

THE α -INVARIANT ON CERTAIN SURFACES WITH SYMMETRY GROUPS

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ABSTRACT. The global holomorphic α -invariant introduced by Tian is closely related to the existence of Kähler-Einstein metrics. We apply the result of Tian, Yau and Zelditch on polarized Kähler metrics to approximate plurisubharmonic functions and compute the α -invariant on $CP^2 \# n\overline{CP^2}$ for $n = 1, 2, 3$.

1. INTRODUCTION

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian [7], Tian and Yau [6] is closely related to the existence of Kähler-Einstein metrics. In his solution of the Calabi conjecture, Yau [12] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with negative or zero first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there are known obstructions such as the Futaki invariant. For a compact Kähler manifold M with positive first Chern class, Tian [7] proved that M admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $CP^2 \# 1\overline{CP^2}$ and $CP^2 \# 2\overline{CP^2}$ [9]. Nevertheless, it would also be interesting to find the estimate of the α invariant for $CP^2 \# 1\overline{CP^2}$ and $CP^2 \# 2\overline{CP^2}$. In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman potential on polarized Kähler manifolds to approximate plurisubharmonic functions and compute the α -invariant of $CP^2 \# n\overline{CP^2}$ for $n = 1, 2, 3$. In the case of $CP^2 \# 2\overline{CP^2}$, it gives an improvement of Abdesselem's result [1]. More precisely, we shall show that:

Theorem 1. $\alpha_G(CP^2 \# 2\overline{CP^2}) = \frac{1}{3}$.

We will give the definitions of the automorphism group G and the α_G -invariant in Section 3.

Let (M, ω) be a compact Kähler manifold, where $\omega = \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j$. We will also prove Tian's conjecture on the generalized Moser-Trudinger inequality in the special case where $\alpha_G(M) > \frac{n}{n+1}$, for $n = \dim M$. Let

$$P(M, \omega) = \left\{ \phi \mid \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0, \sup_M \phi = 0 \right\}.$$

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Let F_ω and J_ω be the functionals defined on $P(M, \omega)$ by

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n - \log\left(\frac{1}{V} \int_M e^{h_\omega - \phi} \omega^n\right),$$

$$J_\omega(\phi) = \frac{\sqrt{-1}}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-i-1}.$$

Assume (M, ω_{KE}) is a Kähler-Einstein manifold with positive first Chern class and $Ric(\omega_{KE}) = \omega_{KE}$. Then for any $\phi \in P(M, \omega_{KE})$, Ding and Tian [2] proved the following inequality of Moser-Trudinger type:

$$\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n}.$$

Tian [10] also conjectured that $\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{(1-\delta)J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n}$ for $\delta > 0$ sufficiently small, if ϕ is perpendicular to Λ_1 , the space of eigenfunctions of ω_{KE} with eigenvalue one.

We shall prove:

Theorem 2. *Let (M, ω) be a Kähler manifold with positive first Chern class. Assume that $\alpha(M) > \frac{n}{n+1}$, so that M admits a Kähler-Einstein metric ω_{KE} , and there exist constants $\delta = \delta(n, \alpha(M))$ and $C = C(n, \lambda_2(\omega_{KE}) - 1, \alpha(M))$ such that for any $\phi \in P(M, \omega_{KE})$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_{\omega_{KE}}(\phi) \geq \delta J_{\omega_{KE}}(\phi) - C.$$

Here $\lambda_2(\omega_{KE})$ is the least eigenvalue of ω_{KE} which is bigger than 1.

2. HOLOMORPHIC APPROXIMATION OF PLURISUBHARMONIC FUNCTIONS

In this section, we will employ the technique in [8, 13] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [13].

Theorem 2.1. *Let M be a compact complex manifold of dimension n and let $(L, h) \rightarrow M$ be a positive Hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m \in \mathbb{N}$, h induces a Hermitian metric h_m on L^m . Let $\{S_0^m, S_1^m, \dots, S_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product:*

$$(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x)) dV_g,$$

where $dV_g = \frac{1}{n!} \omega_g^n$ is the volume form of g . Then there is a complete asymptotic expansion

$$\sum_{i=0}^{d_m-1} \|S_i^m(x)\|_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \dots$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any k ,

$$\left\| \sum_{i=0}^{d_m-1} \|S_i^m(x)\|_{h_m}^2 - \sum_{j < R} a_j(x)m^{n-j} \right\|_{C^k} \leq C_{R,k} m^{n-R}$$

where $C_{R,k}$ depends on R, k and the manifold M .

