TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 1, Pages 45–57
S 0002-9947(04)03484-1
Article electronically published on January 29, 2004

THE α-INVARIANT ON CERTAIN SURFACES WITH SYMMETRY GROUPS

JIAN SONG

Abstract. The global holomorphic α-invariant introduced by Tian is closely related to the existence of Kähler-Einstein metrics. We apply the result of Tian, Yau and Zelditch on polarized Kähler metrics to approximate plurisubharmonic functions and compute the α-invariant on $CP^2\#n\overline{CP^2}$ for $n = 1, 2, 3$.

1. Introduction

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian [7], Tian and Yau [6] is closely related to the existence of Kähler-Einstein metrics. In his solution of the Calabi conjecture, Yau [12] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with negative or zero first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there are known obstructions such as the Futaki invariant. For a compact Kähler manifold M with positive frist Chern class, Tian [7] proved that M admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$ [9]. Nevertheless, it would also be interesting to find the estimate of the α invariant for $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$. In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman potential on polarized Kähler manifolds to approximate plurisubharmonic functions and compute the α-invariant of $CP^2\#n\overline{CP^2}$ for $n = 1, 2, 3$. In the case of $CP^2\#2\overline{CP^2}$, it gives an improvement of Abdesselem's result [1]. More precisely, we shall show that:

Theorem 1. $\alpha_G(CP^2\#2\overline{CP^2}) = \frac{1}{3}$.

We will give the definitions of the automorphism group G and the α_G-invariant in Section 3.

Let (M, ω) be a compact Kähler manifold, where $\omega=\sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j$. We will also prove Tian's conjecture on the generalized Moser-Trudinger inequality in the special case where $\alpha_G(M) > \frac{n}{n+1}$, for $n = \dim M$. Let

$$P(M,\omega) = \left\{ \phi \mid \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0,\ \sup_M \phi = 0 \right\}.$$

Received by the editors May 1, 2003.
2000 *Mathematics Subject Classification.* Primary 53-XX.

Let F_ω and J_ω be the functionals defined on $P(M,\omega)$ by

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - \log(\frac{1}{V}\int_M e^{h_\omega - \phi}\omega^n),$$

$$J_\omega(\phi) = \frac{\sqrt{-1}}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \partial\phi\wedge\overline{\partial}\phi\wedge\omega^i\wedge\omega_\phi^{n-i-1}.$$

Assume (M,ω_{KE}) is a Kähler-Einstein manifold with positive first Chern class and $Ric(\omega_{KE}) = \omega_{KE}$. Then for any $\phi \in P(M,\omega_{KE})$, Ding and Tian [2] proved the following inequality of Moser-Trudinger type:

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \leq Ce^{J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n}.$$

Tian [10] also conjectured that $\frac{1}{V}\int_M e^{-\phi}\omega^n \leq Ce^{(1-\delta)J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n}$ for $\delta > 0$ sufficiently small, if ϕ is perpendicular to Λ_1, the space of eigenfunctions of ω_{KE} with eigenvalue one.

We shall prove:

Theorem 2. *Let (M,ω) be a Kähler manifold with positive first Chern class. Assume that $\alpha(M) > \frac{n}{n+1}$, so that M admits a Kähler-Einstein metric ω_{KE}, and there exist constants $\delta = \delta(n,\alpha(M))$ and $C = C(n,\lambda_2(\omega_{KE})-1,\alpha(M))$ such that for any $\phi \in P(M,\omega_{KE})$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_{\omega_{KE}}(\phi) \geq \delta J_{\omega_{KE}}(\phi) - C.$$

Here $\lambda_2(\omega_{KE})$ is the least eigenvalue of ω_{KE} which is bigger than 1.

2. Holomorphic approximation of plurisubharmonic functions

In this section, we will employ the technique in [8, 13] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [13].

Theorem 2.1. *Let M be a compact complex manifold of dimension n and let $(L,h) \to M$ be a positive Hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m \in N$, h induces a Hermitian metric h_m on L^m. Let $\{S_0^m, S_1^m, ..., S_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M,L^m)$, $d_m = \dim H^0(M,L^m)$, with respect to the inner product:*

$$(S_1,S_2)_{h_m} = \int_M h_m(S_1(x),S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!}\omega_g^n$ is the volume form of g. Then there is a complete asymptotic expansion

$$\sum_{i=0}^{d_m-1}||S_i^m(x)||_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + ...$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any k,

$$\Big|\Big|\sum_{i=0}^{d_m-1}||S_i^m(x)||_{h_m}^2 - \sum_{j<R}a_j(x)m^{n-j}\Big|\Big|_{C^k} \leq C_{R,k}m^{n-R}$$

where $C_{R,k}$ depends on R,k and the manifold M.

