj} \circ b$, where $\mathrm{proj} : S^3 \to SO(3)$ is the quotient map by $\mathbb{Z}_2$. Then η generates $\pi_6(SO(3))$, and

$$\sigma(p, r, \Omega) = (\eta(p, r+\Omega)p,\ r,\ \eta(p, r+\Omega)\Omega)\,,$$

where $r \in \mathbb{R}, \mathrm{Re}(p) = \mathrm{Re}(\Omega) = 0$ and $|p|^2 + r^2 + |\Omega|^2 = 1$ (see also section 3.4).

Remark. Every algebraic computation done in this subsection is immediately applicable to Cayley octonions, since only p, w and their conjugates appear. The only thing not immediately true is the relationship of σ to exotic diffeomorphisms of S^{14} and therefore to exotic 15-spheres. This we do in the next section.

3.3. **The octonionic case.** This section is devoted to the proof of Theorem III.

In the 15-dimensional case, the Gromoll-Meyer construction does not exist. What we do in order to prove $M^{15}_{2,-1} \cong M^{15}_\sigma$ is to inspire ourselves in the formulas appearing in the transitivity $M^7_{2,-1} \cong M^7_{G-M} \cong M^7_\sigma$ and then "eliminate the middleman". This will work because in the formulas for σ, only two quaternions and their conjugates occur.

We begin by explaining the Gromoll-Meyer construction in somewhat more detail than in [7]:

Theorem 2 (Gromoll and Meyer, 1974)**.** *The quotient M^7_{G-M} of the group $Sp(2)$ by the $\bigstar$-action of S^3 above is diffeomorphic to the Milnor exotic sphere $M^7_{2,-1}$.*

Proof. This is proven in [7], but an argument with a somewhat different point of view is helpful in order to understand the proof of Theorem III.

Let us begin with partial trivializations of the bundle $S^3 \to M^7_{G-M} \to S^4$. The projection $M^7_{G-M} \to S^4$ is given by

$$M^7_{G-M} \ni \left[\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right] \mapsto (|d|^2 - |c|^2, -2\bar{d}c)\,.$$

We write S^4 as pairs (z, ρ), where $z \in [-1, 1]$, $\rho \in D^4$, and $z^2 + |\rho|^2 = 1$. Thus z is the height of the sphere S^4, and we think of D^4 as the unit ball of the quaternions.

Define a map $\psi : (S^4 -$ north pole$) \times S^3 \to M^7_{G-M}$ as follows: Let $(x, \xi, k) \in [-1, 1) \times D^4 \times S^3$ with $x^2 + |\xi|^2 = 1$. Note that $x \neq 1$.

$$\psi(x, \xi, k) = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} \frac{\xi k}{\sqrt{1-x}} & -\sqrt{1-x} \\ \sqrt{1-x} \cdot k & \frac{\overline{\xi}}{\sqrt{1-x}} \end{pmatrix} \right].$$

Similarly, define $\phi : (S^4 -$ south pole$) \times S^3 \to M^7_{G-M}$; for (y, ζ, h) with $y \neq -1$ we have

$$\phi(y, \zeta, h) = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} \sqrt{1+y} \cdot h & -\frac{1}{\sqrt{1+y}}\zeta \\ \frac{1}{\sqrt{1+y}}\overline{\zeta} h & \sqrt{1+y} \end{pmatrix} \right].$$

These maps are diffeomorphisms onto their images, but we shall not prove that here.

Set these two classes equal in M^7_{G-M}: there is a unit $s \in S^3$ such that

$$\begin{pmatrix} \frac{s\xi k\overline{s}}{\sqrt{1-x}} & -\sqrt{1-x}s \\ \sqrt{1-x} \cdot sk\overline{s} & \frac{s\overline{\xi}}{\sqrt{1-x}} \end{pmatrix} = \begin{pmatrix} \sqrt{1+y} \cdot h & -\frac{1}{\sqrt{1+y}}\zeta \\ \frac{1}{\sqrt{1+y}}\overline{\zeta} h & \sqrt{1+y} \end{pmatrix}.$$

The bottom right position says $s\overline{\xi} = \sqrt{(1-x)(1+y)}$, but $|\xi| = \sqrt{1-x^2}$, therefore $\sqrt{1-x^2} = \sqrt{(1-x)(1+y)}$ and so $x = y$ and $s = \frac{\xi}{|\xi|}$.

