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STABILITY OF TRANSONIC SHOCK FRONTS IN TWO-DIMENSIONAL EULER SYSTEMS

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ABSTRACT. We study the stability of stationary transonic shock fronts under two-dimensional perturbation in gas dynamics. The motion of the gas is described by the full Euler system. The system is hyperbolic ahead of the shock front, and is a hyperbolic-elliptic composed system behind the shock front. The stability of the shock front and the downstream flow under two-dimensional perturbation of the upstream flow can be reduced to a free boundary value problem of the hyperbolic-elliptic composed system. We develop a method to deal with boundary value problems for such systems. The crucial point is to decompose the system to a canonical form, in which the hyperbolic part and the elliptic part are only weakly coupled in their coefficients. By several sophisticated iterative processes we establish the existence and uniqueness of the solution to the described free boundary value problem. Our result indicates the stability of the transonic shock front and the flow field behind the shock.

1. INTRODUCTION

In this paper we are concerned with the stability of stationary transonic shock fronts under two-dimensional perturbation in gas dynamics. The problem arises in many situation with physical importance. As it is well known, when a supersonic flow passes across a shock front, the normal component of its velocity will be changed from supersonic to subsonic. Such a flow pattern frequently appears in various physical problems of gas dynamics [10], [11], [21]. The stability of the shock front is obviously an important problem that people are concerned with. Viewing the shock front as an object in multidimensional space, many mathematicians studied the stability of shock fronts from various points of view; see [8], [9], [13], [14], [17], [22], [24], etc. In this paper we study the stability of the shock front under perturbation together with the existence of the solution to the nonlinear perturbed problem. The relevant physical problem we will discuss is a compressible flow in a tube. The flow is supersonic upstream and becomes subsonic by passing across a shock front. Previously, with a similar viewpoint, G.Q. Chen and M. Feldman studied the stability of transonic shock fronts by using the model of the potential equation in [7]. The potential equation works in gas dynamics under the assumption that the flow is isentropic and irrotational, so that it offers a good model to study many problems in gas dynamics when only weak shock fronts appear [19]. However,

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when a plane shock is perturbed, the entropy in the flow is not constant and the flow behind the shock is not irrotational. Therefore, it is natural to analyze such problems by using the full Euler system, which offers a more precise description of the inviscid steady flow. Besides, the discussion with the full Euler system will allow us to study more complicated problems involving interaction of shock fronts as we plan to do in the future.

It is inevitable that the study of the stability of transonic shock fronts in the full Euler system will be more difficult, because the whole system exhibits both hyperbolicity and ellipticity. As we will see, behind the shock the system is neither purely hyperbolic nor purely elliptic. The system is a hyperbolic-elliptic composed system indeed, and possesses both real and complex characteristics. To solve a problem for such a system a new treatment is required, though many classical and modern methods for both hyperbolic equations and elliptic equations are well developed. Furthermore, since the location of the perturbed shock front is also to be determined, we will confront a free boundary value problem for a nonlinear hyperbolic-elliptic system. It has been noted by many mathematicians that to develop a theory for solving such free boundary value problems is extremely important in studying multidimensional hyperbolic system of conservation laws (see [2], [3], [23]). In this paper we will develop a method to deal with such problems.

To solve the free boundary problem mentioned above we have to determine the free boundary as well as the solution defined in the regions with the free boundary as a part of their boundaries. The basic strategy which we design is to construct an iterative scheme including two main steps, one being to determine the location of the shock front by solving an ordinary differential equation derived from the Rankine-Hugoniot conditions. The other is to determine the solution in a given region with temporarily fixed boundary, while the domain takes the shock front as part of its temporarily fixed boundary, which may change in different stage of the iteration. Such a procedure is also applied by [4], [5] in solving a shock reflection problem for the UTSD equation. In order to solve the fixed boundary problem of the nonlinear hyperbolic-elliptic system, we transform the system into a canonical characteristic form, which decomposes the principle part of the system into the elliptic part and the hyperbolic part, while these two parts are weakly coupled in their coefficients. The nonlinear problem with such a canonical form can be solved by a Newton iteration scheme (see [15], [16], [20]). At each step in the iteration we have to solve a boundary value problem for a linear hyperbolic-elliptic composed system. Since the linearized system also contains an elliptic part and a hyperbolic part, we introduce an alternative iterative method, which offers a way to deal with these two parts separately. Then the typical method for elliptic equations or hyperbolic equations can be applied respectively, provided careful estimates are established to ensure the convergence of the alternative iteration scheme.

When the solution of the nonlinear hyperbolic-elliptic composed system in fixed domain is obtained, the modification of the approximate free boundary can be easily carried out, because in this step only a scalar ordinary differential equation is involved. Finally, combining the two main steps in our strategy we can obtain the existence of a unique solution to the original free boundary value problem by using Schauder's fixed point theorem. The final conclusion is stated as Theorem 2.1 in Section 2.

The result indicates the stability of the stationary shock front, provided the supersonic flow becomes a subsonic flow behind the shock front and an additional condition in some location downstream away from the shock front is added. In our theorem the additional condition takes the form that the pressure is given. It is more reasonable than the condition "velocity is given" from the physical viewpoint.

In this paper we only consider the stability of shock fronts in two-dimensional space. Due to the more complicated characteristics geometry we leave the discussion on the corresponding three-dimensional problem to the future. In our study (and also in [7]) the flow changes from supersonic to subsonic across a shock front, where a jump of corresponding physical parameters in the flow appears, which is why we call the shock front a transonic shock. In other circumstances, transonic flow was studied earlier when a flow field was changed continuously from supersonic to subsonic. This is also a very difficult subject and results in many mathematically challenging problems, because equations of mixed type are involved (see [1], [18]).

The paper is organized as follows. In Section 2 we give a mathematical formulation of the problem, which is a free boundary value problem for a hyperbolic-elliptic composed system. In this section we also describe the main results and the main idea of the approach that we take. In Section 3 the free boundary value problem is reduced to a fixed boundary value problem of this nonlinear hyperbolic-elliptic composed system and a problem for an ordinary differential equation. The latter is applied to update the free boundary. In Section 4 the nonlinear hyperbolic-elliptic system is transformed to a canonical form, which decomposes the hyperbolic part and the elliptic part in its principal part. Then both the nonlinear system and the boundary conditions are linearized. In Section 5 we solve the linearized boundary value problem of the hyperbolic-elliptic system and establish the required a priori estimates. In Section 6 we give the iteration scheme to solve the nonlinear boundary problem in a domain with fixed boundary. Then in the last section by using the Schauder fixed point theorem we finally prove the existence and uniqueness of the solution to the free boundary value problem, and complete our proof of the stability of the shock front under two-dimensional perturbation.

2. Formulation and main results

Let us consider a compressible flow in a two-dimensional tube with a constant section. The tube is placed parallel to the x-axis, and its section is $0 < y < b$. The velocity of the flow is assumed to be parallel to the wall of the tube. Meanwhile, we also assume that the unperturbed shock is perpendicular to the wall of the tube and located at $x = 0$. The unperturbed flow is separated by a plane shock front located at $x = 0$. The flow field ahead and behind the shock front are denoted by U_-^0 and U_+^0 respectively, where $U = (u, v, p)$ is the parameter of the flow. The state U_-^0 and U_+^0 satisfy $v_-^0 = v_+^0 = 0$, Rankine-Hugoniot conditions as shown later, and the entropy condition $u_-^0 > u_+^0$ or $p_-^0 < p_+^0$.

The stationary Euler system in two-dimensional space can be written as

$$\begin{cases} \nabla \mathbf{m} = 0, \\ \nabla \cdot \left(\dfrac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla p = 0, \end{cases} \tag{2.1}$$

with Bernoulli law

$$\frac{1}{2}\mathbf{u}^2 + i = const., \tag{2.2}$$

where ρ, p, i are the density, pressure, and enthalpy of the fluid, while $\mathbf{u} = (u, v)$ and $\mathbf{m} = \rho\mathbf{u}$ are velocity and momentum vector. The system (2.1) can also be written as

$$\begin{cases} \dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} = 0, \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = 0, \\ u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} = 0. \end{cases} \tag{2.3}$$

In the case of polytropic gas $p = A\rho^\gamma$, and (2.2) takes the form

$$\frac{1}{2}(u^2 + v^2) + \frac{a^2}{\gamma - 1} = const., \tag{2.4}$$

where a is the sonic speed of the fluid.

In the case when a shock front $x = \psi(y)$ appears in the flow field, the Rankine-Hugoniot conditions for the parameters U of the flow field in both sides should be satisfied. That is,

$$\begin{cases} [\rho u] = \psi'[\rho v], \\ [p + \rho u^2] = \psi'[\rho uv], \\ [\rho uv] = \psi'[p + \rho v^2], \end{cases} \tag{2.5}$$

where the bracket means the jump of the corresponding quantities. In addition, the entropy condition means that the density and the pressure will increase when a particle of the flow moves across the shock.

Assume that the upstream flow (ahead of the shock) is somehow perturbed. Then the location of the shock front and the downstream flow (behind the shock) will also be perturbed. The purpose of this paper is to prove such a conclusion: under some reasonable boundary conditions the disturbance of the shock front and the downstream flow is stable with respect to the perturbation of the upstream flow.

Consider (2.3) in the domain $\Omega = (-N_1, N_2)\times(0, b)$. Let us denote the location of the perturbed shock front by $x = \psi(y)$, and denote the perturbed upstream flow and the downstream flow by $U_-(x, y) = (u_-(x, y), v_-(x, y), p_-(x, y))$ and $U_+(x, y) = (u_+(x, y), v_+(x, y), p_+(x, y))$, respectively. Assume that the flow ahead of the shock is supersonic; then it has the property that the downstream part will never influence the upstream part [10, 11]. Therefore, we may assume that the perturbed flow field between $x = -N_1$ and $x = \psi(y)$ is given (the location of the shock front is unknown). The intersection of the shock front and the wall of the tube is assumed to be fixed, so that $\psi(0) = 0$. All boundary conditions on the boundaries of the domain $\Omega_+ = \{\psi(y) < x < N_2, 0 < y < b\}$ are

$$v = 0 \qquad \text{on} \quad y = 0 \quad \text{and} \quad y = b, \tag{2.6}$$

$$p = p_+^0 \qquad \text{on} \quad x = N_2, \tag{2.7}$$

$$R - H \text{ conditions and entropy condition} \qquad \text{on} \quad x = \psi(y). \tag{2.8}$$

Here condition (2.6) means that the wall $y = 0$ and $y = b$ of the tube are impermeable, and the condition (2.7) means that the pressure somewhere at the downstream flow keeps constant. This situation is mostly common in physics. Besides, since the unperturbed flow (U_-^0, U_+^0) satisfies the entropy condition on the shock front, then as its perturbation, $(U_-(x,y), U_+(x,y))$ will also satisfy the entropy condition on the shock front automatically.

