THE $\alpha$-INVARIANT ON CERTAIN SURFACES
WITH SYMMETRY GROUPS

JIAN SONG

Abstract. The global holomorphic $\alpha$-invariant introduced by Tian is closely related to the existence of Kähler-Einstein metrics. We apply the result of Tian, Yau and Zelditch on polarized Kähler metrics to approximate plurisubharmonic functions and compute the $\alpha$-invariant on $\mathbb{C}P^2 \# n\mathbb{C}P^2$ for $n = 1, 2, 3$.

1. Introduction

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian [7], Tian and Yau [6] is closely related to the existence of Kähler-Einstein metrics. In his solution of the Calabi conjecture, Yau [12] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with negative or zero first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there are known obstructions such as the Futaki invariant. For a compact Kähler manifold $M$ with positive first Chern class, Tian [7] proved that $M$ admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \text{dim } M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $\mathbb{C}P^2 \# 1\mathbb{C}P^2$ and $\mathbb{C}P^2 \# 2\mathbb{C}P^2$ [9]. Nevertheless, it would also be interesting to find the estimate of the $\alpha$ invariant for $\mathbb{C}P^2 \# 1\mathbb{C}P^2$ and $\mathbb{C}P^2 \# 2\mathbb{C}P^2$. In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman potential on polarized Kähler manifolds to approximate plurisubharmonic functions and compute the $\alpha$-invariant of $\mathbb{C}P^2 \# n\mathbb{C}P^2$ for $n = 1, 2, 3$. In the case of $\mathbb{C}P^2 \# 2\mathbb{C}P^2$, it gives an improvement of Abdesselem’s result [1]. More precisely, we shall show that:

Theorem 1. $\alpha_G(\mathbb{C}P^2 \# 2\mathbb{C}P^2) = \frac{1}{3}$.

We will give the definitions of the automorphism group $G$ and the $\alpha_G$-invariant in Section 3.

Let $(M, \omega)$ be a compact Kähler manifold, where $\omega = \sqrt{-1}g_{ij}dz_i \wedge d\overline{z}_j$. We will also prove Tian’s conjecture on the generalized Moser-Trudinger inequality in the special case where $\alpha_G(M) > \frac{n}{n+1}$, for $n = \text{dim } M$. Let

$$P(M, \omega) = \left\{ \phi \mid \omega \phi = \omega + \sqrt{-1} \partial \overline{\partial} \phi > 0, \sup_M \phi = 0 \right\}.$$
Let $F_\omega$ and $J_\omega$ be the functionals defined on $P(M, \omega)$ by

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n - \log \left( \frac{1}{V} \int_M e^{h_\omega - \phi} \omega^n \right),$$

$$J_\omega(\phi) = \frac{\sqrt{-1}}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial \phi \wedge \overline{\partial} \phi \wedge \omega^i \wedge \omega_{\omega}^{n-i-1}.$$

Assume $(M, \omega_{KE})$ is a Kähler-Einstein manifold with positive first Chern class and $Ric(\omega_{KE}) = \omega_{KE}$. Then for any $\phi \in P(M, \omega_{KE})$, Ding and Tian [2] proved the following inequality of Moser-Trudinger type:

$$\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{J_\omega(\phi) - \frac{1}{2} \int_M \phi}.$$

Tian [10] also conjectured that $\frac{1}{2} \int_M e^{-\phi} \omega^n \leq C e^{(1-\delta)J_\omega(\phi) - \frac{1}{2} \int_M \phi}$ for $\delta > 0$ sufficiently small, if $\phi$ is perpendicular to $\Lambda_1$, the space of eigenfunctions of $\omega_{KE}$ with eigenvalue one.

We shall prove:

**Theorem 2.** Let $(M, \omega)$ be a Kähler manifold with positive first Chern class. Assume that $\alpha(M) > \frac{n}{n+1}$, so that $M$ admits a Kähler-Einstein metric $\omega_{KE}$, and there exist constants $\delta = \delta(n, \alpha(M))$ and $C = C(n, \lambda_2(\omega_{KE}) - 1, \alpha(M))$ such that for any $\phi \in P(M, \omega_{KE})$ which satisfies $\phi \perp \Lambda_1$, we have

$$F_{\omega_{KE}}(\phi) \geq \delta J_{\omega_{KE}}(\phi) - C.$$

Here $\lambda_2(\omega_{KE})$ is the least eigenvalue of $\omega_{KE}$ which is bigger than 1.

## 2. Holomorphic Approximation of Plurisubharmonic Functions

In this section, we will employ the technique in [8, 13] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [13].

