

CONVERGENCE OF DOUBLE FOURIER SERIES AND W -CLASSES

M. I. DYACHENKO AND D. WATERMAN

ABSTRACT. The double Fourier series of functions of the generalized bounded variation class $\{n/\ln(n+1)\}^*BV$ are shown to be Pringsheim convergent everywhere. In a certain sense, this result cannot be improved. In general, functions of class Λ^*BV , defined here, have quadrant limits at every point and, for $f \in \Lambda^*BV$, there exist at most countable sets P and Q such that, for $x \notin P$ and $y \notin Q$, f is continuous at (x, y) . It is shown that the previously studied class ΛBV contains essentially discontinuous functions unless the sequence Λ satisfies a strong condition.

1. INTRODUCTION

A remarkable variety of definitions of bounded variation have been given for functions of two variables. Here we will discuss generalizations of these definitions along the lines of the notion of Λ -bounded variation (ΛBV) in one variable introduced by Waterman. He used it to extend the Dirichlet-Jordan theorem, and we will investigate the analogous problem for double Fourier series.

For an excellent discussion of ΛBV and its relation to other generalizations of bounded variation, see Avdispahić [1]. For applications to summability and Tauberian theorems, see [2, 8, 11].

Definition 1. Let $\Lambda = \{\lambda_k\}_1^\infty$ be a monotone nondecreasing sequence of positive numbers such that

$$\sum_1^\infty \lambda_k^{-1} = \infty,$$

and let Y denote the class of such sequences. A real function f defined on an interval $[a, b]$ is said to be of Λ -bounded variation, $f \in \Lambda BV([a, b])$, if

$$V_\Lambda(f; [a, b]) = \sup_{\mathcal{I}, n} \sum_1^n \frac{|f(I_k)|}{\lambda_k} = \sup_{\mathcal{I}, n} \sum_1^n \frac{|f(\beta_k) - f(\alpha_k)|}{\lambda_k} < \infty,$$

where \mathcal{I} denotes the class of collections of nonoverlapping intervals $\{I_k = [\alpha_k, \beta_k] \subset [a, b], k = 1, \dots, n\}$.

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Note that functions of Λ -bounded variation are bounded, have right and left limits at every point, and so their discontinuities are at most countable [9].

Function classes, whose definitions depend on the boundedness of sums of the absolute values of interval functions multiplied by weights from sequences such as Λ , have come to be known as *W-classes*.

In [7, 10], Waterman proved the following generalization of the Dirichlet-Jordan theorem.

Theorem A. *If f is a 2π -periodic function, $H = \{n\}_1^\infty$, $T = [-\pi, \pi]$ and $f \in HBV(T)$, then $S[f]$, the Fourier series of f , converges at every point and, if I is a closed interval of points of continuity, then $S[f]$ converges uniformly on I . If $\Lambda BV \setminus HBV \neq \emptyset$, then there is an $f \in \Lambda BV(T)$ such that $S[f]$ diverges at a point.*

A definition of ΛBV for two variables which has been used by Saakjan [5] and Sablin [6] is as follows.

Definition 2. Let $\Lambda \in Y$ and let f be a measurable function on the rectangle $A = [a, b] \times [c, d]$. Then $f \in \Lambda BV(A)$ if and only if

- (1) $f(\cdot, c) \in \Lambda BV([a, b])$ and $f(a, \cdot) \in \Lambda BV([c, d])$, and
- (2) if \mathcal{I}_1 and \mathcal{I}_2 are the sets of finite collections of nonoverlapping intervals $I_k = [\alpha_k, \beta_k]$ and $I_j = [\gamma_j, \delta_j]$ in $[a, b]$ and $[c, d]$ respectively and $f(I_k \times I_j) = f(\alpha_k, \gamma_j) - f(\alpha_k, \delta_j) - f(\beta_k, \gamma_j) + f(\beta_k, \delta_j)$, then

$$V_\Lambda(f; [a, b]) = \sup_{\mathcal{I}_1, \mathcal{I}_2} \sum_k \sum_j \frac{|f(I_k \times I_j)|}{\lambda_k \lambda_j} < \infty.$$

Remark 1. If $\lambda_k \equiv 1$, or what is the same, $\lambda_k = O(1)$, $\Lambda BV(A)$ is the set of functions of Hardy-Krause bounded variation on A .

It is clear that the functions of $\Lambda BV(A)$ are bounded, but the question of continuity is more complicated than in the case of functions of one variable. Dyachenko [3] has proved the following theorem.

