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WELL-POSEDNESS OF THE DIRICHLET PROBLEM FOR THE NON-LINEAR DIFFUSION EQUATION IN NON-SMOOTH DOMAINS

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Abstract. We investigate the Dirichlet problem for the parablic equation

$$u_t = \Delta u^m, m > 0,$$

in a non-smooth domain $\Omega \subset \mathbb{R}^{N+1}, N \geq 2$. In a recent paper [*U.G. Abdulla, J. Math. Anal. Appl., 260, 2 (2001), 384-403*] existence and boundary regularity results were established. In this paper we present uniqueness and comparison theorems and results on the continuous dependence of the solution on the initial-boundary data. In particular, we prove L_1-contraction estimation in general non-smooth domains.

1. Introduction

Consider the equation

$$u_t = \Delta u^m, \tag{1.1}$$

where $u = u(x,t), x = (x_1, \ldots, x_N) \in \mathbb{R}^N, N \geq 2, t \in \mathbb{R}_+, \Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2, m > 0, m \neq 1$. In this paper we continue our study of the Dirichlet problem (DP) for equation (1.1) in a general domain $\Omega \subset \mathbb{R}^{N+1}$. It can be stated as follows: given any continuous function on the parabolic boundary $\mathcal{P}\Omega$ of Ω, find a continuous extension of this function to the closure of Ω which satisfies (1.1) in $\overline{\Omega} \backslash \mathcal{P}\Omega$.

In the recent paper [1] existence and boundary regularity results were established (see Theorem 2.1 of [1]). In the case of classical DP for the linear heat equation ($m = 1$ in (1.1)) existence and boundary regularity results from [1] together with maximum principle imply the existence and uniqueness of the classical solution to DP (see Corollary 2.1 of [1]). For the precise result concerning the solvability of the classical DP we refer to another recent paper by the author [2]. The purpose of this paper is to establish uniqueness and comparison theorems and results on the continuous dependence of the solution on the initial-boundary data for the non-linear DP ($m \neq 1$ in (1.1)). A particular motivation for this work arises from the problem about the evolution of interfaces in problems for porous medium equation ($m > 1$ in (1.1)). Special interest concerns the cases when support of the initial

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data contains a corner or cusp singularity at some points. What about the movement of these kinds of singlularities along the interface? To solve this problem, it is important at the first stage to develop general theory of boundary-value problems in non-cylindrical domains with boundary surfaces which has the same kind of behaviour as the interface. In many cases this may be non-smooth and characteristic. It should be mentioned that in the one-dimensional case Dirichlet and Cauchy-Dirichlet problems for the reaction-diffusion equations in irregular domains were studied in recent papers by the author [3, 4]. Primarily applying this theory a complete description of the evolution of interfaces were presented in other recent papers [5, 6].

We now make precise the meaning of the solution to DP. Let Ω be a bounded open subset of $\mathbb{R}^{N+1}, N \geq 2$. Let the boundary $\partial\Omega$ of Ω consist of the closure of a domain $B\Omega$ lying on $t = 0$, a domain $D\Omega$ lying on $t = T \in (0, \infty)$ and a (not necessarily connected) manifold $S\Omega$ lying in the strip $0 < t \leq T$. Denote

$$\Omega(\tau) = \{(x, t) \in \Omega : t = \tau\}$$

and assume that $\Omega(t) \neq \emptyset$ for $t \in (0, T)$. We shall use the same notation as in [1]: $z = (x, t) = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}, N \geq 2, x = (x_1, \overline{x}) \in \mathbb{R}^N, \overline{x} = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}, |x|^2 = \sum_{i=1}^{N} |x_i|^2, |\overline{x}|^2 = \sum_{i=2}^{N} |x_i|^2$. For a point $z = (x, t) \in \mathbb{R}^{N+1}$ we denote by $B(z; \delta)$ an open ball in $\mathbb{R}^{N+1}$ of radius $\delta > 0$ and with center in z.

Assume that for an arbitrary point $z_0 = (x^0, t_0) \in S\Omega$ (or $(z_0 = (x^0, 0) \in \overline{S\Omega})$ there exists $\delta > 0$ and a continuous function ϕ such that, after a suitable rotation of x-axes, we have

$$\overline{S\Omega} \cap B(z_0, \delta) = \{z \in B(z_0, \delta) : x_1 = \phi(\overline{x}, t)\},$$
$$\text{sign } (x_1 - \phi(\overline{x}, t)) = 1 \text{ for } z \in B(z_0, \delta) \cap \Omega.$$

The set $\mathcal{P}\Omega = \overline{B\Omega} \cup S\Omega$ is called a parabolic boundary of Ω. Furthermore the class of domains with described structure will be denoted by $\mathcal{D}_{0,T}$.

Let $\Omega \in \mathcal{D}_{0,T}$ be given and let ψ be an arbitrary continuous non-negative function defined on $\mathcal{P}\Omega$. DP consists of finding a solution to equation (1.1) in $\Omega \cup D\Omega$ satisfying the initial-boundary condition

$$u = \psi \quad \text{on } \mathcal{P}\Omega. \tag{1.2}$$

We shall follow the following notion of weak solution (super- or subsolution):

Definition 1.1. We shall say that the function $u(x, t)$ is a solution (respectively, super- or subsolution) of DP (1.1), (1.2) if

(a) u is non-negative and continuous in $\overline{\Omega}$, locally Hölder continuous in $\Omega \cup D\Omega$, satisfying (1.2) (respectively satisfying (1.2) with $=$ replaced by $\geq$ or $\leq$),
(b) for any t_0, t_1 such that $0 < t_0 < t_1 \leq T$ and for any domain $\Omega_1 \in \mathcal{D}_{t_0,t_1}$ such that $\overline{\Omega}_1 \subset \Omega \cup D\Omega$ and $\partial B\Omega_1, \partial D\Omega_1, S\Omega_1$ being sufficiently smooth manifolds, the following integral identity holds:

$$\int_{D\Omega_1} uf dx = \int_{B\Omega_1} uf dx + \int_{\Omega_1} (uf_t + u^m \Delta f) dx dt - \int_{S\Omega_1} u^m \frac{\partial f}{\partial \nu} dx dt \tag{1.3}$$

(respectively (1.3) holds with $=$ replaced by $\geq$ or $\leq$), where $f \in C^{2,1}_{x,t}(\overline{\Omega}_1)$ is an arbitrary function (respectively non-negative function) that equals zero on $S\Omega_1$ and ν is the outward-directed normal vector to $\Omega_1(t)$ at $(x,t) \in S\Omega_1$.

DP for the porous medium equation in cylindrical domain with smooth boundary was investigated in [9, 14]. At the moment there is a complete well-established theory of (1.1) (and more general second order nonlinear degenerate and singular parabolic equations and systems) in cylindrical domains due to [7], [9]–[12], [14], [18]–[20]. We refer to survey articles [8, 15, 19] for the complete list of references.

The main result of the previous paper [1] on the existence and boundary regularity of the solution to DP is reformulated below in Theorem 2.1 of Section 2. In [1] a notion of parabolic modulus of left-lower (or left-upper) semicontinuity of the lateral boundary manifold at the given point (Definition 2.1, Section 2) was introduced. It is proved in [1] that the upper (or lower) Hölder condition on it, with critical value of the Hölder exponent being $\frac{1}{2}$ (see Assumption $\mathcal{A}$ in Section 2), is sufficient for the regularity of the boundary point and for the solvability of the DP.

Assumption $\mathcal{A}$ imposes a pointwise restriction to the regularity of the lateral boundary manifold $S\Omega$ in the small neighbourhood of the point $z_0 = (x^0, t_0) \in S\Omega$ which is situated below the hyperplane $t = t_0$. It relates to the parabolic nature of (1.1) and does not depend on m. Furthermore we always suppose that Assumption $\mathcal{A}$ is satisfied at every point $z_0 \in S\Omega$. We introduce in this paper another assumption (Assumption $\mathcal{M}$, Section 2), which plays a crucial role for the uniqueness (Theorem 2.2) of the constructed solution, as well as for the comparison results (Theorem 2.3). Under the same assumption we also prove continuous dependence of the solution on the initial and boundary data (Theorem 2.4, Corollaries 2.2 and 2.3). In particular, we prove L_1-contraction estimations ((2.5), (2.6)). Assumption $\mathcal{M}$ imposes a pointwise geometric restriction to the lateral boundary manifold $S\Omega$ in a small neighbourhood of the point $z_0 = (x^0, t_0) \in S\Omega, t_0 < T$, which is situated upper the hyperplane $t = t_0$. In Section 3 we explain the geometrical meaning of Assumption $\mathcal{M}$. We prove Theorems 2.2–2.4 in Section 4. A key point in the proof of these theorems is the boundary gradient estimates for the solution of the auxiliary linear backward-parabolic problem (see (4.18) and Theorem 5.1 below). To establish these estimates we use a modified version of the techniques proposed in previous paper [1] to prove the boundary regularity. To make it easy for the reader, we present the proof of the key auxiliary Theorem 5.1 (and Corollary 5.1) in a separate section, Section 5.