Let

$$\begin{aligned}\tilde{\omega}_g &= \omega_g + \sqrt{-1}\partial\overline{\partial}\phi > 0,\\ \widetilde{h} &= he^{-\phi}.\end{aligned}$$

Let $\widetilde{h}_m$ be the induced Hermitian metric of $\widetilde{h}$ on L^m, and let $\{\widetilde{S}_0^m, \widetilde{S}_{1,\ldots,}^m \widetilde{S}_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, where $d_m = \dim H^0(M, L^m)$, with respect to the inner product

$$(S_1, S_2)_{\widetilde{h}_m} = \int_M \widetilde{h}_m(S_1(x), S_2(x))dV_{\widetilde{g}}\ .$$

By Theorem 2.1, we have

$$\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m} = \left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{h_m}\right) e^{-m\phi}.$$

Thus

$$\phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right) = -\frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right).$$

As $m \to +\infty$, we obtain for any positive integer R

$$\begin{aligned}&\frac{1}{m}\log\left(\sum_{j<R}\widetilde{a}_j(x)m^{n-j}\right)\\ =\ &\frac{1}{m}\log m^n(\sum_{j<R}\widetilde{a}_j(x)m^{-j})\\ =\ &\frac{n}{m}\log m + \frac{1}{m}\log(1 + O(\frac{1}{m})) \to 0.\end{aligned}$$

Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

Corollary 2.1.

$$\left\|\phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{h_m}\right)\right\|_{C^k} \to 0,\ as\ m \to +\infty.$$

In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of L^m.

3. Proof of Theorem 1

Let M be the blow-up of CP^2 at two points and π be its natural projection. Without loss of generality, we may assume the two points are $p_1 = [0, 1, 0]$ and $p_2 = [0, 0, 1]$. Then M is a subvariety of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0X_1 = Z_1X_0,\ \ Z_0Y_2 = Z_2Y_0,$$

where Z_i, X_j, Y_k are the homogeneous coordinates on CP^2, CP^1 and CP^1, respectively.

Let G be the automorphism group acting on $CP^2 \times CP^1 \times CP^1$ generated by θ_j and permutations τ $(0 \le j \le 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_je^{i\theta}, Z_2] \times [X_0, X_1] \times [Y_0, Y_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_2, Z_1] \times [Y_0, Y_2] \times [X_0, X_1] \ .$$

Let π_0, π_1, π_2 be the projection from $CP^2 \times CP^1 \times CP^1$ onto CP^2, CP^1 and CP^1. Respectively define ω by

$$\begin{aligned} \omega &= \pi_0^*\omega_0 + \pi_1^*\omega_1 + \pi_2^*\omega_2 \\ &= \sqrt{-1}\partial\overline{\partial}\log(|Z_0|^2 + |Z_1|^2 + |Z_2|^2) + \sqrt{-1}\partial\overline{\partial}\log(|X_0|^2 + |X_1|^2) \\ &\quad + \sqrt{-1}\partial\overline{\partial}\log(|Y_0|^2 + |Y_2|^2), \end{aligned}$$

where ω_0, ω_1, ω_2 are the Fubini-Study metrics in CP^2, CP^1 and CP^1. By explicit calculation, it can be shown that the cohomological class of $\omega|_M$ is in the first Chern class of M (see [1]).

Consider the divisor

$$\{[0, Z_1, Z_2] \times CP^1 \times CP^1\} + \{CP^2 \times [1, 0] \times CP^1\} + \{CP^2 \times CP^1 \times [1, 0]\}$$

which defines a line bundle (L, h) on $CP^2 \times CP^1 \times CP^1$. The hermitian metric h is defined by

$$h = \frac{1}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|X_0|^2 + |X_1|^2)(|Y_0|^2 + |Y_2|^2)};$$

then $(L, h)|_M \to M$ defines the anticanonical line bundle on M whose curvature form $-\sqrt{-1}\partial\overline{\partial}\log h$ gives the first Chern class on M.

Since $M\backslash\{\pi^{-1}\{p_1\} \cup \pi^{-1}\{p_2\}\}$ is isomorphic to $CP^2\backslash\{p_1, p_2\}$, if we choose the inhomogeneous coordinates $(z_1, z_2) = [1, z_1, z_2]$ on CP^2, the Kähler metric

$$\omega_{g_0} = \sqrt{-1}\partial\overline{\partial}\log(1 + |z_1|^2 + |z_2|^2) + \sqrt{-1}\partial\overline{\partial}\log(1 + |z_1|^2) + \sqrt{-1}\partial\overline{\partial}\log(1 + |z_2|^2)$$