The upper right position says $s = \frac{\zeta}{\sqrt{(1-x)(1+y)}}$ and from $|\zeta| = \sqrt{1-y^2}$ and $x = y$ it follows that $s = \frac{\zeta}{|\zeta|}$ as well, so $\xi = \zeta$.

Now both positions in the left column imply the same relation

$$h = \frac{\xi^2}{|\xi|^2} \cdot k \cdot \frac{\overline{\xi}}{|\xi|}$$

and thus we have recovered the clutching function T for the associated bundle $S^3 \to M^7_{2,-1} \to S^4$, schematically, $D^4 \times S^3 \cup_T D^4 \times S^3$, with $T(x, \xi, k) = (x, \xi, \frac{\xi^2}{|\xi|^2} \cdot k \cdot \frac{\overline{\xi}}{|\xi|})$. □

Note that from the full set of classes

$$\left[\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right] \in M^7_{2,-1},$$

the ψ-trivialization covers the set such that $c \neq 0$ (and therefore $b \neq 0$), while the ϕ-trivialization covers the set of classes where $a \neq 0$, $d \neq 0$.

We remark that above we have used the associativity of the quaternions in an essential manner. Let us now prove that $M^7_{2,-1}$ is diffeomorphic to M^7_σ. First let us deduce the right formulas using the "middleman" M^7_{G-M}.

So, let

$$\begin{aligned} \psi(x, \xi, k) &= \frac{1}{\sqrt{2}} \left[\begin{pmatrix} \frac{s\xi k\overline{s}}{\sqrt{1-x}} & -\sqrt{1-x}s \\ \sqrt{1-x} \cdot sk\overline{s} & \frac{s\overline{\xi}}{\sqrt{1-x}} \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} \cos(t) + \sin(t)p & -\sin(t)e^{tp}\overline{w} \\ \sin(t)w & \frac{w}{|w|} \left|\cos(t) - \sin(t)p\right| e^{(t-\tau)p} \frac{\overline{w}}{|w|} \end{pmatrix} \right], \end{aligned}$$

where we set $e^{\tau p} = \frac{\cos(t)+\sin(t)p}{|\cos(t)-\sin(t)p|}$. Note that $|\cos(t) - \sin(t)p| = |\cos(t) + \sin(t)p|$.

The modulus of the bottom left position says that $\frac{1}{\sqrt{2}}\sqrt{1-x} = \sin(t)\,|w|$, so $x = 1 - 2\sin^2(t)\,|w|^2$.

The upper right position says $s = e^{tp}\frac{\overline{w}}{|w|}$.

The bottom left position says $k = \frac{w}{|w|}e^{-tp}\frac{w}{|w|}e^{tp}\frac{\overline{w}}{|w|}$.

Put these in the upper left position and observe that

$$|\xi| = \sqrt{2}\sqrt{1-x} \cdot |\cos(t) - \sin(t)p| = 2\sin(t)\,|w|\,|\cos(t) + \sin(t)p|\,,$$

so we get $e^{\tau p}\,|\xi| = e^{tp}\frac{\overline{w}}{|w|}\xi\frac{w}{|w|}e^{-tp}\frac{w}{|w|}$, which implies $\xi = |\xi|\,\frac{w}{|w|}e^{(\tau-t)p}\frac{\overline{w}}{|w|}e^{tp}\frac{\overline{w}}{|w|}$, the same relation that comes out of the bottom right.

