THE $\alpha$-INVARIANT ON CERTAIN SURFACES
WITH SYMMETRY GROUPS

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ABSTRACT. The global holomorphic $\alpha$-invariant introduced by Tian is closely related to the existence of Kähler-Einstein metrics. We apply the result of Tian, Yau and Zelditch on polarized Kähler metrics to approximate plurisubharmonic functions and compute the $\alpha$-invariant on $\mathbb{CP}^2 \# n \mathbb{CP}^2$ for $n = 1, 2, 3$.

1. INTRODUCTION

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian [7], Tian and Yau [6] is closely related to the existence of Kähler-Einstein metrics. In his solution of the Calabi conjecture, Yau [12] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with negative or zero first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there are known obstructions such as the Futaki invariant. For a compact Kähler manifold $M$ with positive first Chern class, Tian [7] proved that $M$ admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $CP^2 \# 1 CP^2$ and $CP^2 \# 2 CP^2$ [9]. Nevertheless, it would also be interesting to find the estimate of the $\alpha$ invariant for $CP^2 \# 1 CP^2$ and $CP^2 \# 2 CP^2$. In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman potential on polarized Kähler manifolds to approximate plurisubharmonic functions and compute the $\alpha$-invariant of $CP^2 \# n CP^2$ for $n = 1, 2, 3$. In the case of $CP^2 \# 2 CP^2$, it gives an improvement of Abdesselem's result [1]. More precisely, we shall show that:

Theorem 1. $\alpha_G(CP^2 \# 2 CP^2) = \frac{1}{3}$.

We will give the definitions of the automorphism group $G$ and the $\alpha_G$-invariant in Section 3.

Let $(M, \omega)$ be a compact Kähler manifold, where $\omega = \sqrt{-1} g_{ji} dz_i \wedge d\overline{z}_j$. We will also prove Tian’s conjecture on the generalized Moser-Trudinger inequality in the special case where $\alpha_G(M) > \frac{n}{n+1}$, for $n = \dim M$. Let

$$P(M, \omega) = \left\{ \phi \mid \omega_{\phi} = \omega + \sqrt{-1} \partial \overline{\partial} \phi > 0, \sup_M \phi = 0 \right\}.$$
Let $F_\omega$ and $J_\omega$ be the functionals defined on $P(M, \omega)$ by

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n - \log \left( \frac{1}{V} \int_M e^{h_\omega - \phi} \omega^n \right),$$

$$J_\omega(\phi) = \frac{\sqrt{-1}}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial\phi \wedge \overline{\partial}\phi \wedge \omega^i \wedge \omega^{n-i-1}.$$  

Assume $(M, \omega_{KE})$ is a Kähler-Einstein manifold with positive first Chern class and $Ric(\omega_{KE}) = \omega_{KE}$. Then for any $\phi \in P(M, \omega_{KE})$, Ding and Tian [2] proved the following inequality of Moser-Trudinger type:

$$\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{\lambda_2(\phi)} - \frac{1}{V} \int_M \phi \omega^n.$$ 

Tian [10] also conjectured that $\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{(1-\delta)\lambda_2(\phi)} - \frac{1}{V} \int_M \phi \omega^n$ for $\delta > 0$ sufficiently small, if $\phi$ is perpendicular to $\Lambda_1$, the space of eigenfunctions of $\omega_{KE}$ with eigenvalue one.

We shall prove:

**Theorem 2.** Let $(M, \omega)$ be a Kähler manifold with positive first Chern class. Assume that $\alpha(M) > \frac{n^2}{n+1}$, so that $M$ admits a Kähler-Einstein metric $\omega_{KE}$, and there exist constants $\delta = \delta(n, \alpha(M))$ and $C = C(n, \lambda_2(\omega_{KE}) - 1, \alpha(M))$ such that for any $\phi \in P(M, \omega_{KE})$ which satisfies $\phi \perp \Lambda_1$, we have

$$F_{\omega_{KE}}(\phi) \geq \delta J_{\omega_{KE}}(\phi) - C.$$ 

Here $\lambda_2(\omega_{KE})$ is the least eigenvalue of $\omega_{KE}$ which is bigger than 1.

**2. Holomorphic approximation of plurisubharmonic functions**

In this section, we will employ the technique in [8, 13] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [13].