Let

$$\begin{aligned}\tilde{\omega}_g &= \omega_g + \sqrt{-1}\partial\bar{\partial}\phi > 0, \\ \tilde{h} &= he^{-\phi}.\end{aligned}$$

Let \tilde{h}_m be the induced Hermitian metric of \tilde{h} on L^m , and let $\{\tilde{S}_0^m, \tilde{S}_1^m, \dots, \tilde{S}_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, where $d_m = \dim H^0(M, L^m)$, with respect to the inner product

$$(S_1, S_2)_{\tilde{h}_m} = \int_M \tilde{h}_m(S_1(x), S_2(x))dV_{\tilde{g}}.$$

By Theorem 2.1, we have

$$\sum_{i=0}^{d_m-1} \|\tilde{S}_i^m(x)\|_{\tilde{h}_m}^2 = \left(\sum_{i=0}^{d_m-1} \|\tilde{S}_i^m(x)\|_{h_m}^2 \right) e^{-m\phi}.$$

Thus

$$\phi - \frac{1}{m} \log \left(\sum_{i=0}^{d_m-1} \|\tilde{S}_i^m(x)\|_{\tilde{h}_m}^2 \right) = -\frac{1}{m} \log \left(\sum_{i=0}^{d_m-1} \|\tilde{S}_i^m(x)\|_{h_m}^2 \right).$$

As $m \rightarrow +\infty$, we obtain for any positive integer R

$$\begin{aligned}& \frac{1}{m} \log \left(\sum_{j < R} \tilde{a}_j(x)m^{n-j} \right) \\ &= \frac{1}{m} \log m^n \left(\sum_{j < R} \tilde{a}_j(x)m^{-j} \right) \\ &= \frac{n}{m} \log m + \frac{1}{m} \log \left(1 + O\left(\frac{1}{m}\right) \right) \rightarrow 0.\end{aligned}$$

Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

Corollary 2.1.

$$\left\| \phi - \frac{1}{m} \log \left(\sum_{i=0}^{d_m-1} \|\tilde{S}_i^m(x)\|_{h_m}^2 \right) \right\|_{C^k} \rightarrow 0, \text{ as } m \rightarrow +\infty.$$

In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of L^m .

3. PROOF OF THEOREM 1

Let M be the blow-up of CP^2 at two points and π be its natural projection. Without loss of generality, we may assume the two points are $p_1 = [0, 1, 0]$ and $p_2 = [0, 0, 1]$. Then M is a subvariety of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0X_1 = Z_1X_0, \quad Z_0Y_2 = Z_2Y_0,$$

where Z_i, X_j, Y_k are the homogeneous coordinates on CP^2, CP^1 and CP^1 , respectively.

Let G be the automorphism group acting on $CP^2 \times CP^1 \times CP^1$ generated by θ_j and permutations τ ($0 \leq j \leq 2$),

$$\theta_j : [Z_0, Z_j, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \rightarrow [Z_0, Z_je^{i\theta}, Z_2] \times [X_0, X_1] \times [Y_0, Y_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \rightarrow [Z_0, Z_2, Z_1] \times [Y_0, Y_2] \times [X_0, X_1].$$

Let π_0, π_1, π_2 be the projection from $CP^2 \times CP^1 \times CP^1$ onto CP^2, CP^1 and CP^1 . Respectively define ω by

$$\begin{aligned} \omega &= \pi_0^* \omega_0 + \pi_1^* \omega_1 + \pi_2^* \omega_2 \\ &= \sqrt{-1} \partial \bar{\partial} \log(|Z_0|^2 + |Z_1|^2 + |Z_2|^2) + \sqrt{-1} \partial \bar{\partial} \log(|X_0|^2 + |X_1|^2) \\ &\quad + \sqrt{-1} \partial \bar{\partial} \log(|Y_0|^2 + |Y_2|^2), \end{aligned}$$

where $\omega_0, \omega_1, \omega_2$ are the Fubini-Study metrics in CP^2, CP^1 and CP^1 . By explicit calculation, it can be shown that the cohomological class of $\omega|_M$ is in the first Chern class of M (see [1]).

Consider the divisor

$$\{[0, Z_1, Z_2] \times CP^1 \times CP^1\} + \{CP^2 \times [1, 0] \times CP^1\} + \{CP^2 \times CP^1 \times [1, 0]\}$$

which defines a line bundle (L, h) on $CP^2 \times CP^1 \times CP^1$. The hermitian metric h is defined by

$$h = \frac{1}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|X_0|^2 + |X_1|^2)(|Y_0|^2 + |Y_2|^2)};$$

then $(L, h)|_M \rightarrow M$ defines the anticanonical line bundle on M whose curvature form $-\sqrt{-1} \partial \bar{\partial} \log h$ gives the first Chern class on M .