can be extended to a Kähler metric g_0 on M which belongs to $c_1(M)$. If we take different inhomogeneous coordinates $(w_0, w_1) = [w_0, w_1, 1]$, the corresponding Kähler metric is

$$\omega_{g_1} = \sqrt{-1}\partial\overline{\partial}\log(1+|w_0|^2+|w_1|^2)+\sqrt{-1}\partial\overline{\partial}\log(1+|w_0|^2)+\sqrt{-1}\partial\overline{\partial}\log(|w_0|^2+|w_1|^2)$$

and we have

$$\begin{aligned} \det g_0 &= \frac{1}{(1 + |z_1|^2 + |z_2|^2)^3} + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_1|^2)} \\ &\quad + \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2(1 + |z_2|^2)} + \frac{1}{(1 + |z_1|^2)^2(1 + |z_2|^2)^2}, \\ \det g_1 &= \frac{1}{(1 + |w_0|^2 + |w_1|^2)^3} + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(|w_0|^2 + |w_1|^2)} \\ &\quad + \frac{1}{(1 + |w_0|^2 + |w_1|^2)^2(1 + |w_0|^2)} + \frac{|w_0|^2}{(1 + |w_0|^2)^2(|w_0|^2 + |w_1|^2)^2}. \end{aligned}$$

Consider the line bundle $(L^N, h_N) \to CP^2 \times CP^1 \times CP^1$. Then

$$\dim H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) = \frac{(N + 1)^3(N + 2)}{2}$$

and $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ is an orthogonal basis for $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N))$.

Let M_1 be the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0X_1 = Z_1X_0,$$

and M_2 the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0 Y_2 = Z_2 Y_0.$$

Then $M = M_1 \cap M_2$.

In view of the short exact sequences

$$\begin{array}{rcl} 0 & \to & \mathcal{O}(L^N - [M_1]) \to \mathcal{O}(L^N) \to \mathcal{O}(L^N|_{M_1}) \to 0, \\ 0 & \to & \mathcal{O}(L^N|_{M_1} - [M]) \to \mathcal{O}(L^N|_{M_1}) \to \mathcal{O}(L^N|_M) \to 0 \end{array}$$

we can choose N sufficiently large so that

$$H^1(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N - [M_1])) = H^1(M_1, \mathcal{O}(L^N|_{M_1} - [M])) = 0.$$

Then $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \to H^0(M_1, \mathcal{O}(L^N|_{M_1})) \to 0$,

$$H^0(M_1, \mathcal{O}(L^N|_{M_1})) \to H^0(M, \mathcal{O}(L^N|_M)) \to 0$$

and thus

$$H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \to H^0(M, \mathcal{O}(L^N|_M)) \to 0.$$

Also we have $Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_1}|_M = Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}$ and

$$\begin{aligned} &||Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}||^2_{h_N} \\ &\qquad = \frac{\left| Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_0^{j_0} Z_1^{j_1} Z_0^{k_0} Z_2^{k_2} \right|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(Z_0|^2 + |Z_1|^2)(Z_0|^2 + |Z_2|^2))^N} \end{aligned}$$

on $CP^2 \backslash \{p_1, p_2\}$. Therefore, $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}|_M\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ contains an orthogonal basis for $H^0(M, \mathcal{O}(L^N|_M))$ with respect to h^N and the G-invariant Kähler metric g on M.

By Corollary 2.1, for any φ in $P_G(M, \omega_g)$, we have on $CP^2 \backslash \{p_1, p_2\}$,

$$\begin{aligned} &\varphi([Z_0, Z_1, Z_2]) \\ &= \lim_{N \to \infty} \frac{1}{N} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |a^{(N)}_{(\varphi) i_0 i_1 i_2 j_0 j_1 k_0 k_2} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^N} \end{aligned}$$

for some coefficients $a^{(N)}_{(\varphi) i_0 i_1 i_2 j_0 j_1 k_0 k_2}$ satisfying $a^{(N)}_{(\varphi) i_0 i_1 i_2 j_0 j_1 k_0 k_2} = a^{(N)}_{(\varphi) i_0 i_2 i_1 k_0 k_2 j_0 j_1}$ due to the group action by G.

Lemma 3.1. *Using the notations above we have*

$$\frac{1}{n} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2))^n} \leq 4$$

for any positive integer n.