**Theorem 2.1.** Let $M$ be a compact complex manifold of dimension $n$ and let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m \in \mathbb{N}$, $h$ induces a Hermitian metric $h_m$ on $L^m$. Let $\{S_g^m, S_1^m, \ldots, S_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product:

$$(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x))dV_g,$$ 

where $dV_g = \frac{1}{m} \omega^n_g$ is the volume form of $g$. Then there is a complete asymptotic expansion

$$\sum_{i=0}^{d_m-1} \|S_i^m(x)\|_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \ldots$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any $k$,

$$\|S_i^m(x)\|_{h_m}^2 - \sum_{j<R} a_j(x)m^{n-j} \leq C_{R,k}m^{n-R}$$

where $C_{R,k}$ depends on $R, k$ and the manifold $M$. 


Let
\[ \tilde{\omega}_g = \omega_g + \sqrt{-1} \partial \bar{\partial} \phi > 0, \]
\[ \tilde{h} = h e^{-\phi}. \]
Let \( \tilde{h}_m \) be the induced Hermitian metric of \( \tilde{h} \) on \( L^m \), and let \( \{ \tilde{S}_0, \tilde{S}_1, \ldots, \tilde{S}_{d_m-1} \} \) be any orthonormal basis of \( H^0(M, L^m) \), where \( d_m = \dim H^0(M, L^m) \), with respect to the inner product
\[ (S_1, S_2)_{\tilde{h}_m} = \int_M \tilde{h}_m(S_1(x), S_2(x)) dV_g. \]
By Theorem 2.1, we have
\[ d_m \sum_{i=0}^{d_m-1} ||\tilde{S}_i(x)||^2_{\tilde{h}_m} \leq \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i(x)||_{h_m}^2 \right) e^{-m \phi}. \]
Thus
\[ \phi - \frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i(x)||^2_{\tilde{h}_m} \right) = - \frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i(x)||^2_{h_m} \right). \]
As \( m \to +\infty \), we obtain for any positive integer \( R \)
\[ \frac{1}{m} \log \left( \sum_{j<R} \tilde{a}_j(x)m^{n-j} \right) \]
\[ = \frac{1}{m} \log m^n \left( \sum_{j<R} \tilde{a}_j(x)m^{-j} \right) \]
\[ = \frac{1}{m} \log m + \frac{1}{m} \log(1 + O(\frac{1}{m})) \to 0. \]
Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

**Corollary 2.1.**
\[ \left\| \phi - \frac{1}{m} \log \left( \sum_{i=0}^{d_m-1} ||\tilde{S}_i(x)||^2_{\tilde{h}_m} \right) \right\|_{C^k} \to 0, \text{ as } m \to +\infty. \]
In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of \( L^m \).

### 3. Proof of Theorem 1

Let \( M \) be the blow-up of \( CP^2 \) at two points and \( \pi \) be its natural projection. Without loss of generality, we may assume the two points are \( p_1 = [0, 1, 0] \) and \( p_2 = [0, 0, 1] \). Then \( M \) is a subvariety of \( CP^2 \times CP^1 \times CP^1 \) defined by the equations
\[ Z_0 X_1 = Z_1 X_0, \quad Z_0 Y_2 = Z_2 Y_0, \]
where \( Z_i, X_j, Y_k \) are the homogeneous coordinates on \( CP^2, CP^1 \) and \( CP^1 \), respectively.
Let \( G \) be the automorphism group acting on \( CP^2 \times CP^1 \times CP^1 \) generated by \( \theta_j \) and permutations \( \tau (0 \leq j \leq 2) \),
\[ \theta_j : [Z_0, Z_j, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_j e^{i\theta}, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \]
can be extended to a Kähler metric and we have

\[ \text{Respectively define } L;h \text{ which defines a line bundle on } CP^2 \times CP^1 \times CP^1. \]

Let \( \omega_0, \omega_1, \omega_2 \) be the projection from \( CP^2 \times CP^1 \times CP^1 \) onto \( CP^2, CP^1 \) and \( CP^1 \). By explicit calculation, it can be shown that the cohomological class of \( \omega \mid_M \) is in the first Chern class of \( M \) (see [1]).

Consider the divisor

\[ \{(0, Z_1, Z_2) \times CP^1 \times CP^1\} + \{CP^2 \times [1,0] \times CP^1\} + \{CP^2 \times CP^1 \times [1,0]\} \]

defines a line bundle \( (L, h) \) on \( CP^2 \times CP^1 \times CP^1 \). The hermitian metric \( h \) is defined by

\[ h = \frac{1}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|X_0|^2 + |X_1|^2)(|Y_0|^2 + |Y_2|^2)}; \]

then \( (L, h) \mid_M \rightarrow M \) defines the anticanonical line bundle on \( M \) whose curvature form \( \sqrt{-1} \partial \bar{\partial} \log h \) gives the first Chern class on \( M \).