2. Statement of main results

Let $\Omega \in \mathcal{D}_{0,T}$ be a given domain and let $z_0 = (x^0, t_0) \in S\Omega$ be a given boundary point. For an arbitrary sufficiently small $\delta > 0$, consider a domain

$$P(\delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < (\delta + t - t_0)^{\frac{1}{2}}, t_0 - \delta < t < t_0\}.$$

Definition 2.1. Let

$$\omega(\delta) = \max(\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) : (\overline{x}, t) \in \overline{P(\delta)}).$$

The function $\omega(\delta)$ is called the parabolic modulus of left-lower semicontinuity of the function ϕ at the point $(\overline{x}^0, t_0)$. For sufficiently small $\delta > 0$ this function is well defined and converge to zero as $\delta \downarrow 0$.

Assumption $\mathcal{A}$. There exists a function $F(\delta)$ which is defined for all positive sufficiently small δ; F is positive with $F(\delta) \to 0+$ as $\delta \downarrow 0$ and

$$\omega(\delta) \leq \delta^{\frac{1}{2}} F(\delta). \tag{2.1}$$

It is proved in [1] that Assumption $\mathcal{A}$ is sufficient for the regularity of the boundary point z_0. Namely, the constructed limit solution takes the boundary value $\psi(z_0)$ at the point $z = z_0$ continuously in $\overline{\Omega}$.

Denote $x_1 = \overline{\phi}(\overline{x}) \equiv \phi(\overline{x}, 0)$.

Definition 2.2. Let

$$\omega_0(\delta) = \max(\overline{\phi}(\overline{x}^0) - \overline{\phi}(\overline{x}) : |\overline{x} - \overline{x}^0| \leq \delta).$$

The function $\omega_0(\delta)$ is called the modulus of lower semicontinuity of the function $x_1 = \overline{\phi}(\overline{x})$ at the point $\overline{x} = \overline{x}^0$.

Assumption $\mathcal{B}$. There exists a function $F_1(\delta)$ which is defined for all positive sufficiently small δ; F_1 is positive with $F_1(\delta) \to 0+$ as $\delta \downarrow 0$ and

$$\omega_0(\delta) \leq \delta F_1(\delta). \tag{2.2}$$

It is proved in [1] that Assumption $\mathcal{B}$ is sufficient for the regularity of the boundary point $z_0 = (x^0, 0) \in \overline{S\Omega}$. Thus the main existence and boundary regularity result of [1] reads:

Theorem 2.1 ([1])**.** *DP* (1.1), (1.2) *is solvable in a domain Ω which satisfies Assumption $\mathcal{A}$ at every point $z_0 \in S\Omega$ and Assumption $\mathcal{B}$ at every point $z_0 = (x^0, 0) \in \overline{S\Omega}$.*

Remark 2.1. It should be mentioned that in [1] we did not include to the definition of the solution (see Definition 1.1 in [1]) the property of local Hölder continuity in $\Omega \cup D\Omega$. However, from the proof given in [1] it may easily be observed that constructed solution has this property, since it is a limit of a sequence of classical solutions $\{u_n\}$ which are uniformly Hölder continuous on every compact subset of $\Omega \cup D\Omega$. We use this property in the proof of uniqueness of the constructed solution as well. For that reason this property is included in the definition of weak solution (super- or subsolution) in this paper.

The following corollary is an easy consequence of Theorem 2.1.

Corollary 2.1. *Let the conditions of Theorem* 2.1 *be satisfied and* $\inf_{\mathcal{P}\Omega} \psi > 0$. *Then there exists a unique classical solution $u \in C(\overline{\Omega}) \cap C^\infty(\Omega \cup D\Omega)$ of the DP.*

Indeed, from the proof of Theorem 2.1 it easily follows that constructed solution satisfies the inequality $\inf_{\Omega} u \geq \inf_{\mathcal{P}\Omega} \psi > 0$. It should be noted that we construct the solution u as a limit of a sequence of classical solutions $\{u_n\}$ to non-degenerate parabolic equation $u_t = div(mu^{m-1}\nabla u)$ and the Hölder norm of u_n is uniformly bounded on every interior compact subset of $\Omega \cup D\Omega$ with sufficiently smooth boundary. Applying classical apriori interior estimations [13, Theorem 10, ch. III], we derive that $C^{2+\alpha,1+\alpha/2}_{x,t}$-norm of u_n(with some $0 < \alpha < 1$) is uniformly bounded on this interior subset. By standard methods we easily derive that the limit solution u has the same regularity. Thus u is a classical solution and its uniqueness follows from the maximum principle. Applying apriori interior estimations from [13, Theorem 10, ch. III] arbitrarily many times we similarly derive that $u \in C(\overline{\Omega}) \cap C^\infty(\Omega \cup D\Omega)$.

Furthermore, **we always suppose in this paper that the conditions of Theorem 2.1 are satisfied.** We are going now to formulate another pointwise restriction at the point $z_0 = (x^0, t_0) \in S\Omega, 0 < t_0 < T$, which plays a crucial role in the proof of uniqueness of the constructed solution. For an arbitrary sufficiently small $\delta > 0$, consider a domain

$$Q(\delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < (\delta + t_0 - t)^{\frac{1}{2}}, t_0 < t < t_0 + \delta\}. \tag{2.3}$$

Our restriction on the behaviour of the funtion ϕ in $Q(\delta)$ for small δ is as follows:

Assumption $\mathcal{M}$. Assume that for all sufficiently small positive δ we have

$$\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) \leq [t - t_0 + |\overline{x} - \overline{x}^0|^2]^\mu \text{ for } (\overline{x}, t) \in \overline{Q(\delta)}, \tag{2.4}$$

where $\mu > \frac{1}{2}$ if $0 < m < 1$, and $\mu > \frac{m}{m+1}$ if $m > 1$.

Furthermore we denote $\nu = \mu - 1$ assuming without loss of generality that $\nu \in (-\frac{1}{2}, 0)$ if $0 < m < 1$ and $\nu \in (-\frac{1}{m+1}, 0)$ if $m > 1$. Assumption $\mathcal{M}$ is of geometric nature. We explain its meaning below in Section 3. In Section 5 (see Theorem 5.1) we prove under Assumption $\mathcal{M}$ a boundary gradient estimate at the point $z = z_0$ for the solution to the linear backward parabolic problem (see (4.18) below). This estimation plays a crucial role for our main results.

Assumption $\mathcal{M}$ is pointwise and related number μ in (2.4) depends on $z_0 \in S\Omega$ and may vary for different points $z_0 \in S\Omega$. For our purposes we need to define "the uniform assumption $\mathcal{M}$" for certain subsets of $S\Omega$.

Definition 2.3. Let $[c, d] \subset (0, T)$ be a given segment and

$$S\Omega_{[c,d]} = S\Omega \cap \{(x, t) : c \leq t \leq d\}.$$

We shall say that Assumption $\mathcal{M}$ is satisfied uniformly in $[c, d]$ if there exists $\delta_0 > 0$ and $\mu > 0$ as in (2.4) such that for $0 < \delta \leq \delta_0$, (2.4) is satisfied for all $z_0 \in S\Omega_{[c,d]}$ with the same μ.