Since $M \setminus \{\pi^{-1}\{p_1\} \cup \pi^{-1}\{p_2\}\}$ is isomorphic to $CP^2 \setminus \{p_1, p_2\}$, if we choose the inhomogeneous coordinates $(z_1, z_2) = [1, z_1, z_2]$ on CP^2 , the Kähler metric

$$\omega_{g_0} = \sqrt{-1} \partial \bar{\partial} \log(1 + |z_1|^2 + |z_2|^2) + \sqrt{-1} \partial \bar{\partial} \log(1 + |z_1|^2) + \sqrt{-1} \partial \bar{\partial} \log(1 + |z_2|^2)$$

can be extended to a Kähler metric g_0 on M which belongs to $c_1(M)$. If we take different inhomogeneous coordinates $(w_0, w_1) = [w_0, w_1, 1]$, the corresponding Kähler metric is

$$\omega_{g_1} = \sqrt{-1} \partial \bar{\partial} \log(1 + |w_0|^2 + |w_1|^2) + \sqrt{-1} \partial \bar{\partial} \log(1 + |w_0|^2) + \sqrt{-1} \partial \bar{\partial} \log(|w_0|^2 + |w_1|^2)$$

and we have

$$\begin{aligned} \det g_0 &= \frac{1}{(1 + |z_1|^2 + |z_2|^2)^3} + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_1|^2)} \\ &\quad + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_2|^2)} + \frac{1}{(1 + |z_1|^2)^2(1 + |z_2|^2)^2}, \\ \det g_1 &= \frac{1}{(1 + |w_0|^2 + |w_1|^2)^3} + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(|w_0|^2 + |w_1|^2)} \\ &\quad + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(1 + |w_0|^2)} + \frac{|w_0|^2}{(1 + |w_0|^2)^2(|w_0|^2 + |w_1|^2)^2}. \end{aligned}$$

Consider the line bundle $(L^N, h_N) \rightarrow CP^2 \times CP^1 \times CP^1$. Then

$$\dim H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) = \frac{(N+1)^3(N+2)}{2}$$

and $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ is an orthogonal basis for $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N))$.

Let M_1 be the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0 X_1 = Z_1 X_0,$$

and M_2 the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0 Y_2 = Z_2 Y_0.$$

Then $M = M_1 \cap M_2$.

In view of the short exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{O}(L^N - [M_1]) \rightarrow \mathcal{O}(L^N) \rightarrow \mathcal{O}(L^N|_{M_1}) \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}(L^N|_{M_1} - [M]) \rightarrow \mathcal{O}(L^N|_{M_1}) \rightarrow \mathcal{O}(L^N|_M) \rightarrow 0 \end{aligned}$$

we can choose N sufficiently large so that

$$H^1(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N - [M_1])) = H^1(M_1, \mathcal{O}(L^N|_{M_1} - [M])) = 0.$$

Then $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \rightarrow H^0(M_1, \mathcal{O}(L^N|_{M_1})) \rightarrow 0$,

$$H^0(M_1, \mathcal{O}(L^N|_{M_1})) \rightarrow H^0(M, \mathcal{O}(L^N|_M)) \rightarrow 0$$

and thus

$$H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \rightarrow H^0(M, \mathcal{O}(L^N|_M)) \rightarrow 0.$$

Also we have $Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_1}|_M = Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}$ and

$$\begin{aligned} &\|Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}\|_{h_N}^2 \\ &= \frac{|Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_0^{j_0} Z_1^{j_1} Z_0^{k_0} Z_2^{k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N} \end{aligned}$$

on $CP^2 \setminus \{p_1, p_2\}$. Therefore, $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}|_M\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ contains an orthogonal basis for $H^0(M, \mathcal{O}(L^N|_M))$ with respect to h^N and the G -invariant Kähler metric g on M .

By Corollary 2.1, for any φ in $P_G(M, \omega_g)$, we have on $CP^2 \setminus \{p_1, p_2\}$,

$$\begin{aligned} &\varphi([Z_0, Z_1, Z_2]) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |a_{(\varphi)_{i_0 i_1 i_2 j_0 j_1 k_0 k_2}}^{(N)} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N} \end{aligned}$$

for some coefficients $a_{(\varphi)_{i_0 i_1 i_2 j_0 j_1 k_0 k_2}}^{(N)}$ satisfying $a_{(\varphi)_{i_0 i_1 i_2 j_0 j_1 k_0 k_2}}^{(N)} = a_{(\varphi)_{i_0 i_2 i_1 k_0 k_2 j_0 j_1}}^{(N)}$ due to the group action by G .

Lemma 3.1. *Using the notations above we have*

$$\frac{1}{n} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^n} \leq 4$$

for any positive integer n .

Proof. On the patch $U_0 = \{Z_0 \neq 0\}$, let $z_1 = \frac{Z_1}{Z_0}$ and $z_2 = \frac{Z_2}{Z_0}$,

$$\begin{aligned}
& \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2 \\
& \frac{1}{n} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^n} \\
& \leq \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} \frac{|z_1^{i_1+j_1} z_2^{i_2+k_2}|^2}{(1 + |z_1|^2 + |z_2|^2)^n (1 + |z_1|^2)^n (1 + |z_2|^2)^n} \right) \\
& \leq \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} \frac{|z_1^{i_1+j_1} z_2^{i_2+k_2}|^2}{1 + |z_1^{i_1+j_1} z_2^{i_2+k_2}|^2} \right) \\
& \leq \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} 1 \right) \\
& = \frac{1}{n} \log \frac{(n+1)^3(n+2)}{2} \leq 4.
\end{aligned}$$

