A LOCAL LIMIT THEOREM
FOR CLOSED GEODESICS AND HOMOLOGY

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Abstract. In this paper, we study the distribution of closed geodesics on a compact negatively curved manifold. We concentrate on geodesics lying in a prescribed homology class and, under certain conditions, obtain a local limit theorem to describe the asymptotic behaviour of the associated counting function as the homology class varies.

0. Introduction

Let \(M\) be a compact smooth Riemannian manifold with first Betti number \(k > 0\) and with negative sectional curvatures. Suppose also that either \(\dim M = 2\) or that \(M\) is \(1/4\)-pinched, i.e., the sectional curvatures all lie in an interval \([-\kappa, -\kappa/4]\), for some \(\kappa > 0\). Such a manifold contains a countable infinity of prime closed geodesics. (We say that a closed geodesic is prime if it is not a nontrivial multiple of another closed geodesic.) In this paper we are interested in how these closed geodesics are distributed with respect to homology.

The homology group \(H_1(M, \mathbb{Z})\) is isomorphic to \(\mathbb{Z}^k \oplus \text{Tor}\), where \(\text{Tor}\) is the (finite) torsion subgroup. In this paper, it will be convenient to consider the torsion-free part of the homology, \(H_1(M, \mathbb{Z})/\text{Tor}\). We shall, in fact, assume that an isomorphism has been fixed and write \(\mathbb{Z}^k\) instead of \(H_1(M, \mathbb{Z})/\text{Tor}\).

For a typical (prime) closed geodesic \(\gamma\) on \(M\), let \(l(\gamma)\) denote its length and \([\gamma] \in H_1(M, \mathbb{Z})/\text{Tor} = \mathbb{Z}^k\) the torsion-free part of its homology class. For \(\alpha \in \mathbb{Z}^k\), define a counting function

\[
\pi(T, \alpha) = \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\}.
\]

Recently, several papers have studied the asymptotics of this function as \(T \to \infty\). In particular, Anantharaman [11] and Pollicott and Sharp [17] have shown that there exist constants \(C_0 > 0\), independent of \(\alpha\), and \(C_n(\alpha)\), \(n \geq 1\), such that, for any \(N \geq 1\), we have the asymptotic expansion

\[
(0.1) \quad \pi(T, \alpha) = \frac{e^{hT}}{T^{k/2 + 1}} \left( C_0 + \frac{C_1(\alpha)}{T} + \frac{C_2(\alpha)}{T^2} + \cdots + \frac{C_N(\alpha)}{T^N} + O \left( \frac{1}{T^{N+1}} \right) \right),
\]

where \(h > 0\) denotes the topological entropy of the geodesic flow over \(M\). (In fact, the expansion in [17] contains some extra terms corresponding to powers of \(T^{-1/2}\); a more careful analysis, as carried out in [11], shows that these terms vanish.) Furthermore, Kotani [11], has studied the dependence of the coefficients \(C_n(\alpha)\) on...
\[ \alpha = (\alpha_1, \ldots, \alpha_k), \] showing that they may be expressed as polynomials of degree 2n in \( \alpha_1, \ldots, \alpha_k \). In the special case of manifolds of constant negative curvature, the expansion (0.1) was obtained by Phillips and Sarnak \[14\] and, independently, Katsuda and Sunada \[2\] obtained the leading term. For manifolds of variable negative curvature (without the pinching condition) the leading term of (0.1) was obtained by Katsuda \[8\], Lalley \[12\] and Pollicott \[15\]. Analogous results for manifolds with cusps have been obtained by Epstein \[6\] and Babillot and Peigne \[3\].

In this note, we take a slightly different view and address the question of the behaviour of \( \pi(T, \alpha) \) when \( \alpha \) is allowed to vary independently of \( T \). We obtain the following "local limit theorem".

**Theorem 1.** Let \( M \) be a compact smooth Riemannian manifold with first Betti number \( k > 0 \) and with negative sectional curvatures. Suppose also that either \( \dim M = 2 \) or that \( M \) is 1/4-pinch. Then there exists a symmetric positive definite real matrix \( D \) such that

\[
\lim_{T \to \infty} \frac{h\sigma^k T^{k/2+1}}{e h T} \pi(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, D^{-1} \alpha \rangle/2T} = 0,
\]

uniformly in \( \alpha \in \mathbb{Z}^k \), where \( \sigma > 0 \) satisfies \( \sigma^{2k} = \det D \).

Here, \( \langle \cdot, \cdot \rangle \) denotes the usual inner product \( \langle x, y \rangle = x_1 y_1 + \cdots + x_k y_k \). As a particular consequence, we recover the leading term of the expansion (0.1), with \( C_0 = h^{-1} \sigma^{-k} (2\pi)^{-k/2} \). Theorem 1 appears not to have been stated even for manifolds of constant negative curvature; although, in that case, the result can be easily deduced from the analysis contained in \[14\].