Our main theorems read:

Theorem 2.2 (Uniqueness)**.** *Assume that there exists a finite number of points $t_i, i = 1, \ldots, k$, such that $t_1 = 0 < t_2 < \cdots < t_k < t_{k+1} = T$ and for the arbitrary compact subsegment $[\delta_1, \delta_2] \subset (t_i, t_{i+1}), i = 1, \ldots, k$, Assumption $\mathcal{M}$ is uniformly satisfied in $[\delta_1, \delta_2]$. Then the solution of the DP is unique.*

Theorem 2.3 (Comparison)**.** *Let u be a solution of DP and let g be a supersolution (respectively subsolution) of DP. Assume that the assumption of Theorem 2.2 is satisfied. Then $u \leq$ (respectively $\geq$) g in $\overline{\Omega}$.*

Theorem 2.4 (Stability or L_1-contraction)**.** *Let the assumption of Theorem 2.2 be satisfied. Let g_1 and g_2 be solutions of DP with inital boundary data ψ_1 and ψ_2, respectively. If $\psi_1 = \psi_2 = \psi$ on $S\Omega$, then for arbitrary $t \in [0, T]$ we have*

$$||g_1 - g_2||_{L_1(\Omega(t))} \leq ||\psi_1 - \psi_2||_{L_1(B\Omega)}. \tag{2.5}$$

Corollary 2.2. *Let the assumption of Theorem 2.4 be satisfied. Then for every t and τ such that $0 \leq \tau \leq t \leq T$ we have*

$$||g_1 - g_2||_{L_1(\Omega(t))} \leq ||g_1 - g_2||_{L_1(\Omega(\tau))}. \tag{2.6}$$

Theorem 2.4 presents a continuous dependence of the solution to DP on the initial data. Concerning continuous dependence on the initial-boundary data we have the following

Corollary 2.3. *Let the assumption of Theorem* 2.2 *be satisfied. Let* u *be a solution of DP. Assume that* $\{\psi_n\}$ *is a sequence of non-negative continuous functions defined on* $\mathcal{P}\Omega$ *and* $\lim_{n\to\infty} \psi_n(z) = \psi(z)$*, uniformly for* $z \in \mathcal{P}\Omega$*. Let* u_n *be a solution of DP* (1.1), (1.2) *with* $\psi = \psi_n$*. Then* $u = \lim_{n\to\infty} u_n$ *in* $\overline{\Omega}$ *and convergence is uniform on compact subsets of* $\Omega \cup D\Omega$*.*

3. Geometric meaning of Assumption $\mathcal{M}$

Assumption $\mathcal{M}$ is of geometric nature. To explain its meaning, for simplicity assume that $N = 2$ and rewrite (2.4) as follows:

$$x_1^0 - x_1 \le [t - t_0 + (x_2 - x_2^0)^2]^\mu \text{ for } (x_2, t) \in \overline{Q(\delta)}, \tag{3.1}$$

where $x_1^0 = \phi(x_2^0, t_0)$ and $x_1 = \phi(x_2, t)$ for $(x_2, t) \in \overline{Q(\delta)}$. Consider the hyperbolic paraboloid $x_1^2 = t + x_2^2$ (Figure 1) in the x_1x_2t-space.

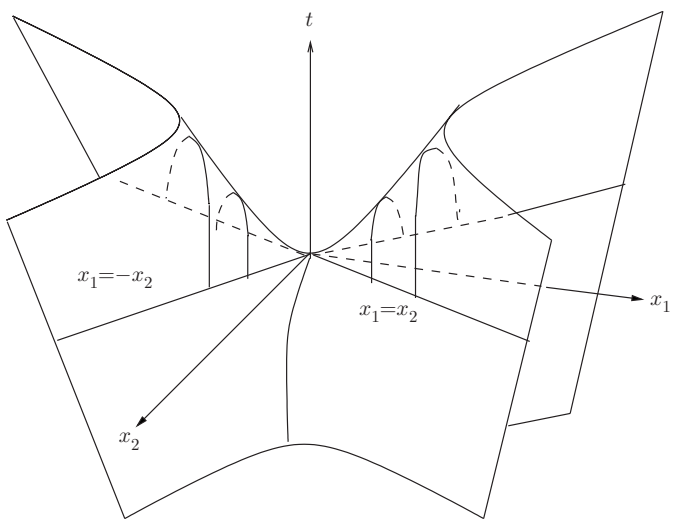

Figure 1. Hyperbolic paraboloid $x_1^2 = t + x_2^2$.

Let M_δ be the piece of it lying in the half-space $\{t \ge 0\}$, between the planes $\{x_1 = 0\}$ and $\{x_1 = -\delta^{\frac{1}{2}}\}$ (see Figure 2).

The projection of M_δ to the plane $\{x_1 = 0\}$ is $\overline{Q_0(\delta)}$, where as $Q_0(\delta)$ we denote $Q(\delta)$ with $N = 2, x_2^0 = 0, t_0 = 0$. The surface M_δ has the following representation:

$$x_1 = \phi(x_2, t) \equiv -\sqrt{x_2^2 + t}, (x_2, t) \in \overline{Q_0(\delta)}. \tag{3.2}$$

Obviously, the function (3.2) satisfies (2.4) with $=$ instead of $\le$ in the critical case when $\mu = \frac{1}{2}$ (we also replace $Q(\delta)$ with $Q_0(\delta)$ in (2.4)). Consider the rigid body displacements of M_δ composed of translations and (or) rotations in x-space and shift along the t-axis.

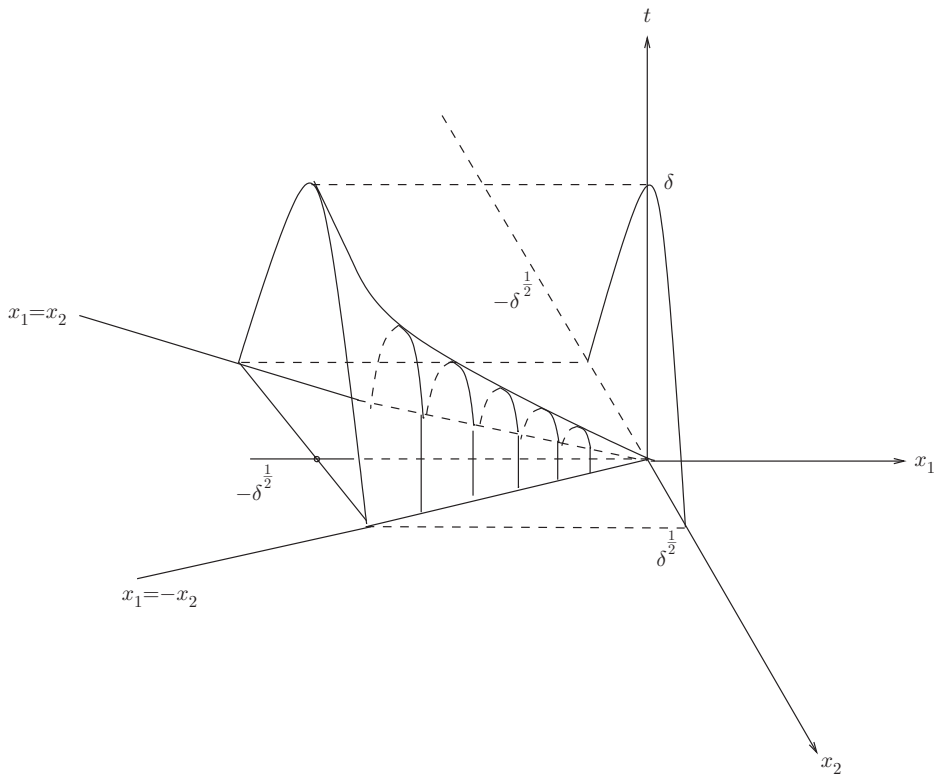

FIGURE 2. Piece M_δ of the hyperbolic paraboloid from Figure 1 lying in the half-space $\{t \geq 0\}$, between the planes $\{x_1 = 0\}$ and $\{x_1 = -\delta^{\frac{1}{2}}\}$.

Let us now consider the critical case of Assumption $\mathcal{M}$ with $\mu = \frac{1}{2}$. Namely, we take $\frac{1}{2}$ in (2.4) and (3.1). Equivalent formulation of this critical assumption may be given as follows:

Assume that after the displacement of the above type, M_δ occupies such a position that its vertex coincides with the point $z_0 = (x^0, t_0) \in S\Omega$, and for all sufficiently small positive δ, it has no common point with Ω.

Similar geometric reformulation of Assumption $\mathcal{M}$ may be given just modifying subsurface M_δ according to the lower restriction imposed on μ. Thus if $0 < m < 1$, then the exterior touching surface is slightly more regular at the vertex point than related subsurface M_δ of the hyperbolic paraboloid. Otherwise speaking, it is slightly more regular than $C^{1,\frac{1}{2}}_{x,t}$ at the vertex point. When m changes from 1 to $+\infty$, the regularity of M_δ increases continuously, with each m being slightly more regular than $C^{\frac{2m}{m+1},\frac{m}{m+1}}_{x,t}$ at the vertex point. Another limit position of M_δ as $m \to +\infty$ (or $\mu \to 1-$) is the upper paraboloid frustrum with vertex at the origin.