To simplify our later discussions we will give an additional assumption on symmetry of the flow field. First let us introduce the following definition.

Definition 2.1. The flow field $U(x,y) = (u(x,y), v(x,y), p(x,y))$ is called properly symmetrical with respect to $y = \frac{b}{2}$ if $u(x,y) = u(x, b-y), v(x,y) = -v(x, b-y)$, $p(x,y) = p(x, b-y)$. Furthermore, if $\psi(y)$ is also symmetrical with respect to $y = \frac{b}{2}$, i.e. $\psi(y) = \psi(b-y)$, then the couple $(\psi(y), U(x,y))$ is called properly symmetrical.

A similar definition of properly symmetrical for $(\psi(y), U(x,y))$ with respect to $y = 0$ can also be given in this way.

Obviously, once we have proved the existence of the solution to the system (2.3) with the boundary conditions (2.6), (2.7), (2.8), and the $C^{2,\alpha}$ norm of the perturbation of U_+ is dominated by $C^{2,\alpha}$ norm of the perturbation of U_-, we are led to the desired conclusion on stability.

Now let us give a precise description of the conclusion in this paper. Since the shock front is unknown, for our convenience we assume that $U_-(x,y)$ is defined in the whole region Ω and properly symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$, and assume that it satisfies the system (2.3), the boundary condition (2.6) and the following estimate:

$$\|u_-(x,y) - u_-^0, v_-(x,y), p_-(x,y) - p_-^0\|_{C^{2,\alpha}(\Omega)} < \epsilon. \tag{2.9}$$

For our convenience we also use the notations

$$O_\epsilon = \{U_-(x,y) : U_-(x,y) \in C^{2,\alpha}(\Omega), \|U_-(x,y) - U_-^0\|_{C^{2,\alpha}(\Omega)} < \epsilon,$$
$$U_-(x,y) \text{ is properly symmetrical with respect to } y = \tfrac{b}{2} \text{ and } y = 0\}.$$

Then the main conclusion of this paper can be stated as follows.

Theorem 2.1. *For any perturbed upstream flow $U_-(x,y) \in O_\epsilon$ with sufficiently small $\epsilon > 0$, there exists a function $\psi(y)$ defined in $(0, b)$ and a triple of functions $U_+(x,y) = (u_+(x,y), v_+(x,y), p_+(x,y))$ defined in Ω_+, properly symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$, satisfying the system* (2.3) *and the boundary conditions* (2.6), (2.7), (2.8). *Moreover,*

$$\|\psi(y)\|_{C^{2,\alpha}(0,b)} < C\epsilon,$$
$$\|u_+(x,y) - u_+^0, v_+(x,y), p_+(x,y) - p_+^0\|_{C^{2,\alpha}(\Omega_+)} < C\epsilon. \tag{2.10}$$

for some constant C, independent of ϵ .

3. Reduction of the free boundary problem

The problem described in Theorem 2.1 is a free boundary value problem. To solve it we will reduce the free boundary value problem to a fixed boundary value problem determining the downstream flow field and a problem of an ordinary differential equation modifying the location of the shock front. The latter is a part of the boundary of the domain where the fixed boundary value problem of (2.3) is treated.

Such an outline is similar to that applied in the papers [5], [6], where the authors discussed the shock reflection problem for the UTSD equation.

Our conclusion in Theorem 2.1 will be finally established by means of Schauder fixed point theorem. To this end we define

$$K_\eta = \{\phi(y) : \phi(y) \in C^{2,\alpha}(0,b), \phi(0) = \phi'(0) = 0, \phi(y) = \phi(b-y), \|\phi\|_{C^{2,\alpha}} \le \eta\}.$$

For any η_0, K_{η_0} is a convex closed set. Taking any $\psi \in K_{\eta_0}$, we can solve a boundary value problem of (2.3) in the domain bounded by $\Gamma_1 : x = \psi(y),\ \Gamma_2 : y = 0,\ \Gamma_3 : y = b,\ \Gamma_4 : x = N_2$. Two boundary conditions on Γ_1 will be assigned, which come from the Rankine-Hugoniot conditions (2.5), while the boundary conditions on $\Gamma_2, \Gamma_3, \Gamma_4$ are given as shown in (2.6), (2.7). The wellposedness of such a fixed boundary value problem will be proved, and then the new value of the flow parameters U on the boundary $x = \psi(y)$ is obtained. By using the function U we update the location of the approximate free boundary. Since the Rankine-Hugoniot conditions essentially contains three relations as shown in (2.5), then besides two relations employed as the boundary conditions of the above boundary value problem, the remaining condition (or its equivalent form) will be used to update the location of the approximate free boundary. The suitable form of the remaining relation in the R-H conditions is an ordinary differential equation given on the interval $(0, b)$. Integrating this equation with a fixed initial value condition we obtain the location of the updated shock front $\Psi(y)$. Denoting the map from $\psi \in K_\eta$ to Ψ by T, we can prove that T is an inner map and is also a compact map. Therefore, the existence of a fixed point of the map T can be obtained according to the Schauder fixed point theorem. The fixed point gives the location of the shock front, which is the free boundary for our free boundary value problem. Hence the solution to the free boundary value problem is easily obtained.

The R-H conditions (2.5) are equivalent to the following three relations:

$$\frac{d\psi}{dy} = \frac{[\rho uv]}{[p + \rho v^2]},$$
$$\frac{[\rho uv]}{[\rho u]} = \frac{[p + \rho v^2]}{[\rho v]},$$
$$\frac{[p + \rho u^2]}{[\rho uv]} = \frac{[\rho uv]}{[p + \rho v^2]}.$$

The first equality will be applied to determine the location of the shock front together with the initial data $\psi(0) = 0$, which means that the intersection of the shock front with the wall is fixed. Therefore, the solution of the initial value problem

$$(I) \ : \ \begin{cases} \dfrac{d\psi}{dy} = \dfrac{[\rho uv]}{[p + \rho v^2]}, \\ \psi(0) = 0, \end{cases} \tag{3.1}$$

play the role to modify the location of the perturbed shock. The problem (3.1) is called problem (I).

On the other hand, by introducing two nonlinear functions from the R-H conditions

$$G_1(U_+, U_-) := [\rho uv][\rho v] - [p + \rho v^2][\rho u],$$
$$G_2(U_+, U_-) := [p + \rho u^2][p + \rho v^2] - [\rho uv]^2,$$

the boundary conditions on $x = \psi(y)$ can be written as

$$G_1(U, U_-) = 0, \quad G_2(U, U_-) = 0. \tag{3.2}$$

We notice that for the unperturbed flow (U_+^0, U_-^0), $G_i(U_+^0, U_-^0) = 0$ $(i = 1, 2)$.

Lemma 3.1. *For any $\psi(y) \in K_\eta$ and $U_-(x, y) \in O_\epsilon$ with $\eta \le \eta_0$, $\epsilon \le \epsilon_0$,*

$$\|G_i(U_+^0, U_-)\|_{C^{2,\alpha}(\Gamma_1)} \le C\epsilon, \tag{3.3}$$

where C only depends on η_0, ϵ_0.

Proof. Since $\psi(y) \in K_{\eta_0}$, $G_i(U_+^0, U_-^0) = 0$, we have $G_i(U_+^0, U_-) = G_i(U_+^0, U_-) - G_i(U_+^0, U_-^0) = (\frac{\partial G_i}{\partial U_-})^* \cdot (U_- - U_-^0)$. Hence (3.3) holds due to $U_-(x, y) \in O_\epsilon$.

For any given $\psi(y) \in K_{\eta_0}$, we can set up a boundary value problem (II) in Ω_+.

$$(II) : \begin{cases} \text{System} \quad (2.3) \quad \text{in} \quad \Omega_+, \\ G_1 = 0, \quad G_2 = 0 \quad \text{on} \quad \Gamma_1, \\ v = 0 \quad \text{on} \quad \Gamma_2, \Gamma_3, \\ p = p_+^0 \quad \text{on} \quad \Gamma_4. \end{cases} \tag{3.4}$$

This is a fixed boundary value problem for a nonlinear hyperbolic-elliptic composed system. In next sections we are going to prove

Theorem 3.1. *Assume that $\psi(y) \in K_\eta$, $U_-(x, y) \in O_\epsilon$ with $\epsilon \le \epsilon_0, \eta \le \eta_0$. Then there is a unique solution $U_+(x, y)$ in Ω_+ of the problem* (II)*, satisfying*

$$\|U_+(x, y) - U_+^0\|_{C^{2,\alpha}(\Omega_+)} < C_1\epsilon, \tag{3.5}$$

where C_1 only depends on ϵ_0, η_0, provided η_0 and ϵ_0 is sufficiently small.

4. Decomposing and linearizing

To discuss problem (II), we first introduce the canonical form of (2.3) by distinguishing its real characteristics and complex characteristics. First we write (2.3) as a symmetric system with the matrix form

$$\begin{pmatrix} \rho u & & 1 \\ & \rho u & \\ 1 & & a^{-2}\rho^{-1}u \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \\ p \end{pmatrix} + \begin{pmatrix} \rho v & & \\ & \rho v & 1 \\ & 1 & a^{-2}\rho^{-1}v \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0, \tag{4.1}$$

or

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0, \tag{4.2}$$

where the matrices A and B have their expressions as shown in (4.1).