**Theorem 2.1.** Let $M$ be a compact complex manifold of dimension $n$ and let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m \in \mathbb{N}$, $h$ induces a Hermitian metric $h_m$ on $L^m$. Let $\{S_0^m, S_1^m, \ldots, S_{d_m-1}^m\}$ be an orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product:

$$(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!} \omega_g^n$ is the volume form of $g$. Then there is a complete asymptotic expansion

$$\sum_{i=0}^{d_m-1} ||S_i^m(x)||_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \ldots$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any $k$,

$$\sum_{i=0}^{d_m-1} ||S_i^m(x)||_{h_m}^2 - \sum_{j<R} a_j(x)m^{n-j}||_{C^k} \leq C_{R,k}m^{n-R}$$

where $C_{R,k}$ depends on $R, k$ and the manifold $M$. 
Let 
\[ \tilde{\omega}_g = \omega_g + \sqrt{-1} \partial \bar{\partial} \phi > 0, \]
\[ \tilde{h} = h e^{-\phi}. \]
Let \( \tilde{h}_m \) be the induced Hermitian metric of \( \tilde{h} \) on \( L^m \), and let \( \{ \tilde{S}_0^m, \tilde{S}_1^m, \ldots, \tilde{S}_{d_m-1}^m \} \) be any orthonormal basis of \( H^0(M, L^m) \), where \( d_m = \dim H^0(M, L^m) \), with respect to the inner product 
\[ (S_1, S_2)_{\tilde{h}_m} = \int_M \tilde{h}_m(S_1(x), S_2(x)) dV_g. \]
By Theorem 2.1, we have 
\[ \sum_{i=0}^{d_m-1} ||\tilde{S}_i^m(x)||^2_{\tilde{h}_m} = \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i^m(x)||^2_{\tilde{h}_m} \right) e^{-m \phi}. \]
Thus 
\[ \phi - \frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i^m(x)||^2_{\tilde{h}_m} \right) = -\frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i^m(x)||^2_{\tilde{h}_m} \right). \]
As \( m \to +\infty \), we obtain for any positive integer \( R \)
\[ \frac{1}{m} \log \left( \sum_{j<R} \tilde{a}_j(x) m^{n-j} \right) = \frac{1}{m} \log \left( \sum_{j<R} \tilde{a}_j(x) m^{n-j} \right) \]
\[ = \frac{n}{m} \log m + \frac{1}{m} \log(1 + O(\frac{1}{m})) \to 0. \]
Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

**Corollary 2.1.**
\[ \left\| \phi - \frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i^m(x)||^2_{\tilde{h}_m} \right) \right\|_{C^k} \to 0, \text{ as } m \to +\infty. \]

In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of \( L^m \).

### 3. Proof of Theorem 1

Let \( M \) be the blow-up of \( CP^2 \) at two points and \( \pi \) be its natural projection. Without loss of generality, we may assume the two points are \( p_1 = [0, 1, 0] \) and \( p_2 = [0, 0, 1] \). Then \( M \) is a subvariety of \( CP^2 \times CP^1 \times CP^1 \) defined by the equations 
\[ Z_0 X_1 = Z_1 X_0, \quad Z_0 Y_2 = Z_2 Y_0, \]
where \( Z_i, X_j, Y_k \) are the homogeneous coordinates on \( CP^2, CP^3 \) and \( CP^1 \), respectively.

Let \( G \) be the automorphism group acting on \( CP^2 \times CP^1 \times CP^1 \) generated by \( \theta_j \) and permutations \( \tau \ (0 \leq j \leq 2) \),
\[ \theta_j : [Z_0, Z_j, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_j e^{i\theta}, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \]

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for \( \theta \in [0, 2\pi) \), and

\[
\tau : [Z_0, Z_1, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \rightarrow [Z_0, Z_2, Z_1] \times [Y_0, Y_2] \times [X_0, X_1].
\]

Let \( \pi_0, \pi_1, \pi_2 \) be the projection from \( CP^2 \cong CP^1 \times CP^1 \) onto \( CP^2, CP^1 \) and \( CP^1 \). Respectively define \( \omega \) by

\[
\omega = \pi_0^* \omega_0 + \pi_1^* \omega_1 + \pi_2^* \omega_2
\]

\[
= \sqrt{-1} \partial \overline{\partial} \log |Z_0|^2 + |Z_1|^2 + |Z_2|^2 + \sqrt{-1} \partial \overline{\partial} \log |X_0|^2 + |X_1|^2 + \sqrt{-1} \partial \overline{\partial} \log |Y_0|^2 + |Y_2|^2,
\]

where \( \omega_0, \omega_1, \omega_2 \) are the Fubini-Study metrics in \( CP^2, CP^1 \) and \( CP^1 \). By explicit calculation, it can be shown that the cohomological class of \( \omega \mid_M \) is in the first Chern class of \( M \) (see [1]).