4. Proofs of Theorems 2.2–2.4

Proof of Theorem 2.2. Suppose that g_1 and g_2 are two solutions of DP. We shall prove uniqueness by proving that

$$g_1 \equiv g_2 \text{ in } \overline{\Omega} \cap \{(x,\tau) : t_j \le \tau \le t_{j+1}\}, j = 1, \dots, k. \tag{4.1}$$

We present the proof of (4.1) only for the case $j = 1$. The proof for cases $j = 2, \dots, k$ coincides with the proof for the case $j = 1$. We prove (4.1) with $j = 1$ by proving that for some limit solution $u = \lim u_n$ the following inequalities are valid:

$$\int\limits_{\Omega(t)} (u(x,t) - g_i(x,t))\omega(x)dx \le 0, i = 1, 2, \tag{4.2}$$

for every $t \in (0, t_2)$ and for every $\omega \in C_0^\infty(\Omega(t))$ such that $|\omega| \le 1$. Obviously, from (4.2) it follows that

$$g_1 = u = g_2 \ \text{ in } \ \overline{\Omega} \cap \{(x,\tau) : t_1 \le \tau < t_2\}, \tag{4.3}$$

which implies (4.1) with $j = 1$ in view of continuity of u, g_1 and g_2 in $\overline{\Omega}$. Since the proof of (4.2) is similar for each i, we shall henceforth let $g = g_i$. Let $t \in (0, t_2)$ be fixed and let $\omega \in C_0^\infty(\Omega(t))$ be an arbitrary function such that $|\omega| \le 1$. To construct the required limit solution, as in [1], we approximate Ω and ψ with a sequence of smooth domains $\Omega_n \in \mathcal{D}_{0,T}$ and smooth positive functions ψ_n. We make a slight modification to the construction of Ω_n and ψ_n. Let Ψ be a non-negative and continuous function in $\mathbb{R}^{N+1}$, which coincides with ψ on $\mathcal{P}\Omega$. This continuation is always possible. Let ψ_n be a sequence of smooth functions such that

$$\max(\Psi; n^{-1}) \le \psi_n \le (\Psi^m + Cn^{-m})^{\frac{1}{m}}, n = 1, 2, \dots, \tag{4.4}$$

where $C > 1$ is a fixed constant. For arbitrary subset $G \subset \mathbb{R}^{N+1}$ and $\rho > 0$, we define

$$O_\rho(G) = \bigcup_{z \in G} B(z, \rho).$$

Since g and Ψ are continuous functions in $\overline{\Omega}$ and $g = \psi$ on $\mathcal{P}\Omega$, for arbitrary n there exists $\rho_n > 0$ such that

$$|g^m(z) - \Psi^m(z)| \le n^{-m} \text{ for } z \in O_{\rho_n}(\overline{S\Omega}) \cap \overline{\Omega}. \tag{4.5}$$

We then assume that Ω_n satisfies the following:

$$\Omega_n \in \mathcal{D}_{0,T}, \overline{\Omega}_n \subseteq \Omega \cup D\Omega, S\Omega_n \subseteq O_{\rho_n}\overline{(S\Omega)}. \tag{4.6}$$

We now formulate assumptions on $S\Omega_n$ near its point z_n, which are direct implications of Assumption $\mathcal{M}$ at the point $z_0 \in S\Omega$. Assume that $S\Omega_n$ in some neighbourhood of its point $z_n = (x_1^{(n)}, \overline{x}^0, t_0)$ is represented by the function $x_1 = \phi_n(\overline{x}, t)$, where $\{\phi_n\}$ is a sequence of sufficiently smooth functions and $\phi_n \to \phi$ as $n \to +\infty$, uniformly in $\overline{Q(\delta_0)}$, where $\delta_0 > 0$ be a sufficiently small fixed number, which does not depend on n. Obviously, we can assume that ϕ_n satisfies Assumption $\mathcal{M}$ (namely (2.4)) at the point $(\overline{x}^0, t_0)$, uniformly with respect to n and with the same exponent μ. Let $\{\delta_n\}$ be some sequence of positive real numbers such that $\delta_n \to 0$

as $n \to +\infty$. Assume also that the sequence $\{\phi_n\}$ is chosen such that, for n being large enough, the following inequality is satisfied:

$$\phi_n(\overline{x}^0, t_0) - \phi_n(\overline{x}, t) \le \delta_n^\nu [t - t_0 + |\overline{x} - \overline{x}^0|^2] \text{ for } (\overline{x}, t) \in \overline{Q(\delta_n)}. \tag{4.7}$$

Obviously, this is possible in view of uniform convergence of ϕ_n to ϕ. For example, if $\phi(\overline{x}, t)$ coincides with its lower bound $\tilde{\phi}(\overline{x}, t) = \phi(\overline{x}^0, t_0) - [t - t_0 + |\overline{x} - \overline{x}^0|^2]^\mu$, for $(\overline{x}, t) \in \overline{Q(\delta_0)}$ (namely (2.4) is satisfied with $=$ instead of $\le$), then for all large n such that $\delta_n < \delta_0$ we first choose $\tilde{\phi_n}$ as follows:

$$\tilde{\phi_n}(\overline{x}, t) = \begin{cases} \phi(\overline{x}^0, t_0) - \delta_n^\nu [t - t_0 + |\overline{x} - \overline{x}^0|^2] \text{ for } (\overline{x}, t) \in \overline{Q(\delta_n)}, \\ \tilde{\phi}(\overline{x}, t) \text{ for } (\overline{x}, t) \in \overline{Q(\delta_0)} \backslash \overline{Q(\delta_n)}. \end{cases}$$

Obviously, $\tilde{\phi_n}$ satisfies (4.7) and converges to ϕ uniformly in $Q(\delta_0)$. Then we easily construct ϕ_n by smoothing $\tilde{\phi_n}$ at the boundary points of $Q(\delta_n)$ satisfying $t - t_0 + |\overline{x} - \overline{x}^0|^2 = \delta_n$. In general, we can do similar construction by taking the function $\tilde{\tilde{\phi}}_n(\overline{x}, t) = \max(\tilde{\phi_n}(\overline{x}, t); \phi(\overline{x}, t))$ instead of $\tilde{\phi_n}(\overline{x}, t)$, which satisfies (4.7) and converges to $\phi(\overline{x}, t))$ as $n \to +\infty$, uniformly in $\overline{Q(\delta_0)}$. Furthermore we will assume that the sequence δ_n is chosen as follows:

$$\delta_n = n^{\frac{\frac{1}{\gamma} - m - \epsilon}{1 + 2\nu}} \text{ with } 0 < \epsilon < (1 + \nu)^{-1} \Big[\nu \Big(m + \frac{1}{\gamma}\Big) + \frac{1}{\gamma}\Big], \tag{4.8}$$

where $\gamma = 1$ if $m > 1$, while if $0 < m < 1$, then γ is chosen such that

$$\frac{1}{m} < \gamma < -\frac{1 + \nu}{\nu m}. \tag{4.9}$$

Let u_n be a classical solution to the following problem:

$$u_t = \Delta u^m, \text{ in } \Omega_n \cup D\Omega_n, \tag{4.10}$$

$$u = \psi_n \text{ on } \mathcal{P}\Omega_n. \tag{4.11}$$

This is a non-degenerate parabolic problem and classical theory [13, 16, 17] implies the existence of a unique $C_{2+\alpha}$ solution. From maximum principle and (4.4) it follows that

$$n^{-1} \le u_n \le M \text{ in } \overline{\Omega}_n, n = 1, 2, \ldots, \tag{4.12}$$

where M is some constant which does not depend on n and $M > \max(\sup_{\mathcal{P}\Omega} \psi, \sup_{\mathcal{P}\Omega_n} \psi_n)$. As in [1], we then prove that for some subsequence n', $u = \lim_{n' \to \infty} u_n'$ is a solution of DP (1.1), (1.2). Furthermore, without loss of generality we write n instead of n'. Take an arbitrary sequence of real numbers $\{\alpha_l\}$ such that

$$0 < \alpha_{l+1} < \alpha_l < t, \alpha_l \downarrow 0 \text{ as } l \to +\infty. \tag{4.13}$$