From $det(B - \lambda A) = 0$ we can determine the eigenvalues λ, which are the roots of

$$det(B - \lambda A) \equiv \rho(v - \lambda u)(a^{-2}(v - \lambda u)^2 - 1 - \lambda^2) = 0. \tag{4.3}$$

The three eigenvalues are

$$\lambda_\pm = \frac{uv \pm a\sqrt{u^2 + v^2 - a^2}}{u^2 - a^2},$$
$$\lambda_3 = \frac{v}{u}.$$

In the subsonic region $u^2+v^2<a^2$ the eigenvalues $\lambda_\pm$ are complex. Write $\lambda_\pm = \lambda_R \pm i\lambda_I$. The real part and imaginary part of $\lambda_\pm$ are

$$\lambda_R = \frac{uv}{u^2-a^2}, \quad \lambda_I = \frac{a\sqrt{a^2-u^2-v^2}}{u^2-a^2}, \tag{4.4}$$

respectively. The left eigenvectors corresponding to $\lambda=\lambda_\pm$ are

$$\ell_\pm = (\lambda_\pm, -1, \rho(v-\lambda_\pm u)).$$

Multiplying the system (4.1) by $\ell=\ell_\pm$ we obtain

$$\ell A(\frac{\partial}{\partial x}+\lambda\frac{\partial}{\partial y})U=0. \tag{4.5}$$

Let us denote $D_R=\frac{\partial}{\partial x}+\lambda_R\frac{\partial}{\partial y}$, $D_I=\lambda_I\frac{\partial}{\partial y}$ and decompose the real part and imaginary part of (4.5). Noticing that

$$(\lambda_R + a^{-2}(uv-\lambda_R u^2)) = \lambda_R(1-\frac{u^2}{a^2})+\frac{uv}{a^2}=0,$$
$$(\lambda_I - \frac{u^2}{a^2}\lambda_I) = -\frac{1}{a}\sqrt{a^2-u^2-v^2},$$

we have

$$\rho v D_R u - \rho u D_R v + hD_I p = 0, \tag{4.6}$$
$$\rho v D_I u - \rho u D_I v - hD_R p = 0, \tag{4.7}$$

where $h=\frac{1}{a}\sqrt{a^2-u^2-v^2}$. On the other hand, the left eigenvector corresponding to λ_3 is

$$\ell_3=(u,v,0).$$

Denoting $D_3 = u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$, and multiplying (4.1) by ℓ_3 we obtain

$$\rho u D_3 u + \rho v D_3 v + D_3 p = 0. \tag{4.8}$$

The equations (4.6), (4.7), (4.8) are the canonical form for the system (4.2). In the new form of the system the hyperbolic part and the elliptic part have been decomposed in the level of principal part, and these two parts are only weakly coupled in their coefficients.

In order to solve problem (II) we have to make linearization of the nonlinear system and the nonlinear boundary conditions at any point in a $C^{2,\alpha}$ neighborhood of U_+^0, and then establish a convergent sequence of approximate solutions by solving the corresponding linear problem. To this end we define the δ-neighborhood Σ_δ of $U_+^0=(u_+^0,v_+^0,p_+^0)$ as

$$\Sigma_\delta = \{U_+(x,y) : U_+(x,y)\in C^{2,\alpha}(\Omega_+), \|U_+(x,y)-U_+^0\|_{C^{2,\alpha}(\Omega_+)} \le \delta,$$
$$U_+(x,y) \text{ is properly symmetrical with respect to } y=\tfrac{b}{2} \text{ and } y=0\}.$$

Now let us linearize problem (II) at $U_+(x,y)\in\Sigma_\delta$ with $\delta\le\delta_0$ for small δ_0. In the sequel we will often simply use U to denote the function $U_+(x,y)$. The linearized system for the perturbation δU is

$$D_R(\rho v\delta u - \rho u \delta v) + D_I(h\delta p) = f_1, \tag{4.9}$$
$$D_I(\rho v\delta u - \rho u \delta v) - D_R(h\delta p) = f_2, \tag{4.10}$$
$$\rho u D_3\delta u + \rho v D_3 \delta v + D_3\delta p = f_3, \tag{4.11}$$

where the terms f_1, f_2, f_3 in the right side are independent of δU, and will be given in the process of solving corresponding nonlinear problems. Meanwhile, the linearization of the boundary conditions (3.2) on Γ_1 are

$$\alpha_1 \delta u + \beta_1 \delta v + \gamma_1 \delta p = \tilde{g}_1, \tag{4.12}$$

$$\alpha_2 \delta u + \beta_2 \delta v + \gamma_2 \delta p = \tilde{g}_2, \tag{4.13}$$

where

$$\begin{gathered}
\alpha_1 = \rho v[\rho v] - \rho[p + \rho v^2], \\
\beta_1 = \rho[\rho uv] + \rho u[\rho v] - 2\rho v[\rho u], \\
\gamma_1 = \frac{uv}{a^2}[\rho v] + \frac{v}{a^2}[\rho uv] - (1 + \frac{v^2}{a^2})[\rho u] - \frac{u}{a^2}[p + \rho v^2], \\
\alpha_2 = 2\rho u[p + \rho v^2] - 2[\rho uv]\rho v, \\
\beta_2 = 2\rho v[p + \rho u^2] - 2[\rho uv]\rho u, \\
\gamma_2 = [p + \rho v^2](1 + \frac{u^2}{a^2}) + [p + \rho u^2](1 + \frac{v^2}{a^2}) - 2[\rho uv]\frac{uv}{a^2}.
\end{gathered}$$

Particularly, when $U_- = U_-^0, U_+ = U_+^0$, we have

$$\begin{gathered}
\alpha_1 = -\rho[p], \beta_1 = 0, \gamma_1 = -\frac{u}{a^2}[p], \\
\alpha_2 = 2[p]\rho u, \beta_2 = 0, \gamma_2 = [p](1 + \frac{u^2}{a^2}).
\end{gathered}$$

In the subsonic region

$$det\begin{pmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{pmatrix} = -[p]^2 det\begin{pmatrix} \rho & \frac{u}{a^2} \\ 2\rho u & 1 + \frac{u^2}{a^2} \end{pmatrix} = -[p]^2\rho(1 - \frac{u^2}{a^2}) < 0$$

at $U_- = U_-^0, U_+ = U_+^0$. Then for any $U \in \Sigma_\delta$ the boundary conditions (4.12), (4.13) on Γ_1 for the linearized problem can be written as

$$\delta u + \tilde{\beta}_1 \delta v = g_1, \tag{4.14}$$

$$\delta p + \tilde{\beta}_2 \delta v = g_2, \tag{4.15}$$

where $\tilde{\beta}_1 = \tilde{\beta}_2 = 0$ for $U_- = U_-^0, U = U_+^0$, and $\|\tilde{\beta}_1, \tilde{\beta}_2\|_{C^{2,\alpha}(\Gamma_1)} < C\delta$ due to $U \in \Sigma_\delta$. Meanwhile, since the boundary conditions on other boundaries are linear, we are led to the following linear problem:

$$(L): \begin{cases} \text{System} \quad (4.9), (4.10), (4.11) \qquad \text{in} \quad \Omega_+, \\ (4.14), (4.15) \qquad \text{on} \quad \Gamma_1, \\ \delta v = 0 \qquad \text{on} \quad \Gamma_2, \Gamma_3, \\ \delta p = 0 \qquad \text{on} \quad \Gamma_4. \end{cases} \tag{4.16}$$

Remark 4.1. Let us describe the symmetry of coefficients and the terms in the right side of the all equalities in (4.16). From their expressions we know that $h, \alpha_1, \gamma_1, \alpha_2, \gamma_2$ are symmetrical with respect to $y = \frac{b}{2}$, and $y = 0$, β_1, β_2 are anti-symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$. It implies that $\tilde{\beta}_1, \tilde{\beta}_2$ are anti-symmetrical. For the terms on the right side of the equalities in (4.16), the functions f_2, f_3, g_1, g_2 should be symmetrical, and the function f_1 should be anti-symmetrical. In the next sections, when we derive any linear problems with the form (L), the terms on the right side must have such symmetry.

Evidently, solving the boundary value problem (L) for the linear hyperbolic-elliptic composed system is a necessary step to solve problem (II).

5. The existence for linearized problem and energy estimates

In order to solve (L) we will first derive a second order elliptic equation from the first two equations of the linearized system. Since

$$[D_R, D_I] = (\frac{\partial \lambda_I}{\partial x} + \lambda_R \frac{\partial \lambda_I}{\partial y} - \lambda_I \frac{\partial \lambda_R}{\partial y})\frac{\partial}{\partial y},$$

we can write

$$[D_R, D_I] = \mu D_I,$$

where $\mu = \frac{1}{\lambda_I}(\frac{\partial \lambda_I}{\partial x} + \lambda_R \frac{\partial \lambda_I}{\partial y} - \lambda_I \frac{\partial \lambda_R}{\partial y})$. Since $U \in \Sigma_\delta$ and μ is a linear form of derivatives of U, we have $\|\mu\|_{C^{1,\alpha}} < C\delta$ with C independent of δ.

Acting D_I on (4.9) and acting D_R on (4.10), then subtracting one from another, we obtain

$$(D_I^2 + D_R^2)(h\delta p) - \mu D_R(h\delta p) = f_*, \tag{5.1}$$

where

$$f_* = \mu f_2 + D_I f_1 - D_R f_2.$$

Obviously, f_* is symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$, and it contains a linear expression of $\nabla(\delta U)$ with coefficients depending on the first derivatives of U, so that

$$\|f_*\|_{C^\alpha(\Omega_+)} \le C(\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)}).$$

The principal symbol of the equation (5.1) is $(\xi + \lambda_R \eta)^2 + \eta^2$, which is positive for $(\xi, \eta) \neq (0,0)$. Then the equation is a second order elliptic equation for δp. Its boundary conditions on Γ_1 and Γ_4 are

$$\delta p = 0 \qquad \text{on} \quad \Gamma_4, \tag{5.2}$$

$$\delta p = g_2 - \tilde{\beta}_2 \delta v \qquad \text{on} \quad \Gamma_1. \tag{5.3}$$

Furthermore, using (4.9) and the fact $v = 0, \delta v = 0, f_1 = 0$ on $\Gamma_{2,3}$, we know $D_I(h\delta p) = 0$ there. Hence the boundary value condition for δp on $\Gamma_{2,3}$ can be written as

$$\frac{\partial}{\partial y}\delta p = 0 \qquad \text{on} \quad \Gamma_2, \Gamma_3, \tag{5.4}$$

because of $\frac{\partial h}{\partial y} = 0$.