Theorem B. *The following conditions are equivalent:*

- (i) for any $f \in \Lambda BV(T^2)$ there exist two at most countable subsets A and B of T such that f is continuous at every point $(x, y) \in T^2$ such that $x \notin A$ and $y \notin B$;
- (ii) for any $f \in \Lambda BV(T^2)$ and any $(x_0, y_0) \in T^2$, $\lim f(x, y)$ exists as $(x, y) \rightarrow (x_0, y_0)$ in each of the open coordinate quadrants;
- (iii) $\sum_1^\infty \lambda_k^{-2} = \infty$.

(The third condition will be called **Condition (*)**.)

Thus we see, for example, that the characteristic function of the triangle $B = \{(x, y) \in [0, 1]^2, 0 \leq y \leq 1 - x\}$ is in $\Lambda BV([0, 1]^2)$ only if Condition (*) does not hold.

If Λ does not satisfy (iii), the requirement of measurability cannot be omitted from Definition 2, for if it were, $\Lambda BV(A)$ would include functions not Lebesgue measurable. Even under the assumption of measurability, we show in Section 2 that $\Lambda BV(A)$ contains an everywhere discontinuous function and, moreover, a function f such that if $g = f$ a.e., then g is a.e. discontinuous.

We will consider the Pringsheim convergence of double Fourier series. If $f \in L(T^2)$ is 2π -periodic in each variable, then

$$S[f] = \sum_{m,n} a_{mn} e^{i(mx+ny)}$$

is its Fourier series, where

$$a_{mn} = a_{mn}(f) = \frac{1}{(2\pi)^2} \int_{T^2} f(x, y) e^{-i(mx+ny)} dx dy.$$

The rectangular partial sums of this series are

$$S_{N_1 N_2}(f; x, y) = \sum_{|k_1| \leq N_1} \sum_{|k_2| \leq N_2} a_{k_1 k_2} e^{i(k_1 x + k_2 y)}$$

with $N_1, N_2 \geq 0$. If $N_1 = N_2$, these are called *square sums*. If

$$S_{N_1 N_2}(f; x, y) \rightarrow \alpha \text{ as } \min(N_1, N_2) \rightarrow \infty,$$

we say that the Fourier series of f converges to α at (x, y) in the Pringsheim sense.

A.A. Saakyan [5] has shown

Theorem C. *If $f \in HBV(T^2)$, then the rectangular partial sums of $S[f]$ are uniformly bounded, and, if for $(x_0, y_0) \in T^2$ the limits of $f(x, y)$ exist as $(x, y) \rightarrow (x_0, y_0)$ in each of the open coordinate quadrants, then $S[f]$ converges (Pringsheim) to the arithmetic mean of these limits.*

This result has been generalized to higher dimensions by A.I. Sablin [6].

As we shall see in §2, an $f \in HBV(T^2)$ need not have a point of continuity. For such a function, Theorem C is inapplicable.

We shall define another W -class such that functions of this class are continuous a.e., and prove that a theorem analogous to Theorem C holds for this class.

Definition 3. Let $\Lambda \in Y$ and let f be a real function on $A = [a, b] \times [c, d]$. We say $f \in \Lambda^*BV(A)$ if

(i) $f(\cdot, c) \in \Lambda BV([a, b])$ and $f(a, \cdot) \in \Lambda BV([c, d])$

and, if Γ is the set of finite collections of nonoverlapping rectangles $A_k = [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset A$ and $f(A_k) = f(\alpha_k, \gamma_k) - f(\alpha_k, \delta_k) - f(\beta_k, \gamma_k) + f(\beta_k, \delta_k)$, then

(ii) $V_\Lambda^*(f; A) = \sup_\Gamma \sum_k \frac{|f(A_k)|}{\lambda_k} < \infty$.

For $f \in \Lambda^*BV(A)$ we set

(iii) $\|f\|_{\Lambda^*} = \|f\|_{\Lambda^*(A)} = |f(a, c)| + V_\Lambda(f(\cdot, c)) + V_\Lambda(f(a, \cdot)) + V_\Lambda^*(f; A)$.

Remark 2. Note that if $f \in \Lambda^*BV(A)$, then

$$V_\Lambda(f(\cdot, y); [a, b]) \leq V_\Lambda^*(f; A) + V_\Lambda(f(\cdot, c); [a, b])$$

for every $y \in [c, d]$. The analogous result holds for the Λ -variation of the restriction of f to the vertical segments.

In §3 we shall prove

Theorem 1. *Let $\Lambda \in Y$ and $A = [a, b] \times [c, d]$. Then, for any $f \in \Lambda^*BV(A)$,*

(i) *there exist at most countable sets $P \subset [a, b]$ and $Q \subset [c, d]$ such that f is continuous at every $(x, y) \in A$ such that $x \notin P$ and $y \notin Q$; and*

(ii) *at every point $(x_0, y_0) \in A$, $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ exists in each open coordinate quadrant.*

In that section we also discuss the relation between $\Lambda^*BV(A)$ and $\Lambda BV(A)$.