Since \( M \setminus \{\pi^{-1}(p_1) \cup \pi^{-1}(p_2)\} \) is isomorphic to \( CP^2 \setminus \{p_1, p_2\} \), if we choose the inhomogeneous coordinates \( (z_1, z_2) = (w_0, w_1) \) on \( CP^2 \), the Kähler metric

\[ \omega_{g_0} = \sqrt{-1} \partial \bar{\partial} \log(1 + |z_1|^2 + |z_2|^2) + \sqrt{-1} \partial \bar{\partial} \log(1 + |z_1|^2 + \sqrt{-1} \partial \bar{\partial} \log(1 + |z_2|^2) \]

can be extended to a Kähler metric \( g_0 \) on \( M \) which belongs to \( c_1(M) \). If we take different inhomogeneous coordinates \( (w_0, w_1) = (w_0, w_1, 1) \), the corresponding Kähler metric is

\[ \omega_{g_1} = \sqrt{-1} \partial \bar{\partial} \log(1 + |w_0|^2 + |w_1|^2) + \sqrt{-1} \partial \bar{\partial} \log(1 + |w_0|^2) + \sqrt{-1} \partial \bar{\partial} \log(|w_0|^2 + |w_1|^2) \]

and we have

\[ \det g_0 = \frac{1}{(1 + |z_1|^2 + |z_2|^2)^3} + \frac{1}{(1 + |z_1|^2 + |z_2|^2)(1 + |z_1|^2)} \]

\[ + \frac{1}{(1 + |z_1|^2 + |z_2|^2)(1 + |z_2|^2)} \]

\[ \det g_1 = \frac{1}{(1 + |w_0|^2 + |w_1|^2)^3} + \frac{1}{(1 + |w_0|^2)(1 + |w_1|^2)} \]

\[ + \frac{1}{(1 + |w_0|^2)(1 + |w_1|^2)} \]

Consider the line bundle \( (L^N, h_N) \rightarrow CP^2 \times CP^1 \times CP^1 \). Then

\[ \dim H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) = \frac{(N + 1)^3(N + 2)}{2} \]

and \( \{Z_0^i Z_1^j Z_2^k X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}\}_{i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = N} \) is an orthogonal basis for \( H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \).

Let \( M_1 \) be the hypersurface of \( CP^2 \times CP^1 \times CP^1 \) defined by the equations

\[ Z_0 X_1 = Z_1 X_0, \]
and $M_2$ the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations
\[ Z_0 Y_2 = Z_2 Y_0. \]

Then $M = M_1 \cap M_2$.

In view of the short exact sequences
\[ 0 \to \mathcal{O}(L^N - [M_1]) \to \mathcal{O}(L^N) \to \mathcal{O}(L^N|_{M_1}) \to 0, \]
\[ 0 \to \mathcal{O}(L^N|_{M_1} - [M]) \to \mathcal{O}(L^N|_{M_1}) \to \mathcal{O}(L^N|_{M}) \to 0 \]
we can choose $N$ sufficiently large so that
\[ H^1(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N - [M_1])) = H^1(M_1, \mathcal{O}(L^N|_{M_1} - [M])) = 0. \]

Then $H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \to H^0(M_1, \mathcal{O}(L^N|_{M_1})) \to 0$,
\[ H^0(M_1, \mathcal{O}(L^N|_{M_1})) \to H^0(M, \mathcal{O}(L^N|_{M})) \to 0 \]
and thus
\[ H^0(CP^2 \times CP^1 \times CP^1, \mathcal{O}(L^N)) \to H^0(M, \mathcal{O}(L^N|_{M})) \to 0. \]

Also we have $Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_2^{k_2} |_{M_1} = Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}$ and
\[
|Z_0^{i_0} < Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_2^{k_2} |_{h_N}^2 = \frac{|Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_0^{j_0} Z_1^{j_1} Z_2^{k_2}|^2}{((Z_0^2 + Z_1^2 + Z_2^2)(Z_0^2 + Z_1^2 + Z_2^2))^N}
\]
on $CP^2 \setminus \{p_1, p_2\}$. Therefore, $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_2^{k_2} |_{M_1} \}_{i_0+i_1+i_2 = j_0+j_1+k_0+k_2 = N}$ contains an orthogonal basis for $H^0(M, \mathcal{O}(L^N|_{M}))$ with respect to $h^N$ and the $G$-invariant Kähler metric $g$ on $M$.