**Remarks.** (i) If we take the torsion part of \( H_1(M, \mathbb{Z}) \) into account, then we need to modify Theorem 1 to read

\[
\lim_{T \to \infty} \frac{h\sigma^k T^{k/2+1}}{e h T} \pi(T, \alpha) - \frac{1}{(\# \text{Tor}) (2\pi)^{k/2}} e^{-\langle \alpha, D^{-1} \alpha \rangle/2T} = 0,
\]

uniformly in \( \alpha \in H_1(M, \mathbb{Z}) \), where \( \alpha_F \in \mathbb{Z}^k \) denotes the torsion-free part of \( \alpha \in H_1(M, \mathbb{Z}) \).

(ii) In Kotani’s formula for the term \( C_n(\alpha)/T^n \) in (0.1), the highest power of \( \alpha \) makes a contribution

\[
\frac{1}{(2\pi)^{k/2} h \sigma^k n!} \left( \frac{\langle \alpha, D^{-1} \alpha \rangle}{2T} \right)^n.
\]

As observed by Kotani in \[11\], formally summing these contributions gives the expression \( e^{-\langle \alpha, D^{-1} \alpha \rangle/2T}/(2\pi)^{k/2} h \sigma^k \).

Theorem 1 should be compared with the results on homology classes varying linearly in \( T \) obtained by Lalley \[12\] and Babillot and Ledrappier \[2\]. Using these results, one can show that, for \( \delta > 0 \) sufficiently small,

\[
\lim_{T \to \infty} \sup_{||\alpha|| \leq \delta T} \left| \frac{T^{k/2+1}}{C(\alpha/T)^{H(\alpha/T)T}} \pi(T, \alpha) - 1 \right| = 0,
\]

where \( H(x) \) is an entropy function satisfying \( H(0) = h, \nabla H(0) = 0 \) and \( \nabla^2 H(0) = -D^{-1} \), and where \( C(x) \) is continuous with \( C(0) = C_0 \). On the other hand, Theorem
1 is equivalent to
\[
\lim_{T \to \infty} \sup_{|\alpha| \leq \delta T} \left| \frac{\log^k T^{k/2+1}}{e^{\alpha T}} \pi(T, \alpha) - \frac{1}{(2\pi)^k/2} e^{-\langle \alpha, D^{-1} \alpha \rangle/2T} \right| = 0
\]
(as the supremum over $|\alpha| > \delta T$ clearly tends to zero). However, even though $H(x) = H(0) - \langle x, D^{-1}x \rangle/2 + O(||x||^3)$, which gives
\[
\exp\{H(\alpha/T)T\} = \exp\{H(0)T - \langle \alpha, D^{-1} \alpha \rangle/2T + O(||\alpha||^3/T^2)\},
\]
the presence of the third order terms means that one cannot deduce (0.3) from (0.2).

The results of [2] and [12] do imply a central limit theorem: for $A \subset \mathbb{R}^k$,
\[
\lim_{T \to \infty} \frac{\# \{ \gamma : l(\gamma) \leq T, [\gamma]/\sqrt{T} \in A \}}{\# \{ \gamma : l(\gamma) \leq T \}} = \frac{1}{(2\pi)^{k/2}} \int_A e^{-\langle x, D^{-1}x \rangle/2} \, dx.
\]

A key ingredient in the proof of Theorem 1 is an understanding of the analytic domain of a family of functions of a complex variable, called $L$-functions, indexed by the characters of $\mathbb{Z}^k$. In the next section, we shall define these functions and discuss their properties. In Section 2, we shall introduce a family of functions $\mathcal{S}_T(t)$, $t \in [-\pi, \pi]^k$, obtained by summing a suitably weighted character $e^{it(\cdot, \cdot)}$ over all (multiple) closed geodesics of length at most $T$, and show that they are related to contour integrals of the corresponding $L$-functions. The results in Section 1 are then used to estimate the sums $\mathcal{S}_T(t)$. In Section 3, we shall use an approach adapted from [18] to transfer information from the $\mathcal{S}_T(t)$ to an auxiliary function $\psi(T, \alpha)$, which is essentially a weighted version of $\pi(T, \alpha)$, and obtain an analogue of Theorem 1 valid for $\psi(T, \alpha)$. In Section 4, we shall complete the proof of Theorem 1 by elementary arguments. In the final section, we shall discuss the application of our method to homologically full Anosov flows, giving a new proof of the first order asymptotic formula for $\pi(T, \alpha)$ (but without uniformity) in that case.

Notation. For given functions $A(T)$ and $B(T)$, we shall write $A(T) \sim B(T)$, as $T \to \infty$, if $\lim_{T \to \infty} A(T)/B(T) = 1$, and $A(T) = O(B(T))$ if $|A(T)| \leq CB(T)$, for some constant $C > 0$.

