CONVERGENCE OF DOUBLE FOURIER SERIES
AND $W$-CLASSES

M. I. DYACHENKO AND D. WATERMAN

Abstract. The double Fourier series of functions of the generalized bounded variation class $\{n/\ln(n+1)\}^*BV$ are shown to be Pringsheim convergent everywhere. In a certain sense, this result cannot be improved. In general, functions of class $\Lambda^*BV$, defined here, have quadrant limits at every point and, for $f \in \Lambda^*BV$, there exist at most countable sets $P$ and $Q$ such that, for $x \notin P$ and $y \notin Q$, $f$ is continuous at $(x,y)$. It is shown that the previously studied class $\Lambda BV$ contains essentially discontinuous functions unless the sequence $\Lambda$ satisfies a strong condition.

1. Introduction

A remarkable variety of definitions of bounded variation have been given for functions of two variables. Here we will discuss generalizations of these definitions along the lines of the notion of $\Lambda$-bounded variation ($\Lambda BV$) in one variable introduced by Waterman. He used it to extend the Dirichlet-Jordan theorem, and we will investigate the analogous problem for double Fourier series.

For an excellent discussion of $\Lambda BV$ and its relation to other generalizations of bounded variation, see Avdispahić [1]. For applications to summability and Tauberian theorems, see [2, 8, 11].

Definition 1. Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a monotone nondecreasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty,$$

and let $Y$ denote the class of such sequences. A real function $f$ defined on an interval $[a, b]$ is said to be of $\Lambda$-bounded variation, $f \in \Lambda BV([a, b])$, if

$$V_\Lambda(f; [a, b]) = \sup_{I, n} \sum_{i=1}^{n} \frac{|f(I_k)|}{\lambda_k} = \sup_{I, n} \sum_{i=1}^{n} \frac{|f(\beta_k) - f(\alpha_k)|}{\lambda_k} < \infty,$$

where $I$ denotes the class of collections of nonoverlapping intervals $\{I_k = [\alpha_k, \beta_k] \subset [a, b], k = 1, \ldots, n\}$. 

Received by the editors March 17, 2003 and, in revised form, September 29, 2003.

2000 Mathematics Subject Classification. Primary 42B05, 26B30; Secondary 26B05.

Key words and phrases. Multiple Fourier series, generalized bounded variation, Waterman classes.

The first author gratefully acknowledges the support of RFFI grant N03-01-00080.
Note that functions of $\Lambda$-bounded variation are bounded, have right and left limits at every point, and so their discontinuities are at most countable \cite{9}.

Function classes, whose definitions depend on the boundedness of sums of the absolute values of interval functions multiplied by weights from sequences such as $\Lambda$, have come to be known as $W$-classes.

In \cite{7} \cite{10}, Waterman proved the following generalization of the Dirichlet-Jordan theorem.

**Theorem A.** If $f$ is a 2$\pi$-periodic function, $H = \{n\}_1^\infty$, $T = [-\pi, \pi]$ and $f \in HBV(T)$, then $S[f]$, the Fourier series of $f$, converges at every point and, if $I$ is a closed interval of points of continuity, then $S[f]$ converges uniformly on $I$. If $\Lambda BV \setminus HBV \neq \emptyset$, there is an $f \in \Lambda BV(T)$ such that $S[f]$ diverges at a point.

A definition of $\Lambda BV$ for two variables which has been used by Saakjan \cite{5} and Sablin \cite{6} is as follows.

**Definition 2.** Let $\Lambda \in Y$ and let $f$ be a measurable function on the rectangle $A = [a, b] \times [c, d]$. Then $f \in \Lambda BV(A)$ if and only if
\begin{enumerate}
  \item $f(\cdot, c) \in \Lambda BV([a, b])$ and $f(a, \cdot) \in \Lambda BV([c, d])$, and
  \item if $I_1$ and $I_2$ are the sets of finite collections of nonoverlapping intervals $I_k = [\alpha_k, \beta_k]$ and $I_j = [\gamma_j, \delta_j]$ in $[a, b]$ and $[c, d]$ respectively and $f(I_k \times I_j) = f(\alpha_k, \gamma_j) - f(\alpha_k, \delta_j) - f(\beta_k, \gamma_j) + f(\beta_k, \delta_j)$, then
  \begin{align*}
  V_\Lambda(f; [a, b]) = \sup_{I_1, I_2} \sum_{k} \sum_{j} \frac{|f(I_k \times I_j)|}{\lambda_k \lambda_j} < \infty.
  \end{align*}
\end{enumerate}

**Remark 1.** If $\lambda_k \equiv 1$, or what is the same, $\lambda_k = O(1)$, $\Lambda BV(A)$ is the set of functions of Hardy-Krause bounded variation on $A$.

It is clear that the functions of $\Lambda BV(A)$ are bounded, but the question of continuity is more complicated than in the case of functions of one variable. Dyachenko \cite{3} has proved the following theorem.