Consider the divisor

\[
\{[0, Z_1, Z_2] \times CP^1 \times CP^1\} + \{CP^2 \times [1, 0] \times CP^1\} + \{CP^2 \times CP^1 \times [1, 0]\}
\]

which defines a line bundle \((L, h)\) on \( CP^2 \times CP^1 \times CP^1 \). The hermitian metric \( h \) is defined by

\[
h = \frac{1}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|X_0|^2 + |X_1|^2)(|Y_0|^2 + |Y_2|^2)};
\]

then \((L, h) \mid_M \rightarrow M\) defines the anticanonical line bundle on \( M \) whose curvature form \(-\sqrt{-1} \partial \overline{\partial} \log h \) gives the first Chern class of \( M \).

Since \( M \setminus \{p_1, p_2\} \) is isomorphic to \( CP^2 \setminus \{p_1, p_2\} \), if we choose the inhomogeneous coordinates \((z_1, z_2) = [z_1, z_2] \) on \( CP^2 \), the Kähler metric

\[
\omega_{g_0} = \sqrt{-1} \partial \overline{\partial} \log (1 + |z_1|^2 + |z_2|^2) + \sqrt{-1} \partial \overline{\partial} \log (1 + |z_1|^2) + \sqrt{-1} \partial \overline{\partial} \log (1 + |z_2|^2)
\]

can be extended to a Kähler metric \( g_0 \) on \( M \) which belongs to \( c_1(M) \). If we take different inhomogeneous coordinates \((w_0, w_1) = [w_0, w_1, 1] \), the corresponding Kähler metric is

\[
\omega_{g_1} = \sqrt{-1} \partial \overline{\partial} \log (1 + |w_0|^2 + |w_1|^2) + \sqrt{-1} \partial \overline{\partial} \log (1 + |w_0|^2) + \sqrt{-1} \partial \overline{\partial} \log (1 + |w_1|^2)
\]

and we have

\[
\det g_0 = \frac{1}{(1 + |z_1|^2 + |z_2|^2)^3} + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_1|^2)} + \frac{1}{(1 + |z_1|^2)^2(1 + |z_2|^2)^2},
\]

\[
\det g_1 = \frac{1}{(1 + |w_0|^2 + |w_1|^2)^3} + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(1 + |w_0|^2)} + \frac{1}{(1 + |w_0|^2)^2(1 + |w_1|^2)^2}.
\]

Consider the line bundle \((L^N, h_N) \rightarrow CP^2 \times CP^1 \times CP^1 \). Then

\[
\dim H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) = \frac{(N + 1)^3(N + 2)}{2}
\]

and \( \{Z_0z_1Z_2x_0x_1x_1y_0y_0y_2y_2\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} \) is an orthogonal basis for \( H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \).

Let \( M_1 \) be the hypersurface of \( CP^2 \times CP^1 \times CP^1 \) defined by the equations

\[Z_0X_1 = Z_1X_0,\]
and $M_2$ the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0 Y_2 = Z_2 Y_0.$$ 

Then $M = M_1 \cap M_2$.

In view of the short exact sequences

$$0 \to \mathcal{O}(L^N - [M_1]) \to \mathcal{O}(L^N) \to \mathcal{O}(L^N|_{M_1}) \to 0,$$

$$0 \to \mathcal{O}(L^N|_{M_1} - [M]) \to \mathcal{O}(L^N|_{M_1}) \to \mathcal{O}(L^N|_{M}) \to 0$$

we can choose $N$ sufficiently large so that

$$H^1(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N - [M_1])) = H^1(M_1, \mathcal{O}(L^N|_{M_1} - [M])) = 0.$$ 

Then $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \to H^0(M_1, \mathcal{O}(L^N|_{M_1})) \to 0$,

$$H^0(M_1, \mathcal{O}(L^N|_{M_1})) \to H^0(M, \mathcal{O}(L^N|_{M})) \to 0$$

and thus

$$H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \to H^0(M, \mathcal{O}(L^N|_{M})) \to 0.$$ 

Also we have $Z_0 Z_2 X_0 X_1 Y_0 Y_2|_{M} = Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}$ and

$$||Z_0^{i_0} Z_2^{i_2} X_0 X_1 Y_0 Y_2||^2_{h_N} = \frac{\left| Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_0^{j_0} Z_1^{j_1} Z_2^{k_2} \right|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N}$$

on $CP^2\backslash \{p_1, p_2\}$. Therefore, $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0 X_1 Y_0 Y_2|_{M}\}_{i_0+i_1+i_2+j_0+j_1+k_0+k_2=N}$ contains an orthogonal basis for $H^0(M, \mathcal{O}(L^N|_{M}))$ with respect to $h^N$ and the $G$-invariant Kähler metric $g$ on $M$.