This inequality also holds on the patch $U_1 = \{Z_1 \neq 0\}$ by continuity, and so the lemma is proved. \square

Lemma 3.2. *There exists $\varepsilon > 0$ such that for any $\varphi \in P_G(M, \omega_g)$ and N , there exist $n > N$, $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ with $i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n$, and $(a_{(\varphi)_{i_0 i_1 i_2 j_0 j_1 k_0 k_2}}^{(n)})^{\frac{1}{n}} > \varepsilon$.*

Proof. Otherwise, for any $\varepsilon > 0$, there exist φ and N , such that for any $n > N$ and any $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ satisfying $i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n$, we have $(a_{(\varphi)_{i_0 i_1 i_2 j_0 j_1 k_0 k_2}}^{(n)})^{\frac{1}{n}} < \varepsilon$. By choosing n large enough and with the lemma above, we have

$$\begin{aligned}
& \varphi([Z_0, Z_1, Z_2]) \\
& \leq \frac{1}{n} \log \frac{\max_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} |a_{(\varphi)_{i_0 i_1 i_2 j_0 j_1 k_0 k_2}}^{(n)}|^2 \sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^n} + \varepsilon \\
& \leq \frac{1}{n} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^n} + 2 \log \varepsilon + \varepsilon \\
& \leq \log \varepsilon + 4.
\end{aligned}$$

Since ε could be arbitrarily small, the above inequality would imply that $\varphi \rightarrow -\infty$ uniformly, which contradicts the fact that $\sup_M \varphi = 0$. \square

Proof of Theorem 1. We use notations as above; since $(a_{(\varphi)_{i_0 i_1 i_2 j_0 j_1 k_0 k_2}}^{(n)})^{\frac{1}{n}} > \varepsilon$, we have

$$\begin{aligned}
& \varphi([Z_0, Z_1, Z_2]) \\
& = \lim_{N \rightarrow \infty} \frac{1}{N} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1+k_0+k_2=N} |a_{(\varphi)_{i_0 i_1 i_2 j_0 j_1 k_0 k_2}}^{(N)} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N} \\
& \geq \frac{1}{N} \log \frac{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2 + |Z_0^{i_0+j_0+k_0} Z_1^{i_2+k_2} Z_2^{i_1+j_1}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N} + \log \varepsilon
\end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{N} \log \frac{|Z_0|^m |Z_1|^{\frac{3}{2}N - \frac{m}{2}} |Z_2|^{\frac{3}{2}N - \frac{m}{2}}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N} + \log \epsilon \\ &\geq \log \frac{|Z_0|^{\frac{2m}{N}} |Z_1|^{3 - \frac{m}{N}} |Z_2|^{3 - \frac{m}{N}}|^2}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2)} + \log \epsilon, \end{aligned}$$

where $i_0 + j_0 + k_0 = m, i_1 + j_1 + i_2 + k_2 = 3N - m$.

On the patch $U_0 = \{Z_0 \neq 0\}$,

$$\begin{aligned} &\int_{U_0 \cap \{0 < |z_1|, |z_2| < 1\}} e^{-\alpha\varphi} \omega_{g_0}^2 \\ &\leq C_1 \int_{0 < |z_1|, |z_2| < 1} e^{-\alpha \log \frac{|Z_0|^{\frac{2m}{N}} |Z_1|^{3 - \frac{m}{N}} |Z_2|^{3 - \frac{m}{N}}}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2)}} \omega_{g_0}^2 \\ &= C_1 \int_{0 < |z_1|, |z_2| < 1} \frac{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)^\alpha (|Z_0|^2 + |Z_1|^2)^\alpha (|Z_0|^2 + |Z_2|^2)^\alpha}{|Z_0|^{\frac{2\alpha m}{N}} |Z_1|^{3\alpha - \frac{\alpha m}{N}} |Z_2|^{3\alpha - \frac{\alpha m}{N}}} \omega_{g_0}^2 \\ &\leq C_2 \int_{0 < |z_1|, |z_2| < 1} \frac{(1 + |z_1|^2 + |z_2|^2)^\alpha (1 + |z_1|^2)^\alpha (1 + |z_2|^2)^\alpha}{|z_1|^{3\alpha - \frac{\alpha m}{N}} |z_2|^{3\alpha - \frac{\alpha m}{N}}} dz_1 \wedge d\bar{z}_1 \\ &\hspace{15em} \wedge dz_2 \wedge d\bar{z}_2 \\ &\leq C_3 \int_{0 < |z_1|, |z_2| < 1} \frac{1}{|z_1|^{3\alpha - \frac{\alpha m}{N}} |z_2|^{3\alpha - \frac{\alpha m}{N}}} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\ &\leq C_3 \int_{0 < |z_1|, |z_2| < 1} \frac{1}{|z_1|^{3\alpha} |z_2|^{3\alpha}} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2, \end{aligned}$$

where C_1, C_2 and C_3 are constants depending only on α and ϵ .