Putting it altogether,

$$\begin{pmatrix} x \\ \xi \\ k \end{pmatrix} = \begin{pmatrix} 1 - 2\sin^2(t)\,|w|^2 \\ 2\sin(t)\frac{w}{|w|}(\cos(t) + \sin(t)p)e^{-tp}\overline{w}e^{tp}\frac{\overline{w}}{|w|} \\ \frac{w}{|w|}e^{-tp}\frac{w}{|w|}e^{tp}\frac{\overline{w}}{|w|} \end{pmatrix} \tag{2}$$

or

$$\begin{pmatrix} x \\ \xi \\ k \end{pmatrix} = \begin{pmatrix} 1 - 2\sin^2(t)\,|w|^2 \\ |\xi|\,\frac{w}{|w|}e^{(\tau-t)p}\frac{\overline{w}}{|w|}e^{tp}\frac{\overline{w}}{|w|} \\ \frac{w}{|w|}e^{-tp}\frac{w}{|w|}e^{tp}\frac{\overline{w}}{|w|} \end{pmatrix},$$

which is a useful form since $|\xi|$ is invariant in many instances.

Proceeding in a similar fashion with the other section, where $y \neq -1$,

$$\begin{aligned} \phi(y, \zeta, h) &= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} \sqrt{1+y} \cdot rh\overline{r} & -\frac{1}{\sqrt{1+y}}r\zeta \\ \frac{1}{\sqrt{1+y}}r\overline{\zeta}h\overline{r} & \sqrt{1+y}r \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} \cos(t) + \sin(t)p & -\sin(t)e^{tp}\overline{w} \\ \sin(t)w & \frac{w}{|w|}\,|\cos(t) - \sin(t)p|\,e^{(t-s)p}\frac{\overline{w}}{|w|} \end{pmatrix}\right]. \end{aligned}$$

We get $r = \frac{w}{|w|}e^{(t-\tau)p}\frac{\bar{w}}{|w|}$, and

$$\begin{pmatrix} y \\ \zeta \\ h \end{pmatrix} = \begin{pmatrix} 1 - 2\sin^2(t)\,|w|^2 \\ 2\sin(t)\frac{w}{|w|}(\cos(t) + \sin(t)p)e^{-tp}\overline{w}e^{tp}\frac{\overline{w}}{|w|} \\ \frac{w}{|w|}e^{(\tau-t)p}\frac{\overline{w}}{|w|}e^{\tau p}\frac{w}{|w|}e^{(t-\tau)p}\frac{\overline{w}}{|w|} \end{pmatrix} \tag{3}$$

or

$$\begin{pmatrix} y \\ \zeta \\ h \end{pmatrix} = \begin{pmatrix} 1 - 2\sin^2(t)\,|w|^2 \\ |\zeta|\,\frac{w}{|w|}e^{(\tau-t)p}\frac{\overline{w}}{|w|}e^{tp}\frac{\overline{w}}{|w|} \\ \frac{w}{|w|}e^{(\tau-t)p}\frac{\overline{w}}{|w|}e^{\tau p}\frac{w}{|w|}e^{(t-\tau)p}\frac{\overline{w}}{|w|} \end{pmatrix}.$$

When $y = x$ we have that $|\zeta| = |\xi|$ and therefore when $(x, \xi) = (y, \zeta)$ we have again that

$$h = \frac{\xi^2}{|\xi|^2} \cdot k \cdot \frac{\overline{\xi}}{|\xi|}\,, \tag{4}$$

as follows easily from the above formulas.

Note now that there are just two quaternions involved in the multiplications, so the associativity is guaranteed even when we replace quaternions by Cayley numbers. With this in mind, we begin the proof of Theorem III in earnest; from now on p and w will be octonions, and we drop the dimension superscript since we are working only in the 15-dimensional case. We shall construct a map $F : M_\sigma \to M_{2,-1}$ in the old fashioned way, that is, by describing the maps in local charts. We prove consistency (that the map does not depend on the charts chosen) and differentiability. Note that the coordinates on the open balls defining M_σ are "polar", i.e. (t, p, w), where $t \in [0, \pi)$ and $|p|^2 + |w|^2 = 1$ (thus the balls that produce M_σ have radius π). This improves readability at the cost of having to give a separate argument for the differentiability at the center of the balls.