Lemma 5.1. *Assume that $U_- \in O_\epsilon, \psi \in K_\eta, U \in \Sigma_\delta$ with $\epsilon \le \epsilon_0, \eta \le \eta_0, \delta \le \delta_0$ sufficiently small, and $f_* \in C^\alpha(\Omega_+)$, $\delta v \in C^{2,\alpha}(\Gamma_1)$ are anti-symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$. Then the problem (5.1)–(5.4) admits a unique solution $\delta p \in C^{2,\alpha}(\Omega_+)$. Moreover, the estimate*

$$\|\delta p\|_{C^{2,\alpha}(\Omega_+)} \le C(\|g_2\|_{C^{2,\alpha}(\Gamma_1)} + \delta\|\delta v\|_{C^{2,\alpha}(\Gamma_1)} + \|f_*\|_{C^\alpha(\Omega_+)}) \tag{5.5}$$

holds, where C depends only on $\eta_0, \delta_0, \epsilon_0$.

Proof. The equation (5.1) is an elliptic equation defined on Ω_+. Since $\psi(y)$ is symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$, we can make an even extension for the function $\psi(y)$ with respect to $y = 0$ and $y = b$, i.e.

$$\psi(kb - y) = \psi(kb + y), \quad k = 0, \pm 1, \pm 2, \cdots.$$

Then $\psi(y)$ becomes a periodic function $\psi^e(y)$ defined on $(-\infty,\infty)$, which is still of $C^{2,\alpha}$. Correspondingly, the domain Ω_+ is also extended to Ω_+^e. Moreover, since $U_-(x,y)$, $U(x,y)$ are properly symmetrical, we can also make an even extension for the functions $u_-(x,y), u(x,y), p_-(x,y), p(x,y)$ with respect to $y=0,b$, and make an odd extension for the functions $v_-(x,y), v(x,y)$. In such a way, we obtain $U_-^e(x,y)$ defined on Ω^e and $U_+^e(x,y)$ defined on Ω_+^e.

Let us also make an even extension for δp with respect to the boundary. Since the coefficients of (5.1) are symmetrical with respect to the boundary and $\frac{\partial}{\partial y}\delta p=0$ on $\Gamma_{2,3}$, then the extension of δp is still C^2 smooth, and satisfies (5.1). Therefore, the problem for determining δp with the form (5.1)–(5.4) in Ω_+ is equivalent to a problem (L^e) in Ω^e which has the form (5.1), (5.2), (5.3) (the condition (5.4) is the consequence of the given symmetry). It turns out that in the process of solving problem (5.1)–(5.4), the corners of Ω_+ will not cause any trouble in our analysis. Therefore, the standard theory of the elliptic boundary value problems is available. The theory indicates that problem (5.1)–(5.4) admits a unique solution and the Schauder estimates hold. Noticing $\tilde\beta_2$ in (4.13) is a small quantity controlled by δ, we are led to (5.5).

We emphasize here that in the above proof (and also below) the constant C is uniform with respect to $U_\pm$, it only depends on $\epsilon_0,\delta_0,\eta_0$, and may change its value in different inequalities. Besides, we always assume $C>1$.

In the estimates (5.5) there appears $\|\delta v\|_{C^{2,\alpha}(\Gamma_1)}$ in its right-hand side. Therefore, in order to apply this estimate to solve (5.1) we have to eliminate the term $\|\delta v\|_{C^{2,\alpha}(\Gamma_1)}$.

Lemma 5.2. *Under the assumptions of Lemma* 5.1, *if* $\delta U=(\delta u,\delta v,\delta p)$ *is a solution of* (L), *then*

$$\|\rho v\delta u-\rho u\delta v\|_{C^{2,\alpha}(\Omega_+)}\le C(\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)}+\|\delta p\|_{C^{2,\alpha}(\Omega_+)}) \tag{5.6}$$

and

$$\|\delta u,\delta v\|_{C^{2,\alpha}(\Gamma_1)}\le C(\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)}+\|g_{1,2}\|_{C^{2,\alpha}(\Gamma_1)}+\delta\|\delta v\|_{C^{2,\alpha}(\Gamma_1)}). \tag{5.7}$$

Proof. From (4.9), (4.10) we have

$$D_R(\rho v\delta u-\rho u\delta v)=f_1-D_I(h\delta p), \tag{5.8}$$

$$D_I(\rho v\delta u-\rho u\delta v)=f_2+D_R(h\delta p). \tag{5.9}$$

Then

$$\begin{aligned}\|\rho v\delta u-\rho u\delta v\|_{C^{2,\alpha}(\Omega_+)}&\le C(\|D_R(\rho v\delta u-\rho u\delta v)\|_{C^{1,\alpha}(\Omega_+)}\\&\qquad+\|D_I(\rho v\delta u-\rho u\delta v)\|_{C^{1,\alpha}(\Omega_+)})\\&\le C(\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)}+\|\delta p\|_{C^{2,\alpha}(\Omega_+)}).\end{aligned}$$

Since $\rho v\delta u-\rho u\delta v$ and $\delta u+\tilde\beta_1\delta v$ are linearly independent, then δu, δv can be expressed by a linear combination of $\rho v\delta u-\rho u\delta v$ and $\delta u+\tilde\beta_1\delta v$. Hence

$$\|\delta u,\delta v\|_{C^{2,\alpha}(\Gamma_1)}\le C(\|g_1\|_{C^{2,\alpha}(\Gamma_1)}+\|\rho v\delta u-\rho u\delta v\|_{C^{2,\alpha}(\Gamma_1)})$$

due to (4.14). In view of (5.5), (5.6) and

$$\|f_*\|_{C^{\alpha}(\Omega_+)}\le C\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)},$$

we obtain (5.7).

Lemma 5.3. *Under the assumptions of Lemma* 5.1*, for any solution* $\delta U=(\delta u,\delta v,\delta p)$ *of* (L)*,*

$$\|\rho u\delta u+\rho v\delta v+\delta p\|_{C^{2,\alpha}(\Omega_+)} \tag{5.10}$$
$$\le C(\|g_{1,2}\|_{C^{2,\alpha}(\Gamma_1)}+\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)}+\|f_3\|_{C^{2,\alpha}(\Omega_+)}+\delta\|\delta U\|_{C^{2,\alpha}(\Omega_+)})$$

holds.

Proof. From (4.11)

$$D_3(\rho u\delta u+\rho v\delta v+\delta p)=f_3-D_3(\rho u)\delta u-D_3(\rho v)\delta v.$$

Since $\rho u\delta u+\rho v\delta v+\delta p, \rho v\delta u-\rho u\delta v, \delta p$ form a basis of $(\delta u,\delta v,\delta p)$, the equation (4.11) can be written as

$$D_3(\rho u\delta u+\rho v\delta v+\delta p)-d_1(\rho u\delta u+\rho v\delta v+\delta p)=d_2(\rho v\delta u-\rho u\delta v)+d_3\delta p+f_3,$$

where d_1,d_2,d_3 are linear forms of DU. When $U\in\Sigma_\delta$, $\|d_i\|_{C^{1,\alpha}(\Omega_+)}\le C$ holds uniformly for $i=1,2,3$.

Therefore, by integrating along the characteristics of D_3 we have

$$\|\rho u\delta u+\rho v\delta v+\delta p\|_{C^{1,\alpha}(\Omega_+)}\le\|\rho u\delta u+\rho v\delta v+\delta p\|_{C^{1,\alpha}(\Gamma_1)} \tag{5.11}$$
$$+C(\|f_3\|_{C^{1,\alpha}(\Omega_+)}+\|\rho v\delta u-\rho u\delta v\|_{C^{1,\alpha}(\Omega_+)}+\|\delta p\|_{C^{1,\alpha}(\Omega_+)}),$$

where C depends on the $C^{1,\alpha}$ norm of d_i, and then it only depends $\epsilon_0,\delta_0,\eta_0$ under the assumptions of Lemma 5.1.

Differentiating (4.11) with respect to x, we obtain

$$\rho uD_3(\delta u)_x+\rho vD_3(\delta v)_x+D_3(\delta p)_x=f_{3x}+q,$$

where $q=[\rho uD_3,D_x]\delta u+[\rho vD_3,D_x]\delta v+[D_3,D_x]\delta p$. Similar to (5.11) we can derive

(5.12)
$$\begin{aligned}\|\rho u(\delta u)_x+\rho v(\delta v)_x+(\delta p)_x\|_{C^{1,\alpha}(\Omega_+)}&\le\|\rho u(\delta u)_x+\rho v(\delta v)_x+(\delta p)_x\|_{C^{1,\alpha}(\Gamma_1)}\\&\quad+C(\|f_{3x}\|_{C^{1,\alpha}(\Omega_+)}+\|\rho v(\delta u)_x-\rho u(\delta v)_x\|_{C^{1,\alpha}(\Omega_+)}\\&\quad+\|(\delta p)_x\|_{C^{1,\alpha}(\Omega_+)}+\|q\|_{C^{1,\alpha}(\Omega_+)})\\&\le C(\|\delta U\|_{C^{2,\alpha}(\Gamma_1)}+\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)}+\|f_3\|_{C^{2,\alpha}(\Omega_+)}\\&\quad+\|g_{1,2}\|_{C^{2,\alpha}(\Gamma_1)}+\delta_0\|\delta U\|_{C^{2,\alpha}(\Omega_+)}),\end{aligned}$$

where C only depends on $\epsilon_0,\eta_0,\delta_0$. In the same way we can establish the estimate for $\|\rho u(\delta u)_y+\rho v(\delta v)_y+(\delta p)_y\|_{C^{1,\alpha}(\Omega_+)}$. Then by using the estimates (5.5), (5.7) we obtain (5.10) immediately.