1. $L$-FUNCTIONS

In order to obtain our main result, we shall need to understand the analytic behaviour of a certain family of functions of a complex variable. We will identify the character group of $\mathbb{Z}^k$ with $[-\pi, \pi]^k$. For $t \in [-\pi, \pi]^k$, define
\[
L(s, t) = \prod_{\gamma} \left( 1 - e^{-s(\gamma) + i(t, [\gamma])} \right)^{-1},
\]
where the product is taken over all prime closed geodesics $\gamma$. This converges for $\Re(s) > h$ and has a meromorphic extension to a strictly larger half-plane [13].

It will be convenient to consider multiple closed geodesics $\gamma' = \gamma^n$, $n \geq 1$. In this case we shall write $l(\gamma') = nl(\gamma)$, $[\gamma'] = n[\gamma]$, and $\Lambda(\gamma') = l(\gamma)$. (Note that $\Lambda$ is analogous to the von Mangoldt function in number theory.)

We are interested in the logarithmic derivative $L'(s, t)/L(s, t)$ of $L(s, t)$. Whenever the summation converges, we have the identity
\[
\frac{L'(s, t)}{L(s, t)} = - \sum_{\gamma'} \Lambda(\gamma') e^{-s(\gamma') + i(t, [\gamma'])}.
\]
We shall make use of the properties of $L'(s,t)/L(s,t)$ described by the following two propositions. These results were obtained in [17] and rely heavily on the techniques of Dolgopyat [4]. We write $U(\delta) = \{ t : ||t|| < \delta \}$.

**Proposition 1** ([17]). For all sufficiently small $\delta > 0$ the following statements are true.

(i) There exists $\epsilon > 0$ and an analytic function $s : U(\delta) \rightarrow (-\infty, h]$, satisfying $s(0) = h$ and $s(t) < h$ for $t \neq 0$, such that

$$\frac{L'(s,t)}{L(s,t)} + \frac{1}{s - s(t)}$$

is analytic in $\text{Re}(s) > h - \epsilon$.

(ii) There exists $\epsilon > 0$ such that, for $t \notin U(\delta)$, $L'(s,t)/L(s,t)$ is analytic in $\text{Re}(s) > h - \epsilon$.

**Proposition 2** ([17]). There exists $\epsilon > 0$, $C > 0$, and $0 < \beta < 1$, such that, for all $t \in [-\pi, \pi]^k$,

$$\left| \frac{L'(s,t)}{L(s,t)} \right| \leq C|\text{Im}(s)|^\beta,$$

for $\text{Re}(s) > 1 - \epsilon$ and $|\text{Im}(s)| \geq 1$.

The function $s(t)$ enjoys the following properties.

**Lemma 1.** $\nabla s(0) = 0$ and $\nabla^2 s(0)$ is strictly negative definite.

We shall write $D = -\nabla^2 s(0)$ and define $\sigma > 0$ by $\sigma^2 = \det D$. The next result is crucial for our subsequent analysis.

**Proposition 3.** There exists $\delta > 0$ such that, for $t \in U(\delta \sigma \sqrt{T})$,

$$\lim_{T \to \infty} e^{(s(t/\sigma \sqrt{T}) - h)T} = e^{-\langle t, D t \rangle / 2\sigma^2}.$$

Furthermore, $|e^{(s(t/\sigma \sqrt{T}) - h)T} - e^{-\langle t, D t \rangle / 2\sigma^2}| \leq 2e^{-\langle t, D t \rangle / 4\sigma^2}$.

**Proof.** Let $f(t) = e^{s(t) - h}$. Then $f(0) = 1$, $\nabla f(0) = \nabla s(0) = 0$, and $\nabla^2 f(0) = \nabla^2 s(0) = -D$. Applying Taylor’s Theorem, we have that, for $||t/\sigma \sqrt{T}|| \leq \delta$,

$$f\left(\frac{t}{\sigma \sqrt{T}}\right) = 1 - \frac{\langle t, D t \rangle}{2\sigma^2 T} + O\left(\frac{||t||^3}{T^{3/2}}\right)$$

(where the implied constant is independent of $t$). The first statement follows from the identity $\lim_{T \to \infty}(1 - x/T)^T = e^{-x}$.

Provided $\delta > 0$ is sufficiently small, for $||u|| \leq \delta$, we have

$$\langle u, D u \rangle / 2 + O(||u||^3) \geq \langle u, D u \rangle / 4.$$

Since $(1 - x/T)^T < e^{-x}$, this gives us $|f(t/\sigma \sqrt{T})| \leq e^{-\langle t, D t \rangle / 4\sigma^2}$. Applying the triangle inequality, we obtain