By Corollary 2.1, for any $\varphi \in P_G(M, \omega_g)$, we have on $CP^2\backslash \{p_1, p_2\}$,

$$\varphi([Z_0, Z_1, Z_2]) = \lim_{N \to \infty} \frac{1}{N} \log \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=N} a_{(\varphi)i_0i_1j_0j_1k_0k_2}^{(N)} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2$$

for some coefficients $a_{(\varphi)i_0i_1j_0j_1k_0k_2}^{(N)}$ satisfying $a_{(\varphi)i_0i_1j_0j_1k_0k_2}^{(N)} = a_{(\varphi)i_0i_1j_0j_1k_0k_2}^{(N)}$ due to the group action by $G$.

**Lemma 3.1.** Using the notations above we have

$$\frac{1}{n} \log \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2$$

$$((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^n \leq 4$$

for any positive integer $n.$
Proof. On the patch $U_0 = \{Z_0 \neq 0\}$, let $z_1 = \frac{Z_1}{Z_0}$ and $z_2 = \frac{Z_2}{Z_0}$,
\[
\frac{1}{n} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n} |Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2 \right) \leq \frac{1}{n} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n} \frac{|Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{(1 + |z_1|^2)^n(1 + |z_2|^2)^n(1 + |z_2|^2)^n} \right)
\]
\[
\leq \frac{1}{n} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n} \frac{|Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{1 + |z_1^{i_1+j_1}z_2^{i_2+k_2}|^2} \right)
\]
\[
= \frac{1}{n} \log \left( \frac{(n+1)^{n+2}}{2} \right) \leq 4.
\]

This inequality also holds on the patch $U_1 = \{Z_1 \neq 0\}$ by continuity, and so the lemma is proved. \hfill \Box

Lemma 3.2. There exists $\varepsilon > 0$ such that for any $\varphi \in P_G(M, \omega_g)$ and $N$, there exist $n > N$, $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ with $i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n$, and $(a^{(n)}_{\varphi})_{i_0,i_1,i_2,j_0,j_1,k_0,k_2} > \varepsilon$.

Proof. Otherwise, for any $\varepsilon > 0$, there exist $\varphi$ and $N$, such that for any $n > N$ and any $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ satisfying $i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n$, we have $(a^{(n)}_{\varphi})_{i_0,i_1,i_2,j_0,j_1,k_0,k_2} < \varepsilon$. By choosing $n$ large enough and with the lemma above, we have
\[
\varphi([Z_0, Z_1, Z_2]) = \max |a^{(n)}_{\varphi})_{i_0,i_1,i_2,j_0,j_1,k_0,k_2}|^2 \leq \frac{1}{n} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n} \frac{|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{(1 + |Z_0|^2 + |Z_1|^2 + |Z_2|^2)^n} + 2 \log \varepsilon + \varepsilon \right)
\]
\[
\leq \frac{1}{n} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n} \frac{|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{(1 + |Z_0|^2 + |Z_1|^2 + |Z_2|^2)^n} + 2 \log \varepsilon + \varepsilon \right)
\]
Since $\varepsilon$ could be arbitrarily small, the above inequality would imply that $\varphi \rightarrow -\infty$ uniformly, which contradicts the fact that $\sup_M \varphi = 0$. \hfill \Box

Proof of Theorem 1. We use notations as above; since $(a^{(n)}_{\varphi})_{i_0,i_1,i_2,j_0,j_1,k_0,k_2} > \varepsilon$, we have
\[
\varphi([Z_0, Z_1, Z_2]) = \lim_{N \to \infty} \frac{1}{N} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = N} |a^{(N)}_{\varphi})_{i_0,i_1,i_2,j_0,j_1,k_0,k_2}|^2 \right)
\]
\[
\geq \frac{1}{N} \log \left( \sum_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = N} \frac{|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{(1 + |Z_0|^2 + |Z_1|^2 + |Z_2|^2)^n} + \log \varepsilon \right)
\]

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\[ \geq \frac{1}{N} \log \left( \frac{|Z_0|^m |Z_1|^{2N-m} |Z_2|^{\frac{2N-m}{2}} }{|Z_0|^2 + |Z_1|^2 + |Z_2|^2} \right)^{\alpha} + \log \epsilon \]
\[ \geq \log \frac{|Z_0|^m |Z_1|^{2N-m} |Z_2|^{\frac{2N-m}{2}} }{|Z_0|^2 + |Z_1|^2 + |Z_2|^2} + \log \epsilon, \]

where \( i_0 + j_0 + k_0 = m, i_1 + j_1 + i_2 + k_2 = 3N - m \).