In §4 we study the convergence of Fourier series of functions of class $\Lambda^*BV(A)$, and prove

Theorem 2. *Let f be a real function on \mathbb{R}^2 which is 2π -periodic in each variable and is in $\Lambda^*BV(T^2)$ with $\Lambda = \{\frac{n}{\ln(n+1)}\}$. Then the rectangular partial sums of $S[f]$ are uniformly bounded and converge at each point to the arithmetic mean of the quadrant limits.*

We also show that, in a certain sense, this result cannot be improved.

Theorem 3. *Let $\Lambda = \{\frac{n}{\ln(n+1)}\xi_n\} \in Y$, where $\xi_n \uparrow \infty$ as $n \rightarrow \infty$. Then there exists a function $f \in \Lambda^*BV(T^2)$ such that the square partial sums of its Fourier series diverge unboundedly at $(0, 0)$.*

2. DISCONTINUOUS FUNCTIONS IN W -CLASSES

We shall require the following lemmas.

Lemma 1. *Let $\Lambda \in Y$ be such that Condition $(*)$ does not hold (i.e., $\sum \lambda_k^{-2} < \infty$) and let $A = [a, b] \times [c, d]$ be a nondegenerate interval. Suppose $E \subset A$ has a connected intersection with every horizontal and vertical line. Then χ_E , the characteristic function of E , is in $\Lambda BV(A)$ and $V_\Lambda(\chi_E; A) < C < \infty$, where C is an absolute constant.*

Proof. Let $\{I_k\}_1^n = \{[\alpha_k, \beta_k]\}$ in $[a, b]$ and $\{J_r\}_1^m = \{[\gamma_r, \delta_r]\}$ in $[c, d]$ be two collections of nonoverlapping intervals. Then, for each $k = 1, 2, \dots, n$, in the sum

$$S = \sum_{k=1}^n \sum_{r=1}^m |\chi_E(I_k \times J_r)| / \lambda_k \lambda_r,$$

there are at most four different $r_{k,j}$ for which $|\chi_E(I_k \times J_{r_{k,j}})| \neq 0$, and in these cases it is either 1 or 2. Let

$$S_j = \sum_{k=1}^n \frac{|\chi_E(I_k \times J_{r_{k,j}})|}{\lambda_k \lambda_{r_{k,j}}}, \quad \text{for } j = 1, 2, 3, 4.$$

Then $S = S_1 + S_2 + S_3 + S_4$, and, as each j can be associated with at most four $r_{k,j}$, we have

$$S_j \leq 2 \sum_{k=1}^n \frac{1}{\lambda_k \lambda_{r_{k,j}}} \leq \sum_{k=1}^n \frac{1}{\lambda_k^2} + \sum_{k=1}^n \frac{1}{\lambda_{r_{k,j}}^2} \leq 5 \sum_{k=1}^n \frac{1}{\lambda_k^2} < C < \infty,$$

and, since the one-dimensional Λ -variation of χ_E on the edges of A is at most $2/\lambda_1$, Lemma 1 is established. \square

Lemma 2. *$A = [a, b] \times [c, d]$ be a nondegenerate interval. There is a sequence of closed rectangles $\{A_i = I_i \times J_i\}$ in A with $I_i \cap I_j = \emptyset$ and $J_i \cap J_j = \emptyset$ for $i \neq j$, with*

$$\sum_{i=1}^{\infty} |I_i| < (b - a)/4 \quad \text{and} \quad \sum_{i=1}^{\infty} |J_i| < (d - c)/4,$$

such that every neighborhood of each point of the following contains some A_i :

$$B = A \setminus \bigcup_{i=1}^{\infty} ((I_i \times [c, d]) \cup ([a, b] \times J_i)).$$

Proof. Choose $n_1 = 1$ and consider the rectangles

$$E_{k,r,1} = \left[a + \frac{(b-a)(k-1)}{2^{n_1}}, a + \frac{(b-a)k}{2^{n_1}} \right] \times \left[c + \frac{(d-c)(r-1)}{2^{n_1}}, c + \frac{(d-c)r}{2^{n_1}} \right],$$

where $k, r = 1, 2, \dots, 2^{n_1}$.