$$|f(t/\sigma \sqrt{T}) - e^{-\langle t, D t \rangle / 2\sigma^2}| \leq e^{-\langle t, D t \rangle / 4\sigma^2} + e^{-\langle t, D t \rangle / 2\sigma^2} \leq 2e^{-\langle t, D t \rangle / 4\sigma^2}.$$
Remark. The function \( s(t) \) has an interpretation in terms of the thermodynamic formalism of the geodesic flow on \( SM \). For a continuous function \( G : SM \to \mathbb{R} \), define its pressure \( P(G) = \sup_{\mu} \{ h_\mu(\phi) + \int Gd\mu \} \), where the supremum is taken over all probability measures invariant under the geodesic flow. We can define a (smooth) function \( F : SM \to \mathbb{R}^k \) with the property that, for each closed geodesic \( \gamma \), \( \int_0^1 F(\gamma(t), \dot{\gamma}(t))dt = [\gamma] \). Then \( \mathbb{R}^k \ni z \mapsto P((z, F)) \) is real analytic and has an analytic extension to a neighbourhood of \( \mathbb{R}^k \) in \( \mathbb{C}^k \). We have that \( s(t) = P((it, F)) \) and that \( D = \nabla^2 P((z, F)) |_{z=0} \) [10], [19].

2. Contour integration

We shall now use the results on \( L \)-functions obtained in the preceding section to examine the behaviour of the summatory function

\[
S_T(t) = \sum_{l(\gamma') \leq T} \Lambda(\gamma')e^{i(t, [\gamma'])},
\]

as \( T \to \infty \). (Here, the ' on the summation sign denotes that the terms with \( l(\gamma') = T \) are counted with weight 1/2.)

We begin by relating \( S_T(t) \) to \( L'(s, t)/L(s, t) \). This is achieved through the following lemma.

Lemma 2 ([20], p. 132], Effective Perron Formula). Define a function \( \theta(y) \) by

\[
\theta(y) = \begin{cases} 
0 & \text{if } 0 < y < 1, \\
\frac{1}{2} & \text{if } y = 1, \\
1 & \text{if } y > 1.
\end{cases}
\]

Then, uniformly for \( d > 0, R > 0 \),

\[
\left| \theta(y) - \frac{1}{2\pi i} \int_{-iR}^{d+iR} \frac{y^s}{s} ds \right| = O \left( \frac{y^d}{1 + R|\log y|} \right).
\]

Set \( d = h + T^{-1} \) and \( R = T^K \) (where \( K > 0 \) will be chosen later). Applying Lemma 2 term-by-term to \( -L'(s, t)/L(s, t) \), we obtain

(2.1) \[
S_T(t) = \frac{1}{2\pi i} \int_{-iR}^{d+iR} \left( -\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{st}}{s} ds + O \left( \sum_{\gamma'} \frac{\Lambda(\gamma')e^{dT}e^{-dT}}{1 + R|T - l(\gamma')|} \right).
\]

We will estimate the big-\( O \) term in this expression. First set \( \epsilon = T^{-M} \) (where \( M > 0 \) will be chosen later) and consider the terms for which \( |T - l(\gamma')| \leq \epsilon \). We will use the following result contained in [10].

Proposition 4 ([16]). There exists \( c < h \) such that

\[
\# \{ \gamma' : l(\gamma') \leq T \} = \int_2^{e^{hT}} \frac{1}{\log u} \, du + O(e^{cT}).
\]

As a consequence, we may write

\[
\# \{ \gamma' : |T - l(\gamma')| \leq \epsilon \} = \int_{e^{hT-h\epsilon}}^{e^{hT+h\epsilon}} \frac{1}{\log u} \, du + O(e^{cT}) = O \left( \frac{\epsilon e^{hT}}{T} \right).
\]
Furthermore, if $|T - l(\gamma')| \leq \epsilon$, then $e^{dT}e^{-d(l(\gamma'))} = O(1)$. Thus,

$$
\sum_{|T - l(\gamma')| \leq \epsilon} \Lambda(\gamma') e^{dT}e^{-d(l(\gamma'))} = O \left( \frac{e^{hT}}{TM} \right).
$$

On the other hand,

$$
\sum_{|T - l(\gamma')| > \epsilon} \Lambda(\gamma') e^{dT}e^{-d(l(\gamma'))} \leq \frac{e^{dT}}{Re} \sum_{\gamma'} \Lambda(\gamma')e^{-d(l(\gamma'))} = O \left( \frac{e^{hT}}{TR - M - 1} \right),
$$

where we have used the estimate

$$
\left| \frac{L'(h + T^{-1}, 0)}{L(h + T^{-1}, 0)} \right| = O(T).
$$

Combining the estimates above, equation (2.1) becomes

$$(2.2) \quad \mathcal{S}_T(t) = \frac{1}{2\pi i} \int_{d-iR}^{d+iR} \frac{s^{2}}{s} ds + O \left( \frac{e^{hT}}{TN} \right).$$

**Lemma 3.** For all $N \geq 1$ we have the following estimates. (The implied constants are independent of $t$.)

(i) For $t \in U(\delta)$,

$$
\mathcal{S}_T(t) = \frac{e^{s(t)T}}{s(t)} + O \left( \frac{e^{hT}}{TN} \right).
$$

(ii) For $t \notin U(\delta)$,

$$
\mathcal{S}_T(t) = O \left( \frac{e^{hT}}{TN} \right).
$$

**Proof.** Choose $h - \epsilon < c < h$ and let $\Gamma$ denote the contour formed by the rectangle with vertices at $d - iR$, $d + iR$, $c + iR$, and $c - iR$, oriented counterclockwise.