On the patch \( U_0 = \{ Z_0 \neq 0 \} \),
\[
\int_{U_0 \cap \{ 0 < |z_1|, |z_2| < 1 \}} e^{-\alpha \varphi_*} \omega_{g_0}^2 
\leq C_1 \int_{0 < |z_1|, |z_2| < 1} e^{-\alpha \log \left( \frac{|z_0|^{2m} |z_1|^{3N-m} |z_2|^{\frac{3N-m}{2}}}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right)} \omega_{g_0}^2 
= C_1 \int_{0 < |z_1|, |z_2| < 1} \frac{(1 + |z_1|^2 + |z_2|^2)^\alpha (1 + |z_1|^2)^\alpha (1 + |z_2|^2)^\alpha}{|z_1|^{3N-m} |z_2|^{\frac{3N-m}{2}}} \omega_{g_0}^2 
\leq C_2 \int_{0 < |z_1|, |z_2| < 1} \frac{1}{|z_1|^{3N-m} |z_2|^{\frac{3N-m}{2}}} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2,
\]

where \( C_1, C_2 \) and \( C_3 \) are constants depending only on \( \alpha \) and \( \epsilon \).

On the patch \( U_2 = \{ Z_2 \neq 0 \} \),
\[
\int_{U_2 \cap \{ 0 < |w_0|, |w_1| \leq 1 \}} e^{-\alpha \varphi_*} \omega_{g_1}^2 
\leq C_4 \int_{0 < |w_0|, |w_1| \leq 1} e^{-\alpha \log \left( \frac{|w_0|^{2m} |w_1|^2}{|w_0|^2 + |w_1|^2} \right)} \omega_{g_1}^2 
= C_4 \int_{0 < |w_0|, |w_1| \leq 1} \frac{(1 + |w_0|^2 + |w_1|^2)^\alpha (1 + |w_0|^2)^\alpha (1 + |w_1|^2)^\alpha}{|w_0|^{3N-m} |w_1|^2} \omega_{g_1}^2 
\leq C_5 \int_{0 < |w_0|, |w_1| \leq 1} \frac{1}{|w_0|^{3N-m} |w_1|^{3N-m}} (|w_0|^2 + |w_1|^2)^{1-\alpha} dw_0 \wedge d\overline{w}_0 
\leq C_6 \int_{0 < |w_0|, |w_1| \leq 1} \frac{1}{s^{3N-m} (s+\overline{s})^{3N-m}} (s+\overline{s})^{1-\alpha} dsdt 
\leq C_6 \int_{s=0}^{1} \int_{t=0}^{1} \frac{1}{s^{3N-\alpha} (s+\overline{s})^{3N-\alpha} (s+\overline{s})^{1-\alpha}} dsdt,
\]

where \( p + q = 1 \) and \( C_4, C_5, C_6 \) are constants depending only on \( \alpha \) and \( \epsilon \).
Case 1: If $1 \leq \frac{m}{N} \leq 3$, we can choose $\alpha < \min\left(\frac{2}{3}, \frac{1}{2} - \frac{p}{q}\right)$ so that
\[
\frac{\alpha m}{N} + (1 - \alpha)p < 1, \\
3\alpha - 1 < 1, \\
\frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1 - \alpha)q < 1.
\]

Case 2: If $0 < \frac{m}{N} < 1$, we can choose $\alpha < \min\left(\frac{2}{3}, \frac{1}{2} - \frac{p}{q}\right)$ so that
\[
\frac{\alpha m}{N} + (1 - \alpha)p < 1, \\
3\alpha - 1 < 1, \\
\frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1 - \alpha)q < 1.
\]

So we could choose any $\alpha < \frac{1}{4}$, which implies that $\alpha_G(M, \omega) \geq \frac{1}{4}$.

Conversely, we choose
\[
\varphi_\varepsilon = \log\left(\frac{|Z_0|^6}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2)} + \varepsilon\right) - \log(1 + \varepsilon)
\in \mathcal{P}_G(M, \omega).
\]

Then we have $\sup_M \varphi_\varepsilon = 0$ and $\varphi_\varepsilon = \log \frac{1}{1 + \varepsilon}$ on the exceptional divisors. Furthermore, we have

\[
\lim_{\varepsilon \to 0} \int_M e^{-\alpha \varphi_\varepsilon} \omega^2 = \infty, \quad \text{for any } \alpha > \frac{1}{3}.
\]

Hence we have shown $\alpha_G(M, \omega) = \frac{1}{4}$.