Proof. On the patch $U_0 = \{Z_0 \neq 0\}$, let $z_1 = \frac{Z_1}{Z_0}$ and $z_2 = \frac{Z_2}{Z_0}$,

$$\begin{aligned}
& \frac{1}{n}\log\frac{\sum\limits_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n} \\
\leq\ & \frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}\frac{|z_1^{i_1+j_1}z_2^{i_2+k_2}|^2}{(1+|z_1|^2+|z_2|^2)^n(1+|z_1|^2)^n(1+|z_2|^2)^n}\right) \\
\leq\ & \frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}\frac{|z_1^{i_1+j_1}z_2^{i_2+k_2}|^2}{1+|z_1^{i_1+j_1}z_2^{i_2+k_2}|^2}\right) \\
\leq\ & \frac{1}{n}\log\left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}1\right) \\
=\ & \frac{1}{n}\log\frac{(n+1)^3(n+2)}{2}\leq 4.
\end{aligned}$$

This inequality also holds on the patch $U_1 = \{Z_1 \neq 0\}$ by continuity, and so the lemma is proved. □

Lemma 3.2. *There exists $\varepsilon > 0$ such that for any $\varphi \in P_G(M,\omega_g)$ and N, there exist $n > N$, $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ with $i_0+i_1+i_2 = j_0+j_1 = k_0+k_2 = n$, and $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} > \varepsilon$.*

Proof. Otherwise, for any $\varepsilon > 0$, there exist φ and N, such that for any $n > N$ and any $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ satisfying $i_0+i_1+i_2=j_0+j_1=k_0+k_2=n$, we have $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} < \varepsilon$. By choosing n large enough and with the lemma above, we have

$$\begin{aligned}
& \varphi([Z_0,Z_1,Z_2]) \\
\leq\ & \frac{1}{n}\log\frac{\max|a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2}|^2\sum\limits_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n}+\varepsilon \\
\leq\ & \frac{1}{n}\log\frac{\sum\limits_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n}|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n}+2\log\varepsilon+\varepsilon \\
\leq\ & \log\varepsilon+4.
\end{aligned}$$

Since ε could be arbitrarily small, the above inequality would imply that $\varphi \to -\infty$ uniformly, which contradicts the fact that $\sup_M \varphi = 0$. □

Proof of Theorem 1. We use notations as above; since $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} > \varepsilon$, we have

$$\begin{aligned}
& \varphi([Z_0,Z_1,Z_2]) \\
=\ & \lim_{N\to\infty}\frac{1}{N}\log\frac{\sum\limits_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}|a^{(N)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2}Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N} \\
\geq\ & \frac{1}{N}\log\frac{|Z_0^{i_0+j_0+k_0}Z_1^{i_1+j_1}Z_2^{i_2+k_2}|^2+|Z_0^{i_0+j_0+k_0}Z_1^{i_2+k_2}Z_2^{i_1+j_1}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N}+\log\epsilon
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{N}\log\frac{|{Z_0}^m Z_1^{\frac{3}{2}N-\frac{m}{2}} Z_2^{\frac{3}{2}N-\frac{m}{2}}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N}+\log\epsilon \\
&\geq \log\frac{|{Z_0}^{\frac{2m}{N}} Z_1^{3-\frac{m}{N}} Z_2^{3-\frac{m}{N}}|^2}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}+\log\epsilon,
\end{aligned}$$

where $i_0+j_0+k_0=m, i_1+j_1+i_2+k_2=3N-m$.

On the patch $U_0=\{Z_0\neq 0\}$,

$$\begin{aligned}
&\int_{U_0\cap\{0<|z_1|,|z_2|<1\}} e^{-\alpha\varphi}\omega_{g_0}^2 \\
\leq\ & C_1\int_{0<|z_1|,|z_2|<1} e^{-\alpha\log\frac{|Z_0|^{\frac{2m}{N}}|Z_1|^{3-\frac{m}{N}}|Z_2|^{3-\frac{m}{N}}}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}}\omega_{g_0}^2 \\
=\ & C_1\int_{0<|z_1|,|z_2|<1}\frac{(|Z_0|^2+|Z_1|^2+|Z_2|^2)^\alpha(|Z_0|^2+|Z_1|^2)^\alpha(|Z_0|^2+|Z_2|^2)^\alpha}{|Z_0|^{\frac{2\alpha m}{N}}|Z_1|^{3\alpha-\frac{\alpha m}{N}}|Z_2|^{3\alpha-\frac{\alpha m}{N}}}\omega_{g_0}^2 \\
\leq\ & C_2\int_{0<|z_1|,|z_2|<1}\frac{(1+|z_1|^2+|z_2|^2)^\alpha(1+|z_1|^2)^\alpha(|1+|z_2|^2)^\alpha}{|z_1|^{3\alpha-\frac{\alpha m}{N}}|z_2|^{3\alpha-\frac{\alpha m}{N}}}dz_1\wedge d\overline{z}_1 \\
& \qquad\qquad \wedge dz_2\wedge d\overline{z}_2 \\
\leq\ & C_3\int_{0<|z_1|,|z_2|<1}\frac{1}{|z_1|^{3\alpha-\frac{\alpha m}{N}}|z_2|^{3\alpha-\frac{\alpha m}{N}}}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
\leq\ & C_3\int_{0<|z_1|,|z_2|<1}\frac{1}{|z_1|^{3\alpha}|z_2|^{3\alpha}}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2,
\end{aligned}$$

where C_1, C_2 and C_3 are constants depending only on α and ϵ.