On the patch $U_2 = \{Z_2 \neq 0\}$,

$$\begin{aligned} &\int_{U_1 \cap \{0 < |w_0|, |w_1| \leq 1\}} e^{-\alpha\varphi} \omega_{g_1}^2 \\ &\leq C_4 \int_{0 < |w_0|, |w_1| \leq 1} e^{-\alpha \log \frac{|Z_0|^{\frac{2m}{N}} |Z_1|^{3 - \frac{m}{N}} |Z_2|^{3 - \frac{m}{N}}}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2)}} \omega_{g_1}^2 \\ &= C_4 \int_{0 < |w_0|, |w_1| \leq 1} \frac{(1 + |w_0|^2 + |w_1|^2)^\alpha (1 + |w_0|^2)^\alpha (|w_0|^2 + |w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}} |w_1|^{3\alpha - \frac{\alpha m}{N}}} \omega_{g_1}^2 \\ &\leq C_5 \int_{0 < |w_0|, |w_1| \leq 1} \frac{(1 + |w_0|^2 + |w_1|^2)^\alpha (1 + |w_0|^2)^\alpha (|w_0|^2 + |w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}} |w_1|^{3\alpha - \frac{\alpha m}{N}} (|w_0|^2 + |w_1|^2)} dw_0 \wedge d\bar{w}_0 \\ &\hspace{15em} \wedge dw_1 \wedge d\bar{w}_1 \\ &\leq C_6 \int_{0 < |w_0|, |w_1| \leq 1} \frac{1}{|w_0|^{\frac{2\alpha m}{N}} |w_1|^{3\alpha - \frac{\alpha m}{N}} (|w_0|^2 + |w_1|^2)^{1-\alpha}} dw_0 \wedge d\bar{w}_0 \wedge dw_1 \wedge d\bar{w}_1 \\ &\leq C_6 \int_{t=0}^1 \int_{s=0}^1 \frac{1}{s^{\frac{\alpha m}{N}} t^{\frac{3}{2}\alpha - \frac{\alpha m}{2N}} (s+t)^{1-\alpha}} ds dt \\ &\leq C_6 \int_{s=0}^1 \frac{1}{s^{\frac{\alpha m}{N}} t^{\frac{3}{2}\alpha - \frac{\alpha m}{2N}} s^{(1-\alpha)p} t^{(1-\alpha)q}} ds dt, \end{aligned}$$

where $p + q = 1$ and C_4, C_5, C_6 are constants depending only on α and ϵ .

Case 1: If $1 \leq \frac{m}{N} \leq 3$, we can choose $\alpha < \min(\frac{2}{3}, \frac{1-p}{3-p})$ so that

$$\begin{aligned} \frac{\alpha m}{N} + (1-\alpha)p &< 1, \\ 3\alpha - 1 &< 1, \\ \frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1-\alpha)q &< 1. \end{aligned}$$

Case 2: If $0 < \frac{m}{N} < 1$, we can choose $\alpha < \min(\frac{2}{3}, \frac{1-q}{3/2-q})$ so that

$$\begin{aligned} \frac{\alpha m}{N} + (1-\alpha)p &< 1, \\ 3\alpha - 1 &< 1, \\ \frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1-\alpha)q &< 1. \end{aligned}$$

So we could choose any $\alpha < \frac{1}{3}$, which implies that $\alpha_G(M, \omega) \geq \frac{1}{3}$. Conversely, we choose

$$\begin{aligned} \varphi_\varepsilon &= \log\left(\frac{|Z_0|^6}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2)} + \varepsilon\right) \\ &\quad - \log(1 + \varepsilon) \\ &\in P_G(M, \omega). \end{aligned}$$

Then we have $\sup_M \varphi_\varepsilon = 0$ and $\varphi_\varepsilon = \log \frac{\varepsilon}{1+\varepsilon}$ on the exceptional divisors. Furthermore, we have

$$\lim_{\varepsilon \rightarrow 0} \int_M e^{-\alpha \varphi_\varepsilon} \omega^2 = \infty, \text{ for any } \alpha > \frac{1}{3}.$$

Hence we have shown $\alpha_G(M, \omega) = \frac{1}{3}$.

We can also apply the above arguments for CP^n ($n \geq 2$), $CP^2 \# 1\overline{CP^2}$ and $CP^2 \# 3\overline{CP^2}$.

(i) Let $M = CP^n$ and let G_n be the automorphism group acting on M , generated by θ_j and permutations $\tau_{i,j}$ ($0 \leq i < j \leq n$),

$$\theta_j : [Z_0, \dots, Z_j, \dots, Z_n] \rightarrow [Z_0, \dots, Z_j e^{i\theta}, \dots, Z_n]$$

for $\theta \in [0, 2\pi)$, and

$$\tau_{i,j} : [Z_0, \dots, Z_i, \dots, Z_j, \dots, Z_n] \rightarrow [Z_0, \dots, Z_j, \dots, Z_i, \dots, Z_n].$$

Theorem 3.1. $\alpha_{G_n}(CP^n) = 1$.