We can also apply the above arguments for $CP^n$ ($n \geq 2$), $CP^2#1\overline{CP^2}$ and $CP^2#3\overline{CP^2}$.

(i) Let $M = CP^n$ and let $G_n$ be the automorphism group acting on $M$, generated by $\theta_j$ and permutations $\tau_{i,j}$ ($0 \leq i < j \leq n$),

\[
\theta_j : [Z_0, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j e^{i\theta}, ..., Z_n]
\]

for $\theta \in [0, 2\pi)$, and

\[
\tau_{i,j} : [Z_0, ..., Z_i, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j, ..., Z_i, ..., Z_n].
\]

**Theorem 3.1.** $\alpha_{G_n}(CP^n) = 1$.

(ii) Let $M$ be the blow-up of $CP^2$ at 3 points which are not collinear. Then we can assume that these 3 points are $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$. Let $G(3)$ be the automorphism group acting on $M$, generated by $\theta_j$ and permutations $\tau_{i,j}$ ($0 \leq i < j \leq 2$),

\[
\theta_j : [Z_0, Z_j, Z_2] \to [Z_0, Z_j e^{i\theta}, Z_2]
\]

for $\theta \in [0, 2\pi)$, and

\[
\tau_{i,j} : [..., Z_i, ..., Z_j, ...] \to [..., Z_j, ..., Z_i, ...].
\]

**Theorem 3.2.** $\alpha_{G(3)}(CP^2#3\overline{CP^2}) = 1$. 

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(iii) Let $M$ be the blow-up of $CP^2$ at one point $[1,0,0]$ and $G(1)$ be the automorphism group acting on $M$, generated by $\theta_j$ and permutations $\tau$ ($0 \leq i \leq 2$),

$$ \theta_j : [Z_0, Z_j, Z_2] \to [Z_0, Z_j e^{i\theta}, Z_2] $$

for $\theta \in [0, 2\pi)$, and

$$ \tau : [Z_0, Z_1, Z_2] \to [Z_0, Z_2, Z_1]. $$

**Theorem 3.3.** $\alpha_{G(1)}(CP^2 \# CP^2) = \frac{1}{2}$.

Also the proof above shows that the sequence of the holomorphic invariants $\{\alpha_{G(m)}(M)\}_m$ defined by Tian [8] on $CP^n$ ($n \geq 2$), $CP^2 \# kCP^2$ ($k = 1, 2, 3$) is stationary.

4. Proof of Theorem 2

In this section, we will prove the generalized Moser-Trudinger inequality on any Kähler manifold $M$ of dimension $n$ whose $(M)$ is greater than $\frac{n^2 + 1}{n+1}$. The following theorem is due to Tian and Zhu [11].

**Theorem 4.1.** Let $(M, \omega)$ be a Kähler-Einstein manifold with $\text{Ric}(\omega) = \omega$; then there exist constants $\delta = \delta(n)$ and $C = C(n, \lambda_2(\omega) - 1) \geq 0$ such that for any $\phi \in P(M, \omega)$ which satisfies $\phi \perp \Lambda_1$, we have

$$ F_\omega(\phi) \geq J_\omega(\phi)^{\delta} - C, $$

which is the same as

$$ \frac{1}{V} \int_M e^{-\phi} \omega^n \leq Ce^{J_\omega(\phi)^{\delta} + \int_M \phi \omega^n - J_\omega(\phi)^{\delta}}. $$

This implies in particular the Moser-Trudinger inequality on $S^2$, which reads

$$ \frac{1}{4\pi} \int_{S^2} e^{-\phi} \omega \leq \exp \left\{ \int_{S^2} |\nabla \phi|^2 \omega - \frac{1}{16} \int_{S^2} \phi \right\}. $$

For any $\phi \in P(M, \omega)$, put $\omega' = \omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi$ and $\text{Ric}(\omega) = \omega + \sqrt{-1} \partial \bar{\partial} h_\omega$. Consider the Monge-Ampère equation

$$ (\omega' + \sqrt{-1} \partial \bar{\partial} \psi)^n = e^{h_\omega - t \psi} \omega^n. $$

We will use the continuity method backwards and let $\phi_t$ be a smooth family which solve the above equation.

The following lemmas are well known [10], but we add the proofs for the sake of completeness.

**Lemma 4.1.** $\text{Ric}(\omega_t) \geq t \omega_t$ and we have equality if and only if $t = 1$, where $\omega_t = \omega + \phi_t$ and $\phi_t$ solves the Monge-Ampère equation at $t$.