Let

$$\Omega_n^l = \Omega_n \cap \{(x, \tau) : \alpha_l < \tau < t\}, \Omega_n^0 = \Omega_n \cap \{(x, \tau) : 0 < \tau < t\},$$
$$S\Omega_n^l = S\Omega_n \cap \{(x, \tau) : \alpha_l < \tau < t\}, S\Omega_n^0 = S\Omega_n \cap \{(x, \tau) : 0 < \tau < t\}.$$

Since u_n is a classical solution of (4.10), it satisfies

$$\int_{\Omega_n(t)} u_n f dx = \int_{\Omega_n(\alpha_l)} u_n f dx + \int_{\Omega_n^l} (u_n f_\tau + u_n^m \Delta f) dx d\tau - \int_{S\Omega_n^l} u_n^m \frac{\partial f}{\partial \nu} dx d\tau \tag{4.14}$$

for arbitrary $f \in C^{2,1}_{x,t}(\overline{\Omega}^l_n)$ that equals zero on $S\Omega^l_n$, and $\nu = \nu(x,\tau)$ is the outward-directed normal vector to $\Omega_n(\tau)$ at $(x,\tau) \in S\Omega^l_n$. Since g is the weak solution of the DP (1.1), (1.2), we also have

$$\int\limits_{\Omega_n(t)} gf dx = \int\limits_{\Omega_n(\alpha_l)} gf dx + \int\limits_{\Omega^l_n} (gf_\tau + g^m \Delta f) dx d\tau - \int\limits_{S\Omega^l_n} g^m \frac{\partial f}{\partial \nu} dx d\tau. \tag{4.15}$$

Subtracting (4.15) from (4.14), we derive

$$\begin{aligned} \int\limits_{\Omega_n(t)} (u_n - g) f dx = & \int\limits_{\Omega_n(\alpha_l)} (u_n - g) f dx - \int\limits_{S\Omega^l_n} (u_n^m - g^m) \frac{\partial f}{\partial \nu} dx d\tau \\ & + \int\limits_{\Omega^l_n} (u_n^{\frac{1}{\gamma}} - g^{\frac{1}{\gamma}})[C_n f_\tau + A_n \Delta f] dx d\tau, \end{aligned} \tag{4.16}$$

where $C_n = 1$ if $m > 1$ (accordingly $\gamma = 1$) and $C_n = B_n$ if $0 < m < 1$, and

$$A_n = m\gamma \int\limits_0^1 (\theta u_n^{\frac{1}{\gamma}} + (1-\theta) g^{\frac{1}{\gamma}})^{m\gamma - 1} d\theta, B_n = \gamma \int\limits_0^1 (\theta u_n^{\frac{1}{\gamma}} + (1-\theta) g^{\frac{1}{\gamma}})^{\gamma - 1} d\theta.$$

The functions A_n and B_n are Hölder continuous in $\overline{\Omega}_n$. From (4.12) and Definition 1.1 it follows that

$$n^{\frac{1-m\gamma}{\gamma}} \le A_n \le \overline{A}, n^{\frac{1-\gamma}{\gamma}} \le B_n \le \overline{B} \text{ for } (x,\tau) \in \overline{\Omega}_n, \tag{4.17}$$

where $\overline{A}$, $\overline{B}$ are some positive constants which do not depend on n. To choose the test function $f = f(x,\tau)$ in (4.16), consider the following problem:

$$C_n f_\tau + A_n \Delta f = 0 \text{ in } \Omega^0_n \cup B\Omega_n, \tag{4.18a}$$

$$f = 0 \text{ on } S\Omega^0_n, \tag{4.18b}$$

$$f = \omega(x) \text{ on } \Omega_n(t). \tag{4.18c}$$

This is the linear non-degenerate backward-parabolic problem. From the classical parabolic theory ([13, 16, 17]) it follows that there exists a unique classical solution $f_n \in C^{2+\beta,1+\beta/2}_{x,\tau}(\overline{\Omega}^0_n)$ with some $\beta > 0$. From the maximum principle it follows that

$$|f_n| \le 1 \quad \text{in } \overline{\Omega^0_n}. \tag{4.19}$$

In the next section we prove boundary gradient estimates for f_n (see Theorem 5.1, Corollary 5.1 and (5.1), (5.2) in Section 5). Here we use the results of Section 5 in order to estimate the right-hand side of (4.16) with $f = f_n(x,\tau)$, which is a solution of the problem (4.18). We have

$$\int\limits_{\Omega_n(t)} (u_n - g)\omega(x) dx = \int\limits_{\Omega_n(\alpha_l)} (u_n - g) f dx - \int\limits_{S\Omega^l_n} (u_n^m - g^m) \frac{\partial f}{\partial \nu} dx d\tau \equiv \mathcal{I}_1 + \mathcal{I}_2. \tag{4.20}$$

By using (4.4)–(4.6), we have

$$|\mathcal{I}_2| \le \sup_{z \in S\Omega^l_n} |\nabla f(z)| \int\limits_{S\Omega^l_n} (|\psi_n^m - \Psi^m| + |\Psi^m - g^m|) dx d\tau \le (C+1) n^{-m} \sup_{z \in S\Omega^l_n} |\nabla f(z)|. \tag{4.21}$$

From (5.2) and (4.21) we derive

$$|\mathcal{I}_2| \leq (C+1)C_1(l)n^{\frac{\epsilon(1+\nu)-\frac{1}{\gamma}-\nu(m+\frac{1}{\gamma})}{1+2\nu}}, \tag{4.22}$$

where ϵ and γ are chosen as in (4.8) and (4.9). Applying (4.19), we have

$$|\mathcal{I}_1| \leq \int\limits_{\Omega_n(\alpha_l)} |u_n - g|dx. \tag{4.23}$$

To estimate the right-hand side, introduce a function

$$u_n^l(x) = \begin{cases} u_n(x,\alpha_l), & x \in \overline{\Omega_n(\alpha_l)}, \\ \psi_n(x,\alpha_l), & x \in \overline{\Omega(\alpha_l)} \backslash \overline{\Omega_n(\alpha_l)}. \end{cases}$$

Obviously $u_n^l(x), x \in \overline{\Omega(\alpha_l)}$ is bounded uniformly with respect to n, l. From (4.23), we have

$$|\mathcal{I}_1| \leq \int\limits_{\Omega(\alpha_l)} |u_n^l - g|dx. \tag{4.24}$$

Since

$$\lim_{n\to+\infty} u_n^l(x) = u(x,\alpha_l) \text{ for } x \ \in \overline{\Omega(\alpha_l)},$$

from Lebesgue's theorem it follows that

$$\lim_{n\to+\infty} \int\limits_{\Omega(\alpha_l)} |u_n^l - g|dx = \int\limits_{\Omega(\alpha_l)} |u(x,\alpha_l) - g(x,\alpha_l)|dx. \tag{4.25}$$

Hence, by using (4.21)–(4.24) in (4.20) and passing to the limit $n \to +\infty$ we have

$$\int\limits_{\Omega(t)} (u-g)\omega(x)dx \leq \int\limits_{\Omega(\alpha_l)} |u-g|dx. \tag{4.26}$$

Let

$$U_l(x) = \begin{cases} u(x,\alpha_l) - g(x,\alpha_l), & x \in \overline{\Omega(\alpha_l)}, \\ 0, & x \notin \overline{\Omega(\alpha_l)}. \end{cases}$$

Obviously, U_l is uniformly bounded with respect to l. Hence, from (4.26) we derive that

$$\int\limits_{\Omega(t)} (u-g)\omega(x)dx \leq \int\limits_{B\Omega} |U_l(x)|dx + C_2 \cdot meas(\Omega(\alpha_l)\backslash B\Omega), \tag{4.27}$$

where the constant C_2 does not depend on l. From Lebesgue's theorem it follows that

$$\lim_{l\to+\infty} \int\limits_{B\Omega} |U_l(x)|dx = 0.$$

Hence, passing to the limit $l \to +\infty$, from (4.27), (4.2) follows. As we explained earlier, from (4.2), (4.1) with $j = 1$ follows. Similarly, we prove (4.1) (step by step) for each $j = 2, \ldots, k$. The theorem is proved.