On the patch $U_2=\{Z_2\neq 0\}$,

$$\begin{aligned}
&\int_{U_1\cap\{0<|w_0|,|w_1|\leq 1\}} e^{-\alpha\varphi}\omega_{g_1}^2 \\
\leq\ & C_4\int_{0<|w_0|,|w_1|\leq 1} e^{-\alpha\log\frac{|Z_0|^{\frac{2m}{N}}|Z_1|^{3-\frac{m}{N}}|Z_2|^{3-\frac{m}{N}}}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}}\omega_{g_1}^2 \\
=\ & C_4\int_{0<|w_0|,|w_1|\leq 1}\frac{(1+|w_0|^2+|w_1|^2)^\alpha(1+|w_0|^2)^\alpha(|w_0|^2+|w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}}\omega_{g_1}^2 \\
\leq\ & C_5\int_{0<|w_0|,|w_1|\leq 1}\frac{(1+|w_0|^2+|w_1|^2)^\alpha(1+|w_0|^2)^\alpha(|w_0|^2+|w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}(|w_0|^2+|w_1|^2)}dw_0\wedge d\overline{w}_0 \\
& \qquad\qquad \wedge dw_1\wedge d\overline{w}_1 \\
\leq\ & C_6\int_{0<|w_0|,|w_1|\leq 1}\frac{1}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}(|w_0|^2+|w_1|^2)^{1-\alpha}}dw_0\wedge d\overline{w}_0\wedge dw_1\wedge d\overline{w}_1 \\
\leq\ & C_6\int_{t=0}^1\int_{s=0}^1\frac{1}{s^{\frac{\alpha m}{N}}t^{\frac{3}{2}\alpha-\frac{\alpha m}{2N}}(s+t)^{1-\alpha}}dsdt \\
\leq\ & C_6\int_{s=0}^1\frac{1}{s^{\frac{\alpha m}{N}}t^{\frac{3}{2}\alpha-\frac{\alpha m}{2N}}s^{(1-\alpha)p}t^{(1-\alpha)q}}dsdt,
\end{aligned}$$

where $p+q=1$ and C_4, C_5, C_6 are constants depending only on α and ϵ.

Case 1: If $1 \leq \frac{m}{N} \leq 3$, we can choose $\alpha < \min(\frac{2}{3}, \frac{1-p}{3-p})$ so that

$$\begin{aligned} \frac{\alpha m}{N} + (1-\alpha)p &< 1, \\ 3\alpha - 1 &< 1, \\ \frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1-\alpha)q &< 1. \end{aligned}$$

Case 2: If $0 < \frac{m}{N} < 1$, we can choose $\alpha < \min(\frac{2}{3}, \frac{1-q}{3/2-q})$ so that

$$\begin{aligned} \frac{\alpha m}{N} + (1-\alpha)p &< 1, \\ 3\alpha - 1 &< 1, \\ \frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1-\alpha)q &< 1. \end{aligned}$$

So we could choose any $\alpha < \frac{1}{3}$, which implies that $\alpha_G(M,\omega) \geq \frac{1}{3}$.

Conversely, we choose

$$\begin{aligned} \varphi_\varepsilon &= \log\Big(\frac{|Z_0|^6}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)} + \varepsilon\Big) \\ &\quad - \log(1+\varepsilon) \\ &\in P_G(M,\omega). \end{aligned}$$

Then we have $\sup_M \varphi_\varepsilon = 0$ and $\varphi_\varepsilon = \log\frac{\varepsilon}{1+\varepsilon}$ on the exceptional divisors. Furthermore, we have

$$\lim_{\varepsilon\to 0}\int_M e^{-\alpha\varphi_\varepsilon}\omega^2 = \infty, \ for\ any\ \alpha > \frac{1}{3}.$$

Hence we have shown $\alpha_G(M,\omega) = \frac{1}{3}$.

We can also apply the above arguments for CP^n $(n \geq 2)$, $CP^2\#1\overline{CP^2}$ and $CP^2\#3\overline{CP^2}$.

(i) Let $M = CP^n$ and let G_n be the automorphism group acting on M, generated by θ_j and permutations $\tau_{i,j}$ $(0 \leq i < j \leq n)$,

$$\theta_j : [Z_0, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j e^{i\theta}, ..., Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau_{i,j} : [Z_0, ..., Z_i, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j, ..., Z_i, ..., Z_n].$$

Theorem 3.1. $\alpha_{G_n}(CP^n) = 1$.