**Proof.**

$$ \text{Ric}(\omega_t) = -\sqrt{-1} \partial \bar{\partial} \log \omega_t^n = -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega_t^n}{\omega^n} + \text{Ric}(\omega) $$

$$ = -\sqrt{-1} \partial \bar{\partial} (h_\omega - t \phi_t) + \omega + \sqrt{-1} \partial \bar{\partial} h_\omega $$

$$ = \omega + t \phi_t = t \omega_t + (1 - t) \omega \geq t \omega_t. $$

$\square$
Lemma 4.2. For any $\phi \in P(M, \omega)$, if the Green's function of $\omega' = \omega + \sqrt{-1} \partial \overline{\partial} \phi$ is bounded from below, we have:

$$-\inf_M \phi \leq \frac{1}{V} \int_M (-\phi) \omega^n + C.$$ 

Proof. Since $\omega + \sqrt{-1} \partial \overline{\partial} \phi = \omega'$ and $\omega' - \sqrt{-1} \partial \overline{\partial} \phi > 0$, we have $\Delta \omega \phi \leq n$, and

$$-\phi = \frac{1}{V} \int_M (\phi) \omega^n + \frac{1}{V} \int_M \Delta \omega' \phi (y) G_{\omega'}(x, y) \omega^n$$
$$\leq \frac{1}{V} \int_M (\phi) \omega^n + \frac{1}{V} \int_M n(G_{\omega'}(x, y) - \inf G_{\omega'}(x, y)) \omega^n$$
$$\leq \frac{1}{V} \int_M (\phi) \omega^n + C.$$ 

Let $(M, \omega)$ be a Kähler-Einstein manifold with $\text{Ric}(\omega) = \omega$ and let $P(M, \omega, K) = \{ \phi \in P(M, \omega) | G_{\omega + \sqrt{-1} \partial \overline{\partial} \phi}(x, y) \geq -K \}$. Then we have:

Proposition 4.1. Let $(M, \omega)$ be a Kähler-Einstein manifold with $\text{Ric}(\omega) = \omega$. If $\alpha(M) > \frac{n}{n+1}$, then there exist constants $\delta(n, \alpha, K)$ and $C(n, \alpha, \lambda_2(\omega) - 1, K)$ such that for any $\phi \in P(M, \omega, K)$, we have

$$F_{\omega}(\phi) \geq \delta J_{\omega}(\phi) - C.$$ 

Proof. Let $\omega' = \omega + \delta \overline{\partial} \phi$, where $\phi \in P(M, \omega, K)$. We have

$$\frac{1}{V} \int_M e^{-\alpha \phi} \omega_n = \frac{1}{V} \int_M e^{-\alpha_1 \phi_1} \phi \omega_n$$
$$\leq \frac{1}{V} \int_M e^{-\alpha_1 \phi_1} \phi \omega_n e^{-\epsilon \inf M \phi},$$

taking $p = \frac{1}{\alpha_1}, q = \frac{1}{1-\alpha_1}$, we have

$$\frac{1}{V} \int_M e^{-\alpha_1 \phi_1} \phi \omega_n \leq \frac{1}{V} \left( \int_M e^{-\alpha_1 \phi_1} \phi \omega_n \right)^{1/p} \left( \int_M e^{-\alpha_2 \phi_2} \omega_n \right)^{1/q}$$

$$= \frac{1}{V} \left( \int_M e^{-\phi_1 \omega_n} \right)^{\alpha_1} \left( \int_M e^{-\alpha_2 \phi_2} \omega_n \right)^{1-\alpha_1}$$

$$\leq Ce^{\alpha_1 J_{\omega}(\phi) - \frac{\alpha_1}{\alpha_1} f_{\omega} \phi \omega_n} \left( \int_M e^{-\alpha_2 \phi_2} \omega_n \right)^{1-\alpha_1}.$$ 

By Lemma 4.2,

$$e^{-\epsilon \inf M \phi} \leq e^{\epsilon \int_M (-\phi) \omega^n + C}$$

$$= e^{\epsilon \int_M (-\phi) - \frac{\alpha_1}{\alpha_1} f_{\omega} \phi \omega_n + C}$$

