

ON STRUCTURALLY STABLE DIFFEOMORPHISMS WITH CODIMENSION ONE EXPANDING ATTRACTORS

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ABSTRACT. We show that if a closed n -manifold M^n ($n \geq 3$) admits a structurally stable diffeomorphism f with an orientable expanding attractor Ω of codimension one, then M^n is homotopy equivalent to the n -torus T^n and is homeomorphic to T^n for $n \neq 4$. Moreover, there are no nontrivial basic sets of f different from Ω . This allows us to classify, up to conjugacy, structurally stable diffeomorphisms having codimension one orientable expanding attractors and contracting repellers on T^n , $n \geq 3$.

INTRODUCTION

Structurally stable diffeomorphisms form an important class of dynamical systems. They comprise an open nonempty set in the space of diffeomorphisms on any closed manifold. For this reason, it is of interest to understand on what manifolds a given class of structurally stable diffeomorphisms satisfying additional conditions exists. In [1], it was proved that Anosov diffeomorphisms on a compact manifold are structurally stable. Several papers have investigated the relationship between the topology of underlying manifolds and the dynamics of codimension one Anosov diffeomorphisms. Franks [12] and Newhouse [37] show that any codimension one Anosov diffeomorphism is conjugate to a hyperbolic torus automorphism (as a consequence, a manifold admitting such diffeomorphisms is homeomorphic to the torus T^n , $n \geq 2$). A simple proof of this Franks-Newhouse theorem that uses foliation theory techniques was recently obtained in [25].

An important class of dynamical systems is made up of the diffeomorphisms satisfying Smale's Axiom A [53], the so-called A -diffeomorphisms. Given such a diffeomorphism, its recurrent behaviour is captured in its nonwandering set, which can be decomposed into invariant topologically transitive pieces (basic sets). One remarkable type of basic sets is an expanding attractor, introduced by Williams [58], [60]. Such an attractor has a complicated topological structure that is combined with a nontrivial recurrence (see section 1). A structurally stable A -diffeomorphism of T^n with an orientable codimension one expanding attractor can be obtained by Smale's surgery ([53], pp. 788-789); it is a codimension one Anosov diffeomorphism (the orientability of an expanding attractor, roughly speaking, means the following: given any arc of a stable manifold and a codimension one unstable manifold, the index of their intersection is the same at every point of intersection; see section 1 for details). Such diffeomorphisms are called *DA-diffeomorphisms*.

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Must any structurally stable C^∞ -diffeomorphism with an orientable codimension one expanding attractor look like a DA -diffeomorphism? This is just the question we consider here for dimensions $\dim M^n = n \geq 3$. Ideologically we follow the papers [12] and [37]. Note that the answer is well-known for the dimension $n = 2$. Namely, starting with a DA -diffeomorphism of T^2 , one can construct structurally stable diffeomorphisms with codimension one orientable expanding attractors on any closed orientable surface of genus $g \geq 1$. (See, e.g., [5], [44], [49]. There is a careful construction of DA -diffeomorphism on T^2 in [31] and [48].)

There are two ways to classify expanding attractors. The first way is to ask, when are the restrictions of two diffeomorphisms to their expanding attractors conjugate? This type of classification was obtained by Williams [60], Theorem A, who proved that the restriction of a diffeomorphism on an expanding attractor of any dimension is conjugate to the shift map of an n -solenoid. The second way is to ask, when are two diffeomorphisms conjugate in neighborhoods of their attractors? The authors of [50] constructed two diffeomorphisms f and g with attractors Ω_f and Ω_g such that $f : \Omega_f \rightarrow \Omega_f$ is conjugate to $g : \Omega_g \rightarrow \Omega_g$ but there is not even a homeomorphism from a neighborhood of Ω_f to a neighborhood of Ω_g taking Ω_f to Ω_g . Thus, these two ways of classification are different. Here we consider the second type of classification. In [15], [16], [17], [18] structurally stable diffeomorphisms with one-dimensional expanding attractors on orientable closed surfaces of genus $g \geq 0$ are classified. In [7] structurally stable diffeomorphisms on orientable closed surfaces of genus $g \geq 0$ are classified. Let us mention the combinatorial description obtained in [61], [62] for one-dimensional expanding attractors on closed surfaces (orientable or non-orientable).

It is well-known that a codimension one basic set (i.e., the topological dimension of the basic set equals $n - 1$) is either an attractor or a repeller. As we know, Smale's question (see [53], p. 785) of whether such nontrivial basic set that is not a codimension one compact submanifold is locally the product of a Cantor set and a manifold is unsolved up to now (in the "codimension one" case). Guckenheimer [22] constructed a structurally stable diffeomorphism of the 5-sphere with a codimension four basic set that is not locally the product of a Cantor set and an interval. Farrell and Jones [11] constructed the orientable basic set of codimension one that is not the total space of a fiber bundle with a manifold for a base space and a Cantor set for fiber. Other interesting examples can be found in [30] and [44].

If a codimension one basic set is an expanding attractor or contracting repeller, then it is locally the product of a Cantor set and a codimension one hyperplane (recall that an expanding attractor has topological dimension equal to the dimension of unstable manifolds of points of the attractor. A contracting repeller of a diffeomorphism f is an expanding attractor for f^{-1}) [43], [60]. In this paper we consider a codimension one basic set that is an expanding attractor or contracting repeller. Certainly, one can consider a contracting repeller instead of an expanding attractor, and vice versa. Therefore we use the notion *pseudotame basic set*, meaning an expanding attractor or contracting repeller.

As we mentioned above, any closed orientable surface of genus $g \geq 1$ admits codimension one orientable pseudotame basic sets [5], [44], [49], [53]. Moreover, any such surface admits a structurally stable diffeomorphism with orientable pseudotame basic sets. As to closed n -manifolds for $n \geq 3$, the first example of a structurally stable DA -diffeomorphism of T^3 with a codimension one orientable

expanding attractor was constructed carefully by Franks and Robinson [13]. This construction was generalized by Plykin [45] for $n \geq 3$ in the framework of the topological classification of codimension one pseudotame basic sets. In [13], one constructs a diffeomorphism f of the connected sum $T^3 \# T^3$ that is quasi-Anosov but not Anosov. Actually, f has two orientable codimension one pseudotame basic sets, one of them an expanding attractor and the other a contracting repeller. Examination of this example shows that f is not structurally stable, because there are intersections of one-dimensional stable and unstable manifolds of points of the expanding attractor and the contracting repeller respectively, and thus the strong transversality condition fails. This raises the natural question of whether there exist structurally stable diffeomorphisms with orientable codimension one pseudotame basic sets on closed n -manifolds ($n \geq 3$) different from T^n .

Let us mention some topological obstructions to manifolds supporting codimension one pseudotame basic sets. In [32], one proved that if a closed n -manifold M^n , $n \geq 3$, admits a codimension one expanding attractor (orientable or not), then M^n has a nontrivial fundamental group. In particular, there are no such diffeomorphisms of the n -sphere S^n , $n \geq 3$. The more general result was obtained later in [45], where one proved that the fundamental group $\pi_1(M^n)$ contains a subgroup isomorphic to the integer lattice \mathbb{Z}^n . To be precise, Plykin [45] showed that if f has $k \geq 1$ copies of codimension one pseudotame basic sets (orientable or not), then $\pi_1(M^n)$ contains the free product of k copies of \mathbb{Z} (for orientable basic sets, this result was obtained in [63]). Plante [42] proved that an orientable codimension one expanding attractor defines a nontrivial element of the first homology group $H_1(M^n)$. Sullivan and Williams [56] showed that the real Čech homology of an oriented attractor (of any codimension) in its top dimension is nontrivial and finite dimensional.

Our following theorem describes the homotopy type of a closed manifold M^n ($n \geq 3$) admitting a structurally stable diffeomorphism with an orientable pseudotame basic set of codimension one. Recall that in our theorems we consider diffeomorphisms to be C^∞ diffeomorphisms, unless stated otherwise. From now on, the nontriviality of a basic set means that the basic set is not a periodic orbit.

Theorem 7.1. *Suppose f is a structurally stable diffeomorphism of a closed n -manifold M^n ($n \geq 3$) and Ω is a codimension one orientable expanding attractor of f . Then:*

- 1) M^n is homotopy equivalent to the n -torus T^n . If $n \neq 4$, then M^n is homeomorphic to T^n .
- 2) Ω is the only nontrivial basic set of f .
- 3) f has at least one repelling periodic point.

If x is a periodic point of f , then the *index* at x is defined to be the dimension of the unstable manifold $W^u(f^p(x))$, where p is the period of x . The index of a periodic orbit equals the index of any point of it.

Corollary 7.1. *Suppose f is a structurally stable diffeomorphism of a closed n -manifold M^n ($n \geq 3$) having an orientable expanding attractor Ω of codimension one. Then the spectral decomposition of f consists of Ω , and a finite nonzero number of repelling periodic orbits of index n , and a finite number (maybe zero) of periodic saddle orbits of index $n - 1$.*

Corollary 7.3. *Suppose f is an A -diffeomorphism of a closed n -manifold M^n ($n \geq 3$) having an orientable expanding attractor of codimension one Ω . Then the following conditions imply that f is not structurally stable:*

- 1) f has a nontrivial basic set different from Ω .
- 2) M^n is not homotopy equivalent to the n -torus.

After Theorem 7.1, it is natural to classify, up to conjugacy, structurally stable diffeomorphisms with orientable pseudotame basic sets of codimension one on the torus T^n . For these diffeomorphisms we introduce the complete invariant of conjugacy, which is a finite collection of periodic orbits of some hyperbolic torus automorphism endowed with presigned natural numbers. Such a collection is called a data set of the hyperbolic automorphism (for details, see section 8).

Theorem 8.2. *Suppose $f_1, f_2 : T^n \rightarrow T^n$ are structurally stable diffeomorphisms having orientable expanding attractors of codimension one Ω_1 and Ω_2 respectively. Then f_1 and f_2 are conjugate if and only if the data sets $\mathcal{D}(f_1, h_1)$ and $\mathcal{D}(f_2, h_2)$ are equivalent, where $h_i : T^n \rightarrow T^n$ ($i = 1, 2$) are continuous maps homotopic to the identity and such that $h_i \circ f_i = A_i \circ h_i$, $(f_i)_* = (A_i)_*$.*

Theorem 8.3. *Let A be a codimension one hyperbolic automorphism of T^n such that the stable manifolds of A are one-dimensional. Given an admissible data set $\{O_j, n_j\}_{j=1}^r$ of A , there is a structurally stable diffeomorphism $f : T^n \rightarrow T^n$ having an orientable expanding attractor of codimension one and such that $\mathcal{D}(f, h) = \{O_j, n_j\}_{j=1}^r$, where $h : T^n \rightarrow T^n$ is a continuous map homotopic to the identity with $h \circ f = A \circ h$ and $f_* = A_*$.*

The first example of a codimension one non-orientable expanding attractor Ω on the two-sphere S^2 was constructed by Plykin [44]. He constructed a structurally stable diffeomorphism $f : S^2 \rightarrow S^2$ with so-called 1-bunches. Roughly speaking, there exists a component of $S^2 - \Omega$ whose accessible boundary consists of a unique (one-dimensional) unstable manifold of a periodic point (see section 1). In [45], codimension one non-orientable expanding attractors with 1-bunches on open n -manifolds, $n \geq 3$, were constructed (in the same paper, Plykin classified up to conjugacy diffeomorphisms which are restricted to some neighborhoods of codimension one expanding attractors). Bearing these examples in mind, a codimension one expanding attractor Ω is called a *codimension one Plykin attractor* whenever Ω has 1-bunches.

The following theorem was proved in collaboration with V. Medvedev.

Theorem 7.3. *Let $f : M^3 \rightarrow M^3$ be a structurally stable diffeomorphism of a closed 3-manifold M^3 . Then the spectral decomposition of f contains neither codimension one non-orientable expanding attractors nor codimension one non-orientable contracting repellers. In particular, f has no codimension one Plykin attractors.*

Corollary 8.1. *Suppose $f_1, f_2 : T^3 \rightarrow T^3$ are structurally stable diffeomorphisms having expanding attractors of codimension one Ω_1 and Ω_2 respectively. Then f_1 and f_2 are conjugate if and only if the data sets $\mathcal{D}(f_1, h_1)$ and $\mathcal{D}(f_2, h_2)$ are equivalent, where $h_i : T^3 \rightarrow T^3$ ($i = 1, 2$) are continuous maps homotopic to the identity and such that $h_i \circ f_i = A_i \circ h_i$ and $(f_i)_* = (A_i)_*$.*

This paper is organized as follows. In section 1 we give the main definitions. We formulate for convenience some well-known results on codimension one basic sets.

We generalize some of these results and give sketches of proofs. In section 2, given a codimension one expanding attractor, we construct characteristic spheres. Here we actually follow the construction of Newhouse [37] for a nontransitive Anosov diffeomorphism. In sections 3 and 4 we prove some technical results we will need later. In section 5 we study the structure of the universal covering for M^n . In this section we prove the following theorem, which is crucial for the proof of item 1) of Theorem 7.1.

Theorem 5.1. *Let f be an A -diffeomorphism of a closed n -manifold M^n ($n \geq 3$) and let Ω be an orientable expanding attractor of codimension one. Suppose that each characteristic sphere $S_{p_i q_i}$ of Ω bounds an n -ball $E_{p_i q_i} \subset M^n$. Then:*

- 1) *Any basic set different from Ω belongs to the union $\bigcup_{i=1}^k E_{p_i q_i}$, where p_i, q_i range over all pairs of associated boundary periodic points of Ω .*
- 2) *Ω is the only orientable codimension one pseudotame basic set of f .*
- 3) *The universal covering \overline{M} of M^n is homeomorphic to Euclidean space \mathbb{R}^n .*
- 4) *The fundamental group $\pi_1(M^n)$ is a free abelian finitely generated group. In particular, M^n is homotopy equivalent to the n -torus T^n .*

In section 6 we prove the following theorem, needed for the proof of the remaining items of Theorem 7.1. In a sense, this theorem can be considered as a sufficient condition for the nonstability of f .

Theorem 6.2. *Let f be an A -diffeomorphism of a closed n -manifold M^n ($n \geq 3$) and Ω an orientable expanding attractor of codimension one. Suppose $\Omega' \neq \Omega$ is a nontrivial basic set of f such that $W^u(\Omega') \cap W^s(\Omega) \neq \emptyset$. Then f is not structurally stable.*

Theorems 7.1 and 7.3 are proved in section 7. The classification Theorems 8.2 and 8.3 are proved in section 8.

1. MAIN DEFINITIONS

We begin by recalling several definitions. Further details may be found in [31], [38], [48], [53].

Let f be a C^∞ diffeomorphism of a closed manifold M^n endowed with some Riemannian metric d . f is said to be an A -diffeomorphism if its nonwandering set $NW(f)$ is hyperbolic and periodic points are dense in $NW(f)$ [53]. The *stable manifold* $W^s(x)$ of a point $x \in NW(f)$ is defined to be the set of points $y \in M^n$ such that $d(f^i x, f^i y) \rightarrow 0$ as $i \rightarrow +\infty$. The *unstable manifold* $W^u(x)$ of x is the stable manifold of x for the diffeomorphism f^{-1} . We shall consider a stable or unstable manifold to be an immersed submanifold of M^n . Stable and unstable manifolds are called *invariant manifolds*. It is well known [27] that invariant manifolds depend continuously on initial conditions on compact sets. By definition, let $W_\varepsilon^s(x) \subset W^s(x)$ (resp. $W_\varepsilon^u(x) \subset W^u(x)$) be the ε -neighborhood of x in the intrinsic topology of the manifold $W^s(x)$ (resp. $W^u(x)$), where $\varepsilon > 0$.

Let $Diff^1(M^n)$ be the space of C^1 diffeomorphisms on M^n endowed with the uniform C^1 topology [26]. A diffeomorphism f is said to be *structurally stable* if there exists a neighborhood U of f in $Diff^1(M^n)$ such that every $g \in U$ is conjugate to f .

Let f be an A -diffeomorphism. We say that f *satisfies the strong transversality condition* if for every $x, y \in NW(f)$, $W^s(x)$ is transverse to $W^u(y)$. According

to Mañé-Robinson's theorem [35], [47], a diffeomorphism f is structurally stable if and only if f is an A -diffeomorphism satisfying the strong transversality condition.

The *spectral decomposition theorem* says that the nonwandering set $NW(f)$ of an A -diffeomorphism f is a finite union of pairwise disjoint f -invariant closed sets $\Omega_1, \dots, \Omega_k$ such that every restriction $f|_{\Omega_i}$ is topologically transitive. These Ω_i are called the *basic sets* of f . Since f is transitive on each basic set Ω_i , it follows that the restrictions of the bundles E^s, E^u to Ω_i have constant dimensions. The dimension $\dim E_{\Omega_i}^u = \dim E_x^u, x \in \Omega_i$, is called *Morse's index* of Ω_i .

A basic set is *nontrivial* if it is not a periodic isolated orbit. Due to [2] and [8], every nontrivial basic set Ω_i is a union of pairwise disjoint closed subsets $\Omega_{i1}, \dots, \Omega_{ih}$ such that

$$f^h(\Omega_{ij}) = \Omega_{ij}, \quad f(\Omega_{ij}) = \Omega_{i,j+1}, \quad \text{where } \Omega_{i,h+1} = \Omega_{i1} \quad (1 \leq j \leq h),$$

and each of the intersections $W_x^s \cap \Omega_{ij}$ and $W_x^u \cap \Omega_{ij}$ is dense in Ω_{ij} for any point $x \in \Omega_{ij}$. These Ω_{ij} are called *C -dense components* of Ω_i . If Ω_i has a unique C -dense component, then we will say that Ω_i is a *C -dense basic set*.

For any $x \in \Omega$, $W^u(x)$ and $W^s(x)$ are immersed submanifolds such that

$$\dim W^u(x) + \dim W^s(x) = n.$$

Moreover, $W^u(x)$ and $W^s(x)$ are homeomorphic to Euclidean space of the corresponding dimension. Therefore both $W^u(x)$ and $W^s(x)$ are endowed with a normal and intrinsic orientation. Hence one can define the index of intersection at each point of $W^u(x) \cap W^s(x)$ [26]. Following [14] and [15], we call a basic set Ω *orientable* if for any $\alpha > 0$ and $\beta > 0$ the index of $W_{x,\alpha}^s \cap W_{x,\beta}^u$ does not depend on a point of intersection.

Following [60], a basic set Ω is called a *codimension one expanding attractor* if $\dim E_{\Omega}^u = \dim \Omega = n - 1$.

Due to [43] and [60], an expanding attractor Ω of codimension one consists of $(n - 1)$ -dimensional unstable manifolds $W_x^u, x \in \Omega$ (thus, $\dim E_{\Omega}^s = 1$). Moreover, Ω is locally homeomorphic to the product of $(n - 1)$ -dimensional Euclidean space and a Cantor set. For given points $x, y \in W^s(z)$ ($z \in \Omega$), denote by $(x, y)^s$ (resp. $[x, y]^s$) an open (resp. closed) arc of $W^s(z)$ with the endpoints x, y .

For the reader's convenience, we give sketches of proofs of some results on codimension one basic sets and codimension one expanding attractors, mainly obtained by Grines [14], [15], Newhouse [37], Plykin [44], and Williams [60]. The reader who is familiar with the cited papers can skip this part of the section.

Lemma 1.1. *Let Ω be a basic set of index $n - 1$ of an A -diffeomorphism $f : M^n \rightarrow M^n, n \geq 2$. If $x \in \Omega$ is a nonperiodic point, then both components of $W^s(x) - x$ intersect $W^u(\Omega)$.*

Proof. Let $x_i = f^{n_i}(x) \rightarrow x_* \in \Omega$ be a convergent sequence ($n_i \rightarrow +\infty$). First we suppose that $W^u(x)$ does not contain periodic points; then we can assume that the sets $W^u(f^{n_i}(x)) = f^{n_i}(W^u(x))$ are disjoint. Due to the local product structure, $W_{\varepsilon}^s(x_2) \cap W_{\varepsilon}^u(x_j)$ is a single point of Ω in some neighborhood of x_* , where $j = 1, 3$. Without loss of generality we can assume that the points $W_{\varepsilon}^s(x_2) \cap W_{\varepsilon}^u(x_1)$ and $W_{\varepsilon}^s(x_2) \cap W_{\varepsilon}^u(x_3)$ belong to different components of $W_{\varepsilon}^s(x_2) - x_2$, because the $W^u(x_i)$ are disjoint. Hence both components of $W^s(x) - x$ intersect Ω .

Suppose now that $W^u(x)$ contains a periodic point, say $x_0 \in \Omega$. Without loss of generality we can assume that x_0 is a fixed point. Since the map $f|_{W^u(x_0)}$ expands

the invariant manifold $W^u(x_0)$, it follows that there is a sequence $n_i \rightarrow +\infty$ such that the distance between $f^{n_i}(x)$ and $f^{n_j}(x)$ in a metric of $W^u(x)$ is more than ε whenever $i \neq j$. We can assume that the sequence $f^{n_i}(x)$ converges to some point $x_* \in \Omega$. In this case, the proof is similar. \square

Taking into account that $W^u(\Omega) = \Omega$ whenever Ω is a codimension one expanding attractor, we get the following lemma as a consequence.

Lemma 1.2. *Let Ω be a codimension one expanding attractor (orientable or not) of an A -diffeomorphism $f : M^n \rightarrow M^n$, $n \geq 2$. Suppose $x \in \Omega$ is a nonperiodic point of f . Then both components of $W^s(x) - x$ intersect Ω .*

The following lemma is a sharpening of Lemma 1.1.

Lemma 1.3. *Let Ω be a basic set of index $n - 1$ of an A -diffeomorphism $f : M^n \rightarrow M^n$, $n \geq 2$. Suppose $W^u(x)$ ($x \in \Omega$) does not contain periodic points. Then both components of $W_\delta^s(x) - x$ intersect $W^u(\Omega)$ for any $\delta > 0$.*

Proof. Let $f^{n_i}(x) \rightarrow x_*$ ($n_i \rightarrow -\infty$) be a convergent sequence. Then $W^u(f^{n_i}(x)) = f^{n_i}(W^u(x))$ are pairwise disjoint. Due to the local product structure in some neighborhood of x_* , both components of $W_\varepsilon^s(f^{n_i}(x)) - f^{n_i}(x)$ intersect $W^u(\Omega)$. Since

$$\lim_{n_i \rightarrow -\infty} \text{diam } f^{-n_i}(W_\varepsilon^s(f^{n_i}(x))) \rightarrow 0,$$

it follows that both components of $W_\delta^s(x) - x$ intersect $W^u(\Omega)$ for any $\delta > 0$. \square

Lemma 1.4. *Suppose Ω is a codimension one expanding attractor (orientable or not) of an A -diffeomorphism $f : M^n \rightarrow M^n$, $n \geq 2$. Then there exists a periodic point $p \in \Omega$ of f such that one of the components of $W^s(p) - p$ does not intersect Ω .*

Proof. Assume the converse. Then by Lemma 1.2, both components of $W^s(x) - x$ intersect Ω for any $x \in \Omega$. Let us show that $W^s(x) \subset \Omega$. Take a convergent sequence $f^{-n_i}(x) \rightarrow y \in \Omega$, $n_i \rightarrow +\infty$. By assumption, both components of $W^s(y) - y$ intersect Ω . As a consequence, both components of $W_\varepsilon^s(f^{-n_i}(x)) - f^{-n_i}(x)$ intersect Ω for some $\varepsilon > 0$. Since f contracts every stable manifold, we see that both components of $W_\delta^s(x) - x$ intersect Ω for any $\delta > 0$. Hence $W^s(x) \subset \Omega$, because Ω is closed. This implies that $\Omega = M^n$, which contradicts the fact that Ω is an attractor. This concludes the proof. \square

Later on, we specify Ω to be a codimension one expanding attractor of an A -diffeomorphism $f : M^n \rightarrow M^n$, $n \geq 2$, unless otherwise arranged. Following [14] and [15], $p \in \Omega$ is called a *boundary point* if at least one component of $W^s(p) - p$ does not intersect Ω . A point is *interior* if it is not a boundary point. Due to Lemma 1.4, boundary points exist. By Lemma 1.2, any boundary point $p \in \Omega$ is periodic.

Lemma 1.5. *Let p be a boundary point of a codimension one expanding attractor Ω . Then there is a unique component of $W^s(p) - p$, denoted by $W_\emptyset^s(p)$, that does not intersect Ω . Moreover, the other component*

$$W_\infty^s(p) \stackrel{\text{def}}{=} W^s(p) - (p \cup W_\emptyset^s(p))$$

contains the subset $W_\infty^s(p) \cap \Omega$, which is dense in Ω_c , where $\Omega_c \subset \Omega$ is the C -dense component of Ω containing p .