**Theorem B.** The following conditions are equivalent:
\begin{enumerate}
  \item for any $f \in \Lambda BV(T^2)$ there exist two at most countable subsets $A$ and $B$ of $T$ such that $f$ is continuous at every point $(x, y) \in T^2$ such that $x \notin A$ and $y \notin B$;
  \item for any $f \in \Lambda BV(T^2)$ and any $(x_0, y_0) \in T^2$, $\lim f(x, y)$ exists as $(x, y) \to (x_0, y_0)$ in each of the open coordinate quadrants;
  \item $\sum_{k=1}^{\infty} \lambda_k^{-2} = \infty$.
\end{enumerate}

(The third condition will be called \textbf{Condition (\ast).})

Thus we see, for example, that the characteristic function of the triangle $B = \{(x, y) : 0 \leq y \leq 1 - x\}$ is in $\Lambda BV([0, 1]^2)$ only if Condition (\ast) does not hold.

If $\Lambda$ does not satisfy (iii), the requirement of measurability cannot be omitted from Definition 2, for if it were, $\Lambda BV(A)$ would include functions not Lebesgue measurable. Even under the assumption of measurability, we show in Section 2 that $\Lambda BV(A)$ contains an everywhere discontinuous function and, moreover, a function $f$ such that if $g = f$ a.e., then $g$ is a.e. discontinuous.
We will consider the Pringsheim convergence of double Fourier series. If \( f \in L(T^2) \) is 2\( \pi \)-periodic in each variable, then

\[
S[f] = \sum_{m,n} a_{mn} e^{i(mx+ny)}
\]

is its Fourier series, where

\[
a_{mn} = a_{mn}(f) = \frac{1}{(2\pi)^2} \int_{T^2} f(x,y)e^{-i(mx+ny)} \, dx \, dy.
\]

The rectangular partial sums of this series are

\[
S_{N_1N_2}(f; x,y) = \sum_{|k_1| \leq N_1} \sum_{|k_2| \leq N_2} a_{k_1k_2} e^{i(k_1x+k_2y)}
\]

with \( N_1, N_2 \geq 0 \). If \( N_1 = N_2 \), these are called square sums. If

\[
S_{N_1N_2}(f; x,y) \to \alpha \text{ as } \min(N_1,N_2) \to \infty,
\]

we say that the Fourier series of \( f \) converges to \( \alpha \) at \((x,y)\) in the Pringsheim sense.

A.A. Saakyan \([5]\) has shown

**Theorem C.** If \( f \in HBV(T^2) \), then the rectangular partial sums of \( S[f] \) are uniformly bounded, and, if for \((x_0, y_0) \in T^2\) the limits of \( f(x,y) \) exist as \((x,y) \to (x_0, y_0)\) in each of the open coordinate quadrants, then \( S[f] \) converges (Pringsheim) to the arithmetic mean of these limits.

This result has been generalized to higher dimensions by A.I. Sablin \([6]\).

As we shall see in §2, an \( f \in HBV(T^2) \) need not have a point of continuity. For such a function, Theorem C is inapplicable.

We shall define another \( W \)-class such that functions of this class are continuous a.e., and prove that a theorem analogous to Theorem C holds for this class.

**Definition 3.** Let \( \Lambda \in Y \) and let \( f \) be a real function on \( A = [a,b] \times [c,d] \). We say \( f \in \Lambda^{*}\text{BV}(A) \) if

(i) \( f(\cdot,c) \in ABV([a,b]) \) and \( f(a,\cdot) \in ABV([c,d]) \)

and, if \( \Gamma \) is the set of finite collections of nonoverlapping rectangles \( A_k = [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subseteq A \) and \( f(A_k) = f(\alpha_k, \gamma_k) - f(\alpha_k, \delta_k) - f(\beta_k, \gamma_k) + f(\beta_k, \delta_k) \), then

(ii) \( V_\Lambda^*(f; A) = \sup_{\Gamma} \sum_k |f(A_k)| < \infty \).

For \( f \in \Lambda^{*}\text{BV}(A) \) we set

(iii) \( \|f\|_{\Lambda^{*}} = \|f\|_{\Lambda^{*}(A)} = |f(a,c)| + V_\Lambda(f(\cdot,c)) + V_\Lambda(f(a,\cdot)) + V_\Lambda^*(f; A) \).

**Remark 2.** Note that if \( f \in \Lambda^{*}\text{BV}(A) \), then

\[
V_\Lambda(f(\cdot,y); [a,b]) \leq V_\Lambda^*(f; A) + V_\Lambda(f(\cdot,c); [a,b])
\]

for every \( y \in [c,d] \). The analogous result holds for the \( \Lambda \)-variation of the restriction of \( f \) to the vertical segments.