Choose an integer $n_2 > 2n_1 + 3$ and, for each choice of k and $r, 1 \leq k, r \leq 2^{n_1}$, choose a rectangle

$$\begin{aligned} E'_{k,r,1} &= [a_{k,r,1}, b_{k,r,1}] \times [c_{k,r,1}, d_{k,r,1}] \\ &= \left[a + \frac{(b-a)(l_k-1)}{2^{n_2}}, a + \frac{(b-a)l_k}{2^{n_2}} \right] \times \left[c + \frac{(d-c)(s_r-1)}{2^{n_2}}, c + \frac{(d-c)s_r}{2^{n_2}} \right] \subset E_{k,r,1} \end{aligned}$$

so that the projections of the chosen rectangles on the coordinate axes do not touch. Clearly

$$\sum_{k,r=1}^{2^{n_1}} (b_{k,r,1} - a_{k,r,1}) < \frac{b-a}{8} \quad \text{and} \quad \sum_{k,r=1}^{2^{n_1}} (d_{k,r,1} - c_{k,r,1}) < \frac{d-c}{8}.$$

Next we consider the rectangles

$$E_{k,r,2} = \left[a + \frac{(b-a)(k-1)}{2^{n_2}}, a + \frac{(b-a)k}{2^{n_2}} \right] \times \left[c + \frac{(d-c)(r-1)}{2^{n_2}}, c + \frac{(d-c)r}{2^{n_2}} \right],$$

where k and r are chosen, $1 \leq k, r \leq 2^{n_2}$, so that

$$E_{k,r,2} \cap \bigcup_{k,r=1}^{2^{n_1}} ([a_{k,r,1}, b_{k,r,1}] \times [c, d]) \cup ([a, b] \times [c_{k,r,1}, d_{k,r,1}]) = \emptyset.$$

We then choose an integer $n_3 > 2n_2 + 4$ and, for each k and $r, 1 \leq k, r \leq 2^{n_2}$, just chosen, we choose a rectangle

$$\begin{aligned} E'_{k,r,2} &= [a_{k,r,2}, b_{k,r,2}] \times [c_{k,r,2}, d_{k,r,2}] \\ &= \left[a + \frac{(b-a)(l_k-1)}{2^{n_3}}, a + \frac{(b-a)l_k}{2^{n_3}} \right] \times \left[c + \frac{(d-c)(s_r-1)}{2^{n_3}}, c + \frac{(d-c)s_r}{2^{n_3}} \right] \\ &\subset E_{k,r,2} \end{aligned}$$

so that the projections of the chosen rectangles on the coordinate axes do not touch. Then

$$\sum_{k,r} (b_{k,r,2} - a_{k,r,2}) < \frac{b-a}{16} \quad \text{and} \quad \sum_{k,r} (d_{k,r,2} - c_{k,r,2}) < \frac{d-c}{16}.$$

Proceeding inductively, we can define $E'_{k,r,j}, j = 1, 2, \dots$, choosing $n_{j+1} > 2n_j + j + 2$ at each step, and, renumbering the $E'_{k,r,j}$ as we wish, $\{A_i\} = \{E'_{k,r,j}\}$ is the required sequence of intervals. \square

We turn now to the principal results of this section.

Proposition 1. *Suppose $\Lambda \in Y, A = [a, b] \times [c, d]$ is a nondegenerate rectangle and Condition (*) does not hold. Then there exists an $f \in \Lambda BV(A)$ which is everywhere discontinuous.*

Proof. Let P and Q be the sets of rationals in $[a, b]$ and $[c, d]$ respectively. Divide the rectangle A into four quarters by passing lines parallel to the axes through the midpoints of the sides. In each of the rectangles $A_i, i = 1, 2, 3, 4$, thus formed, we select $(p_i, q_i), p_i \in P, q_i \in Q$, so that the $\{p_i\}$ and $\{q_i\}$ are distinct. Now we can quarter each A_i and choose one point in each sixteenth not yet containing a chosen point to form $\{(p_i, q_i)\}_5^{16}, p_i \in P, q_i \in Q$, so that $\{p_i\}_1^{16}$ and $\{q_i\}_1^{16}$ are sets of distinct points. Proceeding inductively, we obtain a dense set of points $E = \{(p_i, q_i)\}$ such that $p_i \neq p_j$ and $q_i \neq q_j$ when $i \neq j$. Thus any line parallel to an axis meets E in at most one point and, by Lemma 1, $\chi_E \in \Lambda BV(A)$. \square

The function we have constructed in Proposition 1 is everywhere discontinuous but is equivalent to the function identically equal to zero. The next result shows that in a class $\Lambda BV(A)$ in which Condition (*) does not hold there exist *essentially discontinuous* functions.