Proof of Theorem 2.3. Let us prove the theorem for supersolutions. The proof is similar to the proof of uniqueness. We prove (step by step) that

$$u \le g \text{ in } \overline{\Omega} \cap \{(x,\tau) : t_j \le \tau \le t_{j+1}\}, j = 1, \dots, k. \tag{4.28}$$

We present the proof of (4.28) only for the case $j = 1$. The proof for cases $j = 2, \dots, k$ coincides with the proof for the case $j = 1$. Obviously, to prove (4.28) with $j = 1$ it is enough to prove that for each fixed $t \in (0, t_2)$ the following inequality is valid

$$u \le g \text{ in } \overline{\Omega(t)}. \tag{4.29}$$

Since u and g are continuous in $\overline{\Omega}$, from (4.29), (4.28) with $j = 1$ follows. Suppose on the contrary that $u(x_*, t) > g(x_*, t)$ for some $(x_*, t) \in \Omega(t)$. The continuity of g and u implies that for some $\eta > 0$

$$u(x,t) > g(x,t) \text{ for } |x - x_*| < \eta, \tag{4.30}$$

where $\eta > 0$ is chosen such that $\{(x,t) : |x - x_*| < \eta\} \subset \Omega(t)$. Then we take an arbitrary function $\omega \in C_0^\infty(\Omega(t))$ such that

$$0 \le \omega \le 1 \text{ for } x \in \overline{\Omega(t)},$$
$$\omega > 0 \text{ for } |x - x_*| < \eta;\ \omega = 0 \text{ for } |x - x_*| \ge \eta.$$

Our goal will be achieved if we prove the inequality

$$\int\limits_{\Omega(t)} (u(x,t) - g(x,t))\omega(x)dx \le 0, \tag{4.31}$$

which is a contradiction of our assumption (4.30). Let us prove (4.31). First, we construct a sequence $\{u_n\}$ as in Theorems 2.1 and 2.2. A slight modification is made concerning the choise of the number $\rho_n > 0$ via (4.5). Consider the function $G = \max(\Psi; g)$. Since $\Psi = \psi \le g$ on $\mathcal{P}\Omega$, it may easily be observed that $G = g$ on $\mathcal{P}\Omega$. Obviously, G is a continuous function satisfying

$$\Psi \le G \text{ in } \overline{\Omega}. \tag{4.32}$$

Since g and G are continuous functions in $\overline{\Omega}$ and $g = G$ on $\mathcal{P}\Omega$, for arbitrary n there exists $\rho_n > 0$ such that

$$|g^m(x,\tau) - G^m(x,\tau)| \le n^{-m} \text{ for } (x,\tau) \in O_{\rho_n}(\overline{S\Omega}) \cap \overline{\Omega}. \tag{4.33}$$

We then assume that Ω_n satisfies (4.6) with ρ_n defined via (4.33). As before, there exists a subsequence n' such that $u_{n'}$ converges to the solution of DP (without loss of generality we write n instead of n'). Since u is a unique solution of DP we have $u = \lim u_n$. We then take a sequence of real numbers $\{\alpha_l\}$ as in (4.13). Since g is a supersolution of DP, it satisfies (4.15) with $\ge$ instead of $=$. Substracting this inequality from (4.14), we derive instead of (4.16)

$$\begin{aligned} \int\limits_{\Omega_n(t)} (u_n - g)f dx \le \int\limits_{\Omega_n(\alpha_l)} (u_n - g)f dx - \int\limits_{S\Omega_n^l} (u_n^m - g^m)\frac{\partial f}{\partial \nu} dx d\tau \\ + \int\limits_{\Omega_n^l} (u_n^{\frac{1}{\gamma}} - g^{\frac{1}{\gamma}})[C_n f_\tau + A_n \Delta f] dx d\tau. \end{aligned} \tag{4.34}$$

Instead of f in (4.34) we take the classical solution f_n of problem (4.18). Since $0 \le \omega \le 1$, from the maximum principle it follows that $0 \le f_n \le 1$ in $\overline{\Omega}_n^0$, and hence

$$\frac{\partial f_n}{\partial \nu} \le 0 \text{ in } S\Omega_n^l. \tag{4.35}$$

From (4.18), (4.32), (4.34) and (4.35) we have

(4.36)
$$\int\limits_{\Omega_n(t)} (u_n - g)\omega(x)dx \le \int\limits_{\Omega_n(\alpha_l)} (u_n - g)f dx - \int\limits_{S\Omega_n^l} (\psi_n^m - \Psi^m + G^m - g^m)\frac{\partial f}{\partial \nu} dx d\tau \equiv I_1 + I_2.$$

As in the proof of Theorem 2.2, we estimate I_1 as follows:

$$I_1 \le \int\limits_{\Omega_n(\alpha_l)} (u_n - g)_+ dx \le \int\limits_{\Omega(\alpha_l)} (u_n^l - g)_+ dx, (x)_+ = \max(x; 0),$$

$$\lim_{n\to+\infty} \int\limits_{\Omega(\alpha_l)} (u_n^l - g)_+ dx = \int\limits_{\Omega(\alpha_l)} (U_l(x))_+ dx.$$

Since $U_l(x),\ x \in \Omega(\alpha_l)$, is uniformly bounded with respect to l, we have

$$\int\limits_{\Omega(\alpha_l)} (U_l(x))_+ dx \le \int\limits_{B\Omega} (U_l(x))_+ dx + C_3 meas(\Omega(\alpha_l) \backslash B\Omega),$$

where the constant C_3 does not depend on l. From Lebesgue's theorem it follows that

$$\lim_{l\to+\infty} \int\limits_{B\Omega} (U_l(x))_+ dx = \int\limits_{B\Omega} (u(x,0) - g(x,0))_+ dx = 0,$$

and hence

$$\lim_{l\to+\infty} \int\limits_{\Omega(\alpha_l)} (U_l(x))_+ dx = 0.$$

The estimation of I_2 coincides with that of $\mathcal{I}_2$ given in the previous proof (see (4.21) and (4.22)). The only difference is that in the expression of the function $A_n(x,\tau)$ and $B_n(x,\tau)$ the function g means the supersolution of DP instead of solution. Similarly, by using (4.17), we prove the boundary gradient estimates (5.1), (5.2) for f_n as in Section 5. Applying (5.2), we estimate I_2 as we estimated $\mathcal{I}_2$ in (4.21),(4.22). Hence, from (4.36), the desired inequality (4.29) follows. The proof for supersolutions is completed. The proof for subsolutions is similar. The theorem is proved.

Proof of Theorem 2.4. The proof is similar to the proof of uniqueness Theorem 2.2. As before, let $t \in (0, t_2)$ be fixed and let $\omega \in C_0^\infty(\Omega(t))$ be an arbitrary function such that $|\omega| \le 1$. We approximate Ω and ψ_1 with a sequence of smooth domains $\Omega_n \in \mathcal{D}_{0,T}$ and smooth functions ψ_{1n} as in the proof of Theorems 2.1 and 2.2. Then we construct a sequence $\{u_n\}$ of classical solutions. In view of uniqueness we have $g_1 = \lim\limits_{n\to\infty} u_n$ (as before, we write n instead of subsequence n'). Replacing g with

g_2 in the proof of Theorem 2.2, we derive instead of (4.20)

$$\int\limits_{\Omega_n(t)} (u_n - g_2)\omega(x)dx = \int\limits_{\Omega_n(\alpha_l)} (u_n - g_2) f dx - \int\limits_{S\Omega_n^l} (u_n^m - g_2^m)\frac{\partial f}{\partial \nu} dx d\tau \equiv \mathcal{I}_1 + \mathcal{I}_2. \tag{4.37}$$

Estimating $\mathcal{I}_1$ and $\mathcal{I}_2$ as in (4.21)–(4.25) and passing to the limit $n \to +\infty$ we have instead of (4.26)

$$\int\limits_{\Omega(t)} (g_1 - g_2)\omega(x)dx \le \int\limits_{\Omega(\alpha_l)} |g_1 - g_2| dx. \tag{4.38}$$

Since ω is arbitrary, from (4.38) it follows that

$$||g_1 - g_2||_{L_1(\Omega(t))} \le ||g_1 - g_2||_{L_1(\Omega(\alpha_l))}. \tag{4.39}$$

Estimating the right-hand side of (4.39) as in (4.26) and (4.27), and passing to the limit $l \to +\infty$, from (4.39), (2.5) follows. Since g_1 and g_2 are continuous functions, it easily follows that (2.5) is valid for $t = t_2$ as well. Finally, we can prove (2.5) (step by step) for each $t \in [0, T]$. The theorem is proved.

Corollary 2.2 easily follows from the given proof.