(i) Suppose that $t \in U(\delta)$. By Proposition 1(i) we can choose $c < s(t)$ so that, using the Residue Theorem,

$$
\frac{1}{2\pi i} \int_{\Gamma} \frac{L'(s, t)}{L(s, t)} \frac{e^{sT}}{s} ds = \frac{e^{s(t)T}}{s(t)}.
$$

Using Proposition 2, we also have the following bounds:

(a) \[ \left| \left( \int_{c+iR}^{d+iR} + \int_{c-iR}^{d-iR} \right) \left( \frac{L'(s, t)}{L(s, t)} \right) \frac{e^{sT}}{s} ds \right| = O(R^{2-1}e^{hT}) = O \left( \frac{e^{hT}}{TK(1-\beta)} \right) \]

(b) \[ \left| \int_{c+iR}^{d+iR} \left( \frac{L'(s, t)}{L(s, t)} \right) \frac{e^{sT}}{s} ds \right| = O(R^\beta e^{cT}) = O(T^\beta K e^{cT}). \]

Combining this with (2.2) gives

$$
\mathcal{S}_T(t) = \frac{e^{s(t)T}}{s(t)} + O \left( \frac{e^{hT}}{TN} \right),
$$

where

$$
N = \min\{M, K - M - 1, K(1 - \beta)\}.
$$

Since $K$ and $M$ are arbitrary, we may take $N$ as large as we please.
(ii) Suppose that \( t \notin U(\delta) \). Then, by Proposition 1(ii),
\[
\frac{1}{2\pi i} \int_{\Gamma} \left( -\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{\pi T}}{s} ds = 0.
\]
The result now follows as in the proof of (i).

3. An auxiliary function

In this section, we shall prove a result analogous to Theorem 1 but where \( \pi(T, \alpha) \) is replaced by the auxiliary function
\[
\psi(T, \alpha) = \sum_{\beta(\gamma') \leq T} \Lambda(\gamma'),
\]
which can be related to the sums \( \mathcal{S}_T(t) \) considered in the previous section. We shall adapt an approach used by Rousseau-Egele [18] to examine the quantity \( \sigma^{kT/2} e^{-\pi T} \psi(T, \alpha) \). For \( a > 0 \), write \( I(a) = [-a, a]^k \). Using the orthogonality relationship
\[
\frac{1}{(2\pi)^k} \int_{I(\pi)} e^{-i(t, \alpha)} e^{i(t, \gamma)} dt = \begin{cases} 1 & \text{if } y = \alpha, \\ 0 & \text{if } y \in \mathbb{Z}^k \setminus \alpha, \end{cases}
\]
we have that
\[
\psi(T, \alpha) = \frac{1}{(2\pi)^k} \int_{I(\pi)} e^{-i(t, \alpha) \mathcal{S}_T(t)} dt.
\]
Making the substitution \( t \mapsto t/\sigma \sqrt{T} \), we obtain
\[
\sigma^{kT/2} \psi(T, \alpha) = \frac{1}{(2\pi)^k} \int_{I(\pi \sigma \sqrt{T})} e^{-i(t, \alpha)/\sigma \sqrt{T} \mathcal{S}_T(t/\sigma \sqrt{T})} dt.
\]
The next result is the key to the proof of Theorem 1.

Proposition 5.
\[
\lim_{T \to \infty} \sup_{\alpha \in \mathbb{Z}^k} \left| \int_{U(\delta \sqrt{T})} e^{-i(t, \alpha)/\sigma \sqrt{T} \mathcal{S}_T(t/\sigma \sqrt{T})} dt \right| = 0.
\]
Using the identity
\[
e^{-\alpha, D^{-1}\alpha)/2\sigma^2} dt,
\]
we have established the bound
\[
(2\pi)^k \left| \frac{\sigma^{kT/2}}{e^{\pi T}} \psi(T, \alpha) - \frac{e^{-\alpha, D^{-1}\alpha)/2\sigma^2}}{(2\pi)^k/2} \right|
\leq \int_{U(\delta \sqrt{T})} e^{-i(t, \alpha)/\sigma \sqrt{T} \mathcal{S}_T(t/\sigma \sqrt{T})} - e^{-i(t, \alpha)/\sigma \sqrt{T})} dt
\]
\[
+ \int_{I(\pi \sigma \sqrt{T}) \setminus U(\delta \sqrt{T})} e^{-i(t, \alpha)/\sigma \sqrt{T} \mathcal{S}_T(t/\sigma \sqrt{T})} dt
\]
\[
+ \int_{\mathbb{R}^k \setminus U(\delta \sqrt{T})} e^{-i(t, \alpha)/\sigma \sqrt{T} \mathcal{S}_T(t/\sigma \sqrt{T})} dt
\]
\[
= A_1(T, \alpha) + A_2(T, \alpha) + A_3(T, \alpha).
\]
An easy calculation shows that

$$\lim_{T \to \infty} \sup_{\alpha \in \mathbb{Z}^k} A_3(T, \alpha) = 0,$$

so, to complete the proof of Proposition 5, it remains to estimate $A_1(T, \alpha)$ and $A_2(T, \alpha)$. To do this we shall use the information on $s(t)$ and $S_T(t)$ contained in Proposition 3 and Lemma 3.