Thus define

$$\begin{aligned} f_1(t,p,w) &= \begin{pmatrix} 1 - 2\sin^2(t)\,|w|^2 \\ 2\sin(t)\frac{w}{|w|}(\cos(t) + \sin(t)p)e^{-tp}\overline{w}e^{tp}\frac{\overline{w}}{|w|} \\ \frac{w}{|w|}e^{-tp}\frac{w}{|w|}e^{tp}\frac{\overline{w}}{|w|} \end{pmatrix} = \begin{pmatrix} x \\ \xi \\ k \end{pmatrix} \\ &= \Psi(m),\ m \in M_{2,-1}\,, \\ f_2(t,p,w) &= \begin{pmatrix} 1 - 2\sin^2(t)\,|w|^2 \\ 2\sin(t)\frac{w}{|w|}(\cos(t) + \sin(t)p)e^{-tp}\overline{w}e^{tp}\frac{\overline{w}}{|w|} \\ \frac{w}{|w|}e^{(\tau-t)p}\frac{\overline{w}}{|w|}e^{\tau p}\frac{w}{|w|}e^{(t-\tau)p}\frac{\overline{w}}{|w|} \end{pmatrix} = \begin{pmatrix} y \\ \zeta \\ h \end{pmatrix} \\ &= \Phi(m),\ m \in M_{2,-1}\,. \end{aligned}$$

The maps Ψ and Φ are the 15-dimensional analogs of the trivializations ψ and ϕ defined in the proof of Theorem 2; actually the formulas there greatly help in getting a grasp on Ψ and Φ (e.g. as in the proof of Assertion 2 below). However, since there is no biquotient expression for the 15-dimensional exotic sphere $\Sigma^{15}_{2,-1}$, there are no nice explicit formulae for them. Thus Ψ and Φ are basically markers of the two factors of the identification space $M_{2,-1} = \mathbb{R}^8 \times S^7 \cup_T \mathbb{R}^8 \times S^7$.

The maps f_1, f_2 in principle have as domains the full ball $\mathcal{B}(0,\pi)$, which translates to M_σ minus the south pole, but we must delete from the domain of f_1 (resp. f_2) the points that fail to be in the image of the trivialization Ψ (resp. Φ).

Thus, we have that the domain of f_1 is $\mathcal{B}(0,\pi)$ minus the origin and $w = 0$, which is an embedded $\mathbb{R}^7$ in the ball which produces an embedded 7-sphere $\mathcal{S}_1$ in M_σ after adding the south pole, and the domain of f_2 is $\mathcal{B}(0,\pi)$ minus the set $\{t = \pi/2,\ p = 0\}$, which is an embedded 7-sphere $\mathcal{S}_2$ in the ball and therefore in M_σ.

Thus, let us define $F : M_\sigma \to M_{2,-1}$ by

$$F(\theta) = \begin{cases} \Psi^{-1} \circ f_1(t,p,w), & \theta \notin \mathcal{S}_1, \\ \Phi^{-1} \circ f_2(t,p,w), & \theta \notin \mathcal{S}_2 \text{ and } \theta \neq \text{ south pole}, \\ \Phi^{-1}(1,0,-1), & \theta = \text{ south pole}\,, \end{cases}$$

where $(t,p,w) \in \mathcal{B}(0,\pi)$ correspond to $\theta \in M_\sigma$.

Then, the following implies Theorem III:

Theorem. *The map F defines a diffeomorphism from M_σ onto $M_{2,-1}$.*

The proof follows from the following assertions:

ASSERTION 1: *The map F is well defined.*

ASSERTION 2: *Each of the first two pieces defining F is a diffeomorphism onto its image.*

ASSERTION 3: *The map F is differentiable with a non-singular derivative at each pole.*

Before proving the assertions, let us remark that the innocent-looking Assertion 3 is in fact the crucial one, since it is easy to construct homeomorphisms from the standard sphere S^7 onto exotic spheres which fail to be diffeomorphisms just at the south pole. For example, using the Wiedersehen property of the metric on M^7_σ, we can construct a homeomorphism $H : S^7 \to M^7_\sigma$ defined by $H(\text{south pole of } S^7) =$

south pole of M^7_σ, and for $\theta \neq$ south pole, $H(\theta) = \exp_N(\iota(\exp_N^{-1}(\theta)))$, where N is the north pole on each sphere and ι is some isometry between $T_N S^7$ and $T_N M^7_\sigma$.