Proposition 2. *Suppose $\Lambda \in Y, A = [a, b] \times [c, d]$ is a nondegenerate rectangle and Condition (*) does not hold. Then there is an $f \in \Lambda BV(A)$ such that $g = f$ a.e. implies that g is a.e. discontinuous.*

Proof. Apply Lemma 2 to form the sequence $\{A_i\}$ and the set $B_1 = B$. If $F_1 = \bigcup_1^\infty A_i$ and $f_1 = \chi_{F_1}$, we see that $V_\Lambda(f_1) = C < \infty$, and we observe that $|A \setminus B_1| < |A|/2$. The set $A \setminus B_1$ can be divided into rectangles $\{D_i\}_{i=1}^\infty$ in the natural way. By applying Lemma 2 to each of the rectangles D_i , we obtain for each i a sequence $\{A_{ij}\}_{j=1}^\infty \subset D_i$ with similar properties. Let $F_{2,i} = \bigcup_{j=1}^\infty A_{i,j}$ and $f_{2,i} = \chi_{F_{2,i}}, i = 1, 2, \dots$. Note that $V_\Lambda(f_{2,i}) \leq C$ for every i . If $A_{i,j} = [a_{i,j}, b_{i,j}] \times [c_{i,j}, d_{i,j}]$, set

$$B_{2,i} = D_i \setminus \bigcup_{j=1}^\infty (([a_{i,j}, b_{i,j}] \times [c, d]) \cup ([a, b] \times [c_{i,j}, d_{i,j}])), \quad i = 1, 2, \dots$$

Let us replace i by the symbol i_1 . At the third stage we obtain sets B_{3,i_1,i_2} , and as the induction proceeds we obtain sets $B_{r,i_1,\dots,i_{r-1}}$. Let

$$U = B_1 \cup \left(\bigcup_{r=2}^\infty \left(\bigcup_{i_1,\dots,i_{r-1}=1}^\infty B_{r,i_1,\dots,i_{r-1}} \right) \right);$$

then $|A \setminus U| = 0$. We continue inductively to obtain functions $f_1, \{f_{2,i_1}\}_{i_1=1}^\infty, \dots, \{f_{r,i_1,\dots,i_{r-1}}\}_{i_1,\dots,i_{r-1}=1}^\infty, \dots$, and we renumber these functions to form $\{h_k\}_{k=1}^\infty$. Now, letting

$$f = \sum_{k=1}^\infty 3^{-k} h_k,$$

we have $f \in \Lambda BV(A)$.

Let $g = f$ a.e. and let E be the set of points in A at which g and f are equal. Clearly E is dense in A . We write $\omega(f; (x, y); E)$ for the oscillation of f at (x, y) over the set E and $\omega(g; (x, y); A)$ for the oscillation of g at (x, y) over the set A . Note that

$$\omega(g; (x, y); A) \geq \omega(f; (x, y); E).$$

Consider a point $(x, y) \in U$. Then (x, y) is in some $B_{r,i_1,\dots,i_{r-1}}$. If k is such that $h_k = f_{r,i_1,\dots,i_{r-1}}$, then $\omega(h_k; (x, y); E) = 1$. Note that, since $h_l(A) = \{0, 1\}$, for every

$(s, t) \in A$ and every l , we have either

$$\omega(h_l; (s, t); E) = 0 \text{ or } \omega(h_l; (s, t); E) = 1.$$

Let

$$k_0 = k_0(x, y) = \min\{l : \omega(h_l; (x, y); E) = 1\}.$$

Then

$$\omega(f; (x, y); E) \geq 3^{-k_0} - \sum_{l=k_0+1}^{\infty} 3^{-l} \omega(h_l; (x, y); E) \geq 3^{-k_0} - \sum_{l=k_0+1}^{\infty} 3^{-l} \geq 3^{-k_0-1},$$

implying that g is discontinuous at (x, y) . □

3. CONTINUITY PROPERTIES OF FUNCTIONS OF Λ^*BV

We now turn our attention to the proof of Theorem 1.

Proof. Suppose there are infinitely many points (x_i, y_i) such that $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$, at which the oscillation of f exceeds $1/k$, k a natural number. For a natural number N we can find points (α_i, β_i) and (γ_i, δ_i) , $i = 1, 2, \dots, N$, such that $f(\alpha_i, \beta_i) - f(\gamma_i, \delta_i) > 1/k$ and the sequences $\alpha_1, \gamma_1, \alpha_2, \gamma_2, \alpha_3, \dots$ and $\beta_1, \delta_1, \beta_2, \delta_2, \beta_3, \dots$ are strictly monotone. We will assume them to be increasing; the other cases are handled similarly.

We have

$$\begin{aligned} S_1 &= \sum_{i=0}^N \frac{|f(\alpha_i, \delta_i) - f(\gamma_i, \delta_i)|}{\lambda_i} \\ &\leq \sum_{i=0}^N \frac{|f((\alpha_i, \gamma_i) \times (c, \delta_i))|}{\lambda_i} + \sum_{i=0}^N \frac{|f(\alpha_i, c) - f(\gamma_i, c)|}{\lambda_i} \leq \|f\|_{\Lambda^*} \end{aligned}$$

and, in a similar fashion,

$$S_2 = \sum_{i=0}^N \frac{|f((\gamma_i, \beta_i) - f(\gamma_i, \delta_i))|}{\lambda_i} \leq \|f\|_{\Lambda^*}.$$