Proof. Due to Anosov [2] and Bowen [8], $W^u(x) \cap W^s(y)$ is dense in Ω_c for any point $x, y \in \Omega_c$. It follows that $W_\infty^s(p) \cap \Omega$ is dense in Ω_c , because $W_\emptyset^s(p) \cap \Omega = \emptyset$. \square

Corollary 1.1. *Suppose p is a boundary point of a codimension one expanding attractor Ω and $W_\infty^s(p)$ is a unique component of $W^s(p) - p$ which intersects Ω . Then, for any $\delta > 0$, $W_{\infty, \delta}^s(p) = W_\infty^s(p) \cap W_\delta^s(p)$ intersects Ω . Moreover,*

- *Given a point $x \in W^u(p) - p$, there is a unique arc $(x, y)^s$ such that $(x, y)^s \cap \Omega = \emptyset$ and $y \in \Omega$.*
- *Given a point $x \in W^u(p) - p$, there is the component $W_{\infty, \delta}^s(x)$ of $W_\delta^s(x) - x$, such that $W_{\infty, \delta}^s(x) \cap \Omega \neq \emptyset$ for any $\delta > 0$.*

Proof. This follows from Lemma 1.5 and the theorem on the continuous dependence of invariant manifolds on initial conditions. \square

Lemma 1.6. *A codimension one expanding attractor Ω (orientable or not) has finitely many boundary points.*

Proof. This follows from the local product structure on Ω . \square

The arc $(x, y)^s$ mentioned in Corollary 1.1 (i.e., such that $(x, y)^s \cap \Omega = \emptyset$) will be denoted by $(x, y)_\emptyset^s$.

Lemma 1.7. *Let Ω be a codimension one expanding attractor (orientable or not) of an A -diffeomorphism $f : M^n \rightarrow M^n$, $n \geq 2$. Suppose $W^u(x)$, $x \in \Omega$, does not contain boundary periodic points. Then both components of $W_\delta^s(x) - x$ intersect Ω for any $\delta > 0$.*

Proof. First assume that $W^u(x)$ contains a periodic point $x_0 \in \Omega$. Then x_0 is an interior periodic point. Therefore both components of $W^s(x_0) - x_0$ intersect Ω . Hence both components of $W_\delta^s(x_0) - x_0$ intersect Ω for any $\delta > 0$, because Ω is invariant under f . The theorem on the continuous dependence of invariant manifolds on initial conditions implies the result.

If $W^u(x)$ does not contain periodic points, then the result follows from Lemma 1.3 because $W^u(\Omega) = \Omega$. \square

Lemma 1.8. *Let Ω be a codimension one C -dense expanding attractor (orientable or not) of an A -diffeomorphism $f : M^n \rightarrow M^n$, $n \geq 2$. Suppose that $W^s(x)$, $x \in \Omega$, does not contain boundary periodic points. Then both $W^{s+}(x) \cap \Omega$ and $W^{s-}(x) \cap \Omega$ are dense in Ω , where $W^{s\pm}(x)$ are components of $W^s(x) - x$.*

Proof. Due to the C -density of Ω , $W^s(x) \cap \Omega$ is dense in Ω . By Lemma 1.7 and Corollary 1.1, both $W^{s+}(x) \cap \Omega$ and $W^{s-}(x) \cap \Omega$ contain infinitely many points. First we prove that at least one set $W^{s+}(x) \cap \Omega$ or $W^{s-}(x) \cap \Omega$ is dense in Ω . Given any point $z \in \Omega - W^s(x)$, there is a sequence $x_k \in W^s(x) \cap \Omega$ such that $x_k \rightarrow z$ as $k \rightarrow \infty$. To be definite, assume that all points $x_k \in W^{s+}(x)$. Take any point $y \in \Omega$ and any η -neighborhood $U_\eta(y)$ of y , $\eta > 0$. Since Ω is C -dense, it follows that there is the arc $[z, z_\eta]^s$ with $z_\eta \in U_\eta(y)$. By the theorem on continuous dependence of stable invariant manifolds on initial conditions, there is $\delta > 0$ such that every stable manifold $W^s(x_k)$ passing through a δ -neighborhood $U_\delta(z)$ of z must intersect $U_\eta(y)$ at a point y_k with $[x_k, y_k]^s \subset U_\delta([z, z_\eta]^s)$, where $U_\delta([z, z_\eta]^s)$ is the δ -neighborhood of the arc $[z, z_\eta]^s$. Let us take $\delta > 0$ so that $x \notin U_\delta([z, z_\eta]^s)$. Then $x \notin [x_k, y_k]^s$, and thus $W^{s+}(x) \cap U_\eta(y) \neq \emptyset$. Hence, $W^{s+}(x) \cap \Omega$ is dense in Ω .

It remains to prove that $W^{s-}(x) \cap \Omega$ is dense in Ω . Since $W^{s-}(x) \cap \Omega$ contains infinitely many points and $W^s(x)$ does not contain boundary periodic points, it follows that there exists a sequence $x'_k \in W^{s-}(x) \cap \Omega$ that converges to some point $z' \in \Omega$ as $k \rightarrow \infty$. Moreover, due to Lemma 1.7, one can assume that the points x'_k have no accumulation points in $W^{s-}(x)$ with respect to the interior topology of $W^{s-}(x)$. There are two cases: 1) $z' \notin W^s(x)$; 2) $z' \in W^s(x)$. In the first case, the argument above shows that $W^{s-}(x) \cap \Omega$ is dense in Ω . Let us consider the second case. Denote by $W^{s+}(z')$ the component of $W^s(z') - z'$ that is dense in Ω and intersects with $W^{s+}(x)$. Since the points x'_k have no accumulation points in $W^{s-}(x)$, we see that there are finitely many points x'_1, \dots, x'_{k_0} on the arc $[z', x]^s$. As above, we conclude that $W^{s-}(x) \cap \Omega$ is dense in Ω . \square

Lemma 1.9. *Let Ω be a codimension one expanding attractor (orientable or not) of an A-diffeomorphism $f : M^n \rightarrow M^n$, $n \geq 2$. If a point z belongs to $W^s(\Omega) - \Omega$, then either $z \in (x, y)_\emptyset^s$, where $x \in W^u(p)$ and $y \in W^u(q)$ for some periodic boundary points $p, q \in \Omega$ (maybe $p = q$), or $z \in W_\emptyset^s(p)$ for some periodic boundary point $p \in \Omega$.*

Proof. Due to [28], any point whose forward orbit stays near Ω is asymptotic with a point of Ω . Hence, $z \in W^s(x_0)$ for some $x_0 \in \Omega$. Since Ω is closed, it follows that either $z \in (x, y)^s$ with $x, y \in \Omega$ and $(x, y)^s \cap \Omega = \emptyset$, or z belongs to a component of $W^s(p) - p$ ($p \in \Omega$) which does not intersect Ω . In the last case, by Lemmas 1.2 and 1.7, p is a periodic boundary point and $z \in W_\emptyset^s(p)$. If $z \in (x, y)^s$, then, according to Lemma 1.7, $x \in W^u(p)$ and $y \in W^u(q)$ with periodic boundary points $p, q \in \Omega$. \square

Let $G \subset M$ be an open domain with the boundary $\partial G = \text{clos } G - \text{int } G$. In studying a neighborhood of basic sets, an important feature is which points of ∂G are accessible from within G . A point $x \in \partial G$ is called *accessible* from within G provided there is an open arc a which is in G and x is an endpoint of a . The accessible points form the subset $\delta(G) \subset \partial G$, which is called the *accessible boundary* of G .

Lemma 1.10. *Let Ω be a codimension one expanding attractor (orientable or not) of an A-diffeomorphism $f : M^n \rightarrow M^n$, $n \geq 2$. Given any point $x \in \Omega$, in the interior topology of the stable manifold $W^s(x)$, the intersection $W^s(x) \cap \Omega$ is a Cantor set (i.e., a compact totally disconnected perfect set). Moreover, the accessible points in $W^s(x) \cap \Omega$ from within $W^s(x) - \Omega$ belong to unstable invariant (codimension one) manifolds of periodic boundary points.*

Proof. Since Ω is a compact set and attractor, $W^s(x) \cap \Omega$ is a compact totally disconnected set. Let $\Omega_c \subset \Omega$ be the C -dense component of Ω containing x . Then $W^u(z)$ is dense in Ω_c for any $z \in \Omega_c$ [2], [8]. It follows that $W^s(x) \cap \Omega = W^s(x) \cap \Omega_c$ is a perfect set.

Lemma 1.9 implies that accessible points in $W^s(x) \cap \Omega$ from within $W^s(x) - \Omega$ correspond to unstable invariant manifolds of periodic boundary points of Ω . \square

An unstable manifold $W^u(x) \subset \Omega$, $x \in \Omega$, is called a *boundary unstable manifold* of Ω if $W^u(x)$ contains a boundary periodic point.

Corollary 1.2. *The accessible boundary $\delta(M^n - \Omega)$ of the domain $M^n - \Omega$ is a finite union of boundary unstable (codimension one) manifolds of Ω .*

The boundary unstable manifolds of Ω split into a finite number of so-called bunches in the following way. The pairwise disjoint unstable manifolds $W^u(p_1), \dots, W^u(p_k)$ is said to be a k -bunch if there are points $x_i \in W^u(p_i)$ and arcs

$$(x_i, y_i)_{\emptyset}^s, \quad y_i \in W^u(p_{i+1}), \quad 1 \leq i \leq k, \quad \text{where } p_{k+1} = p_1, y_k \in W^u(p_1),$$

and there are no $(k+1)$ -bunches containing the given one. The boundary periodic points p_1, \dots, p_k are called *associated*.

Let B be a k -bunch of Ω consisting of the boundary unstable manifolds $W^u(p_1), \dots, W^u(p_k)$, where p_i ($i = 1, \dots, k$) are boundary periodic points. By Corollary 1.1, given any point $x \in W^u(p_1)$ with $x \neq p_1$, there is a unique point $y \in W^u(q)$ such that $(x, y)_{\emptyset}^s = (x, y)_{\emptyset}^s$, where $q \in \Omega$ is a boundary periodic point. By definition, put $\varphi_1(x) = y$. Hence, $\varphi_1 : W^u(p_1) - p_1 \rightarrow \Omega$.

Lemma 1.11. *Let B be a k -bunch of Ω consisting of the boundary unstable manifolds $W^u(p_1), \dots, W^u(p_k)$, where p_i ($i = 1, \dots, k$) are boundary periodic points. Let $\varphi_1 : W^u(p_1) - p_1 \rightarrow \Omega$ be the mapping defined above. Then the following statements hold:*

- q is one of the points p_1 or p_2 .
- B is either a 1-bunch (here $q = p_1$) or a 2-bunch (here $q = p_2 \neq p_1$).
- φ_1 is a homeomorphism of $W^u(p_1) - p_1$ onto $W^u(p_j) - p_j$, where j is equal to either 1 or 2.

Proof. By definition of a bunch, $\varphi_1(x_1) = y_1 \in W^u(p_2)$ for some point $x_1 \in W^u(p_1)$. (It is possible that $p_2 = p_1$, i.e., $k = 1$.) Due to the theorem on the continuous dependence of invariant manifolds on initial conditions, φ_1 is a continuous map. Denote by $Dom(x_1) \subset W^u(p_1) - p_1$ the set of points which are mapped to $W^u(p_2)$ via φ_1 . Then $Dom(x_1) \neq \emptyset$. It follows from the local product structure that $Dom(x_1)$ is an open and closed subset of $W^u(p_1) - p_1$. Since $W^u(p_1) - p_1$ is a connected set (recall that $\dim W^u(p_1) \geq 2$), we see that $Dom(x_1) = W^u(p_1) - p_1$. φ_1 is injective because stable invariant manifolds are pairwise disjoint if they do not coincide. Thus, $\varphi_1 : W^u(p_1) - p_1 \rightarrow W^u(p_2) - p_2$ is a homeomorphism.

If $p_2 = p_1$, then B is a 1-bunch. Otherwise, B is a 2-bunch. \square

Corollary 1.3. *If Ω is a codimension one orientable expanding attractor of an A -diffeomorphism $f : M^n \rightarrow M^n$ ($n \geq 3$), then every bunch of Ω is a 2-bunch.*

Proof. Suppose the contrary; then, by Lemma 1.11, there is a 1-bunch $B = W^u(p_1) \in \Omega$. Hence, $\varphi_1(x) \in W^u(p_1)$ for any $x \in W^u(p_1) - p_1$. By Corollary 1.1, the arc $[x, \varphi_1(x)]_{\emptyset}^s$ intersects $W^u(p_1)$ with opposite indexes of intersection at the points $x, \varphi_1(x)$. We get a contradiction to the orientability of Ω . \square

Corollary 1.4. *Suppose Ω is a codimension one orientable expanding attractor of an A -diffeomorphism $f : M^n \rightarrow M^n$ ($n \geq 3$) and $p, q \in \Omega$ are associated periodic points. Then p and q have the same period.*

Thus, the number of boundary periodic points of an orientable pseudotame codimension one basic set is even. These boundary periodic points split into pairs of associated points. Obviously, every boundary unstable manifold belongs to a unique bunch. Note that on every closed surface (two-dimensional manifold), given a natural k , there are codimension one pseudotame basic sets with k -bunches [5].

2. CHARACTERISTIC SPHERES AND CONNECTING CYLINDERS

Throughout this section Ω is an orientable expanding attractor of codimension one. By orientability, any bunch of Ω is a 2-bunch (see Corollary 1.3). Let B_{pq} be the bunch consisting of two unstable manifolds $W^u(p)$ and $W^u(q)$, where the boundary periodic points p and q are associated. By Corollary 1.4, p and q have the same period, denoted by $m = m(p, q)$. Due to Lemma 1.11, given any point $x \in W^u(p)$, there is a unique point $y \in W^u(q)$ such that $(x, y)^s = (x, y)_\emptyset^s$, and vice versa. Let the map

$$\varphi_{pq} : (W^u(p) - p) \cup (W^u(q) - q) \rightarrow (W^u(p) - p) \cup (W^u(q) - q)$$

be given by $\varphi_{pq}(x) = y$ whenever $(x, y)^s = (x, y)_\emptyset^s$. According to the theorem on the continuous dependence of stable and unstable manifolds on initial conditions, φ_{pq} is a homeomorphism. It follows from $f(\Omega) = \Omega$ that

(1)

$$f^{mj} \circ \varphi_{pq}|_{(W^u(p)-p) \cup (W^u(q)-q)} = \varphi_{pq} \circ f^{mj}|_{(W^u(p)-p) \cup (W^u(q)-q)} \text{ for any } j \in \mathbb{Z}.$$

The restriction $f^m|_{W^u(p)}$ has the sole hyperbolic repelling fixed point p . Therefore there is the closed C^1 embedded $(n-1)$ -ball $D_p \subset W^u(p)$ bounded by the C^1 embedded (in $W^u(p)$) $(n-2)$ -sphere $S_p^{n-2} = \partial D_p$ such that the following condition holds:

- $p \in \text{int}(D_p) = D_p - \partial D_p$, $D_p \subset \text{int}(f^m(D_p))$.

Hence, $S_p^{n-2} \cap f^m(S_p^{n-2}) = \emptyset$ and S_p^{n-2} is inside of $f^m(S_p^{n-2})$. Since S_p^{n-2} and $f^m(S_p^{n-2})$ are C^1 embedded spheres, it follows that they satisfy one more condition (see, e.g., [52]):

- $(n-2)$ -spheres S_p^{n-2} and $f^m(S_p^{n-2})$ bound a closed $(n-1)$ -annulus, call it A_p , which is homeomorphic to $\Sigma^{n-2} \times [0, 1]$.

Lemma 2.1. φ_{pq} is extended to the homeomorphism

$$\varphi_{pq} : W^u(p) \cup W^u(q) \rightarrow W^u(p) \cup W^u(q),$$

if we put $\varphi_{pq}(p) = q$ and $\varphi_{pq}(q) = p$.

Proof. Since $\varphi_{pq}|_{(W^u(p)-p)}$ is a homeomorphism, we see that $\varphi_{pq}(S_p^{n-2})$ is the tamely embedded (in $W^u(q)$) $(n-2)$ -sphere S_q^{n-2} . From the generalized Schoenflies theorem [9] it follows that S_q^{n-2} bounds in $W^u(q)$ the closed $(n-1)$ -ball, say $D_q \subset W^u(q)$.

The point q is a unique repelling fixed point of the restriction $f^m|_{W^u(q)}$. Therefore there is $j_0 \in \mathbb{N}$ such that $D_q \subset f^{j_0 m}(D_q)$. From (1) it follows that

$$\varphi_{pq} \circ f^{m j_0}(S_p^{n-2}) = f^{m j_0} \circ \varphi_{pq}(S_p^{n-2}) = f^{j_0 m}(S_q^{n-2}) \subset W^u(q) - D_q.$$

Note that

$$D_p \subset \text{int}(f^m(D_p)) \subset \text{int}(f^{m j_0}(D_p)).$$

Hence, $\varphi_{pq}(D_p - p) = D_q - q$, because $f^{m j_0}(S_p^{n-2}) \subset W^u(p) - D_p$. As a consequence,

$$\varphi_{pq} \circ f^{jm}(D_p - p) = f^{jm} \circ \varphi_{pq}(D_p - p) = f^{jm}(D_q - q).$$

In the same way one can prove that

$$\varphi_{pq} \circ f^{jm}(D_q - q) = f^{jm} \circ \varphi_{pq}(D_q - q) = f^{jm}(D_p - p).$$

This implies the result, because $\bigcap_{j \leq 0} f^{jm}(D_p) = p$ and $\bigcap_{j \leq 0} f^{jm}(D_q) = q$. \square

Corollary 2.1. *The following conditions hold:*

- $q \in \text{int}(D_q)$, $D_q \subset \text{int}(f^m(D_q))$.
- $(n-2)$ -spheres $S_q^{n-2} = \varphi_{pq}(S_p^{n-2})$ and $f^m(S_q^{n-2})$ bound the closed $(n-1)$ -annulus $A_q = \varphi_{pq}(A_p)$, which is homeomorphic to $\Sigma^{n-2} \times [0, 1]$.

Moreover,

$$(2) \quad f^{mj} \circ \varphi_{pq}|_{W^u(p) \cup W^u(q)} = \varphi_{pq} \circ f^{mj}|_{W^u(p) \cup W^u(q)} \text{ for any } j \in \mathbb{Z}.$$

Suppose that points x, z belong to a stable manifold of some point of Ω . Denote by $\lambda(x, z)$ the length of the arc $(x, z)^s$. Actually, the following lemma is a consequence of a local product structure.

Lemma 2.2. *The function $\lambda(x, z)$ varies continuously with respect to both x and z .*

Proof. There exists $j_0 \in \mathbb{N}$ such that the arc $f^{j_0}(x, z)^s = (f^{j_0}(x), f^{j_0}(z))^s$ belongs to a neighborhood of Ω with a local product structure [28]. As a consequence, there is a tube neighborhood $U((x, z)^s)$ of $(x, z)^s$ that is homeomorphic by a Hölder homeomorphism to the product of \mathbb{B}^{n-1} and $(0, 1)$ so that the arcs of stable manifolds correspond to the segments $b \times (0, 1)$, $b \in \mathbb{B}^{n-1}$ [27]. This implies that $\lambda(\cdot, \cdot)$ varies continuously with respect to the both variables x and z . \square

Lemma 2.3. *The set $C_{pq} = \bigcup_{x \in S_p^{n-2}} (x, \varphi_{pq}(x))_\emptyset^s$ is homeomorphic to the open $(n-1)$ -cylinder $\Sigma^{n-2} \times (0, 1)$.*

Proof. Let $\nu : S_p^{n-2} \rightarrow \Sigma^{n-2}$ be an arbitrary homeomorphism. Let $\lambda(x, z)$ be the length of the arc $(x, z)^s \subset (x, y)_\emptyset^s$, $x \in S_p^{n-2}$. Due to Lemma 2.2, $\lambda(x, z)$ varies continuously with respect to both x and z . Moreover, $\lambda(x, \varphi_{pq}(x)) \geq \alpha$ for all $x \in S_p^{n-2}$ and some $\alpha > 0$, since S_p^{n-2} is a compact set. Define the map $\nu_{pq} : C_{pq} \rightarrow \Sigma^{n-2} \times (0, 1)$ as

$$\nu_{pq}(z) = \left(\nu(x), \frac{\lambda(x, z)}{\lambda(x, \varphi_{pq}(x))} \right), \text{ where } z \in (x, \varphi_{pq}(x))_\emptyset^s.$$

This map is one-to-one, because the stable manifolds of Ω are pairwise disjoint. Since $\lambda(x, z)$ is continuous and $S_p^{n-2} \cap S_q^{n-2} = \emptyset$, ν_{pq} is a homeomorphism. \square

Corollary 2.2. *The set*

$$\text{clos } C_{pq} = C_{pq} \cup S_p^{n-2} \cup S_q^{n-2} = \bigcup_{x \in S_p^{n-2}} [x, \varphi_{pq}(x)]_\emptyset^s$$

is homeomorphic to the closed $(n-1)$ -cylinder $\Sigma^{n-2} \times [0, 1]$.

The set $C_{pq} = \bigcup_{x \in \partial D_p} (x, \varphi_{pq}(x))_\emptyset^s$ is called a *connecting cylinder* corresponding to the bunch $B_{pq} = W^u(p) \cup W^u(q)$.

Lemma 2.4. *The set $S_{pq} = D_p \cup D_q \cup C_{pq}$ is homeomorphic to the $(n-1)$ -sphere Σ^{n-1} .*

Proof. By construction, $D_p \cap D_q = \emptyset$, $D_p \cap C_{pq} = \emptyset$, and $D_q \cap C_{pq} = \emptyset$. If we combine this with Corollary 2.2, we get the result. \square

$S_{pq} = D_p \cup D_q \cup C_{pq}$ is called a *characteristic sphere corresponding to the bunch* $B_{pq} = W^u(p) \cup W^u(q)$. Note that neither the characteristic sphere nor the connecting cylinder corresponding to the bunch B_{pq} is unique. Actually, they depend on the choice of the $(n-1)$ -ball D_p .

Recall that $(n-2)$ -spheres S_p^{n-2} and $f^m(S_p^{n-2})$ bound in $W^u(p)$ the closed $(n-1)$ -annulus A_p , which is homeomorphic to $\Sigma^{n-2} \times [0, 1]$. Due to the properties

$$(3) \quad W^u(p) - p = \bigcup_{j \in \mathbb{Z}} f^{jm}(A_p), \quad \text{int}(f^{jm}(A_p)) \cap \text{int}(f^{km}(A_p)) = \emptyset \quad (j \neq k),$$

A_p is called a *fundamental annulus* of $f^m|_{W^u(p)}$. It follows from Corollary 2.1 that $A_q = \varphi(A_p)$ is a fundamental annulus of $f^m|_{W^u(q)}$, i.e.,

$$(4) \quad W^u(q) - q = \bigcup_{j \in \mathbb{Z}} f^{jm}(A_q), \quad \text{int}(f^{lm}(A_q)) \cap \text{int}(f^{km}(A_q)) = \emptyset \quad (l \neq k).$$

By definition, put

$$A_{pq} \stackrel{\text{def}}{=} \bigcup_{x \in A_p} [x, y]_{\emptyset}^s = \bigcup_{x \in A_p} [x, \varphi_{pq}(x)]_{\emptyset}^s = \bigcup_{x \in A_q} [x, \varphi_{pq}(x)]_{\emptyset}^s.$$

Lemma 2.5. *There is a homeomorphism $h_{pq} : A_{pq} \rightarrow A_p \times [0, 1]$ such that*

$$h_{pq}([x, y]_{\emptyset}^s) = \{x\} \times [0, 1] \text{ for any } x \in A_p.$$

Proof. Let $\lambda(x, z)$ be the length of the closed arc $[x, z]^s \subset (x, y)_{\emptyset}^s$. Define h_{pq} as

$$h_{pq}(z) = \left(x, \frac{\lambda(x, z)}{\lambda(x, \varphi_{pq}(x))}\right), \quad \text{where } z \in [x, \varphi_{pq}(x)]^s \subset A_{pq}.$$

Since the arcs $[x, \varphi_{pq}(x)]_{\emptyset}^s, [x', \varphi_{pq}(x')]_{\emptyset}^s$ with $x \neq x'$ are pairwise disjoint, we see that h_{pq} is one-to-one. Due to Lemma 2.2, h_{pq} is a homeomorphism. \square

As a consequence, A_{pq} is homeomorphic to the n -annulus $\Sigma^{n-2} \times [0, 1]^2$. By definition, put

$$D_{pq}(S_{pq}) \stackrel{\text{def}}{=} D_{pq} = \bigcup_{j \geq 0} f^{jm}(A_{pq}).$$

It follows from (3) and (4) that

$$(5) \quad f^{jm}(C_{pq}) \subset D_{pq}, \quad j \geq 0.$$

Lemma 2.6. *We have $D_{pq} = \bigcup_{x \in W^u(p) - \text{int } D_p} [x, \varphi_{pq}(x)]_{\emptyset}^s$.*

Proof. It follows from

$$A_{pq} = \bigcup_{x \in A_p} [x, y]_{\emptyset}^s, \quad \text{and } W^u(p) - \text{int } D_p = \bigcup_{j \geq 0} f^{jm}(A_p)$$

that

$$D_{pq} \subset \bigcup_{x \in W^u(p) - \text{int } D_p} [x, \varphi_{pq}(x)]_{\emptyset}^s.$$

Take $[x', \varphi_{pq}(x')]_{\emptyset}^s$ with $x' \in W^u(p) - \text{int } D_p$. Due to (3), $x' = f^{jm}(x)$ for some $j \geq 0$ and $x \in W^u(p) - \text{int } D_p$. Thus,

$$\bigcup_{x \in W^u(p) - \text{int } D_p} [x, \varphi_{pq}(x)]_{\emptyset}^s \subset D_{pq}. \quad \square$$

Denote by $\pi_p : D_{pq} \rightarrow W^u(p) - \text{int } D_p$ the projection that takes each point $a \in D_{pq}$ to x whenever $a \in [x, y]_\emptyset^s \subset D_{pq}$. According to Lemma 2.6, the map π_p is well-defined because the arcs $[x, \varphi_{pq}(x)]_\emptyset^s$, $[x', \varphi_{pq}(x')]_\emptyset^s$ are pairwise disjoint whenever $x \neq x'$.