Proof of Assertion 1. We need to show that if (t, p, w) belongs to the intersection of the domains of f_1 and f_2, $\Psi^{-1} \circ f_1(t, p, w) = \Phi^{-1} \circ f_2(t, p, w)$. This amounts to showing that for the same (t, p, w), $x = y$, $\xi = \zeta$ and $h = \frac{\xi^2}{|\xi|^2} \cdot k \cdot \frac{\overline{\xi}}{|\xi|}$. But all this follows immediately from formulas 2, 3 and 4.

Proof of Assertion 2. This we do by constructing explicit inverses. Let us consider the case of $\Phi^{-1} \circ f_2$; the case for $\Psi^{-1} \circ f_1$ is similar and we leave it to the reader.

Let us use the aid of the middleman again:

$$\begin{aligned} \phi \begin{pmatrix} y \\ \zeta \\ h \end{pmatrix} &= \frac{1}{\sqrt{2}} \left[\begin{pmatrix} \sqrt{1+y} \cdot r h \bar{r} & -\frac{1}{\sqrt{1+y}} \cdot r\zeta \\ \frac{1}{\sqrt{1+y}} \cdot r\bar{\zeta} h \bar{r} & \sqrt{1+y} \cdot r \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} \cos t + \sin t \cdot p & -\sin t \cdot e^{tp} \cdot \overline{w} \\ \sin t \cdot w & \frac{w}{|w|} \cdot |\cos t - \sin t \cdot p| \cdot e^{(t-\tau)p} \cdot \frac{\overline{w}}{|w|} \end{pmatrix} \right], \end{aligned}$$

where we now must find (t, p, w) in terms of (y, ζ, h). The main difficulty lies in finding who r is in terms of (t, p, w).

From the (1,1) entry, we get $\cos(t) = \sqrt{\frac{1+y}{2}} \mathrm{Re}(h)$, this gives us t. In what follows we will freely use $\sin(t)$ and $\cos(t)$, knowing that they are known differentiable functions of (y, ζ, h) (as we just saw, actually t does not depend on ζ).

Also from the (1,1) entry, we get from the imaginary part: $p = S \cdot r \mathrm{Im}(h) \bar{r}$, where $S = S(y, h)$ is the scalar $\frac{\sqrt{1+y}}{\sqrt{2}\sin(t)}$. Note that since $y \neq -1$, the condition $t = \pi/2$ implies $\mathrm{Re}(h) = 0$, and $p = 0$ implies $\mathrm{Im}(h) = 0$ and thus $t = \pi/2$ and $p = 0$ implies $h = 0$, which is incompatible with $|h| = 1$. This reflects the fact that $t = \pi/2$ and $p = 0$ correspond to the 7-sphere $\mathcal{S}_2$ that is deleted from the domain in this case.

Continuing, we have

$$e^{tp} = r(e^{tS\mathrm{Im}(h)})\bar{r}\,.$$

The (1,2) entry says

$$\sqrt{2}\sin(t) e^{tp} \bar{w} = -\frac{1}{\sqrt{1+y}} r\zeta \;\Rightarrow \sqrt{2}\sin(t) w = \frac{-1}{\sqrt{1+y}} \bar{\zeta} e^{tS\mathrm{Im}(h)} \bar{r}\,.$$

Comparing this with the expression for $\sqrt{2}\sin(t)w$ given by the (2,1) entry, we get

$$\frac{1}{\sqrt{1+y}} r\bar{\zeta} h \bar{r} = \frac{-1}{\sqrt{1+y}} \bar{\zeta} e^{tS\mathrm{Im}(h)} \bar{r}\,,$$

from which it follows that

$$r = -\frac{\bar{\zeta}}{|\zeta|} e^{tS\mathrm{Im}(h)} \bar{h} \frac{\zeta}{|\zeta|}\,.$$