Thus

$$\begin{aligned} V_{\Lambda^*}(f; A) &\geq \sum_{i=0}^N \frac{|f((\alpha_i, \gamma_i) \times (\beta_i, \delta_i))|}{\lambda_i} \\ &\geq \sum_{i=0}^N \frac{|f(\alpha_i, \beta_i) - f(\gamma_i, \delta_i)|}{\lambda_i} - S_1 - S_2 \\ &\geq \frac{1}{k} \sum_{i=0}^N \frac{1}{\lambda_i} - 2 \|f\|_{\Lambda^*}, \end{aligned}$$

which is false for N sufficiently large. Thus all points at which f has an oscillation greater than $1/k$ lie on a finite number of lines parallel to the axes, which establishes the first part of the theorem.

To establish the second part of Theorem 1, we assume that there is a point $p \in A$ such that $f(x, y)$ does not have a limit as $(x, y) \rightarrow p$ within an open coordinate quadrant with vertex p . Without loss of generality we may assume that $p = (0, 0)$ and the quadrant is $\{(x, y) : x > 0, y > 0\}$.

Then there is an $\varepsilon > 0$ such that, for every $\delta > 0$, in every square $(0, \delta)^2$ the oscillation of f is greater than ε . Choose $s, t > 0$. Then, since f is in ΛBV in each variable separately, $\lim_{y \downarrow 0} f(s, y)$ and $\lim_{x \downarrow 0} f(x, t)$ exist. Choose $\delta > 0$ so that the oscillations of $f(s, y)$ and $f(x, t)$ on $0 < y < \delta$ and $0 < x < \delta$ respectively are less than $\varepsilon/8$. Now choose points (x_1, y_1) and (x_2, y_2) in $(0, \delta)^2$ so that

$$|f(x_1, y_1) - f(x_2, y_2)| > \varepsilon/2.$$

Then, letting $P = f(s, t) - f(s, y_1) - f(x_1, t) + f(x_1, y_1)$ and $Q = f(s, t) - f(s, y_2) - f(x_2, t) + f(x_2, y_2)$, we have

$$\begin{aligned} |P - Q| &\geq |f(x_1, y_1) - f(x_2, y_2)| - |f(x_1, t) - f(x_2, t)| - |f(s, y_1) - f(s, y_2)| \\ &> \varepsilon/2 - \varepsilon/8 - \varepsilon/8 = \varepsilon/4, \end{aligned}$$

so that at least one of $|P|$ and $|Q|$ exceeds $\varepsilon/8$, and so we have obtained a rectangle $A_1 \in (0, \delta)^2$ for which $|f(A_1)| > \varepsilon/8$. By choosing our points $(s, t), (x_1, y_1)$ and (x_2, y_2) sufficiently close to the origin, we can repeat this process to obtain a rectangle A_2 which does not overlap A_1 for which $|f(A_2)| > \varepsilon/8$. Thus we can form a sequence $\{A_n\}$ of nonoverlapping intervals in $(0, \delta)^2$ with $|f(A_n)| > \varepsilon/8$. Then

$$\sum_{i=0}^N \frac{|f(A_k)|}{\lambda_k} > \frac{\varepsilon}{8} \sum_{i=0}^N \frac{1}{\lambda_k} \rightarrow \infty \text{ as } N \rightarrow \infty,$$

contradicting our assumption that $f \in \Lambda^*BV(A)$, and completing the proof of Theorem 1. □

It is natural to ask how the classes ΛBV and Λ^*BV are related. This is by no means obvious, although they are clearly the same if $\{\lambda_i\}$ is bounded. There is no loss of generality in assuming the rectangle A to be $[0, 1]^2$.

Proposition 3. *If $\Lambda \in Y$ is an unbounded sequence, then $\Lambda BV \setminus \Lambda^*BV \neq \emptyset$.*

Proof. First we consider the case where $\sum_1^\infty \lambda_i^{-2} < \infty$ and consider χ_E , where

$$E = \{(x, y) \in [0, 1]^2, y \leq 1 - x\}.$$

Lemma 1 implies that $\chi_E \in \Lambda BV$, but from Theorem 1 we have $\chi_E \notin \Lambda^*BV$.

Now assume that Condition (*) holds and $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$. We choose $\alpha_i \searrow 0$ so that

$$\sum_i \frac{\alpha_i}{\lambda_i} = \infty \quad \text{and} \quad \sum_i \frac{\alpha_i}{\lambda_i^2} < \infty.$$

Let

$$f = \sum_1^\infty \alpha_n \chi_{E_n}, \quad \text{where } E_n = \left[\frac{1}{2n}, \frac{1}{2n-1}\right]^2.$$

The rectangles $A_n = [2/(4n+1), 1/2n]^2$ are pairwise disjoint and $|f(A_n)| = \alpha_n$. Hence

$$\sum_{i=1}^N \frac{|f(A_i)|}{\lambda_i} = \sum_{i=1}^N \frac{\alpha_i}{\lambda_i} \rightarrow \infty \text{ as } N \rightarrow \infty,$$

implying $f \notin \Lambda^*BV$.