(ii) Let M be the blow-up of CP^2 at 3 points which are not collinear. Then we can assume that these 3 points are $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$. Let $G(3)$ be the automorphism group acting on M , generated by θ_j and permutations $\tau_{i,j}$ ($0 \leq i < j \leq 2$),

$$\theta_j : [Z_0, Z_j, Z_2] \rightarrow [Z_0, Z_j e^{i\theta}, Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau_{i,j} : [\dots, Z_i, \dots, Z_j, \dots] \rightarrow [\dots, Z_j, \dots, Z_i, \dots].$$

Theorem 3.2. $\alpha_{G(3)}(CP^2 \# 3\overline{CP^2}) = 1$.

(iii) Let M be the blow-up of CP^2 at one point $[1, 0, 0]$ and $G(1)$ be the automorphism group acting on M , generated by θ_j and permutations τ ($0 \leq i \leq 2$),

$$\theta_j : [Z_0, Z_j, Z_2] \rightarrow [Z_0, Z_j e^{i\theta}, Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \rightarrow [Z_0, Z_2, Z_1].$$

Theorem 3.3. $\alpha_{G(1)}(CP^2 \# 1\overline{CP^2}) = \frac{1}{2}$.

Also the proof above shows that the sequence of the holomorphic invariants $\{\alpha_{G,m}(M)\}_m$ defined by Tian [8] on CP^n ($n \geq 2$), $CP^2 \# k\overline{CP^2}$ ($k = 1, 2, 3$) is stationary.

4. PROOF OF THEOREM 2

In this section, we will prove the generalized Moser-Trudinger inequality on any Kähler manifold M of dimension n whose $\alpha(M)$ is greater than $\frac{n}{n+1}$. The following theorem is due to Tian and Zhu [11].

Theorem 4.1. *Let (M, ω) be a Kähler-Einstein manifold with $Ric(\omega) = \omega$; then there exist constants $\delta = \delta(n)$ and $C = C(n, \lambda_2(\omega) - 1) \geq 0$ such that for any $\phi \in P(M, \omega)$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_\omega(\phi) \geq J_\omega(\phi)^\delta - C,$$

which is the same as

$$\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n - J_\omega(\phi)^\delta}.$$

This implies in particular the Moser-Trudinger inequality on S^2 , which reads

$$\frac{1}{4\pi} \int_{S^2} e^{-\phi} \omega \leq e^{\frac{1}{8\pi} \int_{S^2} |\nabla \phi|^2 \omega - \frac{1}{4\pi} \int_{S^2} \phi}.$$

For any $\phi \in P(M, \omega)$, put $\omega' = \omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi$ and $Ric(\omega) = \omega + \sqrt{-1} \partial \bar{\partial} h_\omega$. Consider the Monge-Ampère equation

$$(\omega' + \sqrt{-1} \partial \bar{\partial} \psi)^n = e^{h_\omega - t\psi} \omega'^n.$$

We will use the continuity method backwards and let ϕ_t be a smooth family which solve the above equation.

The following lemmas are well known [10], but we add the proofs for the sake of completeness.

Lemma 4.1. *$Ric(\omega_t) \geq t\omega_t$ and we have equality if and only if $t = 1$, where $\omega_t = \omega + \phi_t$ and ϕ_t solves the Monge-Ampère equation at t .*

Proof.

$$\begin{aligned} Ric(\omega_t) &= -\sqrt{-1} \partial \bar{\partial} \log \omega_t^n = -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega_t^n}{\omega^n} + Ric(\omega) \\ &= -\sqrt{-1} \partial \bar{\partial} (h_\omega - t\phi_t) + \omega + \sqrt{-1} \partial \bar{\partial} h_\omega \\ &= \omega + t\phi_t = t\omega_t + (1-t)\omega \geq t\omega_t. \end{aligned}$$

□

Lemma 4.2. *For any $\phi \in P(M, \omega)$, if the Green's function of $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ is bounded from below, we have:*

$$-\inf_M \phi \leq \frac{1}{V} \int_M (-\phi)\omega'^n + C.$$

Proof. Since $\omega + \sqrt{-1}\partial\bar{\partial}\phi = \omega'$ and $\omega' - \sqrt{-1}\partial\bar{\partial}\phi > 0$, we have $\Delta_{\omega'}\phi \leq n$, and

$$\begin{aligned} -\phi &= \frac{1}{V} \int_M (-\phi)\omega'^n + \frac{1}{V} \int_M \Delta_{\omega'}\phi(y)G_{\omega'}(x, y)\omega'^n \\ &\leq \frac{1}{V} \int_M (-\phi)\omega'^n + \frac{1}{V} \int_M n(G_{\omega'}(x, y) - \inf G_{\omega'}(x, y))\omega'^n \\ &\leq \frac{1}{V} \int_M (-\phi)\omega'^n + C. \end{aligned}$$