It then immediately follows that

$$\begin{aligned} t &= \arccos\left(\sqrt{\frac{1+y}{2}}\mathrm{Re}(h)\right) \\ p &= S(y,h)\frac{\bar{\zeta}}{|\zeta|}e^{tS\mathrm{Im}(h)}\bar{h}\frac{\zeta}{|\zeta|}\mathrm{Im}(h)\frac{\bar{\zeta}}{|\zeta|}he^{-tS\mathrm{Im}(h)}\frac{\zeta}{|\zeta|}\,. \\ w &= \frac{1}{\sin(t)\sqrt{1+y}}\bar{\zeta}e^{tS\mathrm{Im}(h)}\frac{\bar{\zeta}}{|\zeta|}he^{-tS\mathrm{Im}(h)}\frac{\zeta}{|\zeta|}\,. \end{aligned}$$

The only problematic places where this expression might fail to be differentiable are $\zeta = 0$ and $t = 0$. The case $\zeta = 0$ is taken care of by the "commutation is faster than division by zero" principle of Proposition 1: indeed, since $y^2 + |\zeta|^2 = 1$, $\zeta = 0 \Rightarrow y = 1$ (recall that $y = -1$ is not in the domain). Thus as $\zeta \to 0$ and $y \to 1$, and a moments' reflection shows that the expression $e^{tS\mathrm{Im}(h)} \to h$ and thus both $e^{tS\mathrm{Im}(h)}\bar{h}$ and $he^{-tS\mathrm{Im}(h)}$ converge to $|h|^2 = 1$, which commutes with the $\zeta/|\zeta|$. The case $t = 0$, which corresponds to the north pole, is taken care of in the next assertion.

Proof of Assertion 3. First we show that F is differentiable at the north pole. The right coordinates are given by formula 3, and in that formula we replace the polar coordinates (t,p,w), $t \in [0,\pi)$, $|p|^2 + |w|^2 = 1$ with Cartesian coordinates (p,w), $|p|^2 + |w|^2 < \pi$. Thus we replace in formula 3 every appearance of t with $\sqrt{|p|^2 + |w|^2} = |(p,w)|$, p with $p/|(p,w)|$ and w with $w/|(p,w)|$. We get

$$\begin{pmatrix} y \\ \zeta \\ h \end{pmatrix} = \begin{pmatrix} 1 - 2\sin^2(|(p,w)|)\frac{|w|^2}{|p|^2+|w|^2} \\ 2\sin(|(p,w)|)\frac{w}{|w|}(\cos((|(p,w)|) + \sin(|(p,w)|)\frac{p}{|(p,w)|})e^{-p\frac{\overline{w}}{|(p,w)|}}e^{p}\frac{\overline{w}}{|w|} \\ \frac{w}{|w|}e^{(\tau-t)p}\frac{\overline{w}}{|w|}e^{\tau p}\frac{w}{|w|}e^{(t-\tau)p}\frac{\overline{w}}{|w|} \end{pmatrix},$$

and in a long computation similar in spirit to Proposition 1, spelling out the exponentials, the commutation is faster than division by zero, and in the relevant power series of sine and cosine the only surviving terms are the non-negative even powers of $|(p,w)|$. This shows differentiability at $(p,w) = (0,0)$, i.e. at the north pole. Knowing differentiability, one can write the equation above just up to order 1 in $|(p,w)|$, and we get

$$\begin{pmatrix} y \\ \zeta \\ h \end{pmatrix} = \begin{pmatrix} 1 + O(|(p,w)|^2) \\ 2(1+p)\bar{w} + O(|(p,w)|^2) \\ 1 + p + O(|(p,w)|^2) \end{pmatrix},$$

which shows that $\Phi \circ f_2$ is a local diffeomorphism around $(0,0)$.

The key issue is the differentiability at the south pole. This cannot be done with the northern chart of M_σ we have used up to now, which does not cover the south pole. Thus we need to change charts; the southern chart is given by coordinates (t_1,p_1,w_1), with the south pole correspondong to $(t_1,p_1,w_1) = (0,0,0)$, and recall that the change of coordinates is given by

$$(t_1,p_1,w_1) = (\pi - t, \sigma(-p,-w)) = (\pi - t, -\overline{b(p,w)}\,p\,b(p,w), -\overline{b(p,w)}\,w\,b(p,w))\,.$$