Lemma 2.7. *The triple $(D_{pq}, W^u(p) - \text{int } D_p, \pi_p)$ is a trivial fiber bundle with fiber the interval $[0; 1]$.*

Proof. Note that due to Lemma 2.5, the triple $(D_{pq}, W^u(p) - \text{int } D_p, \pi_p)$ is a fiber bundle with fiber the unit interval $[0; 1]$, because $D_{pq} = \bigcup_{j \geq 0} f^{jm}(A_{pq})$ and $A_{pq} \cong A_p \times [0; 1]$ is a fiber bundle with fiber the unit interval $[0; 1]$. In order to prove that this fiber bundle is trivial, we have to construct a bundle map $H_{pq} : D_{pq} \rightarrow (W^u(p) - \text{int } D_p) \times [0, 1]$, which is an isomorphism to the product bundle

$$((W^u(p) - \text{int } D_p) \times [0, 1], W^u(p) - \text{int } D_p, pr_1),$$

where $pr_1 : (W^u(p) - \text{int } D_p) \times [0, 1] \rightarrow W^u(p) - \text{int } D_p$ is the projection on the first factor. Define H_{pq} as

$$H_{pq}(z) = \left(x, \frac{\lambda(x, z)}{\lambda(x, \varphi_{pq}(x))} \right), \quad \text{where } z \in [x, \varphi_{pq}(x)]_\emptyset^s \subset D_{pq}.$$

It immediately follows that H_{pq} takes each $\pi_p^{-1}(x)$ to $\{x\} \times [0, 1]$, where $x \in W^u(p) - \text{int } D_p$. Thus, H_{pq} is a bundle map. Since the arcs $[x, \varphi_{pq}(x)]_\emptyset^s$, $[x', \varphi_{pq}(x')]_\emptyset^s$ with $x \neq x'$ are pairwise disjoint, we see that H_{pq} is one-to-one. Due to Lemma 2.2, H_{pq} is a homeomorphism (here we consider D_{pq} to be endowed with the interior topology). \square

Corollary 2.3. *Each inclusion $S_p = \partial D_p \subset D_{pq}$, $S_q = \partial D_q \subset D_{pq}$ induces the isomorphism*

$$\pi_{n-2}(S_p) \rightarrow \pi_{n-2}(D_{pq}), \quad \pi_{n-2}(S_q) \rightarrow \pi_{n-2}(D_{pq}) \cong \mathbb{Z},$$

respectively. Moreover, both S_p and S_q are homotopy equivalent to D_{pq} .

Proof. Due to Lemma 2.7, $W^u(p) - \text{int } D_p$ is a retract of D_{pq} . To conclude the proof, it remains to note that S_p is a retract of $W^u(p) - \text{int } D_p$ because $W^u(p) \cong \mathbb{R}^{n-1}$. \square

Now we will show that the property of characteristic spheres to bound a ball in M^n depends on an embedding of Ω into M^n but not on the choice of characteristic spheres. Let $S_{pq} = D_p \cup D_q \cup C_{pq}$ and $S'_{pq} = D'_p \cup D'_q \cup C'_{pq}$ be characteristic spheres corresponding to the same 2-bunch B_{pq} .

Lemma 2.8. *The characteristic spheres S_{pq} , S'_{pq} are isotopic in M^n . In particular, S_{pq} bounds a ball in M^n iff S'_{pq} does.*

Proof. Recall that both $S_p = \partial D_p$ and $S'_p = \partial D'_p$ are C^1 embedded in $W^u(p)$ $(n - 2)$ -spheres (in particular, they are tamely embedded). Therefore there is an isotopy $F_t : W^u(p) \rightarrow W^u(p)$, $0 \leq t \leq 1$, taking S_p to S'_p [36]. Moreover, one can construct F_t so that $F_t(p) = p$ for all t , since p is inside of the spheres S_p and S'_p . Hence, F_t is extended to an isotopy of the cylinders C_{pq} , C'_{pq} and the spheres $S_q = \partial D_q$, $S'_q = \partial D'_q$. As a consequence, S_{pq} is isotopic to S'_{pq} . \square

3. PROPERTIES OF CHARACTERISTIC SPHERES

Let $\Omega \subset M^n$ be an orientable codimension one expanding attractor of an A -diffeomorphism $f : M^n \rightarrow M^n$, and let $\{p_i, q_i\}_{i=1}^k$ be pairs of associated points of Ω . Each pair p_i, q_i of associated points corresponds to the 2-bunch $B_{p_i, q_i} = W^u(p_i) \cup W^u(q_i)$. Denote by S_{p_i, q_i} the characteristic sphere corresponding to B_{p_i, q_i} and consisting of the connecting $(n-1)$ -cylinder C_{p_i, q_i} and the closed $(n-1)$ -balls

$$D_{p_i} \subset W^u(p_i), D_{q_i} \subset W^u(q_i), \text{ where } D_{p_i} \subset \text{int } f^{m_i}(D_{p_i}), D_{q_i} \subset \text{int } f^{m_i}(D_{q_i}),$$

and $m_i = m(p_i, q_i)$ is the (same) period of the points p_i, q_i ($i = 1, \dots, k$). The $(n-2)$ -spheres $S_{p_i}^{n-2} = \partial D_{p_i}$, $f^{m_i}(S_{p_i}^{n-2})$ bound in $W^u(p_i)$ a closed $(n-1)$ -annulus, say A_{p_i} , which is homeomorphic to $\Sigma^{n-2} \times [0, 1]$. By definition, put

$$A_{p_i, q_i} = \bigcup_{x \in A_{p_i}} [x, \varphi_{p_i, q_i}(x)]_{\emptyset}^s \cong A_{p_i} \times [0, 1] \cong \Sigma^{n-2} \times [0, 1]^2,$$

where the homeomorphism

$$\varphi_{p_i, q_i} = \varphi_i : W^u(p_i) \cup W^u(q_i) \rightarrow W^u(p_i) \cup W^u(q_i)$$

satisfies property (2) of Corollary 2.1. Due to Lemma 2.6,

$$D_{p_i, q_i} \stackrel{\text{def}}{=} \bigcup_{j \geq 0} f^{jm_i}(A_{p_i, q_i}) = \bigcup_{x \in W^u(p_i) - \text{int } D_{p_i}} [x, \varphi_i(x)]_{\emptyset}^s.$$

Lemma 3.1. *Suppose Ω is a codimension one orientable expanding attractor and $\{p_i, q_i\}_{i=1}^k$ are pairs of associated boundary periodic points of Ω . Then the following properties hold:*

1) *The set*

$$D(\Omega) \stackrel{\text{def}}{=} \bigcup_{i=1}^k \bigcup_{j \geq 0} f^{jm_i}(A_{p_i, q_i}) \cup \Omega = \bigcup_{i=1}^k D_{p_i, q_i} \cup \Omega$$

is the topological n -submanifold of M^n with boundary $\partial D(\Omega) = \bigcup_{i=1}^k S_{p_i, q_i}$. In particular, the family of characteristic spheres $\bigcup_{i=1}^k S_{p_i, q_i}$ divides $M = M^n$. Moreover, any component of $D(\Omega)$ contains a unique C -dense component of Ω and every C -dense component of Ω belongs to a unique component of $D(\Omega)$.

2) *If each characteristic sphere S_{p_i, q_i} bounds a closed ball, say $E_{p_i, q_i} \subset M$, then*

$$M = \bigcup_{i=1}^k E_{p_i, q_i} \cup W^s(\Omega) = \bigcup_{i=1}^k \bigcup_{j \geq 0} f^{jm_i}(E_{p_i, q_i}) \cup \Omega.$$

Moreover, Ω is C -dense.

Proof. Taking an iteration of f , if necessary, we can assume without loss of generality that all periodic boundary points $\{p_i, q_i\}_{i=1}^k$ are fixed. Let us show that the first assertion follows from the second one. Cutting M along the characteristic spheres, we get finitely many topological n -manifolds M_1, \dots, M_l . Suppose M_1 contains boundary unstable manifolds of some boundary fixed points of Ω , and let $\Omega_1 \subset M_1 \cap \Omega$ be the part of Ω that belongs to M_1 and consists of whole unstable manifolds, i.e., if $W^u \cap \Omega_1 \neq \emptyset$, then $W^u \subset M_1$.

Let p be a boundary fixed point and $W^u(p) \subset M_1$. By construction of the characteristic sphere S_{pq} , $(W^u(q) - D_q) \cap M_1 \neq \emptyset$, where q is the boundary fixed point

associated with p . Further, $W^u(q) - D_q$ does not intersect connecting cylinders of Ω , and thus $W^u(q) - D_q$ can't intersect the boundary ∂M_1 of M_1 . Therefore, $W^u(q) \subset M_1$. Thus, Ω_1 contains a boundary unstable manifold $W^u(p)$ if and only if Ω_1 contains the boundary unstable manifold $W^u(q)$, where p, q are associated points. Renumbering associated points of Ω , we can assume that the associated points $p_1, q_1, \dots, p_{k_1}, q_{k_1}$ and their unstable manifolds $W^u(p_1), W^u(q_1), \dots, W^u(p_{k_1}), W^u(q_{k_1})$ are in M_1 . By construction, ∂M_1 contains the $(n - 1)$ -spheres $S_{p_1 q_1}, \dots, S_{p_{k_1} q_{k_1}}$.

Take any unstable manifold $W^u(y) \subset \Omega$ which is different from $W^u(p_i), W^u(q_i), 1 \leq i \leq k_1$, and intersects ∂M_1 . Since any unstable manifold $W^u \subset \Omega$ does not intersect the (open) connecting cylinders of Ω , we see that the intersection $W^u(y) \cap M_1$ is the $(n - 1)$ -ball $D_r \subset \Omega$, where $r \in \Omega$ is the boundary fixed point different from $p_i, q_i, 1 \leq i \leq k_1$. The argument shows that if $W^u(x) \subset \Omega$ is the unstable manifold through $x \in \text{int } M_1$, then $W^u(x) \subset M_1$. Moreover, $W^u(x)$ is in $\text{int } M_1$. Thus, Ω_1 is the union of the unstable manifolds $W^u(p_1), W^u(q_1), \dots, W^u(p_{k_1}), W^u(q_{k_1})$ and the unstable manifolds from Ω that belongs to $\text{int } M_1$.

Now let us consider M_1 to be a subset of M^n . It follows from (3), (4), and (5) that the set

$$\hat{M}_1^+ \stackrel{\text{def}}{=} \text{clos} (f(M_1) - M_1) \cup \text{clos} (M_1 - f(M_1))$$

is a union of annuli of type A_{pq} attached to the boundary ∂M_1 . Hence, M_1 is a retract of both $\hat{M}_1^+ \cup M_1$ and $M_1 - \hat{M}_1^+$. As a consequence, given any $W^u(x) \subset \text{int } M_1$ with $x \in \Omega_1$, there is a path ω in $M_1 - \hat{M}_1^+$ connecting the points x and p_1 , because M_1 is arcwise connected. It follows that $f^{-1}(x) \in M_1$, since $f^{-1}(p_1) = p_1$. Hence, $f^{-1}(W^u(x)) \subset M_1$. Similarly, considering the set

$$\hat{M}_1^- \stackrel{\text{def}}{=} \text{clos} (f^{-1}(M_1) - M_1) \cup \text{clos} (M_1 - f^{-1}(M_1)),$$

one can prove that $f(W^u(x)) \subset M_1$. Since Ω_1 is a union of unstable manifolds, $f^{\pm 1}(\Omega_1) = \Omega_1$, i.e., Ω_1 is invariant under f . The argument above shows that any basic set $\Lambda \subset M_1$ different from Ω is invariant under f , $f(\Lambda) = \Lambda$, because $\hat{M}_1^- \cup \hat{M}_1^+ \subset W^s(\Omega)$, and hence $(\hat{M}_1^- \cup \hat{M}_1^+) \cap \Lambda = \emptyset$.

The boundary ∂M_1 of M_1 consists of $k_2 \geq k_1$ $(n - 1)$ -spheres. Attaching n -balls B_1, \dots, B_{k_2} to M_1 along each $(n - 1)$ -sphere of ∂M_1 , we obtain the closed n -manifold M'_1 . Since

$$\hat{M}_1^- \cup \hat{M}_1^+ = \bigcup_{i=1}^{k_1} f(A_{p_i q_i}) \cup \bigcup_{i=k_1+1}^{k_2} f(A_{p_i q_i}),$$

$f|_{M_1}$ can be extended to the A -diffeomorphism $f_1 : M'_1 \rightarrow M'_1$ so that f_1 has a unique repelling or attracting fixed point O_i inside each attached n -ball B_i ($1 \leq i \leq k_2$). To be precise, the point O_i is repelling for $1 \leq i \leq k_1$ and attracting for $k_1 + 1 \leq i \leq k_2$. The set $\Omega_1 \subset M_1$ becomes the invariant set $\Omega' \subset M'_1$ of f_1 (i.e., Ω' coincides with Ω_1 as a set). Moreover, constructing f_1 carefully, one can get Ω' to be a hyperbolic attractor, because

$$\Omega_1 \cap \partial M_1 = \bigcup_{i=1}^{k_1} (D_{p_i} \cup D_{q_i}).$$

Since Ω' is homeomorphic locally to the product of \mathbb{R}^{n-1} and a Cantor set, Ω' is a codimension one orientable expanding attractor.

By construction, any characteristic sphere of Ω' bounds a ball. Due to the second assertion,

$$M'_1 = \bigcup_{\nu=1}^{k_2} \bigcup_{j \geq 0} f^j(E_{p_\nu q_\nu}) \cup \Omega' = \bigcup_{\nu=1}^{k_2} E_{p_\nu q_\nu} \cup W^s(\Omega')$$

and Ω' is a C -dense basic set. Since

$$\bigcup_{\nu=1}^{k_2} S_{p_\nu q_\nu} = \bigcup_{\nu=1}^{k_2} \partial E_{p_\nu q_\nu} \subset W^s(\Omega'),$$

we have $k_2 = k_1$, as otherwise some connecting cylinders would belong to both $W^s(\Omega')$ and the stable manifold of the corresponding attracting fixed point, which is impossible.

Denote $A_{p_\nu q_\nu} = \text{clos}(f(E_{p_\nu q_\nu}) - E_{p_\nu q_\nu})$, $1 \leq \nu \leq k_1$. The argument above shows that

$$M_1 = \bigcup_{\nu=1}^{k_1} \bigcup_{j \geq 0} f^j(A_{p_\nu q_\nu}) \cup \Omega'$$

is the topological n -submanifold of M^n with the boundary $\partial D(\Omega') = \bigcup_{i=1}^{k_1} S_{p_i q_i}$. Similar reasoning applies to the remaining n -manifolds M_i containing at least one whole boundary unstable manifold of Ω . Note that if $\text{int } M_i$ ($1 \leq i \leq l$) contains an unstable manifold $W^u \subset \Omega$ that belongs to a C -dense component Ω_c of Ω , then any boundary unstable manifold from Ω_c is in M_i because W^u is dense in Ω_c [2], [8]. As a consequence, $\Omega_{c,i} = \Omega_c \subset M_i$, where $\Omega_{c,i} = M_i \cap \Omega$.

What we have proved is that a component M_i ($1 \leq i \leq l$) contains a unique C -dense component $\Omega_{c,i}$ of Ω whenever M_i contains at least one unstable manifold of Ω . Obviously, a union of such components M_i forms the set $D(\Omega)$, which splits into pieces of the kind $M_i = \bigcup_{\nu=1} \bigcup_{j \geq 0} f^j(A_{p_\nu q_\nu}) \cup \Omega_{c,i}$. Thus the second assertion implies the first one.

To conclude the proof, it remains to prove the second assertion. We begin with the equality

$$\bigcup_{i=1}^{k_1} E_{p_i q_i} \cup W^s(\Omega_c) = \bigcup_{i=1}^{k_1} \bigcup_{j \geq 0} f^j(E_{p_i q_i}) \cup \Omega_c \stackrel{\text{def}}{=} M_0,$$

where Ω_c is some C -dense component of Ω . Here we suppose that the associated points $\{p_i, q_i\}_{i=1}^{k_1}$, $k_1 \leq k$, belong to Ω_c . It follows from (3) and (4) that

$$A_{p_i q_i} = \text{clos}(f(E_{p_i q_i}) - E_{p_i q_i}) \subset D_{p_i q_i}.$$

By Lemma 2.6, $D_{p_i q_i} \subset W^s(\Omega_c)$ for $i \leq k_1$. Hence,

$$\bigcup_{i=1}^{k_1} \bigcup_{j \geq 0} f^j(E_{p_i q_i}) \cup \Omega \subset \bigcup_{i=1}^{k_1} E_{p_i q_i} \cup W^s(\Omega_c).$$

To conclude the inverse inclusion, take a point $z \in W^s(\Omega_c) - \Omega_c$. By Lemma 1.9, z belongs either to some arc $(x, y)_\emptyset^s$ with $x \in W^u(p_i)$ and $y \in W^u(q_i)$, or $z \in W_\emptyset^s(p_i)$ for some periodic boundary point $p_i \in \Omega_c$ ($i \leq k_1$). In the last case, $z \in E_{p_i q_i}$. In the case of $z \in (x, y)_\emptyset^s$, we have to consider two possibilities. If $x \in D_{p_i}$, then

$z \in E_{p_i q_i}$. If $x \notin D_{p_i}$, then $z \in D_{p_i q_i}$. Thus in any case, $z \in \bigcup_{j \geq 0} f^j(E_{p_i q_i})$. Hence,

$$W^s(\Omega_c) - \Omega_c \subset \bigcup_{i=1}^{k_1} \bigcup_{j \geq 0} f^j(E_{p_i q_i}).$$

Let us show that the set M_0 is closed and open. Obviously, $W^s(\Omega_c)$ is open. Any characteristic sphere $S_{p_i q_i}$ belongs to $W^s(\Omega_c)$, because $\Omega_c \subset W^s(\Omega_c)$. Hence, $M_0 = \bigcup_{i=1}^{k_1} E_{p_i q_i} \cup W^s(\Omega_c)$ is open.

Take a point $z_0 \in \text{clos } M_0$. Then there is a sequence of points

$$z_s \in \bigcup_{i=1}^{k_1} \bigcup_{j \geq 0} f^j(E_{p_i q_i}) \cup \Omega_c$$

that approach z_0 as $s \rightarrow \infty$. If there are infinitely many points z_s from Ω_c , then $z_0 \in \Omega_c \subset M_0$, because Ω_c is closed. Therefore we can assume that not all points z_s are in Ω_c . Then, by Lemma 1.9, $z_s \in \bigcup_{j \geq 0} f^j(E_{p_i q_i}) - \Omega_c$ for some $1 \leq i \leq k_1$. Hence, $z_s \in f^{j_s}(E_{p_i q_i})$ for some sequence of natural j_s . If the sequence j_s is bounded, then z_0 belongs to a finite union of n -balls of the type $f^{lm}(E_{p_i q_i})$, and thus, $z_0 \in M_0$. If the sequence j_s is unbounded, then $z_s \in (x_s, y_s)_\emptyset^s \subset f^{j_s}(E_{p_i q_i})$. Due to Lemma 1.10, $\lambda(x_s, y_s)_\emptyset^s \rightarrow 0$ as $s \rightarrow \infty$, where $\lambda(\cdot, \cdot)$ is the length of the arc (\cdot, \cdot) . Hence, $\lim_{s \rightarrow \infty} z_s = \lim_{s \rightarrow \infty} x_s$. Since $x_s \in \Omega_c$, $\lim_{s \rightarrow \infty} x_s \in \Omega_c$. Therefore, $z_0 = \lim_{s \rightarrow \infty} z_s \in \Omega_c$. As a consequence, M_0 is closed. Thus, $M = M_0$ because M is connected.

Let us show that $\Omega = \Omega_c$. Suppose not; then Ω has another C -dense component, say Ω_1 , different from Ω_c . Since $M = \bigcup_{i=1}^{k_1} E_{p_i q_i} \cup W^s(\Omega_c)$, it follows that $\Omega_1 \subset \bigcup_{i=1}^{k_1} E_{p_i q_i}$, because stable manifolds are pairwise disjoint. Any unstable manifold $W^u(x)$, $x \in \Omega_1$, is dense in Ω_1 [2], [8]. Hence, Ω_1 is a connected set, since $W^u(x) \subset \Omega_1$. As a consequence, Ω_1 belongs to a unique ball of the union $\bigcup_{i=1}^{k_1} E_{p_i q_i}$. To be definite, assume that $\Omega_1 \subset E_{p_1 q_1}$. Due to [2] and [8], Ω_1 is a codimension one expanding attractor of f^r for some $r \in \mathbb{N}$. Attaching the n -ball \mathbb{B}^n to $E_{p_1 q_1}$ along the boundaries $\partial \mathbb{B}^n = \Sigma^{n-1} \cong \partial E_{p_1 q_1}$, we obtain the n -sphere Σ^n . It follows from (5) that f^r maps the connecting cylinder $C_{p_1 q_1}$ inside of $D_{p_1 q_1}$. Hence, f^r can be extended to Σ^n so that the resulting diffeomorphism $\Sigma^n \rightarrow \Sigma^n$ has a codimension one expanding attractor.

On the other hand, Kollmer [32] proved that a simply connected n -manifold ($n \geq 3$) does not admit a hyperbolic attractor of codimension one (see also [42], where it was proved that a codimension one expanding attractor Λ determines a non-zero homology class if both subbundles E_Λ^s and E_Λ^u in the splitting $T_\Lambda M = E_\Lambda^s \oplus E_\Lambda^u$ are oriented). This contradiction concludes the proof. \square

Corollary 3.1. *Suppose the conditions of Lemma 3.1 hold. Let $\Omega' \neq \Omega$ be another basic set. If the unstable manifold $W^u(z')$ of some point $z' \in \Omega'$ intersects $D_{p_i q_i}$, then $W^u(z')$ intersects the connecting cylinder $C_{p_i q_i}$.*

Proof. Since stable manifolds are pairwise disjoint, $\Omega' \cap D_{p_i q_i} = \emptyset$. Also, $W^u(z')$ is connected and does not intersect $W^u(p_i) \cup W^u(q_i)$. Then the proof follows from Lemma 3.1. \square

4. SOME APPLICATIONS OF THE LAMINATION THEORY

Let \overline{M} be a universal (simply connected) covering space for M , and $\pi : \overline{M} \rightarrow M$ a corresponding cover. Two points $\overline{x}, \overline{y} \in \overline{M}$ are said to be *congruent* if $\pi(\overline{x}) = \pi(\overline{y})$. We will generally identify the fundamental group $\pi_1(M)$ with the group of covering transformations for the cover $\pi : \overline{M} \rightarrow M$. Thus, \overline{x} and \overline{y} are congruent if and only if there is $\alpha \in \pi_1(M)$ such that $\alpha(\overline{x}) = \overline{y}$.