(ii) Let M be the blow-up of CP^2 at 3 points which are not collinear. Then we can assume that these 3 points are $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$. Let $G(3)$ be the automorphism group acting on M, generated by θ_j and permutations $\tau_{i,j}$ $(0 \leq i < j \leq 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \to [Z_0, Z_j e^{i\theta}, Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau_{i,j} : [..., Z_i, ..., Z_j, ...] \to [..., Z_j, ..., Z_i, ...].$$

Theorem 3.2. $\alpha_{G(3)}(CP^2\#3\overline{CP^2}) = 1$.

(iii) Let M be the blow-up of CP^2 at one point $[1,0,0]$ and $G(1)$ be the automorphism group acting on M, generated by θ_j and permutations τ $(0 \le i \le 2)$,

$$\theta_j : [Z_0, Z_j, Z_2] \to [Z_0, Z_j e^{i\theta}, Z_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \to [Z_0, Z_2, Z_1].$$

Theorem 3.3. $\alpha_{G(1)}(CP^2 \# 1\overline{CP^2}) = \frac{1}{2}$.

Also the proof above shows that the sequence of the holomorphic invariants $\{\alpha_{G,m}(M)\}_m$ defined by Tian [8] on CP^n $(n \ge 2)$, $CP^2 \# k\overline{CP^2}$ $(k = 1, 2, 3)$ is stationary.

4. Proof of Theorem 2

In this section, we will prove the generalized Moser-Trudinger inequality on any Kähler manifold M of dimension n whose $\alpha(M)$ is greater than $\frac{n}{n+1}$. The following theorem is due to Tian and Zhu [11].

Theorem 4.1. *Let (M, ω) be a Kähler-Einstein manifold with $Ric(\omega) = \omega$; then there exist constants $\delta = \delta(n)$ and $C = C(n, \lambda_2(\omega) - 1) \ge 0$ such that for any $\phi \in P(M, \omega)$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_\omega(\phi) \ge J_\omega(\phi)^\delta - C,$$

which is the same as

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \le C e^{J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - J_\omega(\phi)^\delta}.$$

This implies in particular the Moser-Trudinger inequality on S^2, which reads

$$\frac{1}{4\pi}\int_{S^2} e^{-\phi}\omega \le e^{\frac{1}{8\pi}\int_{S^2}|\nabla\phi|^2\omega - \frac{1}{4\pi}\int_{S^2}\phi}.$$

For any $\phi \in P(M, \omega)$, put $\omega' = \omega_\phi = \omega + \sqrt{-1}\partial\overline{\partial}\phi$ and $Ric(\omega) = \omega + \sqrt{-1}\partial\overline{\partial}h_\omega$. Consider the Monge-Ampère equation

$$(\omega' + \sqrt{-1}\partial\overline{\partial}\psi)^n = e^{h_\omega - t\psi}\omega'^n.$$

We will use the continuity method backwards and let ϕ_t be a smooth family which solve the above equation.

The following lemmas are well known [10], but we add the proofs for the sake of completeness.

Lemma 4.1. *$Ric(\omega_t) \ge t\omega_t$ and we have equality if and only if $t = 1$, where $\omega_t = \omega + \phi_t$ and ϕ_t solves the Monge-Ampère equation at t.*

Proof.

$$\begin{aligned} Ric(\omega_t) &= -\sqrt{-1}\partial\overline{\partial}\log\omega_t^n = -\sqrt{-1}\partial\overline{\partial}\log\frac{\omega_t^n}{\omega^n} + Ric(\omega) \\ &= -\sqrt{-1}\partial\overline{\partial}\left(h_\omega - t\phi_t\right) + \omega + \sqrt{-1}\partial\overline{\partial}h_\omega \\ &= \omega + t\phi_t = t\omega_t + (1-t)\omega \ge t\omega_t. \end{aligned}$$

□

Lemma 4.2. *For any $\phi \in P(M,\omega)$, if the Green's function of $\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\phi$ is bounded from below, we have:*

$$-\inf_M \phi \leq \frac{1}{V}\int_M (-\phi)\omega'^n + C.$$

Proof. Since $\omega + \sqrt{-1}\partial\overline{\partial}\phi = \omega'$ and $\omega' - \sqrt{-1}\partial\overline{\partial}\phi > 0$, we have $\Delta_{\omega'}\phi \leq n$, and

$$\begin{aligned} -\phi &= \frac{1}{V}\int_M (-\phi)\omega'^n + \frac{1}{V}\int_M \Delta_{\omega'}\phi(y)G_{\omega'}(x,y)\omega'^n \\ &\leq \frac{1}{V}\int_M (-\phi)\omega'^n + \frac{1}{V}\int_M n(G_{\omega'}(x,y) - \inf G_{\omega'}(x,y))\omega'^n \\ &\leq \frac{1}{V}\int_M (-\phi)\omega'^n + C. \end{aligned}$$