Due to the nice properties of b and σ, the inverse change is quite clear:

$$\begin{aligned} (t,p,w) &= (\pi - t_1, \sigma^{-1}(-p_1,w_1)) \\ &= (\pi - t_1, b(p_1,w_1)\,p_1\,\overline{b(p_1,w_1)}, -b(p_1,w_1)\,w_1\,\overline{b(p_1,w_1)})\,. \end{aligned}$$

Thus let us substitute in formula 3. We get

$$\begin{aligned} e^{tp} &= e^{(\pi-t_1)b(p_1,w_1)p_1\overline{b(p_1,w_1)}} = b(p_1,w_1)\, e^{(\pi-t_1)p_1}\,\overline{b(p_1,w_1)},\\ \frac{w}{|w|} &= -b(p_1,w_1)\frac{w_1}{|w_1|}\overline{b(p_1,w_1)},\\ e^{\tau p} &= \frac{\cos(t)+\sin(t)p}{|\cos(t)+\sin(t)p|} = \frac{-\cos(t_1)-\sin(t_1)b(p_1,w_1)p_1\overline{b(p_1,w_1)}}{|-\cos(t_1)-\sin(t_1)b(p_1,w_1)p_1\overline{b(p_1,w_1)}|}\\ &= -b(p_1,w_1)e^{\tau p_1}\overline{b(p_1,w_1)},\ \text{where } \tau(t,p)=\tau(t_1,p_1). \end{aligned}$$

Note that the formulas are expressed in terms of inner automorphism by b; thus when we substitute in (3), a great deal of cancellations take place. Also in factors of the form $e^{(\pi-t_1)p_1} = e^{\pi p_1}e^{(-t_1)p_1}$ the $e^{\pi p_1}$ collaborates with $\frac{w_1}{|w_1|}$ to produce new $b(p_1,w_1)$ when needed. We get the surprising fact that the formula for (y,ζ,h) in terms of (t_1,p_1,w_1) looks almost the same as in terms of (t,p,w); in fact,

$$\begin{pmatrix} y\\ \zeta\\ h\end{pmatrix} = \begin{pmatrix} 1-2\sin^2(t_1)\,|w_1|^2\\ 2\sin(t_1)\frac{w_1}{|w_1|}(\cos(t_1)+\sin(t_1)p_1)e^{-t_1p_1}\overline{w_1}e^{t_1p_1}\frac{\overline{w_1}}{|w_1|}\\ -\frac{w_1}{|w_1|}e^{(\tau-t_1)p_1}\frac{\overline{w_1}}{|w_1|}e^{\tau p_1}\frac{w_1}{|w_1|}e^{(t_1-\tau)p_1}\frac{\overline{w_1}}{|w_1|}\end{pmatrix}$$

(note the sign in front of h!). This is the only difference in the formula, and the south pole given by $(t_1,w_1,p_1)=(0,0,0)$ is mapped to $(y,\zeta,h)=(1,0,-1)$. The exact same proof as the one for the north pole shows that F is a diffeomorphism around the south pole, too.

Assertions 1, 2 and 3 prove Theorem III. □

3.4. **Some remarks about the map σ.** The map σ can also be analyzed as follows: for $(p,w)\in S^{14}, \sigma(p,w) = (\Delta\circ C\circ b(p,w))\begin{pmatrix} p\\ w\end{pmatrix}$, where

$$b(p,w) = \frac{w}{|w|}e^{\pi p}\frac{\overline{w}}{|w|}\in S^7$$

and generates $\pi_{14}S^7$; the map $C: S^7\to SO(7)$

$$C(\alpha)(x) = \alpha x\overline{a}$$

is Cayley conjugation acting on purely imaginary Cayley numbers x and Δ is the diagonal map

$$\Delta(A_{7\times7}) = \begin{pmatrix} A_{7\times7} & & \\ & 1 & \\ & & A_{7\times7}\end{pmatrix}\in SO(15).$$

The homotopy class of C generates the stable group $\pi_7(SO)=\mathbb{Z}$ (see [18]).