By Corollary 2.1, for any $\varphi$ in $P_G(M, \omega_g)$, we have on $CP^2 \setminus \{p_1, p_2\}$,
\[
\varphi([Z_0, Z_1, Z_2]) = \lim_{N \to \infty} \frac{1}{N} \log \sum_{i_0+i_1+i_2 = j_0+j_1+k_0+k_2 = N} |a_{(\varphi, i_0, i_1, i_2, j_0, j_1, k_0, k_2)}^{(N)} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2
\]
for some coefficients $a_{(\varphi, i_0, i_1, i_2, j_0, j_1, k_0, k_2)}^{(N)}$ satisfying $a_{(\varphi, i_0, i_1, i_2, j_0, j_1, k_0, k_2)}^{(N)} = a_{(\varphi, i_0, i_1, i_2, j_0, j_1, k_0, k_2)}^{(N)}$ due to the group action by $G$.

**Lemma 3.1.** Using the notations above we have
\[
\frac{1}{n} \log \sum_{i_0+i_1+i_2 = j_0+j_1+k_0+k_2 = n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2
\]
for any positive integer $n$. 

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Proof. On the patch \( U_0 = \{ Z_0 \neq 0 \} \), let \( z_1 = \frac{Z_1}{Z_0} \) and \( z_2 = \frac{Z_2}{Z_0} \).

\[
\frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0|^{i_1+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2} \right) \\
\leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|z_1|^{i_1+j_1} |z_2|^{i_2+k_2}}{(1 + |z_1|^2 + |z_2|^2)^n (1 + |z_1|^2)^n (1 + |z_2|^2)^n} \right) \\
\leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|z_1|^{i_1+j_1} |z_2|^{i_2+k_2}}{1 + |z_1|^{i_1+j_1} |z_2|^{i_2+k_2}} \right) \\
\leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} 1 \right) \\
= \frac{1}{n} \log \left( n + 1 \right)^{(n+2)} \leq 4.
\]

This inequality also holds on the patch \( U_1 = \{ Z_1 \neq 0 \} \) by continuity, and so the lemma is proved.

Lemma 3.2. There exists \( \varepsilon > 0 \) such that for any \( \varphi \in P_G(M, \omega_g) \) and \( N \), there exist \( n > N \), \( i_0, i_1, i_2, j_0, j_1, k_0, k_2 \) with \( i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n \), and \( (a^{(n)}_{(\varphi)})_{i_0, i_1, i_2, j_0, j_1, k_0, k_2} \geq \varepsilon \).

Proof. Otherwise, for any \( \varepsilon > 0 \), there exist \( \varphi \) and \( N \), such that for any \( n > N \) and any \( i_0, i_1, i_2, j_0, j_1, k_0, k_2 \) satisfying \( i_0 + i_1 + i_2 = j_0 + j_1 = k_0 + k_2 = n \), we have \( (a^{(n)}_{(\varphi)})_{i_0, i_1, i_2, j_0, j_1, k_0, k_2} < \varepsilon \). By choosing \( n \) large enough and with the lemma above, we have

\[
\varphi([Z_0, Z_1, Z_2]) \\
\leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0|^{i_1+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2} \right) + \varepsilon \\
\leq \frac{1}{n} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|Z_0|^{i_1+j_1} Z_1^{i_2+k_2}}{(1 + |Z_0|^2 + |Z_1|^2)^n (1 + |Z_0|^2)^n (1 + |Z_1|^2)^n} \right) + 2 \log \varepsilon + \varepsilon \\
\leq \log \varepsilon + 4.
\]

Since \( \varepsilon \) could be arbitrarily small, the above inequality would imply that \( \varphi \rightarrow -\infty \) uniformly, which contradicts the fact that \( \sup_M \varphi = 0 \).

Proof of Theorem 1. We use notations as above; since \( (a^{(n)}_{(\varphi)})_{i_0, i_1, i_2, j_0, j_1, k_0, k_2} \geq \varepsilon \), we have

\[
\varphi([Z_0, Z_1, Z_2]) \\
= \lim_{N \to \infty} \frac{1}{N} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |a^{(N)}_{(\varphi)}|_{i_0, i_1, i_2, j_0, j_1, k_0, k_2} \right) \\
\geq \frac{1}{N} \log \left( \sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} \frac{|Z_0|^{i_1+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}}{(1 + |Z_0|^2 + |Z_1|^2)^n (1 + |Z_0|^2)^n (1 + |Z_1|^2)^n} \right) + \log \varepsilon
\]
where \( i_0 + j_0 + k_0 = m, i_1 + j_1 + i_2 + k_2 = 3N - m \).