In §3 we shall prove

**Theorem 1.** Let \( \Lambda \in Y \) and \( A = [a,b] \times [c,d] \). Then, for any \( f \in \Lambda^{*}\text{BV}(A) \),

(i) there exist at most countable sets \( P \subseteq [a,b] \) and \( Q \subseteq [c,d] \) such that \( f \) is continuous at every \((x,y) \in A \) such that \( x \notin P \) and \( y \notin Q \); and

(ii) at every point \((x_0,y_0) \in A\), \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) \) exists in each open coordinate quadrant.
In that section we also discuss the relation between \( \Lambda^*BV(A) \) and \( \Lambda BV(A) \).

In §4 we study the convergence of Fourier series of functions of class \( \Lambda^*BV(A) \), and prove

**Theorem 2.** Let \( f \) be a real function on \( \mathbb{R}^2 \) which is \( 2\pi \)-periodic in each variable and is in \( \Lambda^*BV(T^2) \) with \( \Lambda = \{ \frac{n}{\ln(n+1)} \} \). Then the rectangular partial sums of \( S[f] \) are uniformly bounded and converge at each point to the arithmetic mean of the quadrant limits.

We also show that, in a certain sense, this result cannot be improved.

**Theorem 3.** Let \( \Lambda = \{ \frac{n}{\ln(n+1)} \} \in Y \), where \( \xi_n \uparrow \infty \) as \( n \to \infty \). Then there exists a function \( f \in \Lambda^*BV(T^2) \) such that the square partial sums of its Fourier series diverge unboundedly at \( (0,0) \).

### 2. Discontinuous Functions in \( W \)-classes

We shall require the following lemmas.

**Lemma 1.** Let \( \Lambda \in Y \) be such that Condition (*) does not hold (i.e., \( \sum \lambda_k^{-2} < \infty \)) and let \( A = [a,b] \times [c,d] \) be a nondegenerate interval. Suppose \( E \subset A \) has a connected intersection with every horizontal and vertical line. Then \( \chi_E \), the characteristic function of \( E \), is in \( \Lambda BV(A) \) and \( V_\Lambda(\chi_E; A) < C < \infty \), where \( C \) is an absolute constant.

**Proof.** Let \( \{I_k\} = \{[\alpha_k, \beta_k]\} \) in \([a, b]\) and \( \{J_r\} = \{[\gamma_r, \delta_r]\} \) in \([c, d]\) be two collections of nonoverlapping intervals. Then, for each \( k = 1, 2, \ldots, n \), in the sum

\[
S = \sum_{k=1}^{n} \sum_{r=1}^{r} \frac{|\chi_E(I_k \times J_r)|}{\lambda_k \lambda_r},
\]

there are at most four different \( r_{k,j} \) for which \( |\chi_E(I_k \times J_{r_{k,j}})| \neq 0 \), and in these cases it is either 1 or 2. Let

\[
S_j = \sum_{k=1}^{n} \left( \frac{|\chi_E(I_k \times J_{r_{k,j}})|}{\lambda_k \lambda_{r_{k,j}}} \right), \quad \text{for } j = 1, 2, 3, 4.
\]

Then \( S = S_1 + S_2 + S_3 + S_4 \), and, as each \( j \) can be associated with at most four \( r_{k,j} \), we have

\[
S_j \leq 2 \sum_{k=1}^{n} \left( \frac{1}{\lambda_k \lambda_{r_{k,j}}} \right) \leq \sum_{k=1}^{n} \frac{1}{\lambda_k^2} + \sum_{k=1}^{n} \frac{1}{\lambda_{r_{k,j}}^2} \leq 5 \sum_{k=1}^{n} \frac{1}{\lambda_k^2} < C < \infty,
\]

and, since the one-dimensional \( \Lambda \)-variation of \( \chi_E \) on the edges of \( A \) is at most \( 2/\lambda_1 \), Lemma 1 is established.

**Lemma 2.** \( A = [a, b] \times [c, d] \) be a nondegenerate interval. There is a sequence of closed rectangles \( \{A_i = I_i \times J_i\} \) in \( A \) with \( I_i \cap I_j = \emptyset \) and \( J_i \cap J_j = \emptyset \) for \( i \neq j \), with

\[
\sum_{i=1}^{\infty} |I_i| < (b - a)/4 \quad \text{and} \quad \sum_{i=1}^{\infty} |J_i| < (d - c)/4,
\]

such that every neighborhood of each point of the following contains some \( A_i \):

\[
B = A \setminus \bigcup_{i=1}^{\infty} ((I_i \times [c, d]) \cup ([a, b] \times J_i)).
\]
Proof. Choose $n_1 = 1$ and consider the rectangles
\[ E_{k,r,1} = \left[ a + \frac{(b-a)(k-1)}{2^{n_1}}, a + \frac{(b-a)k}{2^{n_1}} \right] \times \left[ c + \frac{(d-c)(r-1)}{2^{n_1}}, c + \frac{(d-c)r}{2^{n_1}} \right], \]
where $k, r = 1, 2, \ldots, 2^{n_1}$.