□

Let (M, ω) be a Kähler-Einstein manifold with $\text{Ric}(\omega) = \omega$ and let $P(M, \omega, K) = \{\phi \in P(M, \omega) \mid G_{\omega + \sqrt{-1}\partial\bar{\partial}\phi}(x, y) \geq -K\}$. Then we have:

Proposition 4.1. *Let (M, ω) be a Kähler-Einstein manifold with $\text{Ric}(\omega) = \omega$. If $\alpha(M) > \frac{n}{n+1}$, then there exist constants $\delta(n, \alpha, K)$ and $C(n, \alpha, \lambda_2(\omega) - 1, K)$ such that for any $\phi \in P(M, \omega, K)$, we have*

$$F_\omega(\phi) \geq \delta J_\omega(\phi) - C.$$

Proof. Let $\omega' = \omega + \partial\bar{\partial}\phi$, where $\phi \in P(M, \omega, K)$. We have

$$\begin{aligned} \frac{1}{V} \int_M e^{-\alpha\phi}\omega^n &= \frac{1}{V} \int_M e^{-(\alpha_1 + \alpha_2 + \varepsilon)\phi}\omega^n \\ &\leq \frac{1}{V} \int_M e^{-(\alpha_1 + \alpha_2)\phi}\omega^n e^{-\varepsilon \inf_M \phi}, \end{aligned}$$

taking $p = \frac{1}{\alpha_1}, q = \frac{1}{1-\alpha_1}$, we have

$$\begin{aligned} \frac{1}{V} \int_M e^{-(\alpha_1 + \alpha_2)\phi}\omega^n &\leq \frac{1}{V} \left(\int_M e^{-\alpha_1 p \phi}\omega^n \right)^{1/p} \left(\int_M e^{-\alpha_2 q \phi}\omega^n \right)^{1/q} \\ &= \frac{1}{V} \left(\int_M e^{-\phi}\omega^n \right)^{\alpha_1} \left(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n \right)^{1-\alpha_1} \\ &\leq C e^{\alpha_1 J_\omega(\phi) - \frac{\alpha_1}{V} \int_M \phi \omega^n} \left(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n \right)^{1-\alpha_1}. \end{aligned}$$

By Lemma 4.2,

$$\begin{aligned} e^{-\varepsilon \inf_M \phi} &\leq e^{\frac{\varepsilon}{V} \int_M (-\phi)\omega'^n + C} \\ &= e^{\varepsilon J_\omega(\phi) - \frac{\varepsilon}{V} \int_M \phi \omega^n + C} \\ &\leq e^{\varepsilon(n+1)J_\omega(\phi) - \frac{\varepsilon}{V} \int_M \phi \omega^n + C}. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} \frac{1}{V} \int_M e^{-\phi} \omega^n &\leq \left(\frac{1}{V} \int_M e^{-\alpha\phi} \omega^n \right)^{\frac{1}{\alpha}} \\ &\leq C e^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha} J_\omega(\phi) - \frac{\alpha_1+\varepsilon}{\alpha V} \int_M \phi \omega^n} \left(\int_M e^{-\frac{\alpha_2}{1-\alpha_1} \phi} \omega^n \right)^{\frac{1-\alpha_1}{\alpha}} \\ &= C e^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha} J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n + \frac{\alpha_2}{V} \int_M (\phi - \sup \phi) \omega^n} \left(\int_M e^{-\frac{\alpha_2}{1-\alpha_1} (\phi - \sup \phi)} \omega^n \right)^{\frac{1-\alpha_1}{\alpha}} \\ &\leq C e^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha} J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n} \left(\int_M e^{-\frac{\alpha_2}{1-\alpha_1} (\phi - \sup \phi)} \omega^n \right)^{\frac{1-\alpha_1}{\alpha}}. \end{aligned}$$

We need to determine $\alpha_1, \alpha_2, \varepsilon$ which satisfy the following conditions:

$$\begin{aligned} \alpha &= \alpha_1 + \alpha_2 + \varepsilon > 1, \\ \alpha &> \alpha_1 + (n+1)\varepsilon, \\ 1 &> \alpha_1. \end{aligned}$$

So we will choose

$$\begin{aligned} \alpha_2 &= n\varepsilon + \varepsilon', \\ \alpha_1 &= 1 - \alpha_2 - \varepsilon + \varepsilon'' = 1 - (n+1)\varepsilon - \varepsilon' + \varepsilon'', \end{aligned}$$

where $\varepsilon, \varepsilon', \varepsilon'' \ll 1$, and $\varepsilon' = o(\varepsilon), \varepsilon'' = o(\varepsilon')$.