On the patch \( U_0 = \{ Z_0 \neq 0 \} \),

\[
\int_{U_0 \cap \{0 < |z_1|, |z_2| < 1\}} e^{-\alpha \varphi} \omega_{g_0}^2 \\
\leq C_1 \int_{0 < |z_1|, |z_2| < 1} e^{-\alpha \log \frac{|Z_0|^{3-\frac{m}{N}} |z_1|^{3-\frac{m}{N}} |z_2|^{3-\frac{m}{N}}}{|Z_1|}|Z_1|^{3-\frac{m}{N}} |Z_2|^{3-\frac{m}{N}} \omega_{g_0}^2 \\
= C_1 \int_{0 < |z_1|, |z_2| < 1} \frac{|Z_0|^{3-\frac{m}{N}} |z_1|^{3-\frac{m}{N}} |z_2|^{3-\frac{m}{N}}}{|Z_1|}|Z_1|^{3-\frac{m}{N}} |Z_2|^{3-\frac{m}{N}} d\sigma_{z_1} \wedge d\bar{z}_1 \\
\leq C_2 \int_{0 < |z_1|, |z_2| < 1} \frac{1}{|z_1|^{\frac{3m}{N}} |z_2|^{\frac{3m}{N}}} d\sigma_{z_1} \wedge d\bar{z}_1 \wedge d\sigma_{z_2} \wedge d\bar{z}_2,
\]

where \( C_1, C_2 \) and \( C_3 \) are constants depending only on \( \alpha \) and \( \epsilon \).

On the patch \( U_2 = \{ Z_2 \neq 0 \} \),

\[
\int_{U_2 \cap \{0 < |w_0|, |w_1| \leq 1\}} e^{-\alpha \varphi} \omega_{g_1}^2 \\
\leq C_4 \int_{0 < |w_0|, |w_1| \leq 1} e^{-\alpha \log \frac{|w_0|^{3-\frac{m}{N}} |w_1|^{3-\frac{m}{N}}}{|w_0|^{\frac{3m}{N}} |w_1|^{3-\frac{m}{N}}} \omega_{g_1}^2 \\
= C_4 \int_{0 < |w_0|, |w_1| \leq 1} \frac{1}{|w_0|^{\frac{3m}{N}} |w_1|^{3-\frac{m}{N}}} (1 + |w_0|^2) \alpha (1 + |w_1|^2)^\alpha d\sigma_{w_0} \wedge d\bar{w}_0 \\
\leq C_5 \int_{0 < |w_0|, |w_1| \leq 1} \frac{1}{|w_0|^{\frac{3m}{N}} |w_1|^{3-\frac{m}{N}}} (1 + |w_0|^2) \alpha (1 + |w_1|^2)^\alpha d\sigma_{w_0} \wedge d\bar{w}_0 \wedge d\sigma_{w_1} \wedge d\bar{w}_1,
\]

where \( p + q = 1 \) and \( C_4, C_5, C_6 \) are constants depending only on \( \alpha \) and \( \epsilon \).
Case 1: If $1 \leq \frac{m}{N} \leq 3$, we can choose $\alpha < \min\left(\frac{2}{3}, \frac{1+m}{2q}\right)$ so that
\[
\frac{\alpha m}{N} + (1 - \alpha)p < 1, \\
3\alpha - 1 < 1, \\
\frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1 - \alpha)q < 1.
\]

Case 2: If $0 < \frac{m}{N} < 1$, we can choose $\alpha < \min\left(\frac{2}{3}, \frac{1-q}{2q}\right)$ so that
\[
\frac{\alpha m}{N} + (1 - \alpha)p < 1, \\
3\alpha - 1 < 1, \\
\frac{3}{2}\alpha - \frac{\alpha m}{2N} + (1 - \alpha)q < 1.
\]

So we could choose any $\alpha < \frac{1}{3}$, which implies that $\alpha_G(M, \omega) \geq \frac{1}{3}$. Conversely, we choose
\[
\phi_\varepsilon = \log\left(\frac{|Z_0|^6}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_0|^2 + |Z_1|^2)(|Z_0|^2 + |Z_2|^2) + \varepsilon}\right) \\
- \log(1 + \varepsilon) \\
\in P_G(M, \omega).
\]

Then we have $\sup_M \phi_\varepsilon = 0$ and $\phi_\varepsilon = \log \frac{\varepsilon}{1+\varepsilon}$ on the exceptional divisors. Furthermore, we have
\[
\lim_{\varepsilon \to 0} \int_M e^{-\alpha \phi_\varepsilon} \omega^2 = \infty, \text{ for any } \alpha > \frac{1}{3}.
\]

Hence we have shown $\alpha_G(M, \omega) = \frac{1}{3}$.

We can also apply the above arguments for $CP^n$ ($n \geq 2$), $CP^2 \# 1CP^2$ and $CP^2 \# 3CP^2$.