Choose an integer $n_2 > 2n_1 + 3$ and, for each choice of $k$ and $r$, $1 \leq k, r \leq 2^{n_1}$, choose a rectangle
\[ E'_{k,r,1} = [a_{k,r,1}, b_{k,r,1}] \times [c_{k,r,1}, d_{k,r,1}] \]
so that the projections of the chosen rectangles on the coordinate axes do not touch. Clearly
\[ \sum_{k,r=1}^{2^{n_1}} (b_{k,r,1} - a_{k,r,1}) < \frac{b-a}{8} \quad \text{and} \quad \sum_{k,r=1}^{2^{n_1}} (d_{k,r,1} - c_{k,r,1}) < \frac{d-c}{8}. \]

Next we consider the rectangles
\[ E_{k,r,2} = \left[ a + \frac{(b-a)(k-1)}{2^{n_2}}, a + \frac{(b-a)k}{2^{n_2}} \right] \times \left[ c + \frac{(d-c)(r-1)}{2^{n_2}}, c + \frac{(d-c)r}{2^{n_2}} \right], \]
where $k$ and $r$ are chosen, $1 \leq k, r \leq 2^{n_2}$, so that
\[ E_{k,r,2} \cap \bigcup_{k,r=1}^{2^{n_1}} ([a_{k,r,1}, b_{k,r,1}] \times [c, d]) \cup ([a, b] \times [c_{k,r,1}, d_{k,r,1}]) = \emptyset. \]

We then choose an integer $n_3 > 2n_2 + 4$ and, for each $k$ and $r$, $1 \leq k, r \leq 2^{n_2}$, just chosen, we choose a rectangle
\[ E'_{k,r,2} = [a_{k,r,2}, b_{k,r,2}] \times [c_{k,r,2}, d_{k,r,2}] \]
\[ \subset E_{k,r,2} \]
so that the projections of the chosen rectangles on the coordinate axes do not touch. Then
\[ \sum_{k,r} (b_{k,r,2} - a_{k,r,2}) < \frac{b-a}{16} \quad \text{and} \quad \sum_{k,r} (d_{k,r,2} - c_{k,r,2}) < \frac{d-c}{16}. \]

Proceeding inductively, we can define $E'_{k,r,j}$, $j = 1, 2, \ldots$, choosing $n_{j+1} > 2n_j + j + 2$ at each step, and, renumbering the $E'_{k,r,j}$ as we wish, \( \{ A_i \} = \{ E'_{k,r,j} \} \) is the required sequence of intervals.

We turn now to the principal results of this section.

**Proposition 1.** Suppose $\Lambda \subset Y$, $A = [a, b] \times [c, d]$ is a nondegenerate rectangle and Condition (\( \ast \)) does not hold. Then there exists an $f \in ABV(A)$ which is everywhere discontinuous.
Proof. Let $P$ and $Q$ be the sets of rationals in $[a, b]$ and $[c, d]$ respectively. Divide the rectangle $A$ into four quarters by passing lines parallel to the axes through the midpoints of the sides. In each of the rectangles $A_i$, $i = 1, 2, 3, 4$, thus formed, we select $(p_i, q_i), p_i \in P, q_i \in Q$, so that the {$p_i$} and {$q_i$} are distinct. Now we can quarter each $A_i$ and choose one point in each sixteenth not yet containing a chosen point to form $\{(p_i, q_i)\}_{i=1}^{16}$, $p_i \in P, q_i \in Q$, so that $\{(p_i)_{i=1}^{16} \text{ and } \{q_i\}_{i=1}^{16}$ are sets of distinct points. Proceeding inductively, we obtain a dense set of points $E = \{(p_i, q_i)\}$ such that $p_i \neq p_j$ and $q_i \neq q_j$ when $i \neq j$. Thus any line parallel to an axis meets $E$ in at most one point and, by Lemma 1, $\chi_E \in \Lambda BV(A)$.

The function we have constructed in Proposition 1 is everywhere discontinuous but is equivalent to the function identically equal to zero. The next result shows that in a class $\Lambda BV(A)$ in which Condition (*) does not hold there exist essentially discontinuous functions.

Proposition 2. Suppose $\Lambda \in Y, A = [a, b] \times [c, d]$ is a nondegenerate rectangle and Condition (*) does not hold. Then there is an $f \in \Lambda BV(A)$ such that $g = f$ a.e. implies that $g$ is a.e. discontinuous.

Proof. Apply Lemma 2 to form the sequence $\{A_i\}$ and the set $B_1 = B$. If $F_1 = \bigcup_{i=1}^{\infty} A_i$ and $f_1 = \chi_{F_1}$, we see that $V_A(f_1) = C < \infty$, and we observe that $|A \setminus B_1| < |A|/2$.