Proof of Corollary 2.3. Since the sequence ψ_n is uniformly bounded, from the proof of Theorem 2.1 of [1] and from [11, 12] it follows that the sequence $\{u_n\}$ is uniformly bounded in $\overline{\Omega}$ and uniformly equicontinuous on every interior compact subset of $\Omega \cup D\Omega$. As in the proof of Theorem 2.1, by a diagonalization argument and the Arzela-Ascoli theorem we may find a subsequence n' and a limit function $\tilde{\vartheta} \in C(\Omega \cup D\Omega)$ such that $u_{n'} \to \tilde{\vartheta}$ as $n' \to +\infty$, pointwise in $\Omega \cup D\Omega$ and the convergence is uniform on compact subsets of $\Omega \cup D\Omega$. Now consider a function ϑ such that $\vartheta = \tilde{\vartheta}$ for $(x,t) \in \Omega \cup D\Omega, \vartheta = \psi$ for $(x,t) \in \mathcal{P}\Omega$. Obviously, the function ϑ is also continuous on $B\Omega$, since the above-mentioned result on the equicontinuity of the sequence $\{u_n\}$ is true up to some neighbourhood of every point $z \in B\Omega$. Our purpose is to prove that $\vartheta = u$ in $\overline{\Omega}$.

Let Ψ_n be a sequence of non-negative, continuous and uniformly bounded functions in $\mathbb{R}^{N+1}$, which coincides with ψ_n on $\mathcal{P}\Omega$. This continuation is always possible. Let $\tilde{\Psi}_n$ be a sequence of smooth functions such that

$$\max(\Psi_n; n^{-1}) \le \tilde{\Psi}_n \le (\Psi_n^m + Cn^{-m})^{\frac{1}{m}}, n = 1, 2, \ldots,$$

where $C > 1$ is a fixed constant. Then we approximate Ω with a sequence of smooth domains $\Omega_n \in \mathcal{D}_{0,T}$ as in the proof of Theorem 2.2 (see also Theorem 2.1 of [1]). The only difference is that to define ρ_n, we replace g and Ψ in (4.5) with u_n and Ψ_n, respectively. Let $\tilde{u}_n$ be a classical solution to the problem (4.10), (4.11), with ψ_n replaced by $\tilde{\Psi}_n$ in (4.11). As in [1], we then prove that some subsequence $\tilde{u}_{n''}$ converges to the solution of DP (1.1), (1.2). From the Uniqueness Theorem 2.2 it follows that $u = \lim\limits_{n\to\infty} \tilde{u}_n$ and the convergence is uniform on compact subsets of $\Omega \cup D\Omega$.

Now we estimate $\tilde{u}_n - u_n$ in order to prove that $\vartheta = u$ in $\overline{\Omega}$. This estimation is similar to that given above in (4.12)–(4.27), where we have to replace u_n and g with $\tilde{u}_n$ and u_n, respectively. From Theorems 2.2, 2.3 and the maximum principle

it easily follows that $0 \le u_n \le M,\ n^{-1} \le \tilde{u}_n \le M$ in $\overline{\Omega}_n$, where the upper bound M does not depend on n. Accordingly the related functions

$$A_n = m\gamma \int_0^1 (\theta \tilde{u}_n^{\frac{1}{\gamma}} + (1-\theta) u_n^{\frac{1}{\gamma}})^{m\gamma - 1} d\theta, B_n = \gamma \int_0^1 (\theta \tilde{u}_n^{\frac{1}{\gamma}} + (1-\theta) u_n^{\frac{1}{\gamma}})^{\gamma - 1} d\theta$$

satisfy (4.17). Thus we can prove the boundary gradient estimates (5.1) and (5.2) for the solution to the problem (4.18), as we did in Section 5. Applying (5.2), the difference $\tilde{u}_n - u_n$ may be estimated as in (4.20)–(4.26). In this context the function g on the right-hand side of (4.25) and in (4.26), (4.27) should be replaced by the limit function ϑ. As before, passing to the limit first with respect to $n' \to +\infty$ and then with respect to $l \to +\infty$ (see (4.25)–(4.27)), we prove that $\vartheta = u$ in $\overline{\Omega}$. Since under the conditions of Corollary 2.3 u is a unique solution of the DP, it follows that the sequence $\{u_n\}$ converges to u. The corollary is proved.

5. Boundary gradient estimates for the linear backward parabolic problem

In this section we establish boundary gradient estimates for problem (4.18).

Theorem 5.1. *Let Assumption $\mathcal{M}$ be satisfied at the point $z_0 = (x_1^0, \overline{x}^0, t_0) \in S\Omega, 0 < t_0 \le t$. Let $z_n = (x_1^{(n)}, \overline{x}^0, t_0) = (x^n, t_0) \in S\Omega_n^0, n = 1, 2, \dots,$ be such a sequence that $z_n \to z_0$ as $n \to \infty$. Then a solution f_n of the problem* (4.18) *satisfies*

$$|\nabla f_n(z_n)| = O\Big(n^{\frac{(1+\nu)(m+\epsilon-\frac{1}{\gamma})}{1+2\nu}}\Big) \text{ as } n \to +\infty, \tag{5.1}$$

where ϵ and γ are chosen as in (4.8), (4.9). *Moreover, the related constant on the right-hand side of* (5.1) *depends on $t_0 > 0$ and may converge to $+\infty$ as $t_0 \downarrow 0$.*

Corollary 5.1. *Let Assumption $\mathcal{M}$ be satisfied uniformly on every compact subsegment of $(0, t]$. Then for every fixed l (see* (4.13)*) there exists a positive constant $C(l)$, which does not depend on n, such that*

$$\sup_{z \in S\Omega_n^l} |\nabla f_n(z)| \le C(l) n^{\frac{(1+\nu)(m+\epsilon-\frac{1}{\gamma})}{1+2\nu}}. \tag{5.2}$$

Obviously, it is enough to consider the case $t_0 < t$, since if $t_0 = t$ then $\nabla f_n(z_n) = \nabla \omega(z_n) = 0$.

Let us now estimate $|\nabla f_n(z_n)|$. Denote $x^n = (x_1^{(n)}, \overline{x}^0) \equiv (\phi_n(\overline{x}^0, t_0), \overline{x}^0)$. Instead of estimating direct $|\nabla f_n(z_n)|$, we are going to estimate

$$[f_n(z_n)] = \sup_{x \in F_n} \frac{|f_n(x, t_0) - f_n(x^n, t_0)|}{|x - x^n|} = \sup_{x \in F_n} \frac{|f_n(x, t_0)|}{|x - x^n|}, \tag{5.3}$$

where F_n is some neighbourhood of z_n in $\Omega_n(t_0)$. Since $\nabla f_n \in C(\overline{\Omega}_n^0)$, we have

$$|\nabla f_n(z_n)| \le [f_n(z_n)]. \tag{5.4}$$

To estimate $[f_n(z_n)]$ we establish a suitable upper estimation for f_n in some neighbourhood of the point z_n. To estimate f_n we use a modified version of the method proposed in [1] to prove a boundary regularity of the solution to the DP (1.1), (1.2).

Consider a function

$$\omega_n(x, \tau) = g(\xi) \equiv log[e - (e-1)\delta_n^{-1-\nu}\xi],$$

where

$$\xi = \phi_n(\overline{x}^0, t_0) + \delta_n^{1+\nu} - x_1 - 2\delta_n^{\nu}[\tau - t_0 + |\overline{x} - \overline{x}^0|^2]$$

(note that δ_n is defined in (4.8)). Then we set

$$V_n = \{z = (x, \tau) : \phi_n(\overline{x}, \tau) < x_1 < \phi_{1n}(\overline{x}, \tau), (\overline{x}, \tau) \in Q(\delta_n)\},$$

$$\phi_{1n}(\overline{x}, \tau) = \phi_n(\overline{x}^0, t_0) + \delta_n^{1+\nu} - 2\delta_n^{\nu}[(\tau - t_0 + |\overline{x} - \overline{x}^0|^2].$$

In the next lemma we clear the structure of ∂V_n.

Lemma 5.1. *The closure of the set*

$$\partial_0 V_n = \partial V_n \cap \{(x, \tau) : \tau > t_0\} \tag{5.5}$$

consists of two boundary surfaces $x_1 = \phi_n(\overline{x}, \tau)$ *and* $x_1 = \phi_{1n}(\overline{x}, \tau)$.