Let $f : M \rightarrow M$ be a diffeomorphism. Since \overline{M} is a universal covering, we see that there is a lift $\overline{f} : \overline{M} \rightarrow \overline{M}$ of f , i.e., $\overline{f} \circ \pi = \pi \circ f$. Note that \overline{f} is a diffeomorphism as well. If f is an A -diffeomorphism, then \overline{f} has the invariant set $\pi^{-1}(NW(f))$ with a local product structure [12], [27], [48]. Let us denote by $w^{u(s)}(\overline{x})$ a lift of $W^{u(s)}(x)$, $x \in NW(f)$, where $\overline{x} \in \overline{M}$ is a lift of x . We get

$$\overline{M} = \bigcup_{\overline{x} \in \pi^{-1}(NW(f))} w^u(\overline{x}) = \bigcup_{\overline{x} \in \pi^{-1}(NW(f))} w^s(\overline{x}),$$

because $M = \bigcup_{x \in NW(f)} W^u(x) = \bigcup_{x \in NW(f)} W^s(x)$. Since each invariant manifold of f is an injectively immersed copy of Euclidean space, it follows that each $w^{u(s)}(\overline{x})$ is also an injectively immersed copy of Euclidean space. Moreover, the families

$$\{w^u(\overline{x})\}_{\overline{x} \in \pi^{-1}(NW(f))} \quad \text{and} \quad \{w^s(\overline{x})\}_{\overline{x} \in \pi^{-1}(NW(f))}$$

are invariant under \overline{f} . Therefore, $w^{u(s)}(\overline{x})$ is said to be the *unstable (stable) invariant manifold* of \overline{f} . Thus, \overline{M} splits into invariant manifolds which are lifts of the invariant manifolds of f .

Suppose $f : M \rightarrow M$ is an A -diffeomorphism with an orientable expanding attractor or contracting repeller Ω of codimension one; then $\overline{\Omega} = \pi^{-1}(\Omega)$ is an invariant hyperbolic set of \overline{f} consisting of codimension one submanifolds smoothly immersed in \overline{M} [12], [27]. Locally $\overline{\Omega}$ is the product of $(n-1)$ -dimensional Euclidean space \mathbb{R}^{n-1} and a Cantor set, because π is a local homeomorphism. Obviously one can define the notions of boundary points (not necessary periodic under \overline{f}), boundary manifolds, and bunches (as a consequence, also associated boundary points) for $\overline{\Omega}$. It is easily shown that the boundary points of $\overline{\Omega}$ are just the points of the preimage $\pi^{-1}(B_\Omega)$, where B_Ω is the set of the boundary points of Ω .

To be definite, assume that Ω is an expanding attractor. We keep the notation of section 2.

Lemma 4.1. *Let $B_{pq} = W^u(p) \cup W^u(q)$ be the bunch of Ω consisting of the unstable manifolds $W^u(p)$, $W^u(q)$ of the associated boundary periodic points p and q . Suppose \overline{p} is a lift of p ; then there is a unique lift \overline{q} of q such that $\overline{B}_{\overline{p}\overline{q}} = w^u(\overline{p}) \cup w^u(\overline{q}) \in \pi^{-1}(B_{pq})$ is a 2-bunch of $\overline{\Omega}$. Moreover, there is the homeomorphism*

$$\overline{\varphi}_{\overline{p}\overline{q}} : (w^u(\overline{p}) - \overline{p}) \cup (w^u(\overline{q}) - \overline{q}) \rightarrow (w^u(\overline{p}) - \overline{p}) \cup (w^u(\overline{q}) - \overline{q}),$$

which covers φ_{pq} and is extended to the homeomorphism

$$\overline{\varphi}_{\overline{p}\overline{q}} : w^u(\overline{p}) \cup w^u(\overline{q}) \rightarrow w^u(\overline{p}) \cup w^u(\overline{q}),$$

if we put $\overline{\varphi}_{\overline{p}\overline{q}}(\overline{p}) = \overline{q}$ and $\overline{\varphi}_{\overline{p}\overline{q}}(\overline{q}) = \overline{p}$.

Proof. Take an arbitrary point $\overline{x} \in w^u(\overline{p}) - \overline{p}$. Since $\pi(w^u(\overline{p}) - \overline{p}) = W^u(p) - p$, we have $x = \pi(\overline{x}) \in W^u(p) - p$. Let $[\overline{x}, \overline{y}]^s$ be the lift of $[x, \varphi_{pq}(x)]_\emptyset^s$. Obviously, $(\overline{x}, \overline{y})^s \cap \overline{\Omega} = \emptyset$ and $\overline{y} \in \overline{\Omega}$. Hence, $(\overline{x}, \overline{y})^s = (\overline{x}, \overline{y})_\emptyset^s$. Recall that B_Ω is the set of

boundary points of Ω . It follows from the definition of boundary points and Lemma 1.7 that there is a point $\bar{q} \in \pi^{-1}(B_\Omega)$. Due to the path lifting theorem (see, e.g., [54], ch. 1), \bar{q} is a lift of the point q . By Lemma 1.11 and Corollary 1.3, $\bar{q} \neq \bar{p}$ and \bar{q} does not depend on the choice of the point $\bar{x} \in w^u(\bar{p}) - \bar{p}$. As a consequence,

$$\overline{B_{\bar{p}\bar{q}}} = w^u(\bar{p}) \cup w^u(\bar{q}) \in \pi^{-1}(B_{pq})$$

is a 2-bunch of $\overline{\Omega}$ which is a lift of the 2-bunch B_{pq} .

The argument shows that given any point $\bar{x} \in w^u(\bar{p}) - \bar{p}$, there is a unique point $\bar{y} \in w^u(\bar{q})$ such that $(\bar{x}, \bar{y})^s = (\bar{x}, \bar{y})_\emptyset^s$, and vice versa. By definition, put $\overline{\varphi_{\bar{p}\bar{q}}}(\bar{x}) = \bar{y}$, $\overline{\varphi_{\bar{p}\bar{q}}}(\bar{y}) = \bar{x}$. Due to the path lifting theorem, $\overline{\varphi_{\bar{p}\bar{q}}}$ is a lift of φ_{pq} . This implies that $\overline{\varphi_{\bar{p}\bar{q}}}$ is extended to the homeomorphism $\overline{\varphi_{\bar{p}\bar{q}}} : w^u(\bar{p}) \cup w^u(\bar{q}) \rightarrow w^u(\bar{p}) \cup w^u(\bar{q})$, if we put $\overline{\varphi_{\bar{p}\bar{q}}}(\bar{p}) = \bar{q}$ and $\overline{\varphi_{\bar{p}\bar{q}}}(\bar{q}) = \bar{p}$, because π is a local homeomorphism. \square

Corollary 4.1. *Suppose B_{pq} is a bunch of Ω ; then $\pi^{-1}(B_{pq})$ splits into pairwise disjoint 2-bunches of $\overline{\Omega}$. Moreover, each bunch of $\pi^{-1}(B_{pq})$ is a lift of B_{pq} .*

Proof. It follows from Lemma 4.1 that $\pi^{-1}(B_{pq})$ consists of 2-bunches of $\overline{\Omega}$. These bunches are pairwise disjoint, because no invariant manifold of $\overline{\Omega}$ has congruent points. \square

Lemma 4.2. *Let $B_{pq} = W^u(p) \cup W^u(q)$ be a bunch of Ω and $\overline{B_{\bar{p}\bar{q}}}$ a lift of B_{pq} which is a bunch of $\overline{\Omega}$, where $\pi(\bar{p}) = p$ and $\pi(\bar{q}) = q$. Suppose S_{pq} is a characteristic sphere corresponding to B_{pq} and consisting of a connecting cylinder C_{pq} and two $(n - 1)$ -balls D_p, D_q . Then there is the characteristic sphere $\overline{S_{\bar{p}\bar{q}}}$ corresponding to $\overline{B_{\bar{p}\bar{q}}}$ and consisting of the connecting cylinder $\overline{C_{\bar{p}\bar{q}}}$ and two $(n - 1)$ -balls $\overline{D_{\bar{p}}}, \overline{D_{\bar{q}}}$ such that $\pi(\overline{D_{\bar{p}}}) = D_p$, $\pi(\overline{D_{\bar{q}}}) = D_q$, and $\pi(\overline{C_{\bar{p}\bar{q}}}) = C_{pq}$. Moreover, $\pi^{-1}(S_{pq})$ splits into pairwise disjoint characteristic spheres which are in one-to-one correspondence to the bunches of $\pi^{-1}(B_{pq})$.*

Proof. It follows from the path lifting theorem [54] that there is an $(n - 1)$ -ball $\overline{D_{\bar{p}}} \subset w^u(\bar{p})$ such that $\bar{p} \in \overline{D_{\bar{p}}}$ and $\overline{D_{\bar{p}}}$ is a lift of D_p . Similarly, there is $(n - 1)$ -ball $\overline{D_{\bar{q}}} \subset w^u(\bar{q})$ such that $\bar{q} \in \overline{D_{\bar{q}}}$ and $\overline{D_{\bar{q}}}$ is a lift of D_q . By Lemma 4.1,

$$\overline{\varphi_{\bar{p}\bar{q}}}(\overline{D_{\bar{p}}}) = \overline{D_{\bar{q}}}, \quad \overline{\varphi_{\bar{p}\bar{q}}}(\overline{D_{\bar{q}}}) = \overline{D_{\bar{p}}}.$$

Hence, the set $\overline{C_{\bar{p}\bar{q}}} \stackrel{\text{def}}{=} \bigcup_{\bar{x} \in \partial \overline{D_{\bar{p}}}} (\bar{x}, \overline{\varphi_{\bar{p}\bar{q}}}(\bar{x}))_\emptyset^s$ is homeomorphic to the open $(n - 1)$ -cylinder $\Sigma^{n-2} \times (0, 1)$, and $\pi(\overline{C_{\bar{p}\bar{q}}}) = C_{pq}$. As a consequence, the union

$$\overline{S_{\bar{p}\bar{q}}} = \overline{D_{\bar{p}}} \cup \overline{D_{\bar{q}}} \bigcup_{\bar{x} \in \partial \overline{D_{\bar{p}}}} (\bar{x}, \bar{y})_\emptyset^s$$

is a characteristic sphere corresponding to $\overline{B_{\bar{p}\bar{q}}}$.

Since no unstable manifold of $\overline{\Omega}$ has congruent points, we see that the restriction of π on each characteristic sphere of $\overline{\Omega}$ is injective. This concludes the proof. \square

Now we recall the notion of a lamination. Let us fix a natural number $1 \leq d \leq n$ and integers $0 \leq l \leq r \leq \infty$. Suppose $N \subset M$ is a closed set. We say that \mathcal{L} is a d -dimensional $C^{r,l}$ -lamination on N , $\text{supp } \mathcal{L} = N$, if N is a union of pairwise disjoint injectively immersed d -manifolds L_α , $N = \bigcup_\alpha L_\alpha$ (called the *leaves*), and for each $x \in \text{supp } \mathcal{L}$ there are a neighborhood $U(x) \subset M$, $x \in U(x)$, and a C^l -diffeomorphism $\varphi_x : U(x) \rightarrow \mathbb{R}^n$ such that φ_x maps every connected component of

the intersection $U(x) \cap L_\alpha$ onto a d -dimensional hyperplane

$$x_{d+1} = c_1, \dots, x_n = c_{n-d} \text{ for some } (c_1, \dots, c_{n-d}) \in \mathbb{R}^{n-d}.$$

The restriction $\varphi_x|_{U(x) \cap L_\alpha}$ is a C^r -diffeomorphism onto the image $\varphi_x(U(x) \cap L_\alpha)$.

Notice that a C^0 -diffeomorphism means a homeomorphism. The pair $(U(x), \varphi_x)$ is called a *laminated chart*. Given any chart $(U(x), \varphi_x)$ and leaf L_α , connected components of the intersection $U(x) \cap L_\alpha$ are called *local leaves*. For $d = n - 1$, \mathcal{L} is called a *codimension one lamination*.

In a sense, the notion of lamination is a weakening of the well-known notion of a foliation. If $\text{supp } \mathcal{L} = M$, then a $C^{r,l}$ -lamination \mathcal{L} is a C^l -foliation. Many definitions from foliation theory can be easily generalized to lamination theory (good references for foliation theory are Lawson's paper [33] and Tamura's book [57]). We presuppose some knowledge of foliations. As a consequence of the definition above, we get the following lemma.

Lemma 4.3. *Let \mathcal{L} be a d -dimensional $C^{r,l}$ -lamination and L_α a leaf of \mathcal{L} . Suppose X is a compact subset of L_α . Then there are a neighborhood $U(X) \subset M$ of $X \subset U(X)$ and a C^l -diffeomorphism $\varphi_X : U(X) \rightarrow \mathbb{R}^n$ such that, given any leaf L_β , φ_X maps every component of the intersection $U(X) \cap L_\beta$ (a local leaf in $U(X)$) onto a d -dimensional hyperplane $x_{d+1} = c_1, \dots, x_n = c_{n-d}$ for some $(c_1, \dots, c_{n-d}) \in \mathbb{R}^{n-d}$.*

Proof. There is a finite cover of the compact set X by laminated charts $(U(x_i), \varphi_i)$. Similarly to the proof of Theorem 5.1 of [57], one can construct a finite set of coherent laminated charts covering X . This implies the result. \square

Corollary 4.2. *If the conditions of Lemma 4.3 hold, then there are a neighborhood $U(X) \subset M$, $X \in U(X)$, and d -dimensional C^l -foliation \mathcal{F} in $U(X)$ such that the local leaves of \mathcal{L} in $U(X)$ are leaves of \mathcal{F} .*

Lemma 4.4. *If \mathcal{L} is a codimension one $C^{r,l}$ -lamination ($1 \leq l \leq r$) and L_α is a simply connected leaf of \mathcal{L} , then the holonomy group of L_α is trivial.*

Proof. Take a point $x_0 \in L_\alpha$ and a smooth closed loop $\gamma \subset L_\alpha$ at x_0 . We assume that γ is endowed with some orientation. Let l be a segment through x_0 which is transverse to \mathcal{L} . Then \mathcal{L} induces the map $G_\gamma : l \cap \mathcal{L} \rightarrow l \cap \mathcal{L}$ corresponding to the positive direction on γ . This map defines the element g_γ of the holonomy group (the reader is referred to [57] for more background on holonomy groups) of the lamination \mathcal{L} .

Since L_α is simply connected, γ is contractible; and we can assume that γ bounds a 2-disk $D \subset L_\alpha$ because L_α is a smoothly immersed submanifold of codimension one. By Corollary 4.2, there is a neighborhood $U(D)$ of D such that \mathcal{L} is embedded in a C^l -foliation. Reeb's stability theorem [46] implies that $G_\gamma = id$ (see Theorems 5.7 and 5.12 in [57] as well). Hence, $g_\gamma = id$. This concludes the proof. \square

Let \mathcal{L} be a codimension one $C^{r,l}$ -lamination ($1 \leq l \leq r$) such that each leaf is endowed with a normal orientation. Suppose L_α is a leaf of \mathcal{L} , and T is an arc or closed curve endowed with some orientation and transversal to L_α . We say that T intersects L_α *oriently* if the index of intersection at every point of $L_\alpha \cap T$ is the same (either $+1$ or -1).

Let $f : M \rightarrow M$ be an A -diffeomorphism with an orientable expanding attractor Ω . Then the unstable manifolds of points of Ω form the codimension one $C^{\infty,1}$ -lamination $\mathcal{L}(\Omega)$ whenever Ω is a codimension one attractor [27],

$$\mathcal{L}(\Omega) = \{W^u(x)\}_{x \in \Omega}, \quad \text{supp } \mathcal{L}(\Omega) = \bigcup_{x \in \Omega} W^u(x).$$

Hence, $\pi^{-1}(\mathcal{L}(\Omega)) = \mathcal{L}(\overline{\Omega})$ is a $C^{\infty,1}$ -lamination on \overline{M} consisting of the unstable manifolds of $\overline{\Omega}$, because π is a local diffeomorphism.

Lemma 4.5. *Let $T \subset M$ be a closed simple curve which is transversal to the leaves of the lamination $\mathcal{L}(\Omega)$. If T intersects oriently all leaves of $\mathcal{L}(\Omega)$, then T is not homotopic to zero in M .*

Proof. Assume the contrary. Then there is a smooth immersion $\psi : D^2 \rightarrow M$ such that $\psi(\partial D^2) = T$, where D^2 is a closed 2-disk. Since $\psi(D^2)$ is a compact set, we see that there is a finite cover $\{U_i\}_{i=1}^k$ of $\psi(D^2)$ with the laminated charts U_i , $\psi(D^2) \subset \bigcup_{i=1}^k U_i$. By Lemma 4.3, $\text{supp } \mathcal{L}(\Omega) \cap U_i$ can be embedded in a C^1 foliation, say \mathcal{F}_i , for every $1 \leq i \leq k$. Now we can use an argument quite similar to the one used by Franks in Lemma 5.1 of [12] to move $\psi(D^2)$ into general position under $\mathcal{L}(\Omega)$. Starting with U_1 , we shall alter $\psi(D^2)$ consecutively, leaving a general position of previous steps. As a result, we obtain $\psi(D^2)$ in general position under $\mathcal{L}(\Omega)$, i.e., $\psi(D^2)$ is transverse to the leaves of $\mathcal{L}(\Omega)$ except for isolated generic tangencies of saddle type or center type. Hence, one get the lamination $\mathcal{L}_{D^2} = \psi^{-1}(\mathcal{L}(\Omega) \cap \psi(D^2))$. Moving $\psi(D^2)$ slightly, one can assume that there are no two singularities which are joined by a leaf of \mathcal{L}_{D^2} .

We now apply Poincaré-Bendixon theory to this situation [6]. By condition, $\mathcal{L}(\Omega) \cap T \neq \emptyset$. Take a point $x_0 \in \partial D^2$ such that $\psi(x_0) \in \mathcal{L}(\Omega) \cap T$. Let l be the leaf crossing ∂D^2 at x_0 , and $[0, \infty) \rightarrow l$ a parametrization of l such that 0 corresponds to x_0 . Since T intersects oriently all leaves of $\mathcal{L}(\Omega)$, we see that l never intersects ∂D^2 after x_0 . Hence, the ω -limit set $\omega(l)$ of l is non-empty and belongs to D^2 . Without loss of generality we can assume that $\omega(l)$ is not a singularity, because \mathcal{L}_{D^2} has a finite number of singularities and the intersection $\mathcal{L}(\Omega) \cap T$ is homeomorphic to a Cantor set (see Lemma 1.10).

Since the support of a lamination is a closed set, $\omega(l)$ is closed and belongs to the lamination \mathcal{L}_{D^2} . Moreover, $\omega(l)$ consists of one-dimensional leaves and singularities of \mathcal{L}_{D^2} [6]. Let $l_0 \subset \omega(l)$ be a one-dimensional leaf. Such a leaf exists, because $\omega(l)$ is not a unique singularity. First, suppose that the limit set $\omega(l_0)$ of l_0 has no singularities. According to [6], $\omega(l_0)$ does not contain one-dimensional leaves except l_0 . Hence, l_0 is a closed leaf, and thus l tends spirally to l_0 . Let L_0 be the leaf of $\mathcal{L}(\Omega)$ with $\psi(l_0) \subset L_0$. Since $l_0 \subset \omega(l)$, $\psi(l_0)$ represents a nontrivial holonomy element of L_0 . But this contradicts Lemma 4.4 because each leaf of $\mathcal{L}(\Omega)$ is simply connected.

Suppose $\omega(l_0)$ contains singularities. Then $\omega(l_0)$ contains only one singularity, say s_0 . Hence, $\omega(l_0)$ is either a separatrix loop of s_0 or is a figure eight with center at s_0 . In either case, $\psi(\omega(l_0))$ represents a nontrivial holonomy element of the leaf that $\psi(\omega(l_0))$ belongs to. Again, this contradicts Lemma 4.4. \square

Corollary 4.3. *Let C be a simple closed curve which is a union of some arc $a \subset W^u(x)$, $x \in \Omega$, and a segment $s \subset W^s(x)$ belonging to some neighborhood of x with a local product structure. Then C is not homotopic to zero in M .*

Proof. By the orientability of Ω , s intersects oriently the leaves of the lamination $\mathcal{L}(\Omega)$. Therefore one can deform C to obtain a simple closed curve T that is homotopic to C and intersects oriently the leaves of the lamination $\mathcal{L}(\Omega)$ [3]. It follows from Lemma 4.5 that T , and as a consequence C , is not homotopic to zero in M . \square

Corollary 4.4. *Suppose Ω' is an orientable codimension one expanding attractor of an A -diffeomorphism $f : M \rightarrow M$. Then there are no closed balls $E \subset M$ such that $\Omega' \subset E$.*

Corollary 4.5. *Suppose Ω is a codimension one orientable expanding attractor of an A -diffeomorphism $f : M \rightarrow M$ and $\{p_i, q_i\}_{i=1}^k$ are pairs of associated boundary periodic points of Ω . If each characteristic sphere S_{p_i, q_i} of Ω bounds a ball in M , then Ω is the only orientable codimension one pseudotame basic set of f .*

5. TOPOLOGICAL STRUCTURE OF UNIVERSAL COVERING

In this section we keep the notation of sections 2 and 4. In particular, Ω is an expanding codimension one orientable attractor of an A -diffeomorphism $f : M \rightarrow M$, and $\bar{f} : \bar{M} \rightarrow \bar{M}$ is a lift of f under the universal covering map $\pi : \bar{M} \rightarrow M$.

Lemma 5.1. *Suppose Ω is a codimension one orientable expanding attractor and $\bar{\Omega} = \pi^{-1}(\Omega)$. Then the following properties hold:*

- 1) *For any $\bar{x}, \bar{y} \in \bar{\Omega}$, there is at most one point in $w^u(\bar{x}) \cap w^s(\bar{y})$.*
- 2) *For any point $\bar{x} \in \bar{\Omega}$, the inclusion $w^u(\bar{x}) \subset \bar{M}$ is an embedding.*
- 3) *For any point $\bar{x} \in \bar{\Omega}$, the inclusion $w^u(\bar{x}) \subset \bar{M}$ is a proper map, i.e., given any compact set $K \subset \bar{M}$, the intersection $K \cap w^u(\bar{x})$ is a compact subset in the topology of the unstable manifold $w^u(\bar{x})$.¹*
- 4) *For any point $\bar{x} \in \bar{\Omega}$, the set $\bar{M} - w^u(\bar{x})$ has two connected component (in other words, $w^u(\bar{x})$ divides \bar{M}).*

Proof. Our proof of the first statement follows the proof of Lemma 5.1 of [12]. Assume the contrary. Then there exist $\bar{x}, \bar{y} \in \bar{\Omega}$ such that $\bar{x}, \bar{y} \in w^u(\bar{x}) \cap w^s(\bar{y})$ and $\bar{x} \neq \bar{y}$. Note that $w^u(\bar{x}) = w^u(\bar{y})$ and $w^s(\bar{x}) = w^s(\bar{y})$ in this case. Let $[\bar{x}, \bar{y}]^u \subset w^u(\bar{x})$ be an arc that joins \bar{x} and \bar{y} in $w^u(\bar{x})$ such that $(\bar{x}, \bar{y})^u \cap (\bar{x}, \bar{y})^s = \emptyset$. Such an arc exists because the one-dimensional stable manifold $w^s(\bar{x})$ intersects the $(n-1)$ -unstable manifold $w^u(\bar{x})$ transversally. Hence, $\bar{A} = [\bar{x}, \bar{y}]^u \cup [\bar{x}, \bar{y}]^s$ is a simple closed curve.

Since \bar{M} is simply connected, \bar{A} is contractible. This implies that $\pi(\bar{A}) = A$ is a simple closed curve, which is homotopic to zero in M because every invariant manifold of $\bar{\Omega}$ has no congruent points. Since the points $x = \pi(\bar{x})$, $y = \pi(\bar{y})$ belong to the same stable manifold, it follows that there is a natural number j such that the arc $[f^j(x), f^j(y)]^s \subset f^j(A)$ belongs to a neighborhood with a local product structure. Obviously, $f^j(A)$ is homotopic to zero. We get a contradiction with Corollary 4.3. This concludes the proof of the first statement.

In fact, the second statement follows from the first one, because any point of $w^u(\bar{x})$ has a neighborhood with a local product structure.

Let us prove item 3). Assume the contrary. Then there is a sequence of points $\bar{z}_k \in w^u(\bar{x})$ such that $\bar{z}_k \rightarrow z \in K$ as $k \rightarrow \infty$, and this sequence is unbounded in

¹Note that item 2) does not imply item 3), because $w^u(\bar{x})$ is not a compact set in \bar{M} .

the topology of $w^u(\bar{x})$. According to [27], $\mathcal{L}(\bar{\Omega})$ is a codimension one lamination consisting of the unstable manifolds of $\bar{\Omega}$. Therefore, $\bar{z} \in \bar{\Omega}$ and there is a neighborhood $U(\bar{z})$ of \bar{z} , which is a laminated chart of $\mathcal{L}(\bar{\Omega})$. Without loss of generality one can assume that $U(\bar{z})$ is endowed with a local product structure. Let \bar{l}_k be a local leaf containing \bar{z}_k in $U(\bar{z})$. Since the sequence \bar{z}_k is unbounded in the topology of $w^u(\bar{x})$, we see that there are at least two different local leaves $l_{k_1} \neq l_{k_2}$. As a consequence, there is an arc $[\bar{a}_1, \bar{a}_2]^s \subset U(\bar{z})$ such that $\bar{a}_i \in l_{k_i}$, $i = 1, 2$. If we join the points \bar{a}_1, \bar{a}_2 with an arc $\bar{l}_{12} \subset w^u(\bar{x})$, we get a closed curve $\bar{C} = \bar{l}_{12} \cup [\bar{a}_1, \bar{a}_2]^s$, which is homotopic to zero because \bar{M} is simply connected. Hence, $\pi(\bar{C})$ is homotopic to zero in M . Again this contradicts Corollary 4.3, and so we complete the proof of item 3).