The set $A \setminus B_1$ can be divided into rectangles $\{D_i\}_{i=1}^{\infty}$ in the natural way. By applying Lemma 2 to each of the rectangles $D_i$, we obtain for each $i$ a sequence $\{A_{ij}\}_{j=1}^{\infty} \subset D_i$ with similar properties. Let $F_{2,i} = \bigcup_{j=1}^{\infty} A_{ij}$ and $f_{2,i} = \chi_{F_{2,i}}$, $i = 1, 2, \ldots$. Note that $V_A(f_{2,i}) \leq C$ for every $i$. If $A_{i,j} = [a_{i,j}, b_{i,j}] \times [c_{i,j}, d_{i,j}]$, set

$$B_{2,i} = D_i \setminus \bigcup_{j=1}^{\infty} \left( ([a_{i,j}, b_{i,j}] \times [c, d]) \cup ([a, b] \times [c_{i,j}, d_{i,j}]) \right), \quad i = 1, 2, \ldots.$$

Let us replace $i$ by the symbol $i_1$. At the third stage we obtain sets $B_{3,i_1,i_2}$, and as the induction proceeds we obtain sets $B_{r,i_1,\ldots,i_{r-1}}$. Let

$$U = B_1 \cup \left( \bigcup_{r=2}^{\infty} \left( \bigcup_{i_1,\ldots,i_{r-1}=1}^{\infty} B_{r,i_1,\ldots,i_{r-1}} \right) \right);$$

then $|A \setminus U| = 0$. We continue inductively to obtain functions $f_1, \{f_{2,i_1}\}_{i_1=1}^{\infty}, \ldots, \{f_{r,i_1,\ldots,i_{r-1}}\}_{i_1,\ldots,i_{r-1}=1}^{\infty}, \ldots$, and we renumber these functions to form $\{h_k\}_{k=1}^{\infty}$. Now, letting

$$f = \sum_{k=1}^{\infty} 3^{-k} h_k ,$$

we have $f \in \Lambda BV(A)$.

Let $g = f$ a.e. and let $E$ be the set of points in $A$ at which $g$ and $f$ are equal. Clearly $E$ is dense in $A$. We write $\omega(f; (x, y); E)$ for the oscillation of $f$ at $(x, y)$ over the set $E$ and $\omega(g; (x, y); A)$ for the oscillation of $g$ at $(x, y)$ over the set $A$. Note that

$$\omega(g; (x, y); A) \geq \omega(f; (x, y); E).$$

Consider a point $(x, y) \in U$. Then $(x, y)$ is in some $B_{r,i_1,\ldots,i_{r-1}}$. If $k$ is such that $h_k = f_{r,i_1,\ldots,i_{r-1}}$, then $\omega(h_k; (x, y); E) = 1$. Note that, since $h_1(A) = \{0, 1\}$, for every
Let \( k_0 = k_0(x, y) = \min \{ l : \omega(h_l; (x, y); E) = 1 \} \). Then
\[
\omega(f; (x, y); E) \geq 3^{-k_0} - \sum_{l=k_0+1}^{\infty} 3^{-l} \omega(h_l; (x, y); E) \geq 3^{-k_0} - \sum_{l=k_0+1}^{\infty} 3^{-l} \geq 3^{-k_0-1},
\]
implicing that \( g \) is discontinuous at \((x, y)\).

3. CONTINUITY PROPERTIES OF FUNCTIONS OF \( \Lambda^*BV \)

We now turn our attention to the proof of Theorem 1.

**Proof.** Suppose there are infinitely many points \((x_i, y_i)\) such that \( x_i \neq x_j \) and \( y_i \neq y_j \) for \( i \neq j \), at which the oscillation of \( f \) exceeds \( 1/k \), \( k \) a natural number. For a natural number \( N \) we can find points \((i, i)\) and \((i, i)\), \( i = 1, 2, \ldots, N \), such that \( f((i, i)) \neq f((i, i)) \) and the sequences \( \alpha_1, \gamma_1, \alpha_2, \gamma_2, \alpha_3, \ldots \) and \( \beta_1, \delta_1, \beta_2, \delta_2, \beta_3, \ldots \) are strictly monotone. We will assume them to be increasing; the other cases are handled similarly.