Clearly, $f(\cdot, 0)$ and $f(0, \cdot)$ are in $\Lambda BV([0, 1])$. Suppose $\{I_i\}_1^N$ and $\{J_i\}_1^N$ are collections of nonoverlapping intervals in $[0, 1]$. For each I_i , there are no more than four values of j such that $f(I_i \times J_j) \neq 0$. Let $j(i)$ denote the smallest. Let $k(i)$

be the smallest of the indices of E_n such that $\chi_{E_n}(I_i \times J_j) \neq 0$ for some j . Then $|f(I_i \times J_j)| \leq 2\alpha_{k(i)}$. Also, each k can appear no more than twice as a $k(i)$ and each j can appear no more than twice as a $j(i)$. Thus

$$S = \sum_{i,j=1}^N \frac{|f(I_i \times J_j)|}{\lambda_i \lambda_j} \leq 8 \sum_{i=1}^N \frac{\alpha_{k(i)}}{\lambda_i \lambda_{j(i)}} \leq 8 \left(\sum_{i=1}^N \frac{\alpha_{k(i)}}{\lambda_i^2} \right)^{\frac{1}{2}} \left(\sum_{i=1}^N \frac{\alpha_{k(i)}}{\lambda_{j(i)}^2} \right)^{\frac{1}{2}} \leq 32 \sum_{i=1}^N \frac{\alpha_i}{\lambda_i^2},$$

which is bounded above independently of N and the choice of $\{I_i\}$ and $\{J_j\}$. Thus $f \in \Lambda BV$. \square

Proposition 4. *If $A = [a, b] \times [c, d]$, $0 < \alpha \leq 1$, and $\Lambda_\alpha = \{n^\alpha\}_{n=1}^\infty$, then*

$$\Lambda_\alpha^* BV \setminus \Lambda_\alpha BV \neq \emptyset.$$

Proof. We will assume once again the $A = [0, 1]^2$. Let

$$f_n = \sum_{k=1}^n \sum_{1 \leq l \leq n/k} (-1)^{k+l} \chi_{[\frac{k-1}{n}, \frac{k}{n}] \times [\frac{l-1}{n}, \frac{l}{n}]}.$$

C will denote a constant, not necessarily the same at each occurrence, and C_α a constant depending on α . The number of terms in the sum defining f_n is not greater than $n \ln(n + 1)$, so for $\alpha \in (0, 1)$,

$$V_{\Lambda_\alpha^*}(f_n) \leq C \sum_{r=1}^{n \ln(n+1)} r^{-\alpha} \leq C_\alpha (n \ln(n + 1))^{1-\alpha}.$$

Similarly,

$$V_{\Lambda_1}(f_n) \leq C \ln(n + 1).$$

On the other hand, for $\alpha \in (0, 1)$,

$$V_{\Lambda_\alpha}(f_n) \geq C \sum_{k=1}^n k^{-\alpha} \left(\sum_{j=1}^{n/k} j^{-\alpha} \right) \geq C_\alpha \sum_{k=1}^n k^{-\alpha} \left(\frac{n}{k} \right)^{1-\alpha} \geq C_\alpha n^{1-\alpha} \ln(n + 1),$$

and similarly,

$$V_{\Lambda_1}(f_n) \geq C(\ln(n + 1))^2.$$

Thus, if

$$f(x, y) = \sum_{k=1}^\infty a_k f_{n_k}(2^k x - 1, 2^k y - 1),$$

where the sequence of coefficients $\{a_k\}$ and the increasing sequence of natural numbers $\{n_k\}$ are appropriately chosen, we will have

$$f \in \Lambda_\alpha^* BV \setminus \Lambda_\alpha BV.$$

\square

The general problem remains open.

4. CONVERGENCE OF DOUBLE FOURIER SERIES

Theorem 2 follows immediately from Theorem C and the following lemma.