Finally, we prove item 4). Take an arc a that intersects $w^u(\bar{x})$ transversally at a unique point, say \bar{x}_a , and belongs to some neighborhood with a local product structure. By item 2), such an arc exists. We have to prove that any arc b connecting the endpoints of a must intersect $w^u(\bar{x})$. Assume the contrary. Without loss of generality one can assume that $\bar{C} = a \cup b$ is a simple smooth curve which is homotopic to zero in \bar{M} . Then there is a smooth immersion $\psi : D^2 \rightarrow M$ such that $\psi(\partial D^2) = \bar{C}$ and $\psi(D^2)$ is transversal to $w^u(\bar{x})$, where D^2 is a closed 2-disk. By item 3), the intersection $w^u(\bar{x}) \cap \psi(D^2)$ is a compact one-dimensional submanifold of $w^u(\bar{x})$. Hence, $w^u(\bar{x}) \cap \psi(D^2)$ consists of finitely many compact curves, each of which is homeomorphic to either a circle or the segment $[0; 1]$. By transversality of the intersection $w^u(\bar{x}) \cap \psi(D^2)$, every curve which is homeomorphic to $[0; 1]$ has endpoints on \bar{C} . This contradicts \bar{x}_a being the unique point of the intersection $w^u(\bar{x}) \cap \bar{C}$. \square

Remark. Another way to prove item 4) of Lemma 5.1 is an application of corollary 4.2 and Lemma 1 of section 3 of [23], where a similar assertion is proved for closed leaves of codimension one foliation.

Recall that for a given boundary point $\bar{p} \in \bar{\Omega}$, there is a unique boundary point $\bar{q} \in \bar{\Omega}$ such that \bar{p} and \bar{q} are associated (Lemma 4.1). Let $\bar{x} \in \bar{\Omega}$. By definition, put

$$w^{s*}(\bar{x}) = \begin{cases} w^s(\bar{x}), & \text{if } w^s(\bar{x}) \text{ does not contain boundary points of } \bar{\Omega}, \\ w^s(\bar{p}) \cup w^s(\bar{q}), & \text{if } w^s(\bar{x}) \text{ contains a boundary point } \bar{p}, \\ & \text{where } \bar{q} \text{ is associated with } \bar{p}. \end{cases}$$

The following lemma is a generalization of item 1) of Lemma 5.1.

Lemma 5.2. *Suppose Ω is a codimension one orientable expanding attractor and $\bar{\Omega} = \pi^{-1}(\Omega)$. Then for any $\bar{x}, \bar{y} \in \bar{\Omega}$, there is at most one point in $w^u(\bar{x}) \cap w^{s*}(\bar{y})$.*

Proof. Assume the contrary. Then there exist two points $\bar{x}, \bar{y} \in \bar{\Omega}$ such that the intersection $w^u(\bar{x}) \cap w^{s*}(\bar{y})$ contains at least two points, say \bar{z}_1, \bar{z}_2 . By Lemma 5.1, $w^{s*}(\bar{y})$ has associated boundary points \bar{p}, \bar{q} so that $w^{s*}(\bar{y}) = w^s(\bar{p}) \cup w^s(\bar{q})$ and $w^u(\bar{x}) \cap w^s(\bar{p}) \neq \emptyset$, $w^u(\bar{x}) \cap w^s(\bar{q}) \neq \emptyset$. Since Ω has finitely many boundary points, we see that there are points $\bar{a} \in w^u(\bar{x})$ arbitrary close to \bar{z}_1 such that $w^s(\bar{a})$ does not contain boundary points of $\bar{\Omega}$. The theorem on the continuous dependence of invariant manifolds on initial conditions implies that $w^s(\bar{a})$ intersects $w^u(\bar{x})$ near the point \bar{z}_2 . This contradicts Lemma 5.1. \square

Lemma 5.3. *Let Ω be a C -dense codimension one orientable expanding attractor and $\overline{\Omega} = \pi^{-1}(\Omega)$. Suppose there are points $\overline{x}, \overline{y}, \overline{z} \in \overline{\Omega}$ such that the following conditions hold:*

- $\overline{y} \in w^u(\overline{x}), \overline{z} \in w^s(\overline{x})$.
- *There is a sequence of points $\overline{x}_i \in [\overline{x}, \overline{z}]^s \cap \overline{\Omega}$ such that $\overline{x}_i \rightarrow \overline{z}$ as $i \rightarrow \infty$.*
- *The intersection $w^u(\overline{x}_i) \cap w^s(\overline{y})$ consists of a unique point \overline{y}_i for every $i \in \mathbb{N}$.*

Then the sequence of points \overline{y}_i is bounded on $w^s(\overline{y})$.

Proof. We follow the proof of Lemma 5.2 in [12]. Suppose the contrary. By Lemma 5.1, $\overline{M} - w^u(\overline{x})$ has two connected components. Denote by $w^{u+}(\overline{x})$ the component of $\overline{M} - w^u(\overline{x})$ which contains \overline{z} . According to Lemma 5.1, $[\overline{x}, \overline{z}]^s \subset w^{u+}(\overline{x})$. As a consequence, all points \overline{x}_i and \overline{y}_i belong to $w^{u+}(\overline{x})$. Therefore all \overline{y}_i belong to a unique component of $w^s(\overline{y}) - \overline{y}$, denoted by $w^{s+}(\overline{y})$, because the intersection $w^u(\overline{x}) \cap w^s(\overline{y})$ consists of the unique point \overline{y} . By our assumption, the sequence of points \overline{y}_i is unbounded on $w^{s+}(\overline{y})$. Hence, $\pi(w^{s+}(\overline{y}))$ has no boundary points of Ω . By Lemma 1.8, $\pi(w^{s+}(\overline{y})) \cap \Omega$ is dense in Ω , because the basic set Ω is C -dense. This shows that $\bigcup_{\eta \in \pi_1(M)} \eta(w^{s+}(\overline{y}))$ is dense in $\overline{\Omega}$. As a consequence, there is $\eta_0 \in \pi_1(M)$ such that $\eta_0(w^{s+}(\overline{y}))$ intersects a local product structure neighborhood of $\overline{x} \in \overline{\Omega}$. Moreover, according to the theorem on the continuous dependence of unstable manifolds on initial conditions, one can choose $\eta_0 \in \pi_1(M)$ so that each unstable manifold $w^u \subset \overline{\Omega}$ intersecting $[\overline{x}, \overline{z}]^s$ also intersects $\eta_0(w^{s+}(\overline{y}))$.

By definition, put $\alpha = \eta_0^{-1} \in \pi_1(M)$. It is clear that $\alpha \neq id$. The argument shows that any unstable manifold intersecting the segment $\alpha([\overline{x}, \overline{z}]^s)$ must intersect $w^{s+}(\overline{y})$.

Let us show that every unstable manifold of $\overline{\Omega}$ intersecting both $\alpha([\overline{x}, \overline{z}]^s)$ and $w^{s+}(\overline{y})$ has to intersect $[\overline{x}, \overline{z}]^s$. Take an unstable manifold $w^u(\overline{a})$, $\overline{a} \in \overline{\Omega}$, so that $w^u(\overline{a}) \cap \alpha([\overline{x}, \overline{z}]^s) \neq \emptyset$, $w^u(\overline{a}) \cap w^{s+}(\overline{y}) \neq \emptyset$. Since the sequence \overline{y}_i is unbounded on $w^s(\overline{y})$, it follows that there are points $\overline{y}_j, \overline{y}_{j+1}$ such that the point $\overline{a}_0 = w^u(\overline{a}) \cap w^{s+}(\overline{y})$ belongs to the arc $(\overline{y}_j, \overline{y}_{j+1})^s$ (it is possible that $\overline{y}_j = \overline{y}$ with $j = 0$). By Lemma 5.1, the points $\overline{y}_j, \overline{y}_{j+1}$ belong to different components of $\overline{M} - w^u(\overline{a})$. Hence the points

$$\overline{x}_j = w^u(\overline{y}_j) \cap [\overline{x}, \overline{z}]^s, \quad \overline{x}_{j+1} = w^u(\overline{y}_{j+1}) \cap [\overline{x}, \overline{z}]^s$$

belong to different components of $\overline{M} - w^u(\overline{a})$ as well (if $j = 0$, then $\overline{x}_j = \overline{x}$). By Lemma 5.1, $w^u(\overline{a}) = w^u(\overline{a}_0)$ intersects the segment $[\overline{x}, \overline{z}]^s$ between the points \overline{x}_j and \overline{x}_{j+1} .

This allows us to define a map

$$\lambda \circ \alpha([\overline{x}, \overline{z}]^s) \cap \overline{\Omega} \rightarrow [\overline{x}, \overline{z}]^s \cap \overline{\Omega}$$

as follows: $\lambda(\overline{a}) = [\overline{x}, \overline{z}]^s \cap w^u(\overline{a})$, where $\overline{a} \in \alpha([\overline{x}, \overline{z}]^s) \cap \overline{\Omega}$. The argument above shows that the map λ is well-defined. By Lemma 5.1, λ is order-preserving and injective. As a consequence, λ is continuous. Let us define the map

$$g : \alpha([\overline{x}, \overline{z}]^s) \cap \overline{\Omega} \rightarrow \alpha([\overline{x}, \overline{z}]^s) \cap \overline{\Omega} \text{ by } g(\overline{a}) = \alpha^{-1} \circ \lambda(\overline{a}), \quad \overline{a} \in \alpha([\overline{x}, \overline{z}]^s) \cap \overline{\Omega}.$$

The map g is continuous and order-preserving because $\alpha(\overline{\Omega}) = \overline{\Omega}$ and λ is continuous and order-preserving. Since $\alpha([\overline{x}, \overline{z}]^s) \cap \overline{\Omega}$ is a compact set, g has a fixed point $\overline{x}_0 \in \alpha([\overline{x}, \overline{z}]^s) \cap \overline{\Omega}$ which can be gotten as a limit $\lim_{k \rightarrow \infty} g^k(\overline{a})$. We have the

quality $\alpha(\bar{x}_0) = \lambda(\bar{x}_0)$, which contradicts the assertion that the unstable manifold $w^u(\bar{x}_0)$ has no congruent points. \square

Lemma 5.4. *Let Ω be a C -dense codimension one orientable expanding attractor and $\bar{\Omega} = \pi^{-1}(\Omega)$. Suppose $\bar{x}, \bar{z} \in \bar{\Omega}$ are points such that $\bar{z} \in w^s(\bar{x}) \cap \bar{\Omega}$. Then $w^{s*}(\bar{y}) \cap w^u(\bar{z}) \neq \emptyset$ for any point $\bar{y} \in w^u(\bar{x})$.*

Proof. By definition, put

$$\mathcal{B} \subset w^u(\bar{x}) = \{\bar{b} \in w^u(\bar{x}) \mid w^{s*}(\bar{b}) \cap w^u(\bar{z}) \neq \emptyset\}.$$

Clearly, $\mathcal{B} \neq \emptyset$ because $\bar{x} \in \mathcal{B}$. According to the theorem on the continuous dependence of stable and unstable manifolds on initial conditions, \mathcal{B} is open in $w^u(\bar{x})$. Let us prove that \mathcal{B} is closed in $w^u(\bar{x})$. Take a sequence of points $\bar{a}_i \in \mathcal{B}$ such that $\bar{a}_i \rightarrow \bar{b}_0$ as $i \rightarrow \infty$. By Lemma 5.2, the intersection $w^{s*}(\bar{a}_i) \cap w^u(\bar{z})$ is a unique point denoted by \bar{c}_i . Denote by $w^{u+}(\bar{x})$ the component of $\bar{M} - w^u(\bar{x})$ which contains \bar{z} . By Lemma 5.1, $[\bar{x}, \bar{z}]^s \subset w^{u+}(\bar{x})$ and, as a consequence, all points \bar{c}_i belong to $w^{u+}(\bar{x})$. Denote $w^{s+}(\bar{b}_0) = w^{s*}(\bar{b}_0) \cap w^{u+}(\bar{x})$.

Suppose that $(\bar{x}, \bar{z})^s \cap \bar{\Omega} = \emptyset$; then $(\bar{x}, \bar{z})^s = (\bar{x}, \bar{z})_{\emptyset}^s$, and $w^u(\bar{x}), w^u(\bar{z})$ form a 2-bunch of $\bar{\Omega}$. By Lemma 4.1, $\bar{\varphi}_{\bar{p}\bar{q}}(\bar{b}_0) \in w^u(\bar{z})$, where $\bar{p} \in w^u(\bar{x}), \bar{q} \in w^u(\bar{z})$ are associated points corresponding to the bunch $w^u(\bar{x}) \cup w^u(\bar{z})$. Hence, $w^{s*}(\bar{b}_0) \cap w^u(\bar{z}) \neq \emptyset$ and $\bar{b}_0 \in \mathcal{B}$.

Suppose now that $(\bar{x}, \bar{z})^s \cap \bar{\Omega} \neq \emptyset$. Due to Lemma 4.1 and Lemma 1.10, we can assume that \bar{z} is not an isolated point on $(\bar{x}, \bar{z})^s \cap \bar{\Omega}$. Therefore there exists a monotone sequence of points $\bar{x}_k \in (\bar{x}, \bar{z})^s \cap \bar{\Omega}$ such that $\bar{x}_k \rightarrow \bar{z}$ as $k \rightarrow \infty$.

Let us show that every unstable manifold $w^u \subset \bar{\Omega}$ intersecting $w^{s+}(\bar{b}_0)$ or $[\bar{a}_i, \bar{c}_i]^s$ intersects $[\bar{x}, \bar{z}]^s$. Note that if $w^u \subset \bar{\Omega}$ intersects $w^{s+}(\bar{b}_0)$, then w^u intersects $[\bar{a}_i, \bar{c}_i]^s$ for i sufficiently large. By Lemma 5.1, the points $\bar{a}_i \in w^u(\bar{x})$ and $\bar{c}_i = w^{s*}(\bar{a}_i) \cap w^u(\bar{z}) \subset w^u(\bar{z})$ belong to different components of $\bar{M} - w^u$. Hence, \bar{x} and \bar{z} belong to different components of $\bar{M} - w^u$ as well. It follows that w^u intersects $[\bar{x}, \bar{z}]^s$. In particular, every unstable manifold that contains boundary points and intersects $w^{s+}(\bar{b}_0)$ or $[\bar{a}_i, \bar{c}_i]^s$ has to intersect $[\bar{x}, \bar{z}]^s$.

Now let us show that every unstable manifold $w^u \subset \bar{\Omega}$ intersecting $[\bar{x}, \bar{z}]^s$ intersects both $w^{s+}(\bar{b}_0)$ and $[\bar{a}_i, \bar{c}_i]^s$. Note that the points \bar{x}, \bar{z} belong to different components of $\bar{M} - w^u$ (Lemma 5.1). As a consequence, the points $\bar{a}_i \in w^u(\bar{x})$ and $\bar{c}_i = w^{s*}(\bar{a}_i) \cap w^u(\bar{z}) \subset w^u(\bar{z})$ belong to different components of $\bar{M} - w^u$ as well. Assume that $w^u \cap [\bar{a}_i, \bar{c}_i]^s = \emptyset$. Then $[\bar{a}_i, \bar{c}_i]^s$ has associated points, say \bar{p}_i and \bar{q}_i . The argument above implies that $w^u(\bar{p}_i)$ and $w^u(\bar{q}_i)$ intersect $[\bar{x}, \bar{z}]^s$. The unstable manifold w^u cannot intersect $[\bar{x}, w^u(\bar{p}_i) \cap w^s(\bar{x})]^s$ or $[w^u(\bar{q}_i) \cap w^s(\bar{z}), \bar{z}]^s$; otherwise w^u would intersect $[\bar{a}_i, \bar{p}_i]^s$ or $[\bar{q}_i, \bar{c}_i]^s$ respectively. Hence, w^u must intersect

$$(w^u(\bar{p}_i) \cap w^s(\bar{x}), w^u(\bar{q}_i) \cap w^s(\bar{z}))^s = (w^u(\bar{p}_i) \cap w^s(\bar{x}), w^u(\bar{q}_i) \cap w^s(\bar{z}))_{\emptyset}^s,$$

which is impossible because of Lemma 4.1.

Thus, $w^{s+}(\bar{b}_0) \cap w^u(\bar{x}_k) \neq \emptyset$. Due to Lemma 5.2, this intersection is a unique point. By definition, put $\bar{y}_k = w^{s+}(\bar{b}_0) \cap w^u(\bar{x}_k)$. By Lemma 5.3, the sequence \bar{y}_k is bounded on $w^{s+}(\bar{b}_0)$. By Lemma 5.1, the sequence \bar{y}_k is monotone because the sequence \bar{x}_k is monotone. Since $\bar{\Omega}$ is a compact set, we see that $\bar{y}_k \rightarrow \bar{y}_* \in \bar{\Omega}$.

Let us prove that $w^u(\bar{y}_*) = w^u(\bar{z})$. Suppose the contrary. Then $w^u(\bar{y}_*)$ intersects $(\bar{x}, \bar{z})^s$ at some point \bar{x}_* . Since $\bar{x}_k \rightarrow \bar{z}$, there are infinitely many points $\bar{x}_k \in (\bar{x}_*, \bar{z})^s$. This contradicts the fact that the \bar{y}_k tend monotonically to \bar{y}_* . Hence,

$w^u(\bar{y}_*) = w^u(\bar{z})$. Therefore, \mathcal{B} is closed. Since $w^u(\bar{x})$ is a connected set, $w^u(\bar{x}) = \mathcal{B}$. \square

Corollary 5.1. *Let Ω be a C -dense codimension one orientable expanding attractor and $\bar{\Omega} = \pi^{-1}(\Omega)$. Suppose $w^{s*}(\bar{x})$ and $w^{s*}(\bar{y})$ intersect a leaf of the lamination $\mathcal{L}(\bar{\Omega})$, where $\bar{x}, \bar{y} \in \bar{\Omega}$. Then $w^{s*}(\bar{x})$ and $w^{s*}(\bar{y})$ intersect the same leaves of $\mathcal{L}(\bar{\Omega})$.*

Proof. By Lemma 4.1 and the theorem on the continuous dependence of leaves on initial conditions, there is a point $\bar{z} \in w^u(\bar{x})$ such that $\pi(w^s(\bar{z}))$ does not contain boundary periodic points of Ω and $w^s(\bar{z}) \cap w^u(\bar{y}) \neq \emptyset$. The result now follows from Lemma 5.4 \square

Lemma 5.5. *Suppose Ω is a codimension one orientable expanding attractor such that each characteristic sphere $S_{p_i q_i}$ of Ω bounds a closed ball $E_{p_i q_i} \subset M$. Let $\bar{B}_{\bar{p}\bar{q}} = w^u(\bar{p}) \cup w^u(\bar{q})$ be a 2-bunch of $\bar{\Omega} = \pi^{-1}(\Omega)$, where \bar{p} and \bar{q} are associated points of $\pi^{-1}(\Omega)$, and let $\mathcal{J}_{\bar{p}\bar{q}} \subset \bar{M}$ be the open domain bounded by $w^u(\bar{p})$ and $w^u(\bar{q})$. Then $\mathcal{J}_{\bar{p}\bar{q}} \cap \bar{\Omega} = \emptyset$, i.e., the domain $\mathcal{J}_{\bar{p}\bar{q}}$ has no points of $\bar{\Omega}$.*

Proof. By Corollary 4.1, $\pi(\bar{B}_{\bar{p}\bar{q}}) = B_{pq} = W^u(p) \cup W^u(q)$ is a bunch of Ω , where $p = \pi(\bar{p})$, $q = \pi(\bar{q})$ are periodic associated points. Let S_{pq} be a characteristic sphere corresponding to B_{pq} . By Lemma 4.2, there is the characteristic sphere $\bar{S}_{\bar{p}\bar{q}}$ which corresponds to $\bar{B}_{\bar{p}\bar{q}}$ and is a lift of S_{pq} . Denote by m the common period of p , q . By condition, S_{pq} bounds the closed ball $E_{pq} \subset M$. The path lifting theorem implies that $\pi^{-1}(E_{pq})$ is a union of closed balls. By item 1) of Lemma 5.1, each closed ball cannot contain another one, and thus these balls are pairwise disjoint. The argument shows that $\pi^{-1}(f^{mj}(E_{pq}))$ is also a union of pairwise disjoint closed balls for any $j \geq 0$. Denote by $\bar{E}_{\bar{p}\bar{q}}^j \subset \pi^{-1}(f^{mj}(E_{pq}))$ the closed ball containing the points \bar{p}, \bar{q} .

By definition, put

$$\mathcal{J}'_{\bar{p}\bar{q}} = \bigcup_{j \geq 0} \bar{E}_{\bar{p}\bar{q}}^j.$$

By Lemma 5.1, $\mathcal{J}_{\bar{p}\bar{q}}$ is a connected set. Hence, $\mathcal{J}_{\bar{p}\bar{q}} \cup w^u(\bar{p}) \cup w^u(\bar{q})$ is a connected set as well. Again by Lemma 5.1, the topological closure of $\mathcal{J}_{\bar{p}\bar{q}}$ is equal to

$$\text{clos } \mathcal{J}_{\bar{p}\bar{q}} = \mathcal{J}_{\bar{p}\bar{q}} \cup w^u(\bar{p}) \cup w^u(\bar{q}),$$

because $w^u(\bar{p})$ and $w^u(\bar{q})$ are embedded submanifolds of \bar{M} .

Evidently, $\mathcal{J}'_{\bar{p}\bar{q}} \subset \text{clos } \mathcal{J}_{\bar{p}\bar{q}}$. Due to the theorem on the continuous dependence of unstable manifolds on initial conditions, $\mathcal{J}'_{\bar{p}\bar{q}}$ is an open subset of $\text{clos } \mathcal{J}_{\bar{p}\bar{q}}$. Let us show that $\mathcal{J}'_{\bar{p}\bar{q}}$ is a closed subset of $\text{clos } \mathcal{J}_{\bar{p}\bar{q}}$. Suppose $\bar{a}_i \in \text{clos } \mathcal{J}_{\bar{p}\bar{q}}$ is a sequence of points such that $\bar{a}_i \rightarrow \bar{a}_*$ as $i \rightarrow \infty$. Then $\bar{a}_i \in \bar{E}_{\bar{p}\bar{q}}^{j(i)}$ for some $j(i) \geq 0$. If the sequence $j(i)$ is bounded, then $\bar{a}_* \in \bar{E}_{\bar{p}\bar{q}}^{j_*}$ for some j_* , because $\bigcup_{j \geq 0} \bar{E}_{\bar{p}\bar{q}}^j$ is a union of increasing balls. If the sequence $j(i)$ is unbounded, then, according to Lemma 3.1, $\bar{a}_* \in \bar{\Omega}$. In both cases, $\bar{a}_* \in \mathcal{J}'_{\bar{p}\bar{q}}$. Hence, $\mathcal{J}'_{\bar{p}\bar{q}}$ is a closed subset of $\text{clos } \mathcal{J}_{\bar{p}\bar{q}}$. Therefore, $\text{clos } \mathcal{J}_{\bar{p}\bar{q}} = \mathcal{J}'_{\bar{p}\bar{q}}$.

By item 1) of Lemma 5.1, there are no points of $\bar{\Omega}$ inside $\bar{E}_{\bar{p}\bar{q}}^j$ for any $j \geq 0$. The desired result follows. \square

Lemma 5.6. *Suppose Ω is a codimension one orientable expanding attractor, and every characteristic sphere $S_{p_i q_i}$ of Ω bounds a ball in M . If the stable manifold*

$W^s(x)$ of a point $x \in \Omega$ does not contain boundary periodic points, then any lift $w^s(\bar{x})$ of $W^s(x)$ intersects all leaves of the lamination $\mathcal{L}(\bar{\Omega})$, where $\bar{x} \in \bar{\Omega} = \pi^{-1}(\Omega)$ is a lift of x .