We have
\[
S_1 = \sum_{i=0}^{N} \frac{|f((\alpha_i, \delta_i)) - f((\gamma_i, \delta_i))|}{\lambda_i} \leq \sum_{i=0}^{N} \frac{|f((\alpha_i, \gamma_i)) \times (\beta_i, \delta_i))|}{\lambda_i} + \sum_{i=0}^{N} \frac{|f((\alpha_i, \epsilon)) - f((\gamma_i, \epsilon))|}{\lambda_i} \leq \|f\|_{\Lambda^*},
\]
and, in a similar fashion,
\[
S_2 = \sum_{i=0}^{N} \frac{|f((\gamma_i, \delta_i)) - f((\delta_i, \delta_i))|}{\lambda_i} \leq \|f\|_{\Lambda^*}. \]

Thus
\[
V_{\Lambda^*}(f; A) \geq \sum_{i=0}^{N} \frac{|f((\alpha_i, \gamma_i)) \times (\beta_i, \delta_i))|}{\lambda_i} \geq \sum_{i=0}^{N} \frac{|f((\alpha_i, \beta_i)) - f((\gamma_i, \delta_i))|}{\lambda_i} - S_1 - S_2 \geq \frac{1}{k} \sum_{i=0}^{N} \frac{1}{\lambda_i} - 2 \|f\|_{\Lambda^*},
\]
which is false for \( N \) sufficiently large. Thus all points at which \( f \) has an oscillation greater than \( 1/k \) lie on a finite number of lines parallel to the axes, which establishes the first part of the theorem.

To establish the second part of Theorem 1, we assume that there is a point \( p \in A \) such that \( f(x, y) \) does not have a limit as \((x, y) \to p\) within an open coordinate quadrant with vertex \( p \). Without loss of generality we may assume that \( p = (0, 0) \) and the quadrant is \( \{(x, y) : x > 0, y > 0\} \).
Then there is an $\varepsilon > 0$ such that, for every $\delta > 0$, in every square $(0, \delta)^2$ the oscillation of $f$ is greater than $\varepsilon$. Choose $s, t > 0$. Then, since $f$ is in $\Lambda BV$ in each variable separately, $\lim_{x \to 0} f(s, y)$ and $\lim_{x \to 0} f(x, t)$ exist. Choose $\delta > 0$ so that the oscillations of $f(s, y)$ and $f(x, t)$ on $0 < y < \delta$ and $0 < x < \delta$ respectively are less than $\varepsilon/8$. Now choose points $(x_1, y_1)$ and $(x_2, y_2)$ in $(0, \delta)^2$ so that

$$|f(x_1, y_1) - f(x_2, y_2)| > \varepsilon/2.$$  

Then, letting $P = f(s, t) - f(s, y_1) - f(x_1, t) + f(x_1, y_1)$ and $Q = f(s, t) - f(s, y_2) - f(x_2, t) + f(x_2, y_2)$, we have

$$|P - Q| \geq |f(x_1, y_1) - f(x_2, y_2)| - |f(x_1, t) - f(x_2, t)| - |f(s, y_1) - f(s, y_2)| > \varepsilon/2 - \varepsilon/8 - \varepsilon/8 = \varepsilon/4,$$

so that at least one of $|P|$ and $|Q|$ exceeds $\varepsilon/8$, and so we have obtained a rectangle $A_1 \in (0, \delta)^2$ for which $|f(A_1)| > \varepsilon/8$. By choosing our points $(s, t), (x_1, y_1)$ and $(x_2, y_2)$ sufficiently close to the origin, we can repeat this process to obtain a rectangle $A_2$ which does not overlap $A_1$ for which $|f(A_2)| > \varepsilon/8$. Thus we can form a sequence $\{A_n\}$ of nonoverlapping intervals in $(0, \delta)^2$ with $|f(A_n)| > \varepsilon/8$. Then

$$\sum_{i=0}^{N} \frac{|f(A_k)|}{\lambda_k} > \frac{\varepsilon}{8} \sum_{i=0}^{N} \frac{1}{\lambda_k} \to \infty \text{ as } N \to \infty,$$

contradicting our assumption that $f \in \Lambda^*BV(A)$, and completing the proof of Theorem 1.

It is natural to ask how the classes $\Lambda BV$ and $\Lambda^*BV$ are related. This is by no means obvious, although they are clearly the same if $\{\lambda_i\}$ is bounded. There is no loss of generality in assuming the rectangle $A$ to be $[0, 1]^2$.

**Proposition 3.** If $\Lambda \in Y$ is an unbounded sequence, then $\Lambda BV \setminus \Lambda^*BV \neq \emptyset$.

**Proof.** First we consider the case where $\sum_{i=1}^{\infty} \lambda_i^{-2} < \infty$ and consider $\chi_E$, where

$$E = \{(x, y) \in [0, 1]^2, y \leq 1 - x\}.$$  

Lemma 1 implies that $\chi_E \in \Lambda BV$, but from Theorem 1 we have $\chi_E \notin \Lambda^*BV$.