**Lemma 4.** There exists $C > 0$ such that, for all sufficiently small $\delta > 0$,

$$\lim_{T \to \infty} \sup_{\alpha \in \mathbb{Z}^k} A_1(T, \alpha) \leq C \left\{ \int_{\mathbb{R}^k} e^{-(t, D_t)/4\sigma^2} \, dt \right\} \delta^2.$$

**Proof.** By Lemma 3, we have that, for $t \in U(\delta \sigma \sqrt{T})$,

$$h e^{-h T} S_T(t/\sigma \sqrt{T}) = \frac{h e^{(s(t/\sigma \sqrt{T}) - h) T}}{s(t/\sigma \sqrt{T})} + O(T^{-(k/2+1)}).$$

Using the analyticity of $s(t)$ and the fact that $\nabla s(0) = 0$, we have

$$\left| e^{(s(t/\sigma \sqrt{T}) - h) T} \left( \frac{h}{s(t/\sigma \sqrt{T})} - 1 \right) \right| \leq C \delta^2 e^{-(t, D_t)/4\sigma^2},$$

for some constant $C > 0$. Thus,

$$A_1(T, \alpha) \leq \int_{U(\delta \sigma \sqrt{T})} \left| e^{(s(t/\sigma \sqrt{T}) - h) T} - e^{-(t, D_t)/2\sigma^2} \right| \, dt$$

$$+ C \delta^2 \int_{U(\delta \sigma \sqrt{T})} e^{-(t, D_t)/4\sigma^2} \, dt + O\left( \frac{1}{T} \right).$$

By Proposition 3, we know that $e^{(s(t/\sigma \sqrt{T}) - h) T}$ converges to $e^{-(t, D_t)/2\sigma^2}$, as $T \to \infty$. Furthermore, we have the estimate

$$\left| e^{(s(t/\sigma \sqrt{T}) - h) T} - e^{-(t, D_t)/2\sigma^2} \right| \leq 2 e^{-(t, D_t)/4\sigma^2}.$$

Hence, applying the Dominated Convergence Theorem, we obtain the desired result.

**Lemma 5.**

$$\lim_{T \to \infty} \sup_{\alpha \in \mathbb{Z}^k} A_2(T, \alpha) = 0.$$

**Proof.** By Lemma 3(ii), for $t \notin U(\delta \sigma \sqrt{T})$,

$$e^{-h T} S_T(t/\sigma \sqrt{T}) = O(T^{-(k/2+1)}),$$

so that $\sup_{\alpha \in \mathbb{Z}^k} A_2(T, \alpha) = O(T^{-1}).$

**Proof of Proposition 5.** Combining the above results we have that

$$\lim_{T \to \infty} \sup_{\alpha \in \mathbb{Z}^k} \left| \frac{h \alpha^{kT/2}}{e^{kT}} \psi(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-(\alpha, D^{-1} \alpha)/2T} \right| \leq C \left\{ \int_{\mathbb{R}^k} e^{-(t, D_t)/4\sigma^2} \, dt \right\} \delta^2.$$

Since this holds for all sufficiently small $\delta > 0$, the proof is complete.
4. Proof of Theorem 1

In this section we will use elementary arguments to deduce Theorem 1 from Proposition 5. Whenever we make a big-\(O\) estimate, the implied constant will be independent of \(\alpha\).

Write
\[
T;\alpha = \sum_{l(\gamma) \leq T, |\gamma| = \alpha} l(\gamma).
\]

An easy argument shows that
\[
\psi(T, \alpha) = \psi^*(T, \alpha) + O(T^2 e^{hT/2}).
\]
Thus, Proposition 5 implies the following.