(i) Let $M = CP^n$ and let $G_n$ be the automorphism group acting on $M$, generated by $\theta_j$ and permutations $\tau_{i,j}$ ($0 \leq i < j \leq n$),
\[
\theta_j : [Z_0, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j e^{i\theta}, ..., Z_n]
\]
for $\theta \in [0, 2\pi)$, and
\[
\tau_{i,j} : [Z_0, ..., Z_i, ..., Z_j, ..., Z_n] \to [Z_0, ..., Z_j, ..., Z_i, ..., Z_n].
\]

**Theorem 3.1.** $\alpha_{G_n}(CP^n) = 1$.

(ii) Let $M$ be the blow-up of $CP^2$ at 3 points which are not collinear. Then we can assume that these 3 points are $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$. Let $G(3)$ be the automorphism group acting on $M$, generated by $\theta_j$ and permutations $\tau_{i,j}$ ($0 \leq i < j \leq 2$),
\[
\theta_j : [Z_0, Z_j] \to [Z_0, Z_j e^{i\theta}, Z_2]
\]
for $\theta \in [0, 2\pi)$, and
\[
\tau_{i,j} : [..., Z_i, ..., Z_j, ...] \to [..., Z_j, ..., Z_i, ...].
\]

**Theorem 3.2.** $\alpha_{G(3)}(CP^2 \# 3CP^2) = 1$. 

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(iii) Let $M$ be the blow-up of $CP^2$ at one point $[1, 0, 0]$ and $G(1)$ be the automorphism group acting on $M$, generated by $\theta_j$ and permutations $\tau$ ($0 \leq i \leq 2$),
\[
\theta_j : [Z_0, Z_j, Z_2] \rightarrow [Z_0, Z_je^{i\theta}, Z_2]
\]
for $\theta \in [0, 2\pi)$, and
\[
\tau : [Z_0, Z_1, Z_2] \rightarrow [Z_0, Z_2, Z_1].
\]

**Theorem 3.3.** $\alpha_{G(1)}(CP^2 \#_1 CP^2) = \frac{1}{2}$.

Also the proof above shows that the sequence of the holomorphic invariants $\{\alpha_{G, m}(M)\}_m$ defined by Tian [8] on $CP^n$ ($n \geq 2$), $CP^2 \#_k CP^2$ ($k = 1, 2, 3$) is stationary.

4. **Proof of Theorem 2**

In this section, we will prove the generalized Moser-Trudinger inequality on any Kähler manifold $M$ of dimension $n$ whose $(\pi, \Lambda_1)$ is greater than $\frac{n+1}{n+2}$. The following theorem is due to Tian and Zhu [11].

**Theorem 4.1.** Let $(M, \omega)$ be a Kähler-Einstein manifold with $Ric(\omega) = \omega$; then there exist constants $\delta = \delta(n)$ and $C = C(n, \lambda_2(\omega) - 1) \geq 0$ such that for any $\phi \in P(M, \omega)$ which satisfies $\phi \perp \Lambda_1$, we have
\[
F_\omega(\phi) \geq J_\omega(\phi)^\delta - C,
\]
which is the same as
\[
\frac{1}{V} \int_M e^{-\phi} \omega^n \leq Ce^{J_\omega(\phi) + \frac{1}{2} \int_M \phi \omega^n - J_\omega(\phi)^\frac{n}{n-1}}.
\]

This implies in particular the Moser-Trudinger inequality on $S^2$, which reads
\[
\frac{1}{4\pi} \int_{S^2} e^{-\phi} \omega \leq e^{\frac{1}{8\pi} \int_{S^2} |\nabla \phi|^2 \omega - \frac{1}{8\pi} \int_{S^2} \phi}.
\]

For any $\phi \in P(M, \omega)$, put $\omega' = \omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi$ and $Ric(\omega) = \omega + \sqrt{-1} \partial \bar{\partial} h_\omega$. Consider the Monge-Ampère equation
\[
(\omega' + \sqrt{-1} \partial \bar{\partial} \psi)^n = e^{h_\omega - t \psi} \omega^m.
\]

We will use the continuity method backwards and let $\phi_t$ be a smooth family which solve the above equation.

The following lemmas are well known [10], but we add the proofs for the sake of completeness.

**Lemma 4.1.** $Ric(\omega_t) \geq t \omega_t$ and we have equality if and only if $t = 1$, where $\omega_t = \omega + \phi_t$ and $\phi_t$ solves the Monge-Ampère equation at $t$.