Proof. By Lemma 3.1, Ω is a C -dense basic set. Hence, $W^s(x) \cap \Omega$ is dense in Ω . Since $W^s(x)$ does not contain boundary periodic points, it follows from Lemma 1.8 that $W^{s+}(x) \cap \Omega$ and $W^{s-}(x) \cap \Omega$ are dense in Ω , where $W^{s\pm}(x)$ are components of $W^s(x) - x$. This implies that there exists a sequence of points $\{\bar{x}_i \in w^s(\bar{x}) \cap \bar{\Omega}\}_{-\infty}^{+\infty}$ such that

$$\bigcup_{i=-\infty}^{+\infty} [\bar{x}_{-i}, \bar{x}_i]^s = w^s(\bar{x}), \quad \dots \subset [\bar{x}_{-i}, \bar{x}_i]^s \subset [\bar{x}_{-i-1}, \bar{x}_{i+1}]^s \subset \dots$$

Denote by $\mathcal{J}_i \subset \bar{M}$ the closed domain bounded by $w^u(\bar{x}_i)$ and $w^u(\bar{x}_{-i})$. By Lemma 5.1, the domains \mathcal{J}_i form a sequence of increasing connected sets $\mathcal{J}_1 \subset \mathcal{J}_2 \subset \dots$, where each inclusion is proper. Therefore, $\bigcup_{i \in \mathbb{Z}} \mathcal{J}_i$ is an open set of \bar{M} , because every \mathcal{J}_i is a topological submanifold. Let us show that $\bigcup_{i \in \mathbb{Z}} \mathcal{J}_i$ is a closed set. Suppose $\bar{y}_i \rightarrow \bar{z}$ as $i \rightarrow \infty$, where $\bar{y}_i \in w^u(\bar{x}_i)$. Then $\bar{z} \in \bar{\Omega}$ because $\bar{\Omega}$ is closed. Hence, $w^{s*}(\bar{z})$ intersects $w^u(\bar{x}_i)$ for i sufficiently large. By Corollary 5.1, $w^u(\bar{z})$ intersects $w^s(\bar{x})$. Since the sequence \bar{x}_i is unbounded on $w^s(\bar{x})$, we have $w^u(\bar{z}) \subset \mathcal{J}_j$ for some $j \in \mathbb{N}$, and so $\bar{z} \in \bigcup_{i \in \mathbb{Z}} \mathcal{J}_i$. Thus, $\bigcup_{i \in \mathbb{Z}} \mathcal{J}_i$ is a closed set. As a consequence, $\bigcup_{i \in \mathbb{Z}} \mathcal{J}_i = \bar{M}$.

It remains to prove that every unstable manifold $w^u \subset \bar{\Omega}$ intersects $w^s(\bar{x})$ whenever $w^u \subset \mathcal{J}_i$ for some $i \in \mathbb{N}$. Suppose the contrary. By Lemma 5.1, \mathcal{J}_i is arcwise connected. Take a path $P \subset \mathcal{J}_i$ from a point $\bar{a} \in w^u(\bar{x}_i)$ to a point $\bar{b} \in w^u$. By definition, put

$$P_a = \{\bar{z} \in P \cap \bar{\Omega} \mid w^u(\bar{z}) \cap w^s(\bar{x}) \neq \emptyset\}.$$

The argument above shows that P_a is a closed subset of P . Therefore there is a point $\bar{a}_* \in P_a$ such that P_a belongs to the closed segment $[\bar{a}, \bar{a}_*] \subset P$ and $(\bar{a}_*, \bar{b}) \cap P_a = \emptyset$. Without loss of generality we can assume that P is transversal to $w^u(\bar{a}_*)$ at the point \bar{a}_* . Then the theorem on the continuous dependence of invariant manifolds on initial conditions implies that $\bar{\Omega}$ does not intersect (\bar{a}_*, \bar{b}) near \bar{a}_* . Hence, $w^u(\bar{a}_*)$ contains a lift of a boundary point, say \bar{p} . Let \bar{q} be the point associated with \bar{p} . Denote by $\mathcal{J}_{\bar{p}\bar{q}}$ the closed domain bounded by $w^u(\bar{p})$ and $w^u(\bar{q})$. By Corollary 5.1, $w^u(\bar{q}) \cap w^s(\bar{x}) \neq \emptyset$. From our definition of the point \bar{a}_* , it follows that $w^u(\bar{q}) \cap (\bar{a}_*, \bar{b}) = \emptyset$. Since $\bar{\Omega}$ does not intersect (\bar{a}_*, \bar{b}) near \bar{a}_* , $(\bar{a}_*, \bar{b}) \subset \mathcal{J}_{\bar{p}\bar{q}}$. Hence, $w^u \subset \mathcal{J}_{\bar{p}\bar{q}}$. On the other hand, according to Lemma 5.5, there are no points of $\bar{\Omega}$ in $\mathcal{J}_{\bar{p}\bar{q}}$. This contradiction concludes the proof. \square

Corollary 5.2. *Let the conditions of Lemma 5.6 hold. Then $w^{s*}(\bar{z})$ intersects all unstable manifolds of $\bar{\Omega}$ for any $\bar{z} \in \bar{\Omega}$.*

Proof. The proof follows from Lemmas 5.5, 5.6 and Corollary 5.1 (note that due to Lemma 3.1, $\pi(\bar{\Omega}) = \Omega$ is C -dense, and so we can apply Lemma 5.5). \square

Lemma 5.7. *Suppose Ω is a codimension one orientable expanding attractor and every characteristic sphere S_{p_i, q_i} of Ω bounds a ball in M . Then \bar{M} is homeomorphic to \mathbb{R}^n .*

Proof. This follows from Lemmas 5.1, 5.5, Corollary 5.2, and the fact that every unstable manifold $w^u(\bar{x}_0)$, $\bar{x}_0 \in \Omega$, is homeomorphic to Euclidean space \mathbb{R}^{n-1} . \square

Theorem 5.1. *Let f be an A -diffeomorphism of a closed n -manifold M^n ($n \geq 3$), and let Ω be an orientable expanding attractor of codimension one. Suppose that each characteristic sphere $S_{p_i q_i}$ of Ω bounds an n -ball $E_{p_i q_i} \subset M^n$. Then:*

- 1) *Any basic set different from Ω belongs to the union $\bigcup_{i=1}^k E_{p_i q_i}$, where p_i, q_i range over all pairs of associated boundary periodic points of Ω .*
- 2) *Ω is the only orientable codimension one pseudotame basic set of f .*
- 3) *The universal covering \overline{M} of M^n is homeomorphic to Euclidean space \mathbb{R}^n .*
- 4) *The fundamental group $\pi_1(M^n)$ is a free abelian finitely generated group. In particular, M^n is homotopy equivalent to the n -torus T^n .*

Proof. Lemma 3.1 implies item 1), because stable manifolds of basic sets are pairwise disjoint. Items 2) and 3) follow from item 2) of Lemma 5.1 and Lemma 5.7 respectively.

By a light modification of the proof of Theorem 5.1 and Corollary 5.2 in [39], one can prove that the fundamental group $\pi_1(M^n)$ is isomorphic to \mathbb{Z}^n . For the reader's convenience we give a sketch of the proof. Take a point $\bar{x} \in \overline{\Omega}$ such that $\pi(w^s(\bar{x}))$ does not contain boundary periodic points of Ω . By Lemmas 5.1 and 5.6, $\overline{w^s(\bar{x})}$ intersects every unstable manifold $\overline{w^u} \subset \overline{\Omega}$ at a unique point. Then each element α of the fundamental group $\pi_1(M^n)$ induces the map $\mathcal{R}(\alpha)$ on $\overline{w^s(\bar{x})} \cap \overline{\Omega}$ as follows:

$$\bar{z} \mapsto w^s(\bar{x}) \cap \alpha(w^u(\bar{z})), \text{ where } \bar{z} \in w^s(\bar{x}) \cap \overline{\Omega}.$$

Let $\mathcal{R}(\pi_1(M^n))$ be the group formed by these maps $\mathcal{R}(\alpha)$, where α ranges over $\pi_1(M^n)$. Obviously, the correspondence $\pi_1(M^n) \rightarrow \mathcal{R}(\pi_1(M^n))$ is a homomorphism.

We consider $w^s(\bar{x})$ to be a curve with some orientation. Take a point $\bar{z}_0 \in w^s(\bar{x}) \cap \overline{\Omega}$. Clearly there are points $\bar{z}_1, \bar{z}_2 \in w^s(\bar{x}) \cap \overline{\Omega}$ such that $\bar{z}_1 < \bar{z}_0 < \bar{z}_2$. It follows from Lemma 5.1 that $w^u(\bar{z}_0)$ belongs to the domain \mathcal{J}_{12} bounded by $w^u(\bar{z}_1)$ and $w^u(\bar{z}_2)$. By Lemmas 5.1 and 5.6, given any $\mathcal{R}(\alpha)$, the intersection $\mathcal{R}(\alpha)(\mathcal{J}_{12}) \cap w^s(\bar{x})$ is an interval, so that the point $\alpha(\bar{z}_0)$ is between the points $\alpha(\bar{z}_1), \alpha(\bar{z}_2)$. Hence, $\mathcal{R}(\alpha)$ is a continuous map. Again by Lemma 5.1, $\mathcal{R}(\alpha)$ is one-to-one. The argument shows that $\mathcal{R}(\alpha)$ is a homeomorphism $\overline{w^s(\bar{x})} \cap \overline{\Omega} \rightarrow \overline{w^s(\bar{x})} \cap \overline{\Omega}$.

Every $\mathcal{R}(\alpha) \in \mathcal{R}(\pi_1(M^n))$, $\alpha \neq id$, is fixed point free, because any $\overline{w^u} \in \overline{\Omega}$ has no congruent points. Hence, $\mathcal{R}(\pi_1(M^n))$ is isomorphic to $\pi_1(M^n)$.

Given an $\mathcal{R}(\alpha) \in \mathcal{R}(\pi_1(M^n))$, let us prove that if $\mathcal{R}(\alpha)(t_0) > t_0$ for some point $t_0 \in w^s(\bar{x}) \cap \overline{\Omega}$, then $\mathcal{R}(\alpha)(t) > t$ for every $t \in w^s(\bar{x}) \cap \overline{\Omega}$. Assume the contrary. Then there is $t'_0 \in w^s(\bar{x}) \cap \overline{\Omega}$ such that $\mathcal{R}(\alpha)(t'_0) < t'_0$. Denote

$$P_u = \{t \in w^s(\bar{x}) \cap \overline{\Omega} \mid \mathcal{R}(\alpha)(t) > t\}, \quad P_l = \{t \in w^s(\bar{x}) \cap \overline{\Omega} \mid \mathcal{R}(\alpha)(t) < t\}.$$

First let us show that both P_u and P_l are closed. Let $t_i \in P_u$ be a sequence such that $t_i \rightarrow t_*$. Then $t_* \in w^s(\bar{x}) \cap \overline{\Omega}$, because the set $w^s(\bar{x}) \cap \overline{\Omega}$ is closed. Since $\mathcal{R}(\alpha)$ is continuous, $\mathcal{R}(\alpha)(t_*) \geq t_*$, and hence $\mathcal{R}(\alpha)(t_*) > t_*$, because $\mathcal{R}(\alpha)$ is fixed point free. It follows that P_u is closed. In the same way one can prove that P_l is a closed set as well.

Without loss of generality we can assume that $t_0 < t'_0$; otherwise one could take $\mathcal{R}(\alpha^{-1})$. Since both P_u and P_l are closed, it follows that $\sup P_u \in P_u$ and $\inf P_l \in P_l$. Moreover, $(\sup P_u, \inf P_l)^s \cap \overline{\Omega} = \emptyset$. By Lemma 1.7, $\sup P_u$ belongs to the unstable manifold, say $w^u(\bar{p})$, with a boundary point \bar{p} of $\overline{\Omega}$. Similarly, $\inf P_l$ belongs to $w^u(\bar{q})$ with a boundary point \bar{q} . By Lemma 4.1, \bar{p} and \bar{q} are associated, since $(\sup P_u, \inf P_l)^s = (\sup P_u, \inf P_l)^s_{\emptyset}$.

Denote by $\mathcal{J}_{\overline{pq}} \subset \overline{M}$ the open domain bounded by $w^u(\overline{p})$ and $w^u(\overline{q})$. By Lemma 5.5, $\mathcal{J}_{\overline{pq}} \cap \overline{\Omega} = \emptyset$, and thus $\alpha(\mathcal{J}_{\overline{pq}}) \cap \overline{\Omega} = \emptyset$. As a consequence, $\mathcal{R}(\alpha)((\sup P_u, \inf P_l)^s)$ is an interval with $\mathcal{R}(\alpha)((\sup P_u, \inf P_l)^s) \cap \overline{\Omega} = \emptyset$. It follows that if

$$\mathcal{R}(\alpha)(\sup P_u) > \inf P_l,$$

then

$$\mathcal{R}(\alpha)(\sup P_l) > \inf P_l.$$

Hence,

$$\mathcal{R}(\alpha)(\sup P_u) = \inf P_l, \quad \alpha(w^u(\overline{p})) = w^u(\overline{q}).$$

Therefore, $\pi(w^u(\overline{p})) = \pi(w^u(\overline{q}))$, and so Ω has a 1-bunch. This contradicts Corollary 1.3.

Thus, $\mathcal{R}(\alpha)(t) > t$ for every $t \in w^s(\overline{x}) \cap \overline{\Omega}$ whenever $\mathcal{R}(\alpha)(t_0) > t_0$ for some point $t_0 \in w^s(\overline{x}) \cap \overline{\Omega}$. This allows us to introduce an order relation on $\mathcal{R}(\pi_1(M^n))$ as follows. By definition, put

$$\mathcal{R}(\alpha) > \mathcal{R}(\beta) \text{ if } \mathcal{R}(\alpha)(\overline{z}) > \mathcal{R}(\beta)(\overline{z}) \text{ for some } \overline{z} \in w^s(\overline{x}) \cap \overline{\Omega}, \quad \alpha, \beta \in \pi_1(M).$$

Note that given any $\mathcal{R}(\alpha)$ and $\mathcal{R}(\beta)$, we have either $\mathcal{R}(\alpha) > \mathcal{R}(\beta)$ or $\mathcal{R}(\alpha) = \mathcal{R}(\beta)$ or $\mathcal{R}(\alpha) < \mathcal{R}(\beta)$. Further, by Lemma 3.1, Ω is C -dense, and so every unstable manifold $W^u(x) \subset \Omega$ is dense in Ω . Hence, $\bigcup_{\alpha \in \pi_1(M^n)} \alpha(w^u)$ is dense in $\overline{\Omega}$. Therefore, $\mathcal{R}(\pi_1(M))$ is an Archimedean ordered group, i.e., given any $\mathcal{R}(\alpha) \neq \mathcal{R}(\beta)$, there is an m such that

$$\mathcal{R}(\alpha)^m > \mathcal{R}(\beta) \text{ or } \mathcal{R}(\alpha)^{-m} > \mathcal{R}(\beta).$$

Due to the Hölder theorem on Archimedean ordered groups, $\mathcal{R}(\pi_1(M))$ is a free abelian group. As $\mathcal{R}(\pi_1(M)) \cong \pi_1(M)$ is finitely generated, we see that $\mathcal{R}(\pi_1(M))$ is isomorphic to an integer lattice \mathbb{Z}^k (see, e.g., [34], ch. 2). Item 3) implies that $\pi_r(M^n) = 0$ for $r \geq 2$, i.e., M is a $K(\pi, 1)$ -space. It follows that $k = n$, because M is a compact manifold. Thus,

$$\pi_r(M^n) = 0 = \pi_r(T^n) \text{ for } r \geq 2, \text{ and } \pi_1(M^n) = \mathbb{Z}^n = \pi_1(T^n).$$

Hence, M^n is weakly homotopy equivalent to T^n , and thus M^n is homotopy equivalent to T^n because M^n is a $K(\pi, 1)$ -space.

Note that we can get the same assertion using Plykin's theorem on invariant foliations ([45], p.93, and Corollary 5.2 in [39]). □

6. ON THE EXISTENCE OF NONTRANSVERSAL INTERSECTIONS

In this section we keep the notation of section 2. In particular, f is an A -diffeomorphism of a closed n -manifold M^n ($n \geq 3$) and Ω is an orientable expanding attractor of codimension one. In the theorem below we consider S_{pq} to be the characteristic sphere corresponding to a 2-bunch $W^u(p) \cup W^u(q)$ of Ω , where $p, q \in \Omega$ are associated boundary periodic points. Recall that S_{pq} is a union of two closed $(n - 1)$ -balls $D_p \subset W^u(p)$, $D_q \subset W^u(q)$, and the connecting cylinder

$$C_{pq} = \bigcup_{x \in \partial D_p, y \in \partial D_q} (x, y)_{\emptyset}^s, \text{ where } p \in \text{int } D_p, q \in \text{int } D_q.$$

Recall also that $D_{pq} = \bigcup_{x \in W^u(p) - \text{int } D_p} [x, \varphi_{pq}(x)]_{\emptyset}^s$.

Theorem 6.1. *Suppose f is a structurally stable diffeomorphism of a closed n -manifold M^n ($n \geq 3$) and Ω is a codimension one orientable expanding attractor of f . Let $\Omega' \neq \Omega$ be a basic set of index $n - 1$ such that $W^u(\Omega') \cap D_{pq} \neq \emptyset$, and let $C \subset D_{pq} \cap W^u(z')$ be a component of the intersection $D_{pq} \cap W^u(z')$, where $z' \in \Omega'$ is a periodic point. Then*

$$W^u(z') \cap C_{pq} = C \cap C_{pq} \neq \emptyset$$

and this intersection consists of a unique $(n - 2)$ -sphere, say S^{n-2} , that is isotopic to both $S_p = \partial D_p$ and $S_q = \partial D_q$ in the cylinder C_{pq} . In particular, S^{n-2} divides C_{pq} into two cylinders, each of whose is homeomorphic to $\Sigma^{n-2} \times [0; 1]$.

Proof. By Mañé's theorem [35], f satisfies the strong transversality condition. Taking into account Corollary 3.1, we see that $W^u(\Omega') \cap D_{pq} \neq \emptyset$ implies $W^u(z') \cap C_{pq} \neq \emptyset$. For the same reason, $C \cap C_{pq} \neq \emptyset$. By the strong transversality condition, $W^u(z') \cap C_{pq}$ locally is an $(n - 2)$ -dimensional submanifold which is transversal to every segment $(x, \varphi_{pq}(x))_{\emptyset}^s$, $x \in S_p = \partial D_p$. Let S be a component of the intersection $C \cap C_{pq}$. We have to prove that S is an $(n - 2)$ -sphere and $W^u(z') \cap C_{pq} = S$.

Recall that according to Lemma 2.7, the triple $(D_{pq}, W^u(p) - \text{int } D_p, \pi_p)$ is a trivial fiber bundle with fiber the interval $[0; 1]$, where $\pi_p : D_{pq} \rightarrow W^u(p) - \text{int } D_p$ is the projection that takes each point $b \in D_{pq}$ to $x \in W^u(p) - \text{int } D_p$ whenever $b \in [x, \varphi_{pq}(x)]_{\emptyset}^s \subset D_{pq}$.

It is convenient to prove the theorem in steps. The end of the proof of a step will be denoted by \diamond . First we prove a technical statement.

Step 6.1. *Let $x_0 \in W^u(p) - p$. There is a neighborhood U_0 of x_0 in $W^u(p)$ with the following property: if $\pi_p^{-1}(x_0)$ intersects the unstable manifold $W^u(\theta)$ of a point θ that belongs to a basic set Θ of index one, then any component of $\pi_p^{-1}(U_0) \cap W^u(\theta)$ is mapped by π_p homeomorphically onto U_0 .*

Proof of Step 6.1. It was proved by Smale [53] that M^n is a union of pairwise disjoint unstable manifolds of basic sets. Therefore, by the strong transversality condition, any point $x \in \pi_p^{-1}(x_0)$ belongs to either an unstable n -manifold or an unstable $(n - 1)$ -manifold. Denote by D_{n-1} (resp. D_n) the subset of $\pi_p^{-1}(x_0)$ which belongs to unstable $(n - 1)$ -manifolds (resp. n -manifolds). Since any unstable n -manifold is an open set of M^n , D_n is an open subset of $\pi_p^{-1}(x_0)$. Hence, D_{n-1} is a compact set. It follows that the angles between the arc $\pi_p^{-1}(x_0)$ and unstable $(n - 1)$ -manifolds through D_{n-1} are separated from zero by a positive constant. This proves the step, because these unstable manifolds through D_{n-1} are locally pairwise disjoint and D_{n-1} is a compact set. \diamond

Below, when we say that a set $A \subset W^u(p) - \text{int } D_p$ is open in $W^u(p) - \text{int } D_p$ we mean the topology induced by the intrinsic topology of $W^u(p)$ on $W^u(p) - \text{int } D_p$ as a subset. Denote by $\pi_p|_C : C \rightarrow W^u(p) - \text{int } D_p$ the restriction of π_p on C .

Step 6.2. *The set $\pi_p(C)$ is open in $W^u(p) - \text{int } D_p$, and $\pi_p|_C$ is a covering map on its image $\pi_p(C)$.*

Proof of Step 6.2. Take a point $x \in \pi_p(C)$. By the strong transversality condition, the intersection $\pi_p^{-1}(x) \cap C$ consists of at most a countable set of points $x_i \in D_{pq}$. Obviously, each x_i is an interior point of $W^u(z')$, and so is an interior point of C in the topology induced by D_{pq} in $C \subset W^u(z') \cap D_{pq}$ as a subset. Hence, $\pi_p|_C(x_i)$ is

an interior point of $\pi_p(C)$, because $\pi_p|_C$ is a local homeomorphism at each of these points x_i . As a consequence, $\pi_p(C)$ is open in $W^u(p) - \text{int } D_p$.

Let $U(x) \subset W^u(p) - \text{int } D_p$ be a neighborhood of x in $W^u(p) - \text{int } D_p$ which satisfies step 6.1, and $U(x_i)$ the component of $C \cap \pi_p^{-1}(U(x))$ that contains the point x_i . Each $U(x_i)$ is a neighborhood of x_i , because $\pi_p|_C$ is a continuous map. It follows from step 6.1 that the neighborhoods $U(x_i)$ are pairwise disjoint and the restriction $\pi_p|_{C \cap U(x_i)}$ is a homeomorphism onto $U(x)$. This completes the proof. \diamond

Obviously, given any set $N \subset W^u(p) - \text{int } D_p$ the restriction $\pi_p|_{\pi_p^{-1}(N)}$ is a trivial fiber bundle with fiber the interval $[0; 1]$. Hence the restriction

$$P \stackrel{\text{def}}{=} \pi_p|_{\text{clos } C_{pq}} : \text{clos } C_{pq} \rightarrow S_p^{n-2}, \quad \text{clos } C_{pq} = C_{pq} \cup S_p^{n-2} \cup S_q^{n-2},$$

is a trivial fiber bundle with fiber the interval $[0; 1]$ as well, because $\text{clos } C_{pq} = \pi_p^{-1}(S_p^{n-2})$.

Step 6.3. $\pi_p|_C : C \rightarrow W^u(p) - \text{int } D_p$ is a surjection i.e., $\pi_p(C) = W^u(p) - \text{int } D_p$. In particular, $P|_S : S \rightarrow S_p^{n-2}$ is a surjection as well, $P(S) = S_p^{n-2}$.

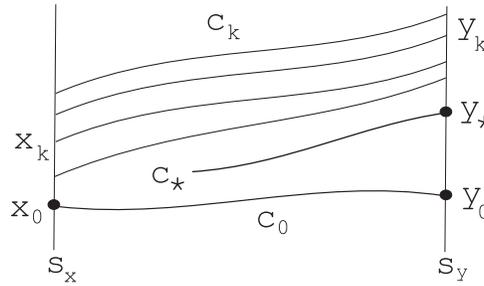
Proof of Step 6.3. Assume the contrary. Then by step 6.2, $\partial\pi_p|_C(C) \neq \emptyset$, and so there is a point $x_0 \in \partial\pi_p|_C(C) - \pi_p|_C(C)$. It follows we shall obtain that there is a sequence of points $a_i \in D_{pq} \cap C$ such that $a_i \rightarrow a_* \in D_{pq} - C$ and $\pi_p|_C(a_i) \rightarrow x_0$ as $i \rightarrow \infty$.

Let $U(x_0) \subset W^u(p) - \text{int } D_p$ be a neighborhood of x_0 in $W^u(p) - \text{int } D_p$ which satisfies step 6.1. Then the neighborhood $U = \pi_p^{-1}(U(x_0))$ is filled by unstable n -manifolds and unstable $(n-1)$ -manifolds necessarily intersecting $\pi_p^{-1}(x_0)$. Since $a_i \rightarrow a_*$, we see that $a_i \in U$ beginning with some index i_0 . Hence, C has to intersect $\pi_p^{-1}(x_0)$. Thus, $x_0 \notin \partial\pi_p|_C(C)$, because $\pi_p|_C$ is a local homeomorphism. This contradiction concludes the proof. \diamond

Step 6.4. If $n \geq 4$, then $\pi_p|_C : C \rightarrow W^u(p) - \text{int } D_p$ is a homeomorphism, and $\pi_p^{-1}(x) \cap C$ consists of a unique point for every $x \in W^u(p) - \text{int } D_p$. In particular, $P|_S : S \rightarrow S_p^{n-2}$ is a homeomorphism and S is an $(n-2)$ -sphere (we recall that S is a component of the intersection $C \cap C_{pq}$).