Lemma 5.4. *Under the assumptions of Lemma* 5.1*, and assuming that* $f_1,f_2,f_3,$ g_1,g_2 *have the symmetry indicated in Remark* 4.1*,* $f_1,f_2\in C^{1,\alpha}(\Omega_+)$*,* $f_3\in C^{2,\alpha}(\Omega_+)$*,* $g_1,g_2\in C^{2,\alpha}(\Gamma_1)$*, then the problem* (L) *admits a unique solution* $\delta U\in C^{2,\alpha}(\Omega_+)$*, which satisfies the estimate*

$$\|\delta U\|_{C^{2,\alpha}(\Omega_+)}<C(\|g_{1,2}\|_{C^{2,\alpha}(\Gamma_1)}+\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)}+\|f_3\|_{C^{2,\alpha}(\Omega_+)}). \tag{5.13}$$

Proof. In order to prove the existence of the solution to problem (L), we introduce an alternative iteration scheme here. Let $\widetilde{\delta U}^{(0)}=0$; we construct a sequence of approximate solutions $\{\widetilde{\delta U}^{(k)}\}$ inductively. Assuming that $\widetilde{\delta U}^{(k)}$ has been obtained, we solve $\widetilde{\delta p}^{(k+1)}$ and $\rho v\widetilde{\delta u}^{(k+1)}-\rho u\widetilde{\delta v}^{(k+1)}$ from the elliptic part of the linearized system and solve $\rho u\widetilde{\delta u}^{(k+1)}+\rho v\widetilde{\delta v}^{(k+1)}+\widetilde{\delta p}^{(k+1)}$ from the hyperbolic part of the linearized system alternatively. The detailed procedure is shown as follows. First, $\widetilde{\delta p}^{(k+1)}$ is defined by

$$(5.14)\qquad \begin{cases} (D_R^2+D_I^2)(h\widetilde{\delta p}^{(k+1)}) - \mu D_R(h\widetilde{\delta p}^{(k+1)}) = f_* \quad \text{in} \quad \Omega_+, \\ \widetilde{\delta p}^{(k+1)} = g_2 - \tilde{\beta}_2\widetilde{\delta v}^{(k)} \quad \text{on} \quad \Gamma_1, \\ \dfrac{\partial}{\partial y}\widetilde{\delta p}^{(k+1)} = 0 \quad \text{on} \quad \Gamma_2, \Gamma_3, \\ \widetilde{\delta p}^{(k+1)} = 0 \quad \text{on} \quad \Gamma_4, \end{cases}$$

where $f_* = D_I f_1 - D_R f_2 + \mu f_2$. Obviously, (5.14) admits a unique solution, which is properly symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$, and satisfies

$$(5.15)\quad \begin{aligned} \|\widetilde{\delta p}^{(k+1)}\|_{C^{2,\alpha}(\Omega_+)} &\le C(\|g_2\|_{C^{2,\alpha}(\Gamma_1)} + \delta\|\widetilde{\delta v}^{(k)}\|_{C^{2,\alpha}(\Gamma_1)} + \|f_*\|_{C^{\alpha}(\Omega_+)}) \\ &\le C(\|g_2\|_{C^{2,\alpha}(\Gamma_1)} + \delta\|\widetilde{\delta v}^{(k)}\|_{C^{2,\alpha}(\Gamma_1)} + \|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)}). \end{aligned}$$

From (5.14), we have

$$\begin{aligned} D_I(f_1 - D_I(h\widetilde{\delta p}^{(k+1)})) - D_R(f_2 + D_R(h\widetilde{\delta p}^{(k+1)})) + \mu(f_2 + D_R(h\widetilde{\delta p}^{(k+1)})) \\ = D_I f_1 - D_R f_2 + \mu f_2 - f_* = 0. \end{aligned}$$

These are exactly the integrability conditions of the system of first order equations

$$(5.16)\qquad \begin{cases} D_R W = f_1 - D_I(h\widetilde{\delta p}^{(k+1)}), \\ D_I W = f_2 + D_R(h\widetilde{\delta p}^{(k+1)}). \end{cases}$$

Hence we can determine a solution W to (5.16), which is anti-symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$ and satisfies $W = 0$ at $(x, y) = (0, 0)$. Let $\rho v\widetilde{\delta u}^{(k+1)} - \rho u\widetilde{\delta v}^{(k+1)} = W$. Then we have

$$(5.17)\qquad D_R(\rho v\widetilde{\delta u}^{(k+1)} - \rho u\widetilde{\delta v}^{(k+1)}) + D_I(h\widetilde{\delta p}^{(k+1)}) = f_1,$$

$$(5.18)\qquad D_I(\rho v\widetilde{\delta u}^{(k+1)} - \rho u\widetilde{\delta v}^{(k+1)}) - D_R(h\widetilde{\delta p}^{(k+1)}) = f_2.$$

Meanwhile, Lemma 5.2 implies

$$(5.19)\quad \|\rho v\widetilde{\delta u}^{(k+1)} - \rho u\widetilde{\delta v}^{(k+1)}\|_{C^{2,\alpha}(\Omega_+)} \le C(\|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)} + \|\widetilde{\delta p}^{(k+1)}\|_{C^{2,\alpha}(\Omega_+)}).$$

Combining (5.19), (5.15) with $\widetilde{\delta u}^{(k+1)} + \tilde{\beta}_1\widetilde{\delta v}^{(k+1)} = g_1$ on Γ_1, we have

$$(5.20)\quad \|\widetilde{\delta U}^{(k+1)}\|_{C^{2,\alpha}(\Gamma_1)} \le C(\|g_{1,2}\|_{C^{2,\alpha}(\Gamma_1)} + \|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)} + \delta\|\widetilde{\delta U}^{(k)}\|_{C^{2,\alpha}(\Omega_+)}).$$

Then we solve $\rho u\widetilde{\delta u}^{(k+1)} + \rho v\widetilde{\delta v}^{(k+1)} + \widetilde{\delta p}^{(k+1)}$ from

$$\rho u D_3\widetilde{\delta u}^{(k+1)} + \rho v D_3\widetilde{\delta v}^{(k+1)} + D_3\widetilde{\delta p}^{(k+1)} = f_3.$$

Similar to Lemma 5.3 the above equation can be written as

$$\begin{aligned} (D_3 - d_1)(\rho u\widetilde{\delta u}^{(k+1)} + \rho v\widetilde{\delta v}^{(k+1)} + \widetilde{\delta p}^{(k+1)}) \\ = d_2(\rho v\widetilde{\delta u}^{(k+1)} - \rho u\widetilde{\rho v}^{(k+1)}) + d_3\widetilde{\delta p}^{(k+1)} + f_3, \end{aligned}$$

where d_1, d_2, d_3 are linear forms of DU as introduced in Lemma 5.3. Integrating along the characteristics of D_3 and using the method in Lemma 5.3 we obtain $\rho u\widetilde{\delta u}^{(k+1)} + \rho v\widetilde{\delta v}^{(k+1)} + \widetilde{\delta p}^{(k+1)}$, which satisfies

$$(5.21)\qquad \begin{aligned} &\|\rho u\widetilde{\delta u}^{(k+1)} + \rho v\widetilde{\delta v}^{(k+1)} + \widetilde{\delta p}^{(k+1)}\|_{C^{2,\alpha}(\Omega_+)} \\ &\le C(\|\widetilde{\delta U}^{(k+1)}\|_{C^{2,\alpha}(\Gamma_1)} + \|g_{1,2}\|_{C^{2,\alpha}(\Gamma_1)} + \|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)} + \|f_3\|_{C^{2,\alpha}(\Omega_+)} \\ &\quad + \delta\|\widetilde{\delta U}^{(k)}\|_{C^{2,\alpha}(\Omega_+)} + \delta\|\widetilde{\delta U}^{(k+1)}\|_{C^{2,\alpha}(\Omega_+)}), \end{aligned}$$

where C only depends on $\epsilon_0, \eta_0, \delta_0$. By taking δ sufficiently small we have

$$(5.22)\qquad \begin{aligned} \|\widetilde{\delta U}^{(k+1)}\|_{C^{2,\alpha}(\Omega_+)} \le & C(\|g_{1,2}\|_{C^{2,\alpha}(\Gamma_1)} + \|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)} + \|f_3\|_{C^{2,\alpha}(\Omega_+)}) \\ & + \frac{1}{2}\|\widetilde{\delta U}^{(k)}\|_{C^{2,\alpha}(\Omega_+)}. \end{aligned}$$

By changing the constant C we are led to the uniform boundedness of $\widetilde{\delta U}^{(k+1)}$, i.e.

$$(5.23)\qquad \|\widetilde{\delta U}^{(k+1)}\|_{C^{2,\alpha}(\Omega_+)} \le C(\|g_{1,2}\|_{C^{2,\alpha}(\Gamma_1)} + \|f_{1,2}\|_{C^{1,\alpha}(\Omega_+)} + \|f_3\|_{C^{2,\alpha}(\Omega_+)}).$$

Due to the compactness of $C^{2,\alpha}(\Omega_+)$ in $C^2(\Omega_+)$, we can select a subsequence from the sequence $\{\widetilde{\delta U}^{(k)}\}$, so that it is convergent in $C^2(\Omega_+)$. It is easy to see that the limit δU of the subsequence satisfies all equations and the boundary conditions in problem (L) and $\delta U \in C^{2,\alpha}(\Omega_+)$. In view of the uniform boundedness of $\widetilde{\delta U}^{(k)}$ in $C^{2,\alpha}(\Omega_+)$, we obtain the estimate (5.13).