**Proof.**
\[
Ric(\omega_t) = -\sqrt{-1} \partial \bar{\partial} \log \omega_t^n = -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega_t^n}{\omega^n} + Ric(\omega)
\]
\[
= -\sqrt{-1} \partial \bar{\partial} (h_\omega - t \phi_t) + \omega + \sqrt{-1} \partial \bar{\partial} h_\omega
\]
\[
= \omega + t \phi_t = t \omega_t + (1 - t) \omega_t \geq t \omega_t.
\]

\[\square\]
Lemma 4.2. For any $\phi \in P(M, \omega)$, if the Green’s function of $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ is bounded from below, we have:

$$-\inf_M \phi \leq \frac{1}{V} \int_M (-\phi)\omega^n + C.$$ 

Proof. Since $\omega + \sqrt{-1}\partial\bar{\partial}\phi = \omega'$ and $\omega' - \sqrt{-1}\partial\bar{\partial}\phi > 0$, we have $\Delta_{\omega'} \phi \leq n$, and

$$-\phi = \frac{1}{V} \int_M (-\phi)\omega^n + \frac{1}{V} \int_M \Delta_{\omega'} \phi(y)G_{\omega'}(x, y)\omega^n$$

$$\leq \frac{1}{V} \int_M (-\phi)\omega^n + \frac{1}{V} \int_M n(G_{\omega'}(x, y) - \inf G_{\omega'}(x, y))\omega^n$$

$$\leq \frac{1}{V} \int_M (-\phi)\omega^n + C.$$ 

Let $(M, \omega)$ be a Kähler-Einstein manifold with $\text{Ric}(\omega) = \omega$ and let $P(M, \omega, K) = \{ \phi \in P(M, \omega) : G_{\omega + \sqrt{-1}\partial\bar{\partial}\phi}(x, y) \geq -K \}$. Then we have:

Proposition 4.1. Let $(M, \omega)$ be a Kähler-Einstein manifold with $\text{Ric}(\omega) = \omega$. If $\alpha(M) > \frac{1}{n+1}$, then there exist constants $\delta(n, \alpha, K)$ and $C(n, \alpha, \lambda_2(\omega) - 1, K)$ such that for any $\phi \in P(M, \omega, K)$, we have

$$F_{\omega}(\phi) \geq \delta J_{\omega}(\phi) - C.$$ 

Proof. Let $\omega' = \omega + \partial\bar{\partial}\phi$, where $\phi \in P(M, \omega, K)$. We have

$$\frac{1}{V} \int_M e^{-\alpha_1^1} \omega^n = \frac{1}{V} \int_M e^{-(\alpha_1^1 + \alpha_2^1 + \epsilon)} \phi \omega^n$$

$$\leq \frac{1}{V} \int_M e^{-(\alpha_1^1 + \alpha_2^1 + \epsilon)} \phi \omega^n e^{-\epsilon \inf_M \phi};$$

taking $p = \frac{1}{\alpha_1^1}, q = \frac{1}{1-\alpha_1^1}$, we have

$$\frac{1}{V} \int_M e^{-(\alpha_1^1 + \alpha_2^1)} \phi \omega^n \leq \frac{1}{V} \left( \int_M e^{-(\alpha_1^1 \phi \omega^n)^{1/p}} \right)^{1/p} \left( \int_M e^{-(\alpha_2^1 \phi \omega^n)^{1/q}} \right)^{1/q}$$

$$= \frac{1}{V} \left( \int_M e^{-\phi \omega^n \alpha_1^1} \right)^{\alpha_2^1} \left( \int_M e^{-\frac{\alpha_2^1}{\alpha_1^1} \phi \omega^n} \right)^{1-\alpha_1^1}$$

$$\leq C e^{\alpha_1^1 J_{\omega}(\phi) - \frac{1}{\alpha_1^1}} f_M \phi \omega^n \left( \int_M e^{-\frac{\alpha_2^1}{\alpha_1^1} \phi \omega^n} \right)^{1-\alpha_1^1}.$$

By Lemma 4.2,

$$e^{-\epsilon \inf_M \phi} \leq e^{\frac{1}{\alpha_1^1} f_M (\phi) \omega^n + C}$$

$$= e^{\frac{1}{\alpha_1^1} J_{\omega}(\phi) - \frac{1}{\alpha_1^1}} f_M \phi \omega^n + C$$

$$\leq e^{\delta(n+1) J_{\omega}(\phi) - \frac{1}{\alpha_1^1}} f_M \phi \omega^n + C.$$ 