Proposition 6.

\[
\lim_{T \to \infty} \sup_{\alpha \in \mathbb{Z}^k} \frac{h\sigma^k T^{k/2}}{e^{hT}} \psi^*(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, D^{-1} \alpha \rangle/2T} = 0.
\]

Finally, we consider \((T;\alpha)\). It is easy to see that
\[
\psi^*(T, \alpha) \leq T \pi(T, \alpha).
\]

For the corresponding lower bound, choose \(\tau > 0\) and set \(\theta = (1 + \tau)^{-1} < 1\). Then
\[
\frac{T \pi(T, \alpha)}{e^{hT}} = \frac{T}{e^{hT}} \sum_{\theta T < l(\gamma) \leq T, |\gamma| = \alpha} 1 + \frac{T \pi(\theta T, \alpha)}{e^{hT}} \leq \frac{1 + \tau}{e^{hT}} \sum_{\theta T < l(\gamma) \leq T, |\gamma| = \alpha} l(\gamma') + \frac{T \pi(\theta T, \alpha)}{e^{hT}} \leq \frac{(1 + \tau) \psi^*(T, \alpha)}{e^{hT}} + \frac{T \# \{ \gamma : l(\gamma) \leq \theta T \}}{e^{hT}}.
\]
Using the estimate \#\{\gamma : l(\gamma) \leq T\} = O(e^{hT}/T) \[13\], we have established
\[
0 \leq \frac{T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{T^{k/2}}{e^{hT}} \psi^*(T, \alpha) \leq \frac{\tau T^{k/2}}{e^{hT}} \psi^*(T, \alpha) + O(T^{k/2} e^{(\theta - 1)hT}),
\]
so that, by applying Proposition 6,
\[
\limsup_{T \to \infty} \sup_{\alpha \in \mathbb{Z}^k} \left| \frac{T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{T^{k/2}}{e^{hT}} \psi^*(T, \alpha) \right| \leq \frac{\tau}{(2\pi)^{k/2} h\sigma^k}.
\]
Since \(\tau > 0\) is arbitrary, this proves Theorem 1.

5. Homologically full Anosov flows

The asymptotic identity (0.1) has been generalized to certain transitive Anosov flows \(\phi : N \to N\), where \(N\) is a compact smooth Riemannian manifold. We now use \(\gamma\) to denote a (prime) periodic orbit of \(\phi\), with least period \(l(\gamma)\). Once again, we write \([\gamma]\) for the torsion-free part of the homology class of \(\gamma\) in \(H_1(N, \mathbb{Z}) \cong \mathbb{Z}^k \oplus \text{Tor}\). We say that \(\phi\) is homologically full if every homology class in \(H_1(N, \mathbb{Z})\) is represented
by a closed orbit. In this case, there exist $\xi \in H^1(N, \mathbb{R})$, $0 < h^* < h$ and $C_0 > 0$ such that

\begin{equation}
\pi(T, \alpha) \sim C_0 e^{-\langle \xi, \alpha \rangle} \frac{e^{h^* T}}{T^{n/2+1}}, \quad \text{as } T \to \infty.
\end{equation}

This result was first proved in [19], drawing on ideas from [10]. An alternative proof was given in [2] and a more precise version is contained in [17]. In this section, we shall sketch a new proof of (5.1), using the techniques discussed above. However, we will not make any claims about uniformity.

\textbf{Remark.} We can define a function $p : H^1(N, \mathbb{R}) \to \mathbb{R}$ by $p(\omega) = P(\omega(X))$, where $\omega$ is a closed 1-form representing the cohomology class $[\omega]$ and $X$ is the vector field tangent to $\phi$. Then $\xi$ and $h^*$ are characterized by the formulae

$$h^* = p(\xi) = \min \{p(\xi') : \xi' \in H^1(N, \mathbb{R})\}.$$ 

We begin by considering a modified family of $L$-functions. We define

\begin{equation}
L(s, t) = \prod (1 - e^{-s(\gamma) + \langle \xi, \alpha \rangle + 2\pi i t(\gamma)} )^{-1},
\end{equation}

which converges for $Re(s) > h^*$. The extension of $L'/L$ to a uniform strip, described in Propositions 1 and 2, is no longer valid; however, the next result provides a weaker substitute. As in the case of closed geodesics, an analysis due to Dolgopyat [5] is crucial here. For $\rho > 0$, write

$$\mathcal{R}(\rho) = \{s : Re(s) > h^* - |Im(s)|^{-\rho}, |Im(s)| \geq 1\}.$$ 

\textbf{Proposition 7 ([17]).} There exists a constant $\rho > 0$ such that, for all sufficiently small $\delta > 0$, the following statements are true.

(i) There exists an analytic function $s : U(\delta) \to \{z \in \mathbb{C} : Re(z) \leq h^*\}$, satisfying $s(0) = h^*$ and $Re(s(t)) < h^*$ for $t \neq 0$ such that

$$\frac{L'(s, t)}{L(s, t)} + \frac{1}{s - s(t)}$$

is analytic in $\mathcal{R}(\rho)$.

(ii) For $t \notin U(\delta)$, $L'(s, t)/L(s, t)$ is analytic in $\mathcal{R}(\rho)$.

\textbf{Proposition 8 ([17]).} There exist $C > 0$ and $\beta > 0$ such that, for all $t \in [-\pi, \pi]^k$,

$$\left| \frac{L'(s, t)}{L(s, t)} \right| \leq C |Im(s)|^\beta,$$

for $s \in \mathcal{R}(\rho)$.