□

Let (M,ω) be a Kähler-Einstein manifold with $\mathrm{Ric}(\omega) = \omega$ and let $P(M,\omega,K) = \{\phi \in P(M,\omega) \mid G_{\omega+\sqrt{-1}\partial\overline{\partial}\phi}(x,y) \geq -K)$. Then we have:

Proposition 4.1. *Let (M,ω) be a Kähler-Einstein manifold with $Ric\,(\omega) = \omega$. If $\alpha(M) > \frac{n}{n+1}$, then there exist constants $\delta(n,\alpha,K)$ and $C(n,\alpha,\lambda_2(\omega)-1,K)$ such that for any $\phi \in P(M,\omega,K)$, we have*

$$F_\omega(\phi) \geq \delta J_\omega(\phi) - C.$$

Proof. Let $\omega' = \omega + \partial\overline{\partial}\phi$, where $\phi \in P(M,\omega,K)$. We have

$$\begin{aligned} \frac{1}{V}\int_M e^{-\alpha\phi}\omega^n &= \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2+\varepsilon)\phi}\omega^n \\ &\leq \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2)\phi}\omega^n e^{-\varepsilon\inf_M \phi}; \end{aligned}$$

taking $p = \frac{1}{\alpha_1}, q = \frac{1}{1-\alpha_1}$, we have

$$\begin{aligned} \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2)\phi}\omega^n &\leq \frac{1}{V}\Big(\int_M e^{-\alpha_1 p\phi}\omega^n\Big)^{1/p}\Big(\int_M e^{-\alpha_2 q\phi}\omega^n\Big)^{1/q} \\ &= \frac{1}{V}\Big(\int_M e^{-\phi}\omega^n\Big)^{\alpha_1}\Big(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n\Big)^{1-\alpha_1} \\ &\leq Ce^{\alpha_1 J_\omega(\phi) - \frac{\alpha_1}{V}\int_M \phi\omega^n}\Big(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n\Big)^{1-\alpha_1}. \end{aligned}$$

By Lemma 4.2,

$$\begin{aligned} e^{-\varepsilon\inf_M \phi} &\leq e^{\frac{\varepsilon}{V}\int_M(-\phi)\omega'^n + C} \\ &= e^{\varepsilon I_\omega(\phi) - \frac{\varepsilon}{V}\int_M \phi\omega^n + C} \\ &\leq e^{\varepsilon(n+1)J_\omega(\phi) - \frac{\varepsilon}{V}\int_M \phi\omega^n + C}. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
\frac{1}{V}\int_M e^{-\phi}\omega^n &\le (\frac{1}{V}\int_M e^{-\alpha\phi}\omega^n)^{\frac{1}{\alpha}} \\
&\le Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{\alpha_1+\varepsilon}{\alpha V}\int_M\phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{\frac{1-\alpha_1}{\alpha}} \\
&= Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{1}{V}\int_M\phi\omega^n+\frac{\alpha_2}{V}\int_M(\phi-\sup\phi)\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n)^{\frac{1-\alpha_1}{\alpha}} \\
&\le Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{1}{V}\int_M\phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n)^{\frac{1-\alpha_1}{\alpha}}.
\end{aligned}$$

We need to determine $\alpha_1,\ \alpha_2,\ \varepsilon$ which satisfy the following conditions:

$$\begin{aligned}
\alpha &= \alpha_1+\alpha_2+\varepsilon>1, \\
\alpha &> \alpha_1+(n+1)\varepsilon, \\
1 &> \alpha_1.
\end{aligned}$$

So we will choose

$$\begin{aligned}
\alpha_2 &= n\varepsilon+\varepsilon', \\
\alpha_1 &= 1-\alpha_2-\varepsilon+\varepsilon''=1-(n+1)\varepsilon-\varepsilon'+\varepsilon'',
\end{aligned}$$

where ε , ε' , $\varepsilon''<<1$, and $\varepsilon'=o(\varepsilon)$, $\varepsilon''=o(\varepsilon')$.