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By Hölder’s inequality,
\[
\frac{1}{V} \int_M e^{-\phi} \omega^n \leq \left( \frac{1}{V} \int_M e^{-\alpha \phi} \omega^n \right)^{\frac{1}{\alpha}}
\]
\[
\leq C e^{\frac{n+1}{2} \int_M \phi \omega^n} \left( \int_M e^{-\frac{n}{1-\alpha} \phi} \omega^n \right)^{1-\alpha}.
\]
\[
= C e^{\frac{n+1}{2} \int_M \phi \omega^n + \frac{n}{2} \int_M (\phi - \sup \phi) \omega^n} \left( \int_M e^{-\frac{n}{1-\alpha} (\phi - \sup \phi)} \omega^n \right)^{1-\alpha}
\]
\[
\leq C e^{\frac{n+1}{2} \int_M \phi \omega^n} \left( \int_M e^{-\frac{n}{1-\alpha} (\phi - \sup \phi)} \omega^n \right)^{1-\alpha}.
\]

We need to determine \(\alpha_1, \alpha_2, \varepsilon\) which satisfy the following conditions:
\[
\alpha = \alpha_1 + \alpha_2 + \varepsilon > 1,
\]
\[
\alpha > \alpha_1 + (n+1)\varepsilon,
\]
\[
1 > \alpha_1.
\]

So we will choose
\[
\alpha_2 = n\varepsilon + \varepsilon',
\]
\[
\alpha_1 = 1 - \alpha_2 - \varepsilon + \varepsilon'' = 1 - (n+1)\varepsilon - \varepsilon' + \varepsilon'',
\]
where \(\varepsilon, \varepsilon', \varepsilon'' \ll 1\) and \(\varepsilon' = o(\varepsilon), \varepsilon'' = o(\varepsilon')\).

Since \(\alpha(M) > \frac{n}{n+1}\), we can choose \(\varepsilon, \varepsilon', \varepsilon''\) small enough; then we have
\[
\frac{\alpha_2}{1-\alpha_1} = \frac{n\varepsilon + \varepsilon'}{(n+1)\varepsilon + \varepsilon' - \varepsilon''} < \alpha(M)
\]
and
\[
\int_M e^{-\frac{n}{1-\alpha_1} (\phi - \sup \phi)} \omega^n < \text{Const}.
\]

Combined with the inequalities above, we have
\[
\frac{1}{V} \int_M e^{-\phi} \omega^n \leq Ce^{(1-\delta)J_\omega(\phi) - \frac{1}{2} \int_M \phi \omega^n}.
\]

This proves the lemma.

**Proof of Theorem 2.** We assume \(\omega\) is the Kähler-Einstein metric of \(M\). For any \(\phi \in P(M, \omega)\), put \(\omega' = \omega + \sqrt{-1} \partial \bar{\partial} \phi\). Consider \((\omega' + \sqrt{-1} \partial \bar{\partial} \psi) = e^{h_{\omega'} + t \psi}\). By solving the Monge-Ampère equation backwards, we get the solutions \(\phi_t\), and \(\phi_1 = -\phi\).

For \(t > \frac{1}{2}\), let \(\omega_t = \omega' + \sqrt{-1} \partial \bar{\partial} \phi_t = \omega + \sqrt{-1} \partial \bar{\partial} (\phi_t - \phi_1)\); by Lemma 4.1,
\[
Ric(\omega_t) \geq \frac{1}{2} \omega_t,
\]
which shows that the Green function of \(\omega_t\) is uniformly bounded from below. Thus by Proposition 4.1 and the calculation in [14] we have
\[
F_\omega(\phi_t - \phi_1) \geq \delta J_\omega(\phi_t - \phi_1) - C
\]
\[
\geq C_1 \text{osc}_M(\phi_t - \phi_1) - C_2,
\]
and consequently,
\[ n(1-t)J_{\omega}(\phi) = n(1-t)J_{\omega}(\phi_1) \geq (1-t)(I_{\omega'}(\phi_1) - J_{\omega'}(\phi_1)) \geq F_{\omega'}(\phi_t) - F_{\omega'}(\phi_1) = F_{\omega}(\phi_t - \phi_1) \geq C_1osc_M(\phi_t - \phi_1) - C_2. \]

Thus we have
\[ F_{\omega}(\phi) = -F_{\omega'}(-\phi) = \int_0^1 (I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t))dt \geq (1-t)(I_{\omega'}(\phi_1) - J_{\omega'}(\phi_1)) \geq \frac{1-t}{n}J_{\omega'}(\phi_1) \geq \frac{1-t}{n}J_{\omega'}(\phi_1) - 2(1-t)(C_1osc_M(\phi_t - \phi_1) - C_2) \geq \frac{1-t}{n}J_{\omega}(\phi) - 2(1-t)^2nC_1J_{\omega}(\phi) - C_3. \]

The theorem follows by choosing \((1-t) < \frac{1}{2n^2C_1}\).

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