Since $\alpha(M) > \frac{n}{n+1}$, we can choose $\varepsilon, \varepsilon', \varepsilon''$ small enough; then we have

$$\frac{\alpha_2}{1-\alpha_1} = \frac{n\varepsilon + \varepsilon'}{(n+1)\varepsilon + \varepsilon' - \varepsilon''} < \alpha(M)$$

and

$$\int_M e^{-\frac{\alpha_2}{1-\alpha_1} (\phi - \sup \phi)} \omega^n < Const.$$

Combined with the inequalities above, we have

$$\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{(1-\delta)J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n}.$$

This proves the lemma. □

Proof of Theorem 2. We assume ω is the Kähler-Einstein metric of M . For any $\phi \in P(M, \omega)$, put $\omega' = \omega + \sqrt{-1} \partial \bar{\partial} \phi$. Consider $(\omega' + \sqrt{-1} \partial \bar{\partial} \psi) = e^{h_{\omega'} + t\psi}$. By solving the Monge-Ampère equation backwards, we get the solutions ϕ_t , and $\phi_1 = -\phi$.

For $t > \frac{1}{2}$, let $\omega_t = \omega' + \sqrt{-1} \partial \bar{\partial} \phi_t = \omega + \sqrt{-1} \partial \bar{\partial} (\phi_t - \phi_1)$; by Lemma 4.1,

$$Ric(\omega_t) \geq \frac{1}{2} \omega_t,$$

which shows that the Green function of ω_t is uniformly bounded from below. Thus by Proposition 4.1 and the calculation in [11] we have

$$\begin{aligned} F_\omega(\phi_t - \phi_1) &\geq \delta J_\omega(\phi_t - \phi_1) - C \\ &\geq C_1 osc_M(\phi_t - \phi_1) - C_2, \end{aligned}$$

and consequently,

$$\begin{aligned}
 n(1-t)J_\omega(\phi) &= n(1-t)J_{\omega'}(\phi_1) \\
 &\geq (1-t)(I_{\omega'}(\phi_1) - J_{\omega'}(\phi_1)) \\
 &\geq F_{\omega'}(\phi_t) - F_{\omega'}(\phi_1) \\
 &= F_\omega(\phi_t - \phi_1) \\
 &\geq C_1 \text{osc}_M(\phi_t - \phi_1) - C_2.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 F_\omega(\phi) &= -F_{\omega'}(-\phi) \\
 &= \int_0^1 (I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t)) dt \\
 &\geq (1-t)(I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t)) \\
 &\geq \frac{1-t}{n} J_{\omega'}(\phi_t) \\
 &\geq \frac{1-t}{n} J_{\omega'}(\phi_1) - 2(1-t)(C_1 \text{osc}_M(\phi_t - \phi_1) - C_2) \\
 &\geq \frac{1-t}{n} J_\omega(\phi) - 2(1-t)^2 n C_1 J_\omega(\phi) - C_3.
 \end{aligned}$$

The theorem follows by choosing $(1-t) < \frac{1}{2n^2 C_1}$.

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REFERENCES

- [1] Ben Abdessellem, A., *Équations de Monge-Ampère d'origine géométrique sur certaines variétés algébriques*, J.Funct. Anal. 149 (1997), 102–134. MR **98i**:32020
- [2] Ding, W and Tian, G., *The generalized Moser-Trudinger Inequality*, Proceedings of Nankai International Conference on Nonlinear Analysis, 1993.
- [3] Lu, Z., *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math. 122 (2000), no. 2, 235–273. MR **2002d**:32034
- [4] Phong, D. H. and Sturm, J., *Algebraic estimates, stability of local zeta functions, and uniform estimates for distribution functions*, Ann. of Math. (2) 152 (2000), no. 1, 277–329. MR **2002f**:11180
- [5] Siu, Y.T., *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. of Math. (2) 127 (1988), no. 3, 585–627. MR **89e**:58032
- [6] Tian, G. and Yau, S.T., *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$* , Comm. Math. Phys. 112 (1987), no.1, 175–203. MR **88k**:32070
- [7] Tian, G., *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$* , Invent. Math. 89 (1987), no. 2, 225–246. MR **88e**:53069
- [8] Tian, G., *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geometry 32 (1990), 99–130. MR **91j**:32031
- [9] Tian, G., *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101 (1990), no. 1, 101–172. MR **91d**:32042
- [10] Tian, G., *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. 130 (1997), no. 1, 1–37. MR **99e**:53065

- [11] Tian, G. and Zhu, X.H., *A nonlinear inequality of Moser-Trudinger type*, Calc. Var. Partial Differential Equations 10 (2000), no. 4, 349–354. MR **2001f**:32044
- [12] Yau, S.T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math. 31 (1978), 339–411. MR **81d**:53045
- [13] Zelditch, S., *Szegő Kernels and a Theorem of Tian*, IMRN1998, No. 6, 317–331. MR **99g**:32055

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