$$\leq e^{\epsilon(n+1) J_{\omega}(\phi) - \frac{\alpha_1}{\alpha_1} f_{\omega} \phi \omega_n + C}.$$
By Hölder’s inequality,
\[
\frac{1}{V} \int_M e^{-\phi} \omega^n \leq \left( \frac{1}{V} \int_M e^{-\alpha \phi} \omega^n \right)^{\frac{1}{\alpha}} \\
\leq C e^{\frac{\alpha + (n+1)}{\alpha} \int_M \phi \omega^n} \left( \int_M e^{-\frac{\alpha}{\alpha-1} \sup \phi} \omega^n \right)^{\frac{\alpha-1}{\alpha}} \\
= C e^{\frac{\alpha + (n+1)}{\alpha} \int_M \phi \omega^n} + \frac{\alpha}{\alpha-1} \int_M (\phi - \sup \phi) \omega^n} \left( \int_M e^{-\frac{\alpha}{\alpha-1} \sup \phi} \omega^n \right)^{\frac{\alpha-1}{\alpha}} \\
\leq C e^{\frac{\alpha + (n+1)}{\alpha} \int_M \phi \omega^n} + \frac{1}{\alpha-1} \int_M \phi \omega^n} \left( \int_M e^{-\frac{\alpha}{\alpha-1} \sup \phi} \omega^n \right)^{\frac{\alpha-1}{\alpha}}.
\]

We need to determine $\alpha_1, \alpha_2, \varepsilon$ which satisfy the following conditions:
\[
\alpha = \alpha_1 + \alpha_2 + \varepsilon > 1, \\
\alpha > \alpha_1 + (n+1)\varepsilon, \\
1 > \alpha_1.
\]

So we will choose
\[
\alpha_2 = n\varepsilon + \varepsilon', \\
\alpha_1 = 1 - \alpha_2 - \varepsilon + \varepsilon'' = 1 - (n+1)\varepsilon - \varepsilon' + \varepsilon'',
\]
where $\varepsilon, \varepsilon', \varepsilon'' < 1$, and $\varepsilon' = o(\varepsilon)$, $\varepsilon'' = o(\varepsilon)$.

Since $\alpha(M) > \frac{n}{n+1}$, we can choose $\varepsilon, \varepsilon', \varepsilon''$ small enough; then we have
\[
\frac{\alpha_2}{1 - \alpha_1} = \frac{n\varepsilon + \varepsilon'}{(n+1)\varepsilon + \varepsilon' - \varepsilon''} < \alpha(M)
\]
and
\[
\int_M e^{-\frac{\alpha}{\alpha-1} \sup \phi} \omega^n < Const.
\]

Combined with the inequalities above, we have
\[
\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{(1-\delta)\int_M \phi \omega^n}.
\]

This proves the lemma. \qed

Proof of Theorem 2. We assume $\omega$ is the Kähler-Einstein metric of $M$. For any $\phi \in P(M, \omega)$, put $\omega' = \omega + \sqrt{-1} \partial \bar{\partial} \phi$. Consider $(\omega' + \sqrt{-1} \partial \bar{\partial} \psi) = e^{h_{\omega'} + t\psi}$. By solving the Monge-Ampère equation backwards, we get the solutions $\phi_t$, and $\phi_1 = -\phi$.

For $t > \frac{1}{2}$, let $\omega_t = \omega' + \sqrt{-1} \partial \bar{\partial} \phi_t = \omega + \sqrt{-1} \partial \bar{\partial} (\phi_t - \phi_1)$; by Lemma 4.1,
\[
\text{Ric}(\omega_t) \geq \frac{1}{2} \omega_t,
\]
which shows that the Green function of $\omega_t$ is uniformly bounded from below. Thus by Proposition 4.1 and the calculation in [1], we have
\[
F_\omega(\phi_t - \phi_1) \geq \delta \int_M (\phi_t - \phi_1) - C \\
\geq C_1 \text{osc}_M (\phi_t - \phi_1) - C_2,
\]
and consequently,
\[
n(1-t)J_\omega(\phi) = n(1-t)J_\omega(\phi_1) \\
\geq (1-t)(J_\omega(\phi_t) - J_\omega(\phi_1)) \\
\geq F_\omega(\phi_t) - F_\omega(\phi_1) \\
= F_\omega(\phi_t - \phi_1) \\
\geq C_1 \text{osc}_M (\phi_t - \phi_1) - C_2.
\]

Thus we have
\[
F_\omega(\phi) = -F_\omega(-\phi) \\
\geq \int_0^1 (I_{\omega_t}(\phi_t) - J_\omega(\phi_t))dt \\
\geq (1-t)(I_{\omega_t}(\phi_t) - J_\omega(\phi_t)) \\
\geq \frac{1-t}{n} J_\omega(\phi_t) \\
\geq \frac{1-t}{n} J_\omega(\phi_1) - 2(1-t)(C_1 \text{osc}_M (\phi_t - \phi_1) - C_2) \\
\geq \frac{1-t}{n} J_\omega(\phi) - 2(1-t)^2 n C_1 J_\omega(\phi) - C_3.
\]

The theorem follows by choosing \((1-t) < \frac{1}{2n^2 C_1}\).

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Department of Mathematics, Columbia University, New York, New York 10027

E-mail address: jsong@math.columbia.edu