From page 107 of [6] we see that $\Sigma^{15}_{(2,-1)}$ (denoted there by $M_3^{15,0}$) generates a subgroup H of index 2 in $\Gamma_{15}\cong\mathbb{Z}_2\oplus\mathbb{Z}_{8128}$ ([2]). It is obvious that the iterates of σ are given by the same formula as in the quaternionic case, and they provide the gluing maps for all elements of the subgroup H.

It would be interesting to find a formula for an element of Γ_{15} that is not a power of σ, and express relations that might exist between these elements and bundles over the (only) exotic 8-sphere with S^7 as fibers.

Acknowledgements

The authors thank U. Abresch, A. Hatcher, L. Mata, T. Püttmann and P. Ontaneda for valuable comments, and the anonymous referee for several suggestions that improved this paper.

References

[1] T.E. Barros and A. Rigas, *The role of commutators in a non cancellation phenomenon*, Math. Jour. Okayama Univ. **43** (2001) 73-93. MR 2003c:57031

[2] G. Brumfiel, *On the homotopy groups of* BPL *and* PL/O, Ann. of Math. (2) **88** (1968) 291-311. MR 38:2775

[3] L.M.Chaves and A. Rigas, *Hopf maps and triality*, Math. Jour. Okayama Univ. **38** (1966) 197-208. MR 99m:55016

[4] M. Davis, *Some group actions on homotopy spheres of dimension seven and fifteen*, Amer. Jour. Math. **104** (1) (1982) 59-90. MR 83g:57027

[5] C. E. Durán, *Pointed Wiedersehen metrics on exotic spheres and diffeomorphisms of S^6*, Geometria Dedicata **88** (2001), 199-210. MR 2002i:57044

[6] J. Eells and N. Kuiper, *An invariant for certain smooth manifolds*, Annali Mat. Pura e Appl. **60** (1962), 93 - 110. MR 27:6280

[7] D. Gromoll and W. Meyer, *An exotic sphere with non-negative sectional curvature*, Ann. of Math. (2)**96** (1972), 413-443. MR 46:8121

[8] P. Hilton and J. Roitberg, *On Principal S^3- bundles over spheres*, Ann. of Math. **90** (1969) 91-107. MR 39:7624

[9] S. T. Hu, *Homotopy theory*, Academic Press, NY, 1959. MR 21:5186

[10] I. M. James, *On H-spaces and their homotopy groups*, Quarterly J. Math. Oxford **(2) 11**, 1960, 161-179. MR 24:A2966

[11] V. Kapovich and W. Ziller, *Biquotients with singly generated rational cohomology*, Preprint 2002.

[12] M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres I*, Ann. of Math. **77** (1963), 504-537. MR 26:5584

[13] J. W. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. **64** (1956), 399-405. MR 18:498d

[14] A. Rigas, *S^3-bundles and exotic actions*, Bull. Soc. Math. France **112** (1984), 69-92. MR 86i:57028

[15] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, 1951. MR 12:522b

[16] I. Tamura, *Homeomorphy classification of total spaces of sphere bundles over spheres*, J. Math. Soc. Japan **10** (1958) 29-43. MR 20:2717

[17] H. Toda, *Composition Methods in the Homotopy Groups of Spheres*, Annals of Math. Studies **49**, 1962. MR 26:777

[18] H. Toda, Y. Saito and T. Yokota, *A note on the generator of $\pi_7 SO(n)$*, Mem. Coll. Sci. Univ. Kyoto Ser.A **30** (1957) 227-230. MR 19:975a

[19] J. H. C. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics 61, Springer-Verlag, 1978. MR 80b:55001

Departamento de Matemáticas, Instituto Venezolano de Investigaciones Científicas, Apartado 21827, Caracas 1020A, Venezuela

E-mail address: cduran@cauchy.ivic.ve

Current address: IMECC-UNICAMP, C.P. 6065, 13083-970, Campinas, SP, Brazil

E-mail address: cduran@ime.unicamp.br

Departamento de Matemáticas Puras y Aplicadas, Universidad Simón Bolívar, Apartado 28000, Caracas 1080, Venezuela

E-mail address: jacob@usb.ve

IMECC-UNICAMP, c.p. 6065, 13083-970, Campinas, SP, Brazil

E-mail address: rigas@ime.unicamp.br