Lemma 3. *Let $\Lambda = \{\frac{n}{\ln(n+1)}\}_{n=1}^{\infty}$, $A = [a, b] \times [c, d]$. Then $\Lambda^*BV(A) \subset HBV(A)$.*

Proof. Suppose $f \in \Lambda^*BV(A)$. Obviously $f(a, \cdot)$ and $f(\cdot, c)$ are in HBV . Let $\{I_i\}_{i=1}^N$ and $\{J_j\}_{j=1}^M$ be systems of nonoverlapping intervals in $[a, b]$ and $[c, d]$ respectively, and let $\Delta_{i,j} = |f(I_i \times J_j)|$. We enumerate the pairs (i, j) , $i \in [1, N]$ and $j \in [1, M]$, as follows: assign 1 to $(1, 1)$, and 2 and 3 to $(1, 2)$ and $(2, 1)$. Next we enumerate the (i, j) such that $i \cdot j = 3$ in any order, and so on. Let $\mu(i, j)$ denote the index corresponding to (i, j) . For a given n , the number of (i, j) with $\mu(i, j) \geq 1$ and $i \cdot j \leq n$ is not greater than

$$\sum_{i=1}^n \frac{n}{i} \leq n \ln(n+1),$$

implying that, for these pairs, $\mu(i, j) \leq n \ln(n+1)$, and so

$$\lambda_{\mu(i,j)} \leq \frac{n \ln(n+1)}{\ln(n \ln(n+1) + 1)} \leq 2n.$$

Thus, if $i \cdot j = n$, we have

$$\frac{\Delta_{i,j}}{i \cdot j} \leq \frac{2\Delta_{i,j}}{\lambda_{\mu(i,j)}}$$

for all (i, j) . Thus

$$\sum_{i,j=1}^{i=N, j=M} \frac{\Delta_{i,j}}{i \cdot j} \leq 2 \sum_{i,j=1}^{i=N, j=M} \frac{\Delta_{i,j}}{\lambda_{\mu(i,j)}} \leq V_{\Lambda^*}(f),$$

which establishes Lemma 3. \square

Remark 3. It is easily seen that $\Lambda^*BV(A)$ is a Banach space with $\|f\|_{\Lambda^*}$, as given in Definition 3(iii), as norm.

We turn now to the proof of Theorem 3.

Proof. Let N be a positive integer and $M = [N/2]$. For positive integers m and n such that $m \cdot n \leq M$, let

$$A_{mn} = \left[\frac{\pi(m-1)}{N + \frac{1}{2}}, \frac{\pi m}{N + \frac{1}{2}} \right) \times \left[\frac{\pi(n-1)}{N + \frac{1}{2}}, \frac{\pi n}{N + \frac{1}{2}} \right)$$

and set

$$g_N(x, y) = \sum_{m \cdot n \leq M} (-1)^{m+n} \chi_{A_{mn}}(x, y).$$

If A is a closed rectangle with sides parallel to the axes, then $g_N(A) \neq 0$ only when $A \cap A_{mn}$ contains exactly one vertex of A . Since the number of A_{mn} is not greater than $M \ln(M+1)$, we have

$$V_{\Lambda^*}(g_N) \leq \sum_{r=1}^{4M \ln(M+1)} \frac{\ln(r+1)}{r \xi_r} = o(\ln^2(M+1)),$$

and so

$$\|g_N\|_{\Lambda^*} = o(\ln^2(M+1)) = \eta_N \ln^2(M+1),$$

where $\eta_N = o(1)$ as $N \rightarrow \infty$, and, if

$$h_N = \frac{g_N}{\eta_N \ln^2(M+1)},$$

then $\{\|h_N\|_{\Lambda^*}\}$ is bounded.

If we now consider the square partial sums of the Fourier series of h_n at $(0, 0)$, we have

$$\begin{aligned} \pi^2 S_{NN}[h_N; (0, 0)] &= \frac{1}{\eta_N \ln^2(M+1)} \sum_{m \cdot n \leq M} (-1)^{m+n} \iint_{A_{mn}} D_N(s) D_N(t) ds dt \\ &\geq \frac{4}{\eta_N \ln^2(M+1) \pi^2} \sum_{m \cdot n \leq M} \frac{1}{m \cdot n} \geq C \frac{1}{\eta_N} \frac{\ln^2(M+1)}{\ln^2(M+1)} \rightarrow \infty \end{aligned}$$

as $N \rightarrow \infty$, C being an absolute constant. Applying the Banach-Steinhaus theorem, we see that there must be an $f \in \Lambda^*BV$ such that $\{S_{NN}[f; (0, 0)]\}$ diverges unboundedly. \square

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PROFESSOR OF THE CHAIR OF THEORY OF FUNCTIONS AND FUNCTIONAL ANALYSIS, DEPARTMENT OF MATHEMATICS AND MECHANICS, MOSCOW STATE UNIVERSITY, VOROBYEVI GORI, GZ, MOSCOW, RUSSIA 119992

E-mail address: dyach@mail.ru

RESEARCH PROFESSOR, FLORIDA ATLANTIC UNIVERSITY (PROFESSOR EMERITUS, SYRACUSE UNIVERSITY), 7739 MAJESTIC PALM DRIVE, BOYNTON BEACH, FLORIDA 33437

E-mail address: fourier@adelphia.net