In order to prove the uniqueness we only need to indicate that the solution to (4.16) with homogeneous right side terms is zero. That is, assuming that $\delta u, \delta v, \delta p$ satisfies

$$(5.24)\qquad \begin{cases} D_R(\rho v\delta u - \rho u\delta v) + D_I(h\delta p) = 0, \\ D_I(\rho v\delta u - \rho u\delta v) - D_R(h\delta p) = 0, \\ \rho u D_3\delta u + \rho v D_3\delta v + \delta p = 0, \end{cases}$$

$$\begin{aligned} \delta u + \tilde\beta_1\delta v = 0 \quad &\text{on} \quad \Gamma_1, \\ \delta p + \tilde\beta_2\delta v = 0 \quad &\text{on} \quad \Gamma_1, \\ \delta v = 0 \quad &\text{on} \quad \Gamma_2, \Gamma_3, \\ \delta p = 0 \quad &\text{on} \quad \Gamma_4, \end{aligned}$$

we have to prove $\delta u = \delta v = \delta p = 0$. Indeed, δp satisfies

$$(5.25)\qquad \begin{cases} (D_R^2 + D_I^2)(h\delta p) - \mu D_R(h\delta p) = 0 \quad \text{in} \quad \Omega_+, \\ \delta p = -\tilde\beta_2\delta v \quad \text{on} \quad \Gamma_1, \\ \dfrac{\partial}{\partial y}\delta p = 0 \quad \text{on} \quad \Gamma_2, \Gamma_3, \\ \delta p = 0 \quad \text{on} \quad \Gamma_4. \end{cases}$$

Then

$$(5.26)\qquad \|h\delta p\|_{C^{2,\alpha}(\Omega_+)} \le C\|\tilde\beta_2\delta v\|_{C^{2,\alpha}(\Gamma_1)} \le C\delta\|\delta v\|_{C^{2,\alpha}(\Gamma_1)}.$$

Meanwhile, denote by D_t the differential operator along the tangential direction of Γ_1, we have

$$\|\delta v\|_{C^{2,\alpha}(\Gamma_1)} = \|D_t \delta v\|_{C^{1,\alpha}(\Gamma_1)} \tag{5.27}$$
$$= \|\frac{1}{\rho u}[D_t(\rho v\delta u - \rho u \delta v) + D_t(\rho u)\delta v - D_t(\rho v)\delta u - \rho v D_t(\delta u)]\|_{C^{1,\alpha}(\Gamma_1)}.$$

By using the system (5.24), $D_t(\rho v\delta u - \rho u\delta v)$ can be written as a linear combination of $D_R\delta p$ and $D_I\delta p$. Then, in view of $U \in \Sigma_\delta$ and the boundary condition $\delta u + \tilde\beta_1\delta v = 0$ on Γ_1, we have

$$\|\delta v\|_{C^{2,\alpha}(\Gamma_1)} \le C(\|D(h\delta p)\|_{C^{1,\alpha}(\Gamma_1)} + \delta\|\delta v\|_{C^{2,\alpha}(\Gamma_1)}).$$

Applying (5.26) we have

$$\|\delta v\|_{C^{2,\alpha}(\Gamma_1)} \le C\delta\|\delta v\|_{C^{2,\alpha}(\Gamma_1)}$$

with another constant C. Since $\delta \le \delta_0$ can be sufficiently small, we are led to $\delta v = 0$ on Γ, and consequently, $\delta u = \delta p = 0$ on Γ_1. Therefore, the solution δp of (5.25) is zero in Ω_+, and then $\delta u = \delta v = 0$ in Ω_+ can also be obtained easily.

Remark 5.1. The a priori estimate (5.13) indicate that the linearized problems are stable. Moreover, we emphasize once more that all constants in the above estimates are uniform for $U \in \Sigma_{\delta_0}$.

6. The existence for nonlinear problem (II)

Returning to nonlinear problem (II), we are going to prove Theorem 3.1. To this end we use the Newton iteration scheme to establish a sequence of approximate solutions. The approximate equations in the scheme are

$$\begin{aligned}
&D_R^{(n)}(\rho^{(n)}v^{(n)}\delta u^{(n+1)} - \rho^{(n)}u^{(n)}\delta v^{(n+1)}) + D_I^{(n)}(h^{(n)}\delta p^{(n+1)})\\
&= \quad D_R^{(n)}(\rho^{(n)}v^{(n)})\delta u^{(n)} - D_R^{(n)}(\rho^{(n)}u^{(n)})\delta v^{(n)} + D_I^{(n)}(h^{(n)})\delta p^{(n)},\\
&D_I^{(n)}(\rho^{(n)}v^{(n)}\delta u^{(n+1)} - \rho^{(n)}u^{(n)}\delta v^{(n+1)}) - D_R^{(n)}(h^{(n)}\delta p^{(n+1)})\\
&= \quad D_I^{(n)}(\rho^{(n)}v^{(n)})\delta u^{(n)} - D_I^{(n)}(\rho^{(n)}u^{(n)})\delta v^{(n)} - D_R^{(n)}(h^{(n)})\delta p^{(n)},\\
&\rho^{(n)}u^{(n)}D_3^{(n)}\delta u^{(n+1)} + \rho^{(n)}v^{(n)}D_3^{(n)}\delta v^{(n+1)} + D_3^{(n)}\delta p^{(n+1)} = 0,
\end{aligned}$$

where $D_R^{(n)}, D_I^{(n)}, D_3^{(n)}$ are the corresponding operators obtained from D_R, D_I, D_3 by replacing U in their coefficients by $U^{(n)}$. The above system can be simply denoted as

$$\mathbf{F}^{(n)}\delta U^{(n+1)} = \mathbf{f}^{(n)} \tag{6.1}$$

with $\mathbf{f} = (f_1, f_2, f_3)$.

The approximate boundary conditions on Γ_1 are

$$\alpha_i^{(n)}\delta u^{(n+1)} + \beta_i^{(n)}\delta v^{(n+1)} + \gamma_i^{(n)}\delta p^{(n+1)}$$
$$= -G_i(U^n, U_-) + \alpha_i^{(n)}\delta u^{(n)} + \beta_i^{(n)}\delta v^{(n)} + \gamma_i^{(n)}\delta p^{(n)},$$

where $\alpha_i^{(n)}, \beta_i^{(n)}, \gamma_i^{(n)}$ are the corresponding functions obtained by replacing U in $\alpha_i, \beta_i, \gamma_i$ by $U^{(n)}$. The linear operator with these coefficients is denoted by $\mathbf{G}_i^{(n)}$; then we have

$$\mathbf{G}_i^{(n)}\delta U^{(n+1)} = -G_i^{(n)} + \mathbf{G}_\mathbf{i}^{(\mathbf{n})}\delta U^{(n)} \quad (= g_i^{(n)}). \tag{6.2}$$

Therefore, the iteration scheme can be given as follows:

$$U^{(0)} = U^0_+, \quad \delta U^{(0)} = 0.$$

For any n, when $U^{(n)}$ has been given, $\delta U^{(n+1)}$ is the solution to the following problem:

$$(6.3) \qquad (L^{(n)}) : \begin{cases} \mathbf{F}^{(n)} \delta U^{(n+1)} = \mathbf{f}^{(n)} \quad \text{in} \quad \Omega_+, \\ \mathbf{G}_i^{(n)} \delta U^{(n+1)} = -G_i^{(n)} + \mathbf{G}_i^{(n)} \delta U^{(n)} \quad \text{on} \quad \Gamma_1, \quad i = 1, 2, \\ \delta v^{(n+1)} = 0 \quad \text{on} \quad \Gamma_2, \Gamma_3, \\ \delta p^{(n+1)} = 0 \quad \text{on} \quad \Gamma_4, \end{cases}$$

which has the form of (L), and $f_i^{(n)}, g_i^{(n)}$ are symmetry as indicated in Remark 4.1. Furthermore, we take

$$(6.4) \qquad U^{(n+1)} = U^{(0)} + \delta U^{(n+1)}.$$

Lemma 6.1. *Under the assumptions of Theorem* 3.1 *one can construct a sequence* $\{U^{(n)}\}$ *of approximate solutions to problem* (II) *by using the scheme* (6.3), (6.4), *and the solution* $\{U^{(n)}\}$ *is properly symmetrical with respect to* $y = \frac{b}{2}$ *and* $y = 0$, *uniformly bounded in* $C^{2,\alpha}(\Omega_+)$. *Moreover, the sequence is convergent.*

Proof. Let us start with the initial term $U^{(0)} = U^0_+$. For any n the problem (6.3) admits a solution $\delta U^{(n+1)}$, if $U^{(n)} \in \Sigma_\delta$ with $\delta \le \delta_0$. Indeed, for the right side of (6.1), (6.2), we have

$$\|{f_1}^{(n)}, f_2^{(n)}\|_{C^{1,\alpha}(\Omega_+)} \le C\delta \|\delta U^{(n)}\|_{C^{2,\alpha}(\Omega_+)}, \quad f_3^{(n)} = 0.$$

Moreover,

$$g_i^{(n)} = -G_i^{(0)} + (G_i^{(0)} - G_i^{(n)} + \mathbf{G}_i^{(n)} \delta U^{(n)}).$$

Lemma 3.1 indicates $\|G_i^{(0)}\|_{C^{2,\alpha}(\Gamma_1)} \le C\epsilon$. Meanwhile

$$\|G_i^{(0)} - G_i^{(n)} + \mathbf{G}_i^{(n)} \delta U^{(n)}\|_{C^{2,\alpha}(\Gamma_1)} \le C \|\delta U^{(n)}\|^2_{C^{2,\alpha}(\Gamma_1)}.$$

Then from Lemma 5.4 we know that problem $(L^{(n)})$ has a unique solution $\delta U^{(n+1)}$, which satisfies

$$(6.5) \qquad \begin{aligned} \|\delta U^{(n+1)}\|_{C^{2,\alpha}(\Omega_+)} &\le C(\|g_{1,2}^{(n)}\|_{C^{2,\alpha}(\Gamma_1)} + \|f_{1,2}^{(n)}\|_{C^{1,\alpha}(\Omega_+)}) \\ &\le C(\epsilon + \delta \|\delta U^{(n)}\| + \|\delta U^{(n)}\|^2_{C^{2,\alpha}(\Omega_+)}), \end{aligned}$$

where the constant C only depends on $\delta_0, \epsilon_0, \eta_0$. From (6.5) one can easily obtain the estimate $\|\delta U^{(n)}\|_{C^{2,\alpha}(\Omega_+)} < \delta$ inductively, provided $\delta < \dfrac{1}{4C}, \epsilon < \dfrac{\delta}{2C}$, and $\epsilon_0, \eta_0, \delta_0$ are sufficiently small.