Although the function $s(t)$ is now complex valued, it is still the case that $\nabla s(0) = 0$ and that $\nabla^2 s(0)$ is real and strictly negative definite. Moreover, again writing $\mathcal{D} = -\nabla^2 s(0)$ and $\sigma^{2k} = \det \mathcal{D}$, the function $e^{s(t) - h^*}$ still satisfies the conclusions of Proposition 3.

We shall now mimic the arguments of Section 2. However, the weaker bounds on $L'(s, t)/L(s, t)$ force us to use a more complicated auxiliary function. For $n \geq 0$, define

$$\psi_n(T, \alpha) = e^{\langle \xi, \alpha \rangle} \sum_{\substack{t(\gamma) \leq T, \gamma \in \alpha}} \Lambda(\gamma') \left(e^{h^* T} - e^{h^* t(\gamma')}\right)^n.$$
Then we have the identity
\[ \sigma^k T^{k/2} \psi_n(T, \alpha) = \frac{1}{(2\pi)^k} \int_{I(\pi \sigma \sqrt{T})} e^{-i(t, \alpha)/\sigma \sqrt{T}} S^*_T(t/\sigma \sqrt{T}) dt, \]
where
\[ S^*_T(t) = \sum_{\ell(\gamma') \leq T} \Lambda(\gamma') e^{(\ell, \gamma') + i(t, \gamma')} \left( e^{h^* T} - e^{h^* t} \right)^n. \]

In order to estimate the function \( S^*_T(t) \) we need the following identity, for \( d > h^* \),
\[ S^*_T(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \left( -\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{(s+n)T}}{s(s+1) \cdots (s+n)} ds, \]
where we have used the formula
\[ \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{y^s}{s(s+1) \cdots (s+n)} ds = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{\zeta(n)} & \text{if } y \geq 1. \end{cases} \]

Choose \( 0 < \epsilon < 1/\rho \) and set \( R = T^\epsilon \) and \( d = h^* + T^{-1} \). Then replacing the integral in (5.3) with the truncated integral \( \int_{d-iR}^{d+iR} \) introduces an error of order \( O(e^{(h^*+n)T}/T^n) \). Using the estimates
(a) \[ \left| \left( \int_{h^* - R^{-1} + iR}^{h^* - R^{-1} - iR} + \int_{h^* - R^{-1} + iR}^{h^* - R^{-1} - iR} \right) \left( -\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{(s+n)T}}{s(s+1) \cdots (s+n)} ds \right| = O(R^\beta e^{(h^*+n)T}) = O(e^{(h^*+n)T} T^{-\epsilon(\rho+n+1-\beta)}); \]
(b) \[ \left| \int_{h^* - R^{-1} \pm R}^{h^* - R^{-1} \mp R} \left( -\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{(s+n)T}}{s(s+1) \cdots (s+n)} ds \right| = O(R^{\beta+1} e^{(h^*+n-R^{-1})T}) = O(T^{\beta+1} e^{(h^*+n)T} e^{-T^{-\epsilon+\rho}}), \]
we may repeat the proof of Lemma 3 to obtain the following lemma.

**Lemma 6.** Setting \( N = \min\{cn, \epsilon(\rho+n+1-\beta)\} \), we have the following. (The implied constants are independent of \( t \).)

(i) For \( t \in U(\delta) \),
\[ S^*_T(t) = \frac{e^{(s(t)+n)T}}{s(t)(s(t)+1) \cdots (s(t)+n)} + O \left( \frac{e^{(h^*+n)T}}{T^N} \right). \]

(ii) For \( t \notin U(\delta) \), \( S^*_T(t) = O(e^{(h^*+n)T}/T^N) \).

Provided \( n \) is sufficiently large such that \( N > k/2 \), we may repeat the arguments used in the proof of Proposition 5 to obtain
\[ \lim_{T \to \infty} \sup_{\alpha \in \mathbb{R}^k} \left| \prod_{j=0}^n \left( h^* + j \right)^{\sigma^k T^{k/2}} \frac{1}{e^{(h^*+n)T}} \psi_n(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, D^{-1} \alpha \rangle/2T} \right| = 0. \]
From this it immediately follows that
\[ \psi_n(T, \alpha) \sim \frac{1}{(2\pi)^{k/2} \sigma^k} \prod_{j=0}^n \frac{1}{(h^* + j)} \frac{e^{(h^*+n)T}}{T^{k/2}}, \quad \text{as } T \to \infty. \]
The asymptotic formula

\[ \psi_0(T, \alpha) \sim \frac{1}{(2\pi)^{k/2}h^{k/2} \sigma^{k/2} T^{k/2}}, \quad \text{as } T \to \infty, \]

now follows by a standard inductive argument (cf. p. 35 of [7]). Finally, (5.1) may be deduced as in Section 3. (Note that one needs the \textit{a priori} estimate \( \limsup_{T \to \infty} (\pi(T, \alpha))^{1/T} \leq e^{k^*} \), which follows from the convergence of (5.2) for \( \Re(s) > h^* \).)

References


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