Since $\alpha(M)>\frac{n}{n+1}$, we can choose ε, ε' , ε'' small enough; then we have

$$\frac{\alpha_2}{1-\alpha_1}=\frac{n\varepsilon+\varepsilon'}{(n+1)\varepsilon+\varepsilon'-\varepsilon''}<\alpha(M)$$

and

$$\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n<Const.$$

Combined with the inequalities above, we have

$$\frac{1}{V}\int_M e^{-\phi}\omega^n\le Ce^{(1-\delta)J_\omega(\phi)-\frac{1}{V}\int_M\phi\omega^n}.$$

This proves the lemma. □

Proof of Theorem 2. We assume ω is the Kähler-Einstein metric of M. For any $\phi\in P(M,\omega)$, put $\omega'=\omega+\sqrt{-1}\partial\overline{\partial}\phi$. Consider $(\omega'+\sqrt{-1}\partial\overline{\partial}\psi)=e^{h_{\omega'}+t\psi}$. By solving the Monge-Ampère equation backwards, we get the solutions ϕ_t, and $\phi_1=-\phi$.

For $t>\frac{1}{2}$, let $\omega_t=\omega'+\sqrt{-1}\partial\overline{\partial}\phi_t=\omega+\sqrt{-1}\partial\overline{\partial}(\phi_t-\phi_1)$; by Lemma 4.1,

$$Ric(\omega_t)\ge\frac{1}{2}\omega_t,$$

which shows that the Green function of ω_t is uniformly bounded from below. Thus by Proposition 4.1 and the calculation in [11] we have

$$\begin{aligned}
F_\omega(\phi_t-\phi_1) &\ge \delta J_\omega(\phi_t-\phi_1)-C \\
&\ge C_1 osc_M(\phi_t-\phi_1)-C_2,
\end{aligned}$$

and consequently,

$$\begin{aligned} n(1-t)J_\omega(\phi) &= n(1-t)J_{\omega'}(\phi_1) \\ &\geq (1-t)(I_{\omega'}(\phi_1) - J_{\omega'}(\phi_1)) \\ &\geq F_{\omega'}(\phi_t) - F_{\omega'}(\phi_1) \\ &= F_\omega(\phi_t - \phi_1) \\ &\geq C_1 osc_M(\phi_t - \phi_1) - C_2. \end{aligned}$$

Thus we have

$$\begin{aligned} F_\omega(\phi) &= -F_{\omega'}(-\phi) \\ &= \int_0^1 (I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t))dt \\ &\geq (1-t)(I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t)) \\ &\geq \frac{1-t}{n} J_{\omega'}(\phi_t) \\ &\geq \frac{1-t}{n} J_{\omega'}(\phi_1) - 2(1-t)(C_1 osc_M(\phi_t - \phi_1) - C_2) \\ &\geq \frac{1-t}{n} J_\omega(\phi) - 2(1-t)^2 n C_1 J_\omega(\phi) - C_3. \end{aligned}$$

The theorem follows by choosing $(1-t) < \frac{1}{2n^2 C_1}$.

Acknowledgements

The author deeply thanks his advisor, Professor D.H. Phong for his constant encouragement and help. He also thanks Professor G. Tian for his suggestion on this work. This paper is part of the author's future Ph.D. thesis in the Mathematical Department of Columbia University.

References

[1] Ben Abdesselem, A., *Équations de Monge-Ampère d'origine géométrique sur certaines variétés algébriques*, J.Funct. Anal. 149 (1997), 102–134. MR **98i:**32020

[2] Ding, W and Tian, G., *The generalized Moser-Trudinger Inequality*, Proceedings of Nankai International Conference on Nonlinear Analysis, 1993.

[3] Lu, Z., *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math. 122 (2000), no. 2, 235–273. MR **2002d:**32034

[4] Phong, D. H. and Sturm, J., *Algebraic estimates, stability of local zeta functions, and uniform estimates for distribution functions*, Ann. of Math. (2) 152 (2000), no. 1, 277–329. MR **2002f:**11180

[5] Siu, Y.T., *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. of Math. (2) 127 (1988), no. 3, 585–627. MR **89e:**58032

[6] Tian, G. and Yau, S.T., *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$*, Comm. Math. Phys. 112 (1987), no.1, 175–203. MR **88k:**32070

[7] Tian, G., *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math. 89 (1987), no. 2, 225–246. MR **88e:**53069

[8] Tian, G., *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geometry 32 (1990), 99–130. MR **91j:**32031

[9] Tian, G., *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101 (1990), no. 1, 101–172. MR **91d:**32042

[10] Tian, G., *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. 130 (1997), no. 1, 1–37. MR **99e:**53065

[11] Tian, G. and Zhu, X.H., *A nonlinear inequality of Moser-Trudinger type*, Calc. Var. Partial Differential Equations 10 (2000), no. 4, 349–354. MR **2001f:**32044

[12] Yau, S.T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math. 31 (1978), 339–411. MR **81d:**53045

[13] Zelditch, S., *Szegö Kernels and a Theorem of Tian*, IMRN1998, No. 6, 317–331. MR **99g:**32055

Department of Mathematics, Columbia University, New York, New York 10027
E-mail address: jsong@math.columbia.edu