Consider the convergence of the sequence $\{U^n\}$. (6.3) implies

(6.6)

$$\mathbf{F}^{(n+1)}(U^{(n+2)} - U^{(n+1)}) = (\mathbf{f}^{(n+1)} - \mathbf{f}^{(n)}) - (\mathbf{F}^{(n+1)} - \mathbf{F}^{(n)})U^{(n+1)} \quad \text{in} \quad \Omega_+,$$

(6.7)

$$\mathbf{G}_i^{(n+1)}(U^{(n+2)} - U^{(n+1)}) = -G_i^{(n+1)} + G_i^{(n)} + \mathbf{G}_i^{(n)}(U^{(n+1)} - U^{(n)}) \quad \text{on} \quad \Gamma_i,$$

$$(6.8) \qquad U^{(n+2)} - U^{(n+1)} = 0 \quad \text{on} \quad \Gamma_2, \Gamma_3,$$

$$(6.9) \qquad p^{(n+2)} - p^{(n+1)} = 0 \quad \text{on} \quad \Gamma_4.$$

From Lemma 5.4 we have

$$
\begin{aligned}
(6.10)\qquad & \|U^{(n+2)} - U^{(n+1)}\|_{C^{2,\alpha}(\Omega_+)} \\
& \le C\ (\sum_{i=1,2} (\| - G_i^{(n+1)} + G_i^{(n)} + \mathbf{G}_i^{(n)}(U^{(n+1)} - U^{(n)})\|_{C^{2,\alpha}(\Gamma_1)} \\
& \quad + \|(\mathbf{F}^{(n+1)} - \mathbf{F}^{(n)})U^{(n+1)}\|_{C^{1,\alpha}(\Omega_+)} + \|\mathbf{f}^{(n+1)} - \mathbf{f}^{(n)}\|_{C^{1,\alpha}(\Omega_+)}).
\end{aligned}
$$

Notice that as both $U^{(n+1)}$ and $U^{(n)}$ are in Σ_δ, we have

$$\|\mathbf{f}^{(n+1)} - \mathbf{f}^{(n)}\|_{C^{1,\alpha}(\Omega_+)} \le C\delta\|U^{(n+1)} - U^{(n)}\|_{C^{2,\alpha}(\Omega_+)}.$$

Since any term in the coefficients of the operator $\mathbf{F}^{(n+1)} - \mathbf{F}^{(n)}$ contains the components of $(u^{(n+1)} - u^{(n)}, v^{(n+1)} - v^{(n)}, p^{(n+1)} - p^{(n)})$, then

$$\|(\mathbf{F}^{(n+1)} - \mathbf{F}^{(n)})U^{(n+1)}\|_{C^{1,\alpha}(\Omega_+)} \le C\delta\|U^{(n+1)} - U^{(n)}\|_{C^{2,\alpha}(\Omega_+)}.$$

Furthermore,

$$\| - G_i^{(n+1)} + G_i^{(n)} + \mathbf{G_i}^{(n)}(U^{(n+1)} - U^{(n)})\|_{C^{2,\alpha}(\Gamma_1)} \le C\|U^{n+1)} - U^{(n)}\|^2_{C^{2,\alpha}(\Gamma_1)}.$$

Therefore, when δ_0 is sufficiently small, we can inductively establish the estimate

$$\|U^{(n+2)} - U^{(n+1)}\|_{C^{2,\alpha}(\Omega_+)} \le \frac{1}{2}\|U^{(n+1)} - U^{(n)}\|_{C^{2,\alpha}(\Omega_+)},$$

which implies the convergence of $\{U^{(n)}\}$.

Proof of Theorem 3.1. As shown in Lemma 6.1 we have established a convergent sequence $\{U^{(n)}\}$ of approximate solutions. It is easy to see that the limit U of the sequence $\{U^{(n)}\}$ satisfies (4.9)–(4.11), and the boundary condition

$$G_i(U, U_-) = 0 \qquad \text{on} \quad \Gamma_1$$

as well as $v = 0$ on $\Gamma_{2,3}$ and $p = p_0^+$ on Γ_4. This means that U is the solution to problem (II).

Denote $\bar{U} = U - U_+^0$. Then $\bar{U} = (\bar{u}, \bar{v}, \bar{p})$ satisfies

$$
\begin{aligned}
D_R(\rho v\bar{u} - \rho u\bar{v}) + D_I(h\bar{p}) &= \overline{f_1}, \\
D_I(\rho v\bar{u} - \rho u\bar{v}) - D_R(h\bar{p}) &= \overline{f_2},
\end{aligned}
$$

where

$$\overline{f_1} = D_R(\rho v)\bar{u} - D_R(\rho u)\bar{v} + (D_I u)\bar{p}, \quad \overline{f_2} = D_I(\rho v)\bar{u} - D_I(\rho u)\bar{v} - (D_I h)\bar{p},$$

satisfying

$$\|\overline{f_{1,2}}\|_{C^{1,\alpha}(\Omega_+)} \le C\delta\|\bar{U}\|_{C^{2,\alpha}(\Omega_+)}.$$

Furthermore, from $G_i(U, U_-) = 0,\ G_i(U_+^0, U_-^0) = 0$ we obtain

$$
\begin{aligned}
0 &= G_i(U, U_-) - G_i(U_+^0, U_-^0) \\
&= G_i(U, U_-) - G_i(U_+^0, U_-) + G_i(U_+^0, U_-) - G_i(U_+^0, U_-^0) \\
&= \alpha_i^*\bar{u} + \beta_i^*\bar{v} + \gamma_i^*\bar{p} + (\tfrac{\partial G_i}{\partial U_-})^*(U_- U_-^0),
\end{aligned}
$$

where $\alpha_i^*, \beta_i^*, \gamma_i^*$ have the expressions as shown in Section 4, and take value at some point on the segment connecting U and U_+^0 in the space (u, v, p). In addition,

$$\|(\frac{\partial G_i}{\partial U_-})^*(U_- - U_-^0)\|_{C^{2,\alpha}(\Gamma_1)} \le C\epsilon.$$

Hence we have

$$\bar{u} + \tilde{\beta}_1^* \bar{v} = \bar{g}_1, \quad \bar{p} + \tilde{\beta}_2^* \bar{v} = \bar{g}_2 \quad \text{on} \quad \Gamma_1,$$

where

$$\|\bar{g}_1, \bar{g}_2\|_{C^{2,\alpha}(\Gamma_1)} \leq C\epsilon.$$

Notice that $\bar{f}_1, \bar{f}_2, \bar{g}_1, \bar{g}_2$ have the symmetry indicated in Remark 4.1. Then by using Lemma 5.4 we obtain

$$\begin{aligned} \|\bar{U}\|_{C^{2,\alpha}(\Omega_+)} &\leq C(\|\bar{g}_{1,2}\|_{C^{2,\alpha}(\Gamma_1)} + \|\bar{f}_{1,2}\|_{C^{1,\alpha}(\Omega_+)}) \\ &\leq C(\epsilon + \delta \|\bar{U}\|_{C^{2,\alpha}(\Omega_+)}). \end{aligned} \tag{6.11}$$

When δ is small, we obtain (3.5).

7. The existence for the free boundary value problem

To solve the original free boundary problem, we turn to problem (I).

Lemma 7.1. *For given $\psi(y) \in K_{\eta_0}$ and $U(x,y) \in C^{2,\alpha}(\Omega_+)$, there is a unique solution $\Psi(y)$, symmetrical with respect to $y = \frac{b}{2}$ and $y = 0$, satisfying*

$$\begin{cases} \dfrac{d\Psi}{dy} = \dfrac{\rho uv - \rho_- u_- v_-}{p + \rho v^2 - p_- - \rho_- v_-^2}, \\ \Psi(0) = 0. \end{cases} \tag{7.1}$$

Moreover, $\Psi(y)$ satisfies

$$\|\Psi(y)\|_{C^{3,\alpha}(0,b)} \leq C\epsilon. \tag{7.2}$$

Proof. First we extend $U(x,y)$ from Ω_+ to the whole Ω by a fixed manner, such that the extension (still denoted by $U(x,y)$) is a $C^{2,\alpha}(\Omega)$ function satisfying

$$\|U(x,y)\|_{C^{2,\alpha}(\Omega)} \leq 2\|U(x,y)\|_{C^{2,\alpha}(\Omega_+)}.$$

Then by solving problem (7.1) we obtain $\Psi(y)$. Here what we have to check is the symmetry of $\Psi(y)$ with respect to $y = \frac{b}{2}$ and $\Psi'(0) = 0$. Since $\psi(y)$ and $U_-(x,y)$ are properly symmetrical with respect to $y = \frac{b}{2}$, then the solution to problem (L) is properly symmetrical, because f_1 is anti-symmetrical and f_2, f_3, g_1, g_2 are symmetrical. The solution to problem (II) obtained by using the Newton iteration scheme is also properly symmetrical with respect to $y = \frac{b}{2}$. Therefore, the right-hand side of (7.1) is anti-symmetrical. This implies that the solution $\Psi(y)$ is symmetrical with respect to $y = \frac{b}{2}$. Correspondingly, $\Psi'(0) = 0$ holds due to $v(x,0) = v_-(x,0) = 0$.

To prove (7.2) we write the right side of (7.1) as $H(U, U_-)$. In view of $H(U_+^0, U_-^0) = 0$, we have

$$H(U, U_-) = H(U, U_-) - H(U, U_-^0) + H(U, U_-^0) - H(U_+^0, U_-^0).$$

Therefore,

$$\|\Psi(y)\|_{C^{3,\alpha}(0,b)} \leq C(\|U - U_+^0\|_{C^{2,\alpha}(\Omega)} + \|U - U_-^0\|_{C^{2,\alpha}(\Omega)})$$

because of (7.1). Then Theorem 3.1 implies (7.2) with possibly different constant C.

The proof of Theorem 2.1. Having established the existence of the solution to problems (I), (II) and the corresponding estimates, we can now prove the existence of the solution to the original free boundary problem. For given $U_-(x,y) \in O_\epsilon$ with $\epsilon \leq \epsilon_0$, by taking $\psi(y) \in K_{\eta_0}$, we solve problem (II) in Ω_+ to obtain $U(x,y) \in C^{2,\alpha}(\Omega_+)$.