Now assume that Condition $(\ast)$ holds and $\lambda_i \to \infty$ as $i \to \infty$. We choose $\alpha_i \searrow 0$ so that

$$\sum_i \frac{\alpha_i}{\lambda_i} = \infty \quad \text{and} \quad \sum_i \frac{\alpha_i}{\lambda_i} < \infty.$$  

Let

$$f = \sum_{i=1}^{\infty} \alpha_i \chi_{E_n}, \quad \text{where } E_n = \left[\frac{1}{2n}, \frac{1}{2n} + 1\right]^2.$$  

The rectangles $A_n = [2/(4n + 1), 1/2n]^2$ are pairwise disjoint and $|f(A_n)| = \alpha_n$. Hence

$$\sum_{i=1}^{N} \frac{|f(A_i)|}{\lambda_i} = \sum_{i=1}^{N} \frac{\alpha_i}{\lambda_i} \to \infty \text{ as } N \to \infty,$$

implying $f \notin \Lambda^*BV$.

Clearly, $f(c, 0)$ and $f(0, c)$ are in $\Lambda BV([0, 1])$. Suppose $\{I_i\}_i^N$ and $\{J_j\}_j^N$ are collections of nonoverlapping intervals in $[0, 1]$. For each $I_i$, there are no more than four values of $j$ such that $f(I_i \times J_j) \neq 0$. Let $j(i)$ denote the smallest. Let $k(i)$
be the smallest of the indices of $E_n$ such that $\chi_{E_n}(I_i \times J_j) \neq 0$ for some $j$. Then $|f(I_i \times J_j)| \leq 2\alpha_{k(i)}$. Also, each $k$ can appear no more than twice as a $k(i)$ and each $j$ can appear no more than twice as a $j(i)$. Thus

$$S = \sum_{i,j=1}^{N} \frac{|f(I_i \times J_j)|}{\lambda_i \lambda_j} \leq 8 \sum_{i=1}^{N} \frac{\alpha_{k(i)}}{\lambda_i \lambda_j(i)} \leq 8 \left( \sum_{i=1}^{N} \frac{\alpha_{k(i)}}{\lambda_i^2} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} \frac{\alpha_{j(i)}}{\lambda_j^2} \right)^{\frac{1}{2}} \leq 32 \sum_{i=1}^{N} \frac{\alpha_{i}}{\lambda_i^2},$$

which is bounded above independently of $N$ and the choice of $f(I_i)$ and $f(J_j)$. Thus $f \in \Lambda BV$.

**Proposition 4.** If $A = [a,b] \times [c,d]$, $0 < \alpha \leq 1$, and $\Lambda_{\alpha} = \{n^\alpha\}_{n=1}^\infty$, then

$$\Lambda_{\alpha}^*BV \setminus \Lambda_{\alpha}BV \neq \emptyset.$$

**Proof.** We will assume once again the $A = [0,1]^2$. Let

$$f_n = \sum_{k=1}^{n} \sum_{1 \leq j \leq n/k} (-1)^{k+i} \chi_{(\frac{k-1}{n}, \frac{j-1}{n}) \times (\frac{k-1}{n}, \frac{j-1}{n})}.$$  

$C$ will denote a constant, not necessarily the same at each occurrence, and $C_\alpha$ a constant depending on $\alpha$. The number of terms in the sum defining $f_n$ is not greater than $n \ln(n+1)$, so for $\alpha \in (0, 1)$,

$$V_{\Lambda_{\alpha}^*}(f_n) \leq C \sum_{r=1}^{n \ln(n+1)} r^{-\alpha} \leq C_\alpha (n \ln(n+1))^{1-\alpha}.$$  

Similarly,

$$V_{\Lambda_{\alpha}^*}(f_n) \leq C \ln(n+1).$$

On the other hand, for $\alpha \in (0, 1)$,

$$V_{\Lambda_{\alpha}}(f_n) \geq C \sum_{k=1}^{n} k^{-\alpha} \left( \sum_{j=1}^{n/k} j^{-\alpha} \right) \geq C_\alpha \sum_{k=1}^{n} k^{-\alpha} (\frac{n}{k})^{1-\alpha} \geq C_\alpha n^{1-\alpha} \ln(n+1),$$

and similarly,

$$V_{\Lambda_{\alpha}}(f_n) \geq C (\ln(n+1))^2.$$  

Thus, if

$$f(x,y) = \sum_{k=1}^{\infty} a_k f_{n_k}(2^k x - 1, 2^k y - 1),$$

where the sequence of coefficients $\{a_k\}$ and the increasing sequence of natural numbers $\{n_k\}$ are appropriately chosen, we will have

$$f \in \Lambda_{\alpha}^*BV \setminus \Lambda_{\alpha}BV.$$

The general problem remains open.
4. Convergence of double Fourier series

Theorem 2 follows immediately from Theorem C and the following lemma.

**Lemma 3.** Let $\Lambda = \{ \frac{n}{\ln(n+1)} \}_{n=1}^{\infty}, A = [a, b] \times [c, d]$. Then $\Lambda^*BV(A) \subset HBV(A)$.