Proof of Step 6.4. It follows from steps 6.2 and 6.3 that $\pi_p|_C : C \rightarrow W^u(p) - \text{int } D_p$ is a cover. Since $n \geq 4$, $\pi_1(S_p^{n-2}) = \pi_1(W^u(p) - \text{int } D_p) = 0$. Hence, $\pi_p|_C : C \rightarrow W^u(p) - \text{int } D_p$ is a trivial cover (see, e.g., [54], section 13.9), i.e., C is homeomorphic to the product of $W^u(p) - \text{int } D_p$ and an at most countable set of points. Taking into account that C is connected, it follows that the fiber of $\pi_p|_C$ is trivial as well, i.e., consists of a unique point. Therefore $\pi_p|_C$ is a homeomorphism $C \rightarrow W^u(p) - \text{int } D_p$. By construction of $\pi_p|_C$, $S \cap \pi_p|_C^{-1}(x)$ is a unique point for any $x \in W^u(p) - \text{int } D_p$. \diamond

Before we prove the similar statement for $n = 3$ we have to prove that some special family of curves on the cylinder C_{pq} satisfies the theorem on continuous dependence on initial conditions. Recall that, due to the strong transversality condition, C_{pq} intersects 2-dimensional and 3-dimensional unstable invariant manifolds only when $n = 3$. Denote by $D_2 \subset C_{pq}$ the set of the intersections C_{pq} with 2-dimensional unstable invariant manifolds. Again, due to the strong transversality condition, any 2-dimensional unstable invariant manifold intersects C_{pq} in a curve. Such curves form a family, denoted by \mathcal{F} .

FIGURE 1. The family \mathcal{F} .

Step 6.5. *The family \mathcal{F} satisfies the theorem on continuous dependence on initial conditions.*

Proof of Step 6.5. Take a curve $c_0 \in \mathcal{F}$ and any points $x_0, y_0 \in c_0$. Denote by s_x (resp. s_y) the fiber of the fiber bundle π_p through x_0 (resp. y_0), shown in Figure 1.

Let $x_k \in D_2$ be any sequence of points converging to x_0 as $k \rightarrow \infty$, and denote by $c_k \in \mathcal{F}$ the curve through x_k . By step 6.3, every curve c_k intersects both s_x and s_y . Therefore without loss of generality we can assume that $x_k \in s_x$ and the points x_k converge to x_0 monotonically on s_x . Let $[x_k, y_k] \subset c_k$ be the segment with endpoints x_k and $y_k \in s_y$ such that $\pi_p([x_k, y_k]) = \pi_p([x_0, y_0])$ for any k ; see Figure 1, where the segment $[x_0, y_0] \subset c_0$ is mapped by π_p onto $\pi_p([x_0, y_0])$ homeomorphically. Then the sequence y_k is monotone on s_y . We have to prove that $y_k \rightarrow y_0$ as $k \rightarrow \infty$.

Suppose the contrary. Then, passing to a subsequence if necessary, one can assume that $y_k \rightarrow y_*$ as $k \rightarrow \infty$ for some point $y_* \in s_y$ that is between y_1 and y_0 , $y_* \neq y_0$. Since D_2 is a compact set, $y_* \in D_2$, and thus there is a curve, say $c_* \in \mathcal{F}$, through y_* . By step 6.3, c_* intersects s_x . Take the segment $[x_*, y_*] \subset c_*$ such that $\pi_p([x_*, y_*]) = \pi_p([x_0, y_0])$, where $x_* \in s_x$. Then $x_* \neq x_0$, since $y_* \neq y_0$. This is impossible because the sequence y_k is monotone on s_y , and thus there are no points y_k on the segment $[y_*, y_0] \subset s_y$ between y_* , y_0 . \diamond

Step 6.6. *Every component S of $C \cap C_{pq}$ is an $(n-2)$ -sphere. Moreover, $S \cap P^{-1}(x)$ is a unique point for any $x \in S_p^{n-2}$.*

Proof of Step 6.6. By step 6.4, it remains to consider the case $n = 3$. Rename $S = c$ and assume that the curve $c \subset W^u(z') \cap C_{pq}$ is not homeomorphic to a circle. Hence, c has a point of accumulation, say $m_0 \in C_{pq}$, i.e., there is a sequence of points $m_i \in c$ such that $m_i \rightarrow m_0$ as $i \rightarrow \infty$. Take a neighborhood $U(m_0) \subset C_{pq}$ of m_0 that is a trivial bundle over the open segment $P(U(m_0))$, $P(m_0) \in P(U(m_0))$, with fibers the arcs of stable manifolds of points $x \in P(U(m_0))$. Let $W_0^s \subset U(m_0)$ be the component of $W^s(P(m_0)) \cap U(m_0)$ through m_0 . Denote by $K(m_i)$ the component of $c \cap U(m_0)$ containing the point m_i . By step 6.3, we can assume that $K(m_i)$ intersects W_0^s . Since $K(m_i)$ is transversal to the fibers of the trivial fiber bundle $U(m_0)$, $K(m_i) \cap W_0^s$ is a unique point. Then without loss of generality one can assume that $K(m_i) \cap W_0^s = m_i$. Moreover, we can assume that the components $K(m_i)$ are pairwise disjoint, because $m_i \rightarrow m_0$ as $i \rightarrow \infty$. Denote by $C(m_i, m_{i+1})$ the arc of c with endpoints m_i, m_{i+1} and by $C_{i,i+1}$ the closed curve which is the union of $C(m_i, m_{i+1})$ and the segment $(m_i, m_{i+1}) \subset W_0^s$. Since $C(m_i, m_{i+1})$ is transversal to

the fibers of the fiber bundle $P : \text{clos } C_{pq} \rightarrow S_p^{n-2}$ and $P(m_i) = P(m_{i+1})$, it follows that $C_{i,i+1}$ is not homotopic to zero on the cylinder $\text{clos } C_{pq}$ and $P(C_{i,i+1}) = S_p^{n-2}$. By the strong transversality condition, $C(m_i, m_{i+1})$ does not intersect $P^{-1}(P(m_0))$ except at the points m_i, m_{i+1} . The argument shows that the sequence $\{m_i\}_{i=1}^\infty$ converges monotonically to m_0 , and so c tends spirally to m_0 .

According to [28] and [53], $m_0 \in W^u(\theta)$ for some point θ of a basic set Θ . By the strong transversality condition, $\dim W^u(\theta) \geq 2$. Since m_0 is a point of accumulation of points $m_i \in W^u(z')$ with $\dim W^u(z') = 2$, we have $\dim W^u(\theta) = 2$. Hence the intersection $W^u(\theta) \cap \text{clos } C_{pq}$ contains the curve c_0 through m_0 .

By step 6.5, c_0 belongs to the limit set of the curve c . The argument of Poincaré-Bendixon theory (see, e.g., [3], [6]) shows that c_0 is a closed curve, since c and c_0 are transversal to any arc $P^{-1}(a)$, $a \in S_p^1$. Therefore, c_0 bounds a disc on $W^u(\theta)$.

Let us show that θ is a periodic point. If $\Theta = \Omega'$, then there is nothing to prove (recall that z' is a periodic point and unstable invariant manifolds are pairwise disjoint). So we assume that $\Theta \neq \Omega'$. By the theorem on continuous dependence of invariant manifolds on initial conditions, there is $\alpha > 0$ such that any unstable manifold $W^u(\eta)$, $\eta \in \Theta$, meeting $W_\alpha^s(\theta)$ must intersect the cylinder $\text{clos } C_{pq}$ through a closed curve near c_0 . Since c tends spirally to c_0 , it follows that at least one component of the set $W_\alpha^s(\theta) - \theta$ has no intersections with any unstable invariant manifold $W^u(\eta)$, $\eta \in \Theta$, because $\Theta \neq \Omega'$. Due to Lemma 1.3, θ is a periodic point.

Using the argument of the proof of Lemma 1.5 in [40], one get that $W^u(z')$ has to meet $W_\alpha^s(\theta) - \theta$. The λ -lemma and the theorem on continuous dependence of invariant manifolds on initial conditions implies that $W^u(z')$ has to intersect the cylinder $\text{clos } C_{pq}$ by closed curves near m_0 . This contradiction concludes the proof that c is a closed curve.

It remains to remark that the strong transversality condition implies that $c \cap P^{-1}(x)$ is a unique point for any $x \in S_p^1$. \diamond

Note that all we need in steps 6.2, 6.3, 6.4, and 6.6 was the existence of the set $\pi_p^{-1}(S_p^{n-2}) \stackrel{\text{def}}{=} \text{clos } C_{pq}$, i.e., S_p^{n-2} is any $(n-2)$ -sphere tamely embedded in $W^u(p)$ that is not through the point p . In the sequel, we specify S_p^{n-2} to be a boundary of the $(n-1)$ -ball D_p belonging to the characteristic sphere S_{pq} .

Step 6.7. *The $(n-2)$ -sphere $S = C \cap C_{pq}$ is isotopic to both S_p^{n-2} and S_q^{n-2} in the cylinder $\text{clos } C_{pq}$. In particular, the inclusion $S \subset D_{pq}$ induces the monomorphism (injective homomorphism) $\pi_{n-2}(S) \rightarrow \pi_{n-2}(D_{pq}) \cong \mathbb{Z}$. Furthermore, S divides $\text{clos } C_{pq}$ into two subcylinders, each of whose is homeomorphic to the cylinder $\Sigma^{n-2} \times [0; 1]$.*

Proof of Step 6.7. Recall that P is a trivial fiber bundle with fiber the interval $[0; 1]$. By step 6.3 and the strong transversality condition, $P|_S : S \rightarrow S_p^{n-2}$ is a homeomorphism. In particular, every fiber of the fiber bundle $(\text{clos } C_{pq}, S_p^{n-2}, P)$ intersects S at a unique point. Hence there is an isotopy of S into S_p^{n-2} along the fibers of the fiber bundle $(\text{clos } C_{pq}, S_p^{n-2}, P)$. Similarly, S is isotopic to S_q^{n-2} . Since S intersects every fiber of the trivial fiber bundle $(\text{clos } C_{pq}, S_p^{n-2}, P)$ at a unique point and the fiber is the interval $[0; 1]$, S divides $\text{clos } C_{pq}$ into two cylinders, each of which is homeomorphic to $\Sigma^{n-2} \times [0; 1]$. Combining this with Corollary 2.3, we obtain that $\pi_{n-2}(S) \rightarrow \pi_{n-2}(D_{pq})$ is a monomorphism. \diamond

By the theorem on the continuous dependence of invariant manifolds on initial conditions, each $(n-2)$ -sphere S of the intersection $W^u(z') \cap C_{pq}$ is tamely embedded in both $W^u(z')$ and C_{pq} . Hence, S bounds an open $(n-1)$ -ball in $W^u(z')$, denoted by B_S^{n-1} [9]. An $(n-2)$ -sphere $S \subset W^u(z') \cap C_{pq}$ and the corresponding $(n-1)$ -ball $B_S^{n-1} \subset W^u(z')$ are called *minimal*, if there are no points of $W^u(z') \cap C_{pq}$ inside B_S^{n-1} . By the strong transversality condition, every compact subset $K \subset W^u(z')$ intersects finitely many spheres of $W^u(z') \cap C_{pq}$. Hence minimal spheres and corresponding minimal balls exist.

Step 6.8. *Let $B_S^{n-1} \subset W^u(z')$ be a minimal $(n-1)$ -ball bounded by an $(n-2)$ -sphere $S \subset W^u(z') \cap C_{pq}$. Then $B_S^{n-1} \cap D_{pq} = \emptyset$.*

Proof of Step 6.8. Suppose the contrary. Then $B_S^{n-1} \subset D_{pq}$, because B_S^{n-1} is minimal. Therefore, S defines a trivial element of the group $\pi_{n-2}(D_{pq})$. On the other hand, according to step 6.7, S defines a nontrivial element of $\pi_{n-2}(D_{pq})$. This contradiction concludes the proof. \diamond

There are two possibilities:

- 1) There exists a minimal $(n-1)$ -ball $B' \subset W^u(z')$ containing the point z' .
- 2) No minimal ball contains z' .

Consider case 1). Without loss of generality one can assume that $f(z') = z'$, $f(p) = p$, and $f(q) = q$; otherwise we take a some iteration f^k . Moreover, we shall suppose that the restriction $f|_{W^u(z')}$ is an orientation-preserving diffeomorphism.

Recall that a homeomorphism $\psi : M \rightarrow M$ is *stable in the Bing sense*, if ψ can be represented as a superposition $\psi = \psi_1 \circ \dots \circ \psi_k$, where every ψ_i , $1 \leq i \leq k$, is fixed inside of some open domain $D_i \subset M$. It is well known that an orientation-preserving diffeomorphism is stable in the Bing sense.

Since $f(C_{pq}) \subset D_{pq}$, we have $f(S) \cap S = \emptyset$, where the $(n-2)$ -sphere S bounds B' on $W^u(z')$. Hence, S is inside of $f(S)$, because $z' \in B'$ is the hyperbolic source of the restriction $f|_{W^u(z')}$. Taking [10] into account, we get that the $(n-2)$ -spheres $S, f(S)$ bound a closed annulus, say $A \subset W^u(z')$, homeomorphic to $\Sigma^{n-2} \times [0; 1]$, because the diffeomorphism $f|_{W^u(z')}$ is stable in the Bing sense. Combining this with $f(S) \cap S = \emptyset$, we get that A is a fundamental annulus of $f|_{W^u(z')}$, i.e.,

$$\bigcup_{j \in \mathbb{Z}} f^j(A) = W^u(z') - z', \quad \text{int } f^i(A) \cap \text{int } f^l(A) = \emptyset, \quad i \neq l.$$

Moreover, $\bigcup_{j \geq 0} f^j(A) = W^u(z') - B'$. In particular, $B' \cap A = \emptyset$.

Step 6.9. *Suppose case 1) holds, i.e., S is a minimal sphere such that z' is inside S and A is a fundamental annulus of $f|_{W^u(z')}$ bounded by $(n-2)$ -spheres S and $f(S)$. Then $A \subset D_{pq}$. To be precise, $A \cap \text{int } D_{pq} = \text{int } A \cup f(S)$.*

Proof of Step 6.9. Let $h : \Sigma^{n-2} \times [0; 1] \rightarrow A$ be an orientation-preserving homeomorphism such that $h(\Sigma^{n-2} \times \{0\}) = S$ and $h(\Sigma^{n-2} \times \{1\}) = f(S)$. By the strong transversality condition, the restriction

$$P_A \stackrel{\text{def}}{=} \pi_p|_{A \cap D_{pq}} : A \cap D_{pq} \rightarrow W^u(p) - \text{int } D_p$$

is a local homeomorphism. Since $B' \cap D_{pq} = \emptyset$ (see step 6.8) and $B' \cap A = \emptyset$, we see that there is an $\varepsilon > 0$ such that $h(\Sigma^{n-2} \times [0; \varepsilon]) \subset D_{pq}$. Therefore,

$$P_A(h(\Sigma^{n-2} \times [0; \varepsilon])) \subset W^u(p) - \text{int } D_p.$$

Besides, one can choose $\varepsilon > 0$ so that P_A takes the annulus $h(\Sigma^{n-2} \times [0; \varepsilon])$ onto $P_A(h(\Sigma^{n-2} \times [0; \varepsilon]))$ homeomorphically, because P_A is a local homeomorphism and $P_A|_S = P|_S$ is a homeomorphism $S \rightarrow S_p^{n-2}$ (see step 6.6). Hence for any $0 \leq t < \varepsilon$, the $(n-2)$ -sphere $P_A(h(\Sigma^{n-2} \times \{t\}))$ is inside the $(n-2)$ -sphere

$$P_A(h(\Sigma^{n-2} \times \{\varepsilon\}))$$

on the unstable manifold $W^u(p)$.

Let $E \subset [0; 1]$ be the set of parameters t_0 satisfying the following properties:

- 1) $h(\Sigma^{n-2} \times [0; t_0]) \subset D_{pq}$ and $P_A(h(\Sigma^{n-2} \times [0; t_0])) \subset W^u(p) - \text{int } D_p$.
- 2) P_A maps the $(n-1)$ -annulus $h(\Sigma^{n-2} \times [0; t_0])$ onto $P_A(h(\Sigma^{n-2} \times [0; t_0]))$ homeomorphically.
- 3) Given any $0 \leq t < t_0$, the $(n-2)$ -sphere $P_A(h(\Sigma^{n-2} \times \{t\}))$ is inside the $(n-2)$ -sphere $P_A(h(\Sigma^{n-2} \times \{t_0\}))$ on $W^u(p)$. In particular,

$$P_A(h(\Sigma^{n-2} \times [0; t_0])) \subset W^u(p) - \text{int } D_p.$$

The argument above shows that E is a nonempty set of $[0; 1]$, because $[0; \varepsilon] \subset E$. The strong transversality condition implies that E is an open subset of $[0; 1]$. Furthermore, if $t \in E$, then $[0, t] \subset E$. Thus, E is an interval in $[0; 1]$ with the left endpoint 0. Let $t_* = \sup E$. Obviously, $t_* > 0$. By continuity of h , $h(\Sigma^{n-2} \times \{t_*\}) \subset D_{pq}$ because D_{pq} is closed. Moreover, $h(\Sigma^{n-2} \times \{t_*\}) \subset \text{int } D_{pq}$ since $[0; \varepsilon] \subset E$ for $\varepsilon > 0$, and so $t_* \geq \varepsilon$.

Let us show that P_A maps $h(\Sigma^{n-2} \times \{t_*\})$ onto $P_A(h(\Sigma^{n-2} \times \{t_*\}))$ as a homeomorphism. For $n \geq 4$, this follows from step 6.4. It remains to prove this fact for $n = 3$. Suppose not. Then there are points $a, b \in \Sigma^1 \times \{t_*\}$ such that

$$P_A(h(a)) = P_A(h(b)) \stackrel{\text{def}}{=} x_0, \quad a \neq b.$$

It follows that $h(\Sigma^1 \times [0; t_*])$ intersects the arc $\pi_p^{-1}(x_0)$ at two points $h(a), h(b)$. By the strong transversality condition, there are 2-balls $D_a \subset W^u(z')$, $D_b \subset W^u(z')$ containing the points a, b respectively and such that $D_a \cap D_b = \emptyset$. Moreover, we can assume that π_p projects D_a and D_b homeomorphically onto $\pi_p(D_a)$ and $\pi_p(D_b)$ respectively, and $D_a \cup D_b \subset \text{int } D_{pq}$ because $h(a), h(b) \subset \text{int } D_{pq}$.

Take t , $0 < t < t_*$, close enough to t_* so that the simple closed curve $h(\Sigma^1 \times \{t\})$ intersects both D_a and D_b . Since $P_A(D_a)$ and $P_A(D_b)$ are 2-balls and $P_A(h(\Sigma^1 \times \{t\}))$ bounds the 2-ball B_0 in $W^u(p)$, it follows that

$$B_0 \cup P_A((D_a) \cup P_A((D_b))) \stackrel{\text{def}}{=} Q$$

contains a simple closed curve, say S_0^1 , through x_0 . Furthermore, one can construct S_0^1 so that $S_0^1 \subset Q - D_p$. Then $\pi_p^{-1}(S_0^1)$ is a well defined set homeomorphic to a closed annulus $\Sigma^1 \times [0; 1]$. By construction, the intersection $\pi_p^{-1}(S_0^1) \cap W^u(z')$ contains the component S_0 through the points $h(a), h(b)$. This contradicts step 6.6. Thus, P_A maps $h(\Sigma^{n-2} \times \{t_*\})$ onto $P_A(h(\Sigma^1 \times \{t_*\}))$ homeomorphically for any $n \geq 3$.

Taking into account the strong transversality condition, we obtain $t_* \in E$. As a consequence, $t_* = 1$. It follows that $P_A(h(\Sigma^{n-2} \times [0; 1])) \subset W^u(p) - \text{int } D_p$. Hence, $A \cap \text{int } D_{pq} = \text{int } A \cup f(S)$. \diamond

Denote $W' = W^u(z') - B'$. Step 6.9 implies that $W' \subset D_{pq} \subset W^s(\Omega)$, because $\bigcup_{j \geq 0} f^j(A) = W^u(z') - B'$ and $f^j(D_{pq}) \subset D_{pq}$ for any $j \geq 0$. Hence, $C = W'$ is a

unique component of the intersection $D_{pq} \cap W^u(z')$ and $S = C \cap C_{pq}$ is a unique component of $W^u(z') \cap C_{pq}$. This concludes the proof of the theorem in case 1).

Let us consider case 2), i.e., no minimal ball contains z' . We actually prove that this case does not take place.

Step 6.10. *Suppose case 2) holds, i.e., no minimal ball contains z' . Then any path $\omega \subset W^u(z')$ connecting z' and a minimal sphere must intersect spheres of the intersection $W^u(z') \cap C_{pq}$ at interior points of ω ,*

$$\text{int } \omega \cap (W^u(z') \cap C_{pq}) \neq \emptyset.$$

Proof of Step 6.10. Let ω be a path with endpoints $z', z'' \in S$, where $S \subset W^u(z') \cap C_{pq}$ is a minimal sphere. Assume the contrary. Then $\omega \cap (W^u(z') \cap C_{pq}) = z''$. By step 6.8, $B_S \cap D_{pq} = \emptyset$, where $B_S \subset W^u(z')$ is the minimal $(n-1)$ -ball bounded by S . Hence, by the strong transversality condition, some exterior of S near S belongs to D_{pq} . Therefore, $\omega \subset D_{pq}$, and as a consequence, $z' \subset D_{pq}$. This is impossible. \diamond

Step 6.11. *Suppose that no minimal ball contains z' (i.e., case 2) holds). Then there is a compact domain $Q \subset W^u(z')$ bounded by spheres of $W^u(z') \cap C_{pq}$ (one sphere of ∂Q denoted by S_{ext} , is exterior and the others are inside of S_{ext}) such that $Q \subset D_{pq}$.*

Proof of Step 6.11. It is sufficient to prove that there exist a sphere $S' \subset W^u(z') \cap C_{pq}$ and a minimal ball $B'' \subset W^u(z')$ inside S' such that there is no sphere of $W^u(z') \cap C_{pq}$ which is inside of S' and contains B'' in its interior. Indeed, if we take the region $R \subset W^u(z')$ bounded by $S' \stackrel{\text{def}}{=} S_{ext}$ and delete from R all balls bounded by spheres of $W^u(z') \cap C_{pq}$ which are inside of S' , then we obtain the desired domain Q , because Q contains some exterior of B'' that is in D_{pq} , due to step 6.8, and Q does not contain spheres of $W^u(z') \cap C_{pq}$ except the boundary ∂Q .

Take any minimal ball $B' \subset W^u(z')$. By step 6.10, there is a sphere $S \subset W^u(z') \cap C_{pq}$ that separates z' and B' . It follows that either z' or B' is inside S . In the last case, there is nothing to prove, because one could take a smallest sphere that bounds a region with B' and is not the boundary of B' . By steps 6.10, 6.8, and the strong transversality condition, such a sphere exists.

So it remains to consider the case when z' is in the ball B bounded by S . By the strong transversality condition and compactness of $B \cup S$, B contains finitely many spheres of the intersection $W^u(z') \cap C_{pq}$. Therefore we can assume that there are no spheres inside S surrounding z' . Since no minimal ball contains z' , B is not a minimal ball. Then B contains a minimal ball, say B'' , bounded by the sphere $S'' = \partial B''$ and such that B'' is inside S . By step 6.8, $B'' \cap D_{pq} = \emptyset$. Obviously, $z' \notin D_{pq}$. By step 6.10, there is a sphere surrounding B'' but not the point z' . We get the case considered above. This concludes the proof. \diamond

Let $Q \subset W^u(z')$ be a compact domain that satisfies the conditions of step 6.11. Then

$$Q \cap C_{pq} = Q \cap D_{pq} = \partial Q,$$

because $\partial Q \subset W^u(z') \cap C_{pq}$. By definition, put

$$P_Q \stackrel{\text{def}}{=} \pi_p|_Q : Q \rightarrow W^u(p) - \text{int } D_p.$$

By the strong transversality condition, $P_Q|_{\text{int } Q}$ is a local homeomorphism. Hence,

$$P_Q|_{\text{int } Q} : \text{int } Q \rightarrow P_Q(\text{int } Q)$$

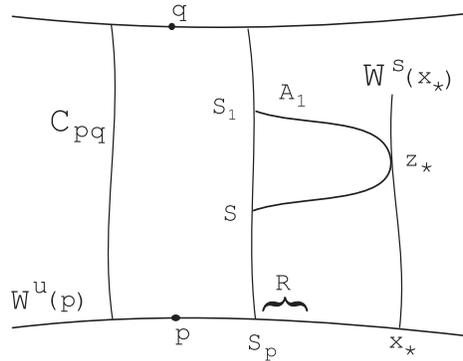


FIGURE 2. z_* is the point at which P_Q fails to be a local homeomorphism.

is a covering map. The strong transversality condition implies that there is an annulus $A_1 \subset Q$ with $\partial A_1 \subset \partial Q$ such that the annulus $P_Q(A_1) = R$ has the boundary component S_p , as shown in Figure 2. It follows that $\partial P_Q(Q) - S_p \neq \emptyset$, because Q and $P_Q(Q)$ are compact sets. Take a point $x_* \in \partial P_Q(Q) - S_p$. There is a point $z_* \in Q$ such that $P_Q(z_*) = x_*$, because $P_Q(Q)$ is a compact set, and thus $\partial P_Q(Q) \subset P_Q(Q)$. Since $x_* \in \partial P_Q(Q) - S_p$, $z_* \in \text{int } Q$. It follows that $P_Q|_{\text{int } Q}$ fails to be a local homeomorphism at the point z_* . What we have actually proved is that case 2) implies a tangency of invariant manifolds. This is impossible when f is a structurally stable diffeomorphism. This contradiction completes the proof of the theorem. \square

The main result of this section is the following theorem.