Then by solving (7.1) we obtain $\Psi(y) \in C^{3,\alpha}(0,b)$ and $\|\Psi(y)\|_{C^{3,\alpha}(0,b)} \leq C\epsilon$. Therefore, when ϵ is small enough, the map T from $\psi \in K_{\eta_0}$ to Ψ is an inner and compact map. Since K_{η_0} is a convex and closed set, then the map T has a fixed point $\psi^*(y)$ by Schauder's fixed point theorem. Correspondingly, we obtain a domain $\Omega^* = \{\psi^*(y) < x < N_2, 0 < y < b\}$ and a solution $U^*(x,y)$ to (II) defined in Ω^*. The couple $(\psi^*(y), U^*(x,y))$ is exactly the solution of our original free boundary problem, satisfying

$$\|\psi^*(y)\|_{C^{3,\alpha}(0,b)} \leq C\epsilon_0, \|U^*(x,y) - U_+^0\|_{C^{2,\alpha}(\Omega_+)} \leq C\epsilon_0,$$

because of (7.2) and (3.5).

The remaining part is to prove uniqueness. Suppose that the free boundary value problem (2.3), (2.6), (2.7), (2.8) has two solutions. The corresponding free boundaries are $\Gamma_{1A} : x = \psi_A(y)$ and $\Gamma_{1B} : x = \psi_B(y)$, respectively, while the solution $U_A(x,y)$ is defined in $\Omega_{+A} = \{\psi_A(y) < x < N_2, 0 < y < b\}$, and the solution $U_B(x,y)$ is defined in $\Omega_{+B} = \{\psi_B(y) < x < N_2, 0 < y < b\}$. Since $U_A(x,y)$ and $U_B(x,y)$ satisfy (2.3) and the requirement in Theorem 2.1, we have to prove $\psi_A(y) = \psi_B(y)$ and $U_A(x,y) = U_B(x,y)$.

Introducing the coordinates transformation π:

$$\tilde{y} = y, \tilde{x} = \psi_A(y) + \frac{N_2 - \psi_A(y)}{N_2 - \psi_B(y)}(x - \psi_B(y)),$$

it maps the domain Ω_{+B} into Ω_{+A}, maps Γ_{1B} into Γ_{1A}, and maps $\Gamma_{2,3,4}$ into themselves, respectively. Under the transformation π the solution U_B becomes $\tilde{U}_B = U_B \circ \pi^{-1}$, which is also defined in Ω_{+A}. Next we prove $\tilde{U}_B(x,y) = U_A(x,y)$.

Viewing $U_A(x,y)$ and $\tilde{U}_B(x,y)$ as the solutions of a boundary value problem with fixed boundaries we can write

$$\begin{aligned}
\mathbf{F}_{U_A}[U_A] &= 0, \\
G_i(U_A, U_-(\psi_A(y), y)) &= 0, \qquad i = 1, 2, \\
\mathbf{F}_{\tilde{U}_B}[\tilde{U}_B] &= 0, \\
G_i(\tilde{U}_B, U_-(\psi_B(y), y)) &= 0, \qquad i = 1, 2.
\end{aligned}$$

Hence

$$\begin{aligned}
&G_i(U_A, U_-(\psi_A(y), y)) - G_i(\tilde{U}_B, U_-(\psi_B(y), y)) \\
&= G_i(U_A, U_-(\psi_A(y), y)) - G_i(\tilde{U}_B, U_-(\psi_A(y), y)) + G_i(\tilde{U}_B, U_-(\psi_A(y), y)) \\
&\qquad - G_i(\tilde{U}_B, U_-(\psi_B(y), y)) \\
&= \frac{\partial G_i}{\partial U_+}(U_+^*, U_-(\psi_A(y), y))(U_A - \tilde{U}_B) + G_i(\tilde{U}_B, U_-(\psi_A(y), y)) \\
&\qquad - G_i(\tilde{U}_B, U_-(\psi_B(y), y))
\end{aligned}$$

holds on Γ_A.

Denoting $\tilde{U}_B - U_A$ by U_{BA}, i.e. $u_{BA} = \tilde{u}_B - u_A, v_{BA} = \tilde{v}_B - v_A, p_{BA} = \tilde{p}_B - p_A$, and denoting U_A by U, we have

$$F_{U_A}[U_{BA}] = F_{U_A}[\tilde{U}_B] - F_{\tilde{U}_B}[\tilde{U}_B], \quad (= f_{BA}) \tag{7.3}$$

$$\begin{aligned}
&\frac{\partial G}{\partial U_+}(U_+^*, U_-(\psi_A(y), y))U_{BA} \\
&\qquad = G_i(U_A, U_-(\psi_A(y), y)) - G_i(\tilde{U}_B, U_-(\psi_B(y), y)),
\end{aligned} \tag{7.4}$$

where

$$\|f_{BA}\|_{C^\alpha(\Omega_{+A})} \le C\|U_{BA}\cdot D\tilde{U}_B\|_{C^\alpha(\Omega_{+A})} \le C\epsilon\|U_{BA}\|_{C^\alpha(\Omega_{+A})}. \tag{7.5}$$

(7.4) can be written as

$$u_{BA} + \tilde\beta_1 v_{BA} = g_{BA1}, \tag{7.6}$$

$$p_{BA} + \tilde\beta_2 v_{BA} = g_{BA2}, \tag{7.7}$$

where

$$\|g_{BAi}\|_{C^{1,\alpha}(0,b)} \le C\|U_-\circ\pi^{-1} - U_-\|_{C^{1,\alpha}(0,b)} \le C\epsilon\|\psi_B(y) - \psi_A(y)\|_{C^{1,\alpha}(0,b)} \tag{7.8}$$

for $i = 1, 2$. Moreover,

$$\|\psi_B(y) - \psi_A(y)\|_{C^{1,\alpha}(0,b)} = \|\int_0^y (\frac{d\psi_B}{dy} - \frac{d\psi_A}{dy})dy\|_{C^{1,\alpha}(0,b)}$$
$$\le \int_0^b \|H(\tilde U_B, U_-\circ\pi^{-1}) - H(U_A, U_-)\|_{C^{1,\alpha}(0,b)}dy.$$

This implies that

$$\|\psi_B(y) - \psi_A(y)\|_{C^{1,\alpha}(0,b)} \le C\epsilon\|U_{BA}\|_{C^{1,\alpha}(\Omega_{+A})}; \tag{7.9}$$

then we have

$$\|g_{BAi}\|_{C^{1,\alpha}(0,b)} \le C\epsilon\|U_{BA}\|_{C^{1,\alpha}(\Omega_{+A})}. \tag{7.10}$$

By using the same technique as in Section 5, ρ_{BA} satisfies the following problem:

$$\begin{aligned}
(D_I^2 + D_R^2)(hp_{BA}) - \mu D_R(hp_{BA}) &= D_I(f_{BA})_1 - D_R(f_{BA})_2 + \mu(f_{BA})_2 \quad \text{in} \quad \Omega_{+A},\\
p_{BA} &= g_{BA2} - \tilde\beta_2 v_{BA} \quad \text{on} \quad \Gamma_{1A},\\
\frac{\partial}{\partial y}(h\ p_{BA}) &= 0 \quad \text{on} \quad \Gamma_2, \Gamma_3,\\
p_{BA} &= 0 \quad \text{on} \quad \Gamma_4.
\end{aligned} \tag{7.11}$$

Therefore, by using the maximum principle and the estimates for the weak solutions to second order elliptic equation (see [12]) we have

$$\|h\ p_{BA}\|_{C^{1,\alpha}(\Omega_{+A})} \le C(\|g_{BA2} - \tilde\beta_2 v_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})} + \|f_{BA}\|_{C^\alpha(\Omega_{+A})}). \tag{7.12}$$

In view of

$$\begin{aligned}
\|v_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})} &\le \|\frac{1}{u}(uv_{BA} - vu_{BA} + vu_{BA})\|_{C^{1,\alpha}(\Gamma_{1A})}\\
&\le C(\|uv_{BA} - vu_{BA}\|_{C^{1,\alpha}(\Gamma_1A)} + \|v\tilde\beta_1 v_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})} + \|g_{BA1}\|_{C^{1,\alpha}(\Gamma_{1A})}\\
&\le C(\|hp_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})} + \|f_{BA}\|_{C^\alpha(\Gamma_{1A})} + \|g_{BA1}\|_{C^{1,\alpha}(\Gamma_{1A})} + \epsilon\|v_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})}).
\end{aligned} \tag{7.13}$$

Combining (7.12) with (7.13) we have

$$\|hp_{BA}\|_{C^{1,\alpha}(\Omega_{+A})} \le C(\epsilon\|U_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})} + \|f_{BA}\|_{C^\alpha(\Omega_{+A})} + \epsilon\|g_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})}). \tag{7.14}$$

Again use the first two equations in (7.3) we know $\|uv_{BA} - vu_{BA}\|_{C^{1,\alpha}(\Omega_{+A})}$, $\|U_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})}$ are also controlled by the right side of (7.14).

Finally from the third equation of the system in (7.3) we have

$$(D_3 - d_1)(\rho uu_{BA} + \rho vv_{BA} + p_{BA}) = d_2(\rho vu_{BA} - \rho uv_{BA}) + d_3 p_{BA}.$$

Then

$$\|\rho u u_{BA} + \rho v v_{BA} + p_{BA}\|_{C^{1,\alpha}(\Omega_{+A})} \le \|\rho u u_{BA} + \rho v v_{BA} + p_{BA}\|_{C^{1,\alpha}(\Gamma_{1A})} + C(\|\rho v u_{BA} - \rho u v_{BA}\|_{C^{1,\alpha}(\Omega_{1A})} + \|p_{BA}\|_{C^{1,\alpha}(\Omega_{1A})}), \tag{7.15}$$

and it can also be controlled by the right side of (7.14). Noticing that $p_{BA}, \rho u v_{BA} - \rho v u_{BA}, \rho u u_{BA} + \rho v v_{BA} + p_{BA}$ form a basis of all linear combinations of $\mu_1 u_{BA} + \mu_2 v_{BA} + \mu_3 p_{BA})$, then by combining the estimates (7.5), (7.8) we obtain

$$\|U_{BA}\|_{C^{1,\alpha}(\Omega_{+A})} \le C\epsilon \|U_{BA}\|_{C^{1,\alpha}(\Omega_{+A})},$$

which implies $U_{BA} = 0$, provided ϵ is taken small enough.

Obviously, we can then obtain $\psi_A = \psi_B$, and $U_A = \tilde{U}_B = U_B$ immediately. This is the uniqueness. Then the proof of Theorem 2.1 is complete, and we are led to the desired stability of the transonic shock fronts.

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