**Proof.** Suppose $f \in \Lambda^*BV(A)$. Obviously $f(a, \cdot)$ and $f(\cdot, c)$ are in $HBV$. Let $\{I_i\}_{i=1}^{N}$ and $\{J_j\}_{j=1}^{M}$ be systems of nonoverlapping intervals in $[a, b]$ and $[c, d]$ respectively, and let $\Delta_{i,j} = |f(I_i \times J_j)|$. We enumerate the pairs $(i, j)$, $i \in [1, N]$ and $j \in [1, M]$, as follows: assign 1 to $(1, 1)$, 2 and 3 to $(1, 2)$ and $(2, 1)$. Next we enumerate the $(i, j)$ such that $i \cdot j = 3$ in any order, and so on. Let $\mu(i, j)$ denote the index corresponding to $(i, j)$. For a given $n$, the number of $(i, j)$ with $\mu(i, j) \geq 1$ and $i \cdot j \leq n$ is not greater than

$$\sum_{i=1}^{n} \frac{n}{i} \leq n \ln(n + 1),$$

implying that, for these pairs, $\mu(i, j) \leq n \ln(n + 1)$, and so

$$\lambda_{\mu(i, j)} \leq \frac{n \ln(n + 1)}{\ln(n \ln(n + 1) + 1)} \leq 2n.$$

Thus, if $i \cdot j = n$, we have

$$\frac{\Delta_{i,j}}{i \cdot j} \leq \frac{2\Delta_{i,j}}{\lambda_{\mu(i, j)}},$$

for all $(i, j)$. Thus

$$\sum_{i,j=1}^{i=N, j=M} \frac{\Delta_{i,j}}{i \cdot j} \leq 2 \sum_{i,j=1}^{i=N, j=M} \frac{\Delta_{i,j}}{\lambda_{\mu(i, j)}} \leq V_{\Lambda^*}(f),$$

which establishes Lemma 3. \hfill $\Box$

**Remark 3.** It is easily seen that $\Lambda^*BV(A)$ is a Banach space with $\|f\|_{\Lambda^*}$, as given in Definition 3(iii), as norm.

We turn now to the proof of Theorem 3.

**Proof.** Let $N$ be a positive integer and $M = [N/2]$. For positive integers $m$ and $n$ such that $m \cdot n \leq M$, let

$$A_{mn} = \left[ \frac{\pi(m-1)}{N + \frac{1}{2}}, \frac{\pi m}{N + \frac{1}{2}} \right] \times \left[ \frac{\pi(n-1)}{N + \frac{1}{2}}, \frac{\pi n}{N + \frac{1}{2}} \right]$$

and set

$$g_{N}(x, y) = \sum_{m,n \leq M} (-1)^{m+n} \chi_{A_{mn}}(x, y).$$

If $A$ is a closed rectangle with sides parallel to the axes, then $g_{N}(A) \neq 0$ only when $A \cap A_{mn}$ contains exactly one vertex of $A$. Since the number of $A_{mn}$ is not greater than $M \ln(M + 1)$, we have

$$V_{\Lambda^*}(g_{N}) \leq \sum_{r=1}^{4M \ln(M+1)} \frac{\ln(r+1)}{r \xi_{r}} = o(\ln^{2}(M + 1)), $$

and so

$$\|g_{N}\|_{\Lambda^*} = o(\ln^{2}(M + 1)) = \eta_{N} \ln^{2}(M + 1),$$
where \( \eta_N = o(1) \) as \( N \to \infty \), and, if

\[
h_N = \frac{g_N}{\eta_N \ln^2(M + 1)},
\]

then \( \{ \| h_N \|_{A^r} \} \) is bounded.

If we now consider the square partial sums of the Fourier series of \( h_n \) at \( (0, 0) \), we have

\[
\pi^2 S_{NN} [h_N; (0, 0)] = \frac{1}{\eta_N \ln^2(M + 1)} \sum_{m \cdot n \leq M} (-1)^{m+n} \int_{A_{m \cdot n}} D_N(s)D_N(t)dsdt \geq \frac{4}{\eta_N \ln^2(M + 1)\pi^2} \sum_{m \cdot n \leq M} \frac{1}{m \cdot n} \geq C \frac{1}{\eta_N \ln^2(M + 1)} \to \infty
\]

as \( N \to \infty \), \( C \) being an absolute constant. Applying the Banach-Steinhaus theorem, we see that there must be an \( f \in A^r BV \) such that \( \{ S_{NN}[f; (0, 0)] \} \) diverges unboundedly.

References


Professor of the Chair of Theory of Functions and Functional Analysis, Department of Mathematics and Mechanics, Moscow State University, Vorobyevy Gori, GZ, Moscow, Russia 119992

E-mail address: dyach@mail.ru

Research Professor, Florida Atlantic University (Professor Emeritus, Syracuse University), 7739 Majestic Palm Drive, Boynton Beach, Florida 33437

E-mail address: fourier@adelphia.net