Proof. From (2.4) for ϕ_n it follows that

$$\phi_{1n}(\overline{x}, \tau) - \phi_n(\overline{x}, \tau) = \phi_n(\overline{x}_0, t_0) - \phi_n(\overline{x}, \tau) - \delta_n^{1+\nu} \leq 0$$

$$\text{for } |\overline{x} - \overline{x}^0| = (\delta_n + t_0 - \tau)^{\frac{1}{2}}, t_0 \leq \tau \leq t_0 + \delta_n, \tag{5.6}$$

and the assertion of lemma immediately follows. The lemma is proved.

It is natural to call $\overline{\partial_0 V_n}$ the backward-parabolic boundary of V_n. The latter means that $\overline{\partial_0 V_n}$ is a parabolic boundary of the transformed domain V_n after change of the variable τ with $-\tau$. In the next lemma we estimate f_n via the barrier function ω_n on the backward-parabolic boundary $\overline{\partial_0 V_n}$ of V_n. A special structure of V_n, established in Lemma 5.1, plays a crucial role in the proof of this lemma.

Lemma 5.2. *If n is large enough, then*

$$f_n(x, \tau) \leq \omega_n(x, \tau) \text{ on } \overline{\partial_0 V_n}. \tag{5.7}$$

Proof. We have

$$\omega_n|_{x_1 = \phi_{1n}(\overline{x}, \tau)} = g(0) = 1.$$

Hence, from (4.19) it follows that (5.7) is valid on the part of $\overline{\partial_0 V_n}$ with $x_1 = \phi_{1n}(\overline{x}, t)$. Then we observe that

$$\omega_n|_{x_1 = \mathcal{H}_n(\overline{x}, \tau)} = 0,$$

$$\omega_n \geq 0 \text{ for } x_1 \geq \mathcal{H}_n(\overline{x}, \tau), \tag{5.8}$$

where

$$\mathcal{H}_n(\overline{x}, \tau) = \phi_n(\overline{x}^0, t_0) - 2\delta_n^{\nu}[\tau - t_0 + |\overline{x} - \overline{x}^0|^2].$$

Hence, from (4.7) it follows that

$$\mathcal{H}_n(\overline{x}, \tau) \leq \phi_n(\overline{x}, \tau) \text{ for } (\overline{x}, \tau) \in \overline{Q(\delta_n)}. \tag{5.9}$$

From (5.8), (5.9) it follows that

$$\omega_n \geq 0 \text{ for } (x, \tau) \in \overline{\partial V_n} \cap \{(x, \tau) : x_1 = \phi_n(\overline{x}, \tau)\} \tag{5.10}$$

and hence (5.7) is also valid on the part of $\overline{\partial_0 V_n}$ with $x_1 = \phi_n(\overline{x}, \tau)$. The lemma is proved.

Lemma 5.3. *If n is large enough, then*

$$L\omega_n \equiv -C_n(x, \tau)\omega_{n_\tau} - A_n(x, \tau)\Delta\omega_n > 0 \text{ for } (x, \tau) \in V_n. \tag{5.11}$$

Proof. First, we easily derive that

$$0 \le \xi \le \delta_n^{1+\nu} \text{ for } (x,\tau) \in \overline{V}_n. \tag{5.12}$$

The right-hand side of (5.12) follows from (5.8)–(5.10), while the left-hand side is a consequence of the inequality $x_1 \le \phi_{1n}(\overline{x},\tau)$. Let us transform $L\omega_n$:

$$L\omega_n = (2C_n + 4A_n(N-1))\delta_n^{\nu} g'(\xi) - A_n(1 + 16\delta_n^{2\nu}|\overline{x} - \overline{x}^0|^2)g''(\xi). \tag{5.13}$$

Obviously, we have

$$1 - e \le \delta_n^{1+\nu} g'(\xi) \le \frac{1-e}{e}, -(e-1)^2 \le \delta_n^{2(1+\nu)} g'' \le -\frac{(e-1)^2}{e^2} \text{ for } 0 \le \xi \le \delta_n^{1+\nu}. \tag{5.14}$$

Thus from (5.13), (4.17) and (5.14) it follows that

$$L\omega_n \ge -(2\overline{C} + 4\overline{A}(N-1))(e-1)\delta_n^{-1} + n^{\frac{1-m\gamma}{\gamma}} e^{-2}(e-1)^2 \delta_n^{-2(1+\nu)} \text{ in } V_n,$$

where $\overline{C} = 1$ if $m > 1$, and $\overline{C} = \overline{B}$ if $0 < m < 1$. Hence, the assertion of the lemma is a consequence of our choice of the sequence δ_n via (4.8). The lemma is proved.

By the standard maximum principle from Lemma 5.1, (5.7) and (5.11) it follows that

$$f_n \le \omega_n \text{ in } \overline{V}_n$$

Since (4.18a) is linear, we also derive that $f_n \ge -\omega_n$ in $\overline{V}_n$ and hence

$$|f_n| \le \omega_n \text{ in } \overline{V}_n. \tag{5.15}$$

Now by using (5.15) we can estimate $[f_n(z_n)]$ from (5.3) letting $F_n = \overline{V}_n \cap \{(x,\tau) : \tau = t_0\}$ and keeping in mind that $f_n(x^n, t_0) = \omega_n(x^n, t_0) = 0$:

$$[f_n(z_n)] \le \sup_{x \in F_n} \frac{\omega_n(x,t_0)}{|x - x^n|} = \sup_{x \in F_n} \frac{\omega_n(x,t_0) - \omega_n(x^n,t_0)}{|x - x^n|} \le \sup_{x \in F_n} |\nabla \omega_n(x,t_0)|. \tag{5.16}$$

We have

$$|\omega_{n_{x_1}}| \le (e-1)\delta_n^{-1-\nu}, \ |\omega_{n_{x_i}}| \le 4(e-1)\delta_n^{-\frac{1}{2}}, i = 2, \dots, N \text{ in } \overline{F}_n.$$

From (5.4), (5.16) it follows that

$$|\nabla f_n(z_n)| = O(\delta_n^{-1-\nu}) \text{ as } n \to +\infty. \tag{5.17}$$

From (5.17) and (4.8), (5.1) follows. Theorem 5.1 is proved.

Corollary 5.1 follows from the given proof of Theorem 5.1 by using Definition 2.3 from Section 2. Indeed, in view of the condition of Corollary 5.1, for each fixed l (or $\alpha_l \in (0,t)$ from (4.13)) Assumption $\mathcal{M}$ is satisfied uniformly in $[\alpha_l, t]$. The related numbers μ and δ_0 (see Definition 2.3) may depend on l and t, but do not depend on the points $z \in S\Omega \cap \{(x,\tau) : \alpha_l \le \tau \le t\}$. It may be easily seen that under this condition neither the largeness of n, which is required in the proof of Lemmas 5.1–5.3, nor the right-hand sides of (5.16), (5.17) vary for different points $z_n \in S\Omega_n^l$. Hence, under the condition of Corollary 5.1, (5.2) easily follows.

6. Appendix

At some stage of the long and difficult referring process one of the referees claimed that my uniqueness and comparison results are a consequence of the results in Feireisl, Petzeltova and Simondon ([FPS]), "Admissible solutions for a class of nonlinear parabolic problems with non-negative data", Proceedings of Royal Society of Edinburgh, 131A (2001), 857-883. After studying this report together with my reply the final referee made the following conclusion: "... if the referee raises the point that Abdulla's uniqueness and comparison results are a consequence of the results in [FPS], it means that many other readers may have the same false impression."

For that reason and also following the recommendation of the final referee and the editor I address this point below.

In the above-mentioned paper [FPS] the notion of admissible solution, which is the adaptation of the well-known notion of viscosity solution to the case of porous-medium kind equations, was introduced. Roughly speaking, admissible solutions are solutions which satisfy a comparison principle. Accordingly, admissible solution of the DP will be unique in view of its definition. By using a simple analysis one can show that the limit solution of the DP (1.1), (1.2) which I constructed in [1] is an admissible solution. However, this does not solve the problem of the uniqueness of the weak solution to DP. The question must be whether every weak solution in the sense of Definition 1.1 is an admissible solution. It is not possible to answer this question staying in the "admissible framework" and one should take as a starting point the integral identity (1.3). In fact, the uniqueness Theorem 2.2 addresses exactly this question and one can express its proof as follows: if there are two weak solutions of the DP, then one can construct a limit solution (or admissible solution) which coincides with both of them provided that Assumption $\mathcal{M}$ is satisfied. Under the same conditions, Theorems 2.3 and 2.4 address a general comparison theorem for arbitrary weak super- and subsolutions and important L_1-contraction estimation for weak solutions.

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