Theorem 6.2. *Let f be an A -diffeomorphism of a closed n -manifold M^n ($n \geq 3$), and Ω an orientable expanding attractor of codimension one. Suppose $\Omega' \neq \Omega$ is a nontrivial basic set of f such that $W^u(\Omega') \cap W^s(\Omega) \neq \emptyset$. Then f is not structurally stable.*

Proof. Taking into account the Mañé-Robinson theorem [35], [47], it is sufficient to prove that f does not satisfy the strong transversality condition. Therefore we can assume that Morse’s index of every basic set Θ with $W^u(\Theta) \cap W^s(\Omega) \neq \emptyset$ is not less than $n - 1$. Otherwise there is nothing to prove, since $\dim E_x^s = 1$, $x \in \Omega$.

Suppose the theorem is not correct; then any stable manifold $W^s(x)$, $x \in \Omega$, intersects transversally any unstable manifold $W^u(y)$, $y \in NW(f)$. By the condition of the theorem and the paper [28], $W^s(z) \cap W^u(z') \neq \emptyset$ for some points $z \in \Omega$, $z' \in \Omega'$. Hence the unstable manifold $W^u(z')$ is either $(n - 1)$ -dimensional or n -dimensional. If $W^u(z')$ is n -dimensional, then z' is a periodic point, and so Ω' is trivial. Thus, $W^u(z')$ is $(n - 1)$ -dimensional. Without loss of generality, we can assume that the points z and z' are periodic, because periodic points are dense in basics sets.

Take a point $a \in W^s(z) \cap W^u(z')$. Obviously, no point of $W^s(z) \cap W^u(z')$ is in Ω , because Ω is an attractor and consists of unstable manifolds of its points. It follows from Lemma 1.9 that either $a \in (x_a, y_a)_\emptyset^s \subset W^s(z)$, where $x_a \in W^u(p)$, $y_a \in W^u(q)$ for some associated boundary periodic points $p, q \in \Omega$, or $a \in W_\emptyset^s(p)$ for some periodic boundary point $p \in \Omega$. Taking into account the theorem on the

continuous dependence of invariant manifolds on initial conditions, we can assume that $a \in (x_a, y_a)_{\emptyset}^s \subset W^s(z)$.

Let S_{pq} be a characteristic sphere corresponding to the 2-bunch $W^u(p) \cup W^u(q)$. Recall that S_{pq} is a union of two closed $(n-1)$ -balls $D_p \subset W^u(p)$, $D_q \subset W^u(q)$ and the connecting cylinder

$$C_{pq} = \bigcup_{x \in \partial D_p, y \in \partial D_q} (x, y)_{\emptyset}^s, \text{ where } p \in \text{int } D_p, q \in \text{int } D_q.$$

By construction of a characteristic sphere (see section 2), we can choose S_{pq} so that

$$(x_a, y_a)_{\emptyset}^s \in D_{pq}, \text{ where } D_{pq} = \bigcup_{x \in W^u(p) - \text{int } D_p} [x, \varphi_{pq}(x)]_{\emptyset}^s.$$

Thus, $D_{pq} \cap W^u(z') \neq \emptyset$. By Corollary 3.1, $W^u(z')$ intersects C_{pq} .

By Theorem 6.1, the intersection $W^u(z') \cap D_{pq}$ consists of a unique component $C \subset D_{pq} \cap W^u(z')$. Moreover,

$$W^u(z') \cap C_{pq} = C \cap C_{pq} = S^{n-2}$$

is an $(n-2)$ -sphere that is isotopic to both $S_p = \partial D_p$ and $S_q = \partial D_q$ in the cylinder C_{pq} . Denote by $B' \subset W^u(z')$ the open $(n-1)$ -ball bounded by S^{n-2} in $W^u(z')$.

Denote $W' = W^u(z') - B'$, and let Ω'_c be the C -dense component of Ω' containing the point z' . Without loss of generality we can assume that $f(\Omega'_c) = \Omega'_c$ (see [2], [8]). Since $W^u(z') \cap C_{pq} = S^{n-2}$, $W' \subset D_{pq} \subset W^s(\Omega)$. Hence, $W' \cap \Omega'_c = \emptyset$. It follows from the nontriviality of Ω'_c that the intersection $W^u(z') \cap \Omega'_c$ must be dense in Ω'_c . Hence,

$$W^u(z') \cap \Omega'_c = (B' \cup W') \cap \Omega'_c = B' \cap \Omega'_c$$

is dense in Ω'_c . On the other hand, $B' \cap \Omega'_c$ is nowhere dense in Ω'_c , because $\text{clos } B'$ is a compact set. This contradiction concludes the proof. \square

Corollary 6.1. *Suppose f is a structurally stable A -diffeomorphism of a closed n -dimensional manifold M^n ($n \geq 3$), and Ω is an orientable expanding attractor of codimension one. Let B be a 2-bunch consisting of two unstable manifolds $W^u(p)$ and $W^u(q)$, where $p, q \in \Omega$ are associated boundary periodic points. Let S_{pq} be the characteristic sphere corresponding to B which is the union of a connecting cylinder C_{pq} and two $(n-1)$ -balls $D_p \subset W^u(p)$, $D_q \subset W^u(q)$. Suppose Θ is a basic set of index $n-1$ such that $W^u(z) \cap C_{pq}$ ($z \in \Theta$) is nonempty. Then Θ is a trivial basic set, z is a periodic point, and $W^u(z) \cap C_{pq}$ consists of a unique $(n-2)$ -sphere, say S^{n-2} , that is isotopic to both $S_p = \partial D_p$ and $S_q = \partial D_q$ in the cylinder C_{pq} . In particular, S^{n-2} divides C_{pq} into two cylinders that are homeomorphic to $\Sigma^{n-2} \times [0; 1]$.*

Corollary 6.2. *Suppose f is a structurally stable A -diffeomorphism of a closed n -dimensional manifold M^n ($n \geq 3$), and Ω is an orientable expanding attractor of codimension one. Let B be a 2-bunch consisting of two unstable manifolds $W^u(p)$ and $W^u(q)$, where $p, q \in \Omega$ are associated boundary periodic points. Then both $W_{\emptyset}^s(p)$ and $W_{\emptyset}^s(q)$ belong to n -dimensional unstable manifolds of repelling periodic points.*

Proof. Suppose the contrary; then $W^u(z) \cap (W_{\emptyset}^s(p) \cup W_{\emptyset}^s(q)) \neq \emptyset$ for some point $z \in \Theta$, where the basic set Θ is of index $n-1$. To be definite, suppose that $W^u(z) \cap W_{\emptyset}^s(p) \neq \emptyset$. Due to Theorem 6.2, Θ is trivial, i.e., z is a periodic point.

As a consequence, $W^u(z)$ must intersect $W_\emptyset^s(p)$ arbitrarily close to p . We keep the notation of Corollary 6.1. According to the λ -lemma, $W^u(z)$ must intersect the connecting cylinder C_{pq} in infinitely many components. This contradicts Corollary 6.1. \square

7. ON A TOPOLOGICAL STRUCTURE OF M^n

The main result of this section is the following theorem.

Theorem 7.1. *Suppose f is a structurally stable diffeomorphism of a closed n -manifold M^n ($n \geq 3$), and Ω is a codimension one orientable expanding attractor of f . Then:*

- 1) M^n is homotopy equivalent to the n -torus T^n . If $n \neq 4$, then M^n is homeomorphic to T^n .
- 2) Ω is the only nontrivial basic set of f .
- 3) f has at least one repelling periodic point.

Proof. Take a bunch B of Ω . Since Ω is orientable, B is a 2-bunch consisting of unstable manifolds $W^u(p)$ and $W^u(q)$, where p and q are associated boundary periodic points of Ω ; see Corollary 1.3. Let S_{pq} be the characteristic sphere corresponding to B and consisting of two $(n-1)$ -balls $D_p \subset W^u(p)$, $D_q \subset W^u(q)$, and connecting $(n-1)$ -cylinder $C_{pq} \subset W^s(\Omega)$. According to Theorem 5.1, in order to prove the homotopical equivalence of M^n and T^n , we must show that S_{pq} bounds a ball.

Taking some iteration f^k if necessary, we can assume without loss of generality that p and q are fixed points. Then Corollary 6.2 implies that $W_\emptyset^s(p)$ and $W_\emptyset^s(q)$ belong to unstable n -dimensional manifolds of repelling fixed points, say α_p and α_q respectively. As a consequence, p and q belong to the accessible boundaries $\delta(W^u(\alpha_p))$ and $\delta(W^u(\alpha_q))$ respectively. Due to the theorem on the continuous dependence of unstable manifolds on initial conditions, $W^u(p)$ and $W^u(q)$ belong to $\delta(W^u(\alpha_p))$ and $\delta(W^u(\alpha_q))$ respectively as well. Hence, $W^u(\alpha_p)$ and $W^u(\alpha_q)$ have to intersect the cylinder C_{pq} . Obviously, the intersections $W^u(\alpha_p) \cap C_{pq}$ and $W^u(\alpha_q) \cap C_{pq}$ are open subsets of C_{pq} .

Note that C_{pq} can intersect n -dimensional or $(n-1)$ -dimensional unstable manifolds only, since f is structurally stable and satisfies the strong transversality condition [35]. If C_{pq} has no intersections with $(n-1)$ -dimensional unstable manifolds, then the connectedness of C_{pq} implies $W^u(\alpha_p) = W^u(\alpha_q)$ and $\alpha_p = \alpha_q \stackrel{\text{def}}{=} \alpha$. Hence, $C_{pq} \subset W^u(\alpha)$.

For $\varepsilon > 0$ sufficiently small, according to the local product structure on D_p , the set

$$N(D_p) \stackrel{\text{def}}{=} \bigcup_{x \in D_p} W_\varepsilon^s(x)$$

is a tubular domain, which we can consider to be a trivial fiber bundle

$$(N(D_p), D_p, N(D_p)) \rightarrow D_p \cong (D_p \times (-1; +1), D_p, D_p \times (-1; +1)) \rightarrow D_p$$

with the fiber $W_\varepsilon^s(x)$ over every point $x \in D_p$. Clearly, D_p divides $N(D_p)$ into two domains. One of them, say $N^+(D_p)$, intersects both C_{pq} and $W_\emptyset^s(p)$. Combining this with the inclusion $D_p \subset \delta(W^u(\alpha))$, we see that

$$\text{clos } N^+(D_p) \subset D_p \cup W^u(\alpha) \subset \delta(W^u(\alpha)) \cup W^u(\alpha).$$

Bearing in mind the natural isomorphism of the fiber bundles $N(D_p)$ and $D_p \times (-1; +1)$, denote

$$D_{p,\varepsilon} \cong D_p \times \left\{ \frac{1}{2} \right\} \subset N^+(D_p), \quad C_{p,\varepsilon} = \left(\bigcup_{x \in \partial D_p} (x, y)_{\emptyset, \varepsilon}^s \right) \cap C_{pq} = N^+(D_p) \cap C_{pq}.$$

It is easy to see that $D_{p,\varepsilon}$, D_p , and $C_{p,\varepsilon}$ bound the n -ball $B_p \subset W^u(\alpha)$, and the union $C_{p,\varepsilon} \cup D_p$ is an $(n-1)$ -ball on the characteristic sphere S_{pq} .

Let $D_{q,\varepsilon}$, $C_{q,\varepsilon}$ be the similar ball and cylinder constructed in some tubular domain $N(D_q)$ near D_p . Denote

$$C_{pq,\varepsilon} \stackrel{\text{def}}{=} C_{pq} - (C_{p,\varepsilon} \cup C_{q,\varepsilon}).$$

Then $S'_{pq} = C_{pq,\varepsilon} \cup D_{p,\varepsilon} \cup D_{q,\varepsilon}$ is an $(n-1)$ -sphere isotopic to S_{pq} in $W^u(\alpha)$. Note that $D_p \subset \delta(W^u(\alpha))$ and $D_q \subset \delta(W^u(\alpha))$ are isotopic to $D_{p,\varepsilon}$ and $D_{q,\varepsilon}$ in $B_p \subset W^u(\alpha)$ and $B_q \subset W^u(\alpha)$ respectively.

Without loss of generality we can assume that S'_{pq} is a tamely embedded sphere. Therefore, S'_{pq} bounds a closed n -ball, say B'_{pq} , in $W^u(\alpha)$, because $W^u(\alpha)$ is homeomorphic to \mathbb{R}^n . It follows that S_{pq} also bounds the closed n -ball $B'_{pq} \cup B_p \cup B_q$ in $W^u(\alpha)$. Thus we've proved that if C_{pq} has no intersections with $(n-1)$ -dimensional unstable manifolds, then S_{pq} bounds an n -ball in M^n .

Suppose now that C_{pq} intersects $(n-1)$ -dimensional unstable manifolds. Due to Theorem 6.2, each of them is an unstable manifold of some saddle periodic point P_i . Moreover, the orbit of every P_i forms a trivial basic set of index $n-1$. There are finitely many of these points, because there are finitely many basic sets. Thus we have finitely many periodic saddle points P_1, \dots, P_k such that $W^u(P_i) \cap C_{pq} \neq \emptyset$, $1 \leq i \leq k$.

By Corollary 6.1, given a point P_i , the intersection $W^u(P_i) \cap C_{pq}$ consists of a unique $(n-2)$ -sphere, say S_i^{n-2} , that is isotopic to both $S_p = \partial D_p$ and $S_q = \partial D_q$ in the cylinder C_{pq} . In particular, S_i^{n-2} divides C_{pq} into two cylinders that are homeomorphic to $\Sigma^{n-2} \times [0; 1]$. Therefore we can rename the points P_i if necessary, putting them in order from

$$S_0^{n-2} = W^u(p) \cap \partial C_{pq} = \partial D_p$$

to

$$S_{k+1}^{n-2} = W^u(q) \cap \partial C_{pq} = \partial D_q.$$

Rename α_p and α_q as α_0 and α_k respectively.

Every sphere $S_i^{n-2} = W^u(P_i) \cap C_{pq}$ is tamely embedded in $W^u(P_i)$, and thus it bounds the closed $(n-1)$ -ball $D_i \subset W^u(P_i)$, because $W^u(P_i) \cong \mathbb{R}^{n-1}$, $i = 1, \dots, k$. Since S_i^{n-2} and S_{i+1}^{n-2} are isotopic in C_{pq} , it follows that they bound the domain $C_i \subset C_{pq}$ homeomorphic to the open cylinder $\Sigma^{n-2} \times (0; 1)$, $i = 0, \dots, k$. Then the union

$$S_i \stackrel{\text{def}}{=} D_i \cup C_i \cup D_{i+1}$$

is an $(n-1)$ -sphere tamely embedded in M^n . By Corollary 6.2, every cylinders C_i belongs to an n -dimensional unstable manifold of a repelling periodic point, say α_i . By passing to some iteration of f , we can assume that all points $p = P_0, P_1, \dots, P_k, P_{k+1} = q$, and $\alpha_0, \dots, \alpha_k$ are fixed.

Due to the theorem on the continuous dependence of invariant manifolds on initial conditions, the inclusions $C_i \subset W^u(\alpha_i)$ and $C_{i-1} \subset W^u(\alpha_{i-1})$ imply

$$D_i \subset \delta(W^u(\alpha_i)) \cap \delta(W^u(\alpha_{i-1})) \text{ for } i = 1, \dots, k.$$

Hence, $W^s(P_i) \cap W^u(\alpha_i) \neq \emptyset$ and $W^s(P_i) \cap W^u(\alpha_{i-1}) \neq \emptyset$. Note that P_i is a saddle fixed point of index $n - 1$. Therefore, $W^s(P_i)$ is a one-dimensional curve divided by P_i into two curves $W^{s-}(P_i)$ and $W^{s+}(P_i)$. Each of them can't intersect unstable $(n - 1)$ -manifolds; otherwise C_{i-1} or C_i would intersect unstable $(n - 1)$ -manifolds as well, which is impossible. Thus,

$$W^{s-}(P_i) \cup W^{s+}(P_i) \subset W^u(\alpha_i) \cup W^u(\alpha_{i-1}).$$

Using an argument quite similar to the one above for the case $k = 0$, one can prove that the $(n - 1)$ -sphere S_0 bounds an n -ball $B_0 \subset W^u(\alpha_0)$. By the local product structure on Ω , given any point $x \in D_0 - P_0$, the arc $[x, \varphi_{pq}(x)]_0^s$ enters into B_0 , and has to leave B_0 because $\varphi_{pq}(x)$ is outside of B_0 (recall that $D_0 = D_p$ and $P_0 = p$). Let $\varphi_{01}(x)$ be the first intersection of $[x, \varphi_{pq}(x)]_0^s$ with $S_0 = \partial B_0$ after x . By orientability of Ω , $\varphi_{01}(x) \notin D_0$. Hence, $\varphi_{01}(x) \in D_1 - P_1$. Since stable manifolds are pairwise disjoint, we see that

$$\varphi_{01} : D_0 - P_0 \rightarrow D_1 - P_1$$

is a one-to-one map. It follows that $W^{s-}(P_1)$ and $W^{s+}(P_1)$ do not both intersect $D_1 - P_1$. Therefore there is a unique component of $W^s(P_1) - P_1$, say $W^{s-}(P_1)$, which belongs to $B_0 \cap W^u(\alpha_0)$. Another component $W^{s+}(P_1)$ belongs to $W^u(\alpha_1)$. In particular, $\alpha_0 \neq \alpha_1$.

Continuing this line of reasoning, we see that every $(n - 1)$ -sphere S_i bounds an n -ball $B_i \subset W^u(\alpha_i)$, and the points $\alpha_0, \dots, \alpha_k$ are pairwise disjoint. As a consequence, the open balls $\text{int } B_0, \dots, \text{int } B_k$ are pairwise disjoint as well. It follows that the characteristic sphere S_{pq} bounds the open n -ball

$$\bigcup_{i=1}^k \text{int } B_i \cup \bigcup_{i=1}^k \text{int } D_i \stackrel{\text{def}}{=} B_{pq}^n.$$

By Theorem 5.1, M^n is homotopy equivalent to the n -torus T^n .

Due to Hsiang and Wall [29], any closed n -manifold ($n \geq 5$) homotopy equivalent to a torus is homeomorphic to a torus. So to conclude the proof of item 1), it remains to consider the case $n = 3$. By Lemma 2 in [51], M^3 is irreducible (i.e., every tamely embedded 2-sphere in M^3 bounds a ball) because the universal cover is homeomorphic to \mathbb{R}^3 . Due to Stallings's theorem [55], if there is an epimorphism $\pi_1(M^3) \rightarrow \mathbb{Z}$ with kernel $\neq \mathbb{Z}_2$, then M^3 is a fiber bundle over S^1 with fiber a closed surface T . Obviously such an epimorphism exists, because $\pi_1(M^3)$ is isomorphic to \mathbb{Z}^3 .

The homotopy exact sequence

$$\pi_3(S^1) \rightarrow \pi_2(T) \rightarrow \pi_2(M^3) \rightarrow \pi_2(S^1) \rightarrow \pi_1(T) \rightarrow \pi_1(M^3) \rightarrow \pi_1(S^1) \rightarrow 0$$

of the fiber bundle $(M^3, S^1, M^3 \rightarrow S^1)$ is the following:

$$0 \rightarrow \pi_2(T) \rightarrow \pi_2(M^3) \rightarrow 0 \rightarrow \pi_1(T) \rightarrow \mathbb{Z}^3 \rightarrow \mathbb{Z}^1.$$

Hence, $\pi_2(T) = \pi_2(M^3) = 0$ and $\pi_1(T)$ is mapped injectively into the free abelian group $\pi_1(M^3)$, and thus $\pi_1(T)$ is free abelian as well. Since $\pi_2(T) = \pi_2(M^3) = 0$, T is the 2-torus $T^2 = S^1 \times S^1$.

Now M^3 is homeomorphic to the factor-space $(T \times \mathbb{R}^1)/g$, where $(m, x+1) \in T \times \mathbb{R}^1$ is equivalent to $(g(m), x)$ under some homeomorphism $g : T \rightarrow T$. We can take three generators a, b, c of $\pi_1(M^3)$ so that $a, b \in \pi_1(T)$ and the following relations hold:

$$ab = ba, \quad g_*(a) = c^{-1}ac, \quad g_*(b) = c^{-1}bc,$$

where $g_* : \pi_1(T) \rightarrow \pi_1(T)$ is the homomorphism induced by g [24]. Since $\pi_1(T)$ is abelian, we have $g_*(a) = a$ and $g_*(b) = b$. This implies that g is isotopic to the identity of T . Thus, M^3 is homeomorphic to $T \times S^1 = T^2 \times S^1 = T^3$. This concludes the proof of item 1).

Item 2) follows from item 1) of Theorem 5.1. By Corollary 6.2, there is at least one repelling periodic point inside of characteristic spheres. This concludes the proof. \square

Corollary 7.1. *Suppose f is a structurally stable diffeomorphism of a closed n -manifold M^n ($n \geq 3$) having an orientable expanding attractor of codimension one Ω . Then the spectral decomposition of f consists of Ω , a finite nonzero number of repelling periodic orbits of index n , and a finite number (maybe zero) of periodic saddle orbit of index $n - 1$.*

Corollary 7.2. *Suppose f is a structurally stable diffeomorphism of a closed n -manifold M^n ($n \geq 3$) having an orientable expanding attractor Ω of codimension one. Let S_{pq} be a characteristic sphere corresponding to a 2-bunch $B = W^u(p) \cup W^u(q)$ of Ω , where $p, q \in \Omega$ are associated boundary periodic points. Then S_{pq} bounds an n -ball containing $k_0 + 1$ repelling periodic points $\alpha_1, \dots, \alpha_{k_0+1}$ ($k_0 \geq 0$) of index n and k_0 saddle periodic points P_1, \dots, P_k of index $n - 1$. Moreover, the following conditions hold:*

1) *The union*

$$a_{pq} = \{p\} \cup W_\emptyset^s(p) \cup \alpha_1 \cup W^s(P_1) \cup P_1 \cup \alpha_2 \cup \dots \cup P_k \cup \alpha_{k+1} \cup W_\emptyset^s(q) \cup \{q\}$$

is a compact arc with no self-intersections and with the endpoints p, q .

2) *On the arc a_{pq} , the repelling periodic points of index n alternate with the saddle periodic points of index $n - 1$ (including p and q).*

Corollary 7.3. *Suppose f is an A -diffeomorphism of a closed n -manifold M^n ($n \geq 3$) having an orientable expanding attractor Ω of codimension one. Then the following conditions imply that f is not structurally stable:*

1) *f has a nontrivial basic set different from Ω .*

2) *M^n is not homotopy equivalent to the n -torus.*

Now we consider non-orientable codimension one expanding attractors and contracting repellers. Recall that structurally stable diffeomorphisms with non-orientable codimension one expanding attractors exist on any closed 2-manifolds beginning with the 2-sphere [44]. In contrast, we show that if a closed 3-manifold M^3 admits a non-orientable codimension one expanding attractor, then M^3 must be non-orientable. The following theorem, proved by V. Medvedev and E. Zhuzhoma, is a crucial step in the proof of Theorem 7.3.

Theorem 7.2. *Let $f : M^3 \rightarrow M^3$ be an A -diffeomorphism of a closed 3-manifold M^3 . If f has a non-orientable codimension one expanding attractor Ω , then M^3 is non-orientable.*

Proof. Due to Theorem 2.1 of [45], Ω has 1-bunches. Let $W^u(m_0) \subset \Omega$ be an unstable manifold forming a 1-bunch of Ω , where m_0 is a boundary periodic point. Without loss of generality we can assume that m_0 is a fixed point of f . According to Lemma 2.1 and Lemma 1.11, there is a homeomorphism $\varphi : W^u(m_0) \rightarrow W^u(m_0)$ such that $\varphi^2 = id$ and m_0 is the only fixed point of φ . It is convenient to prove the theorem in steps. The end of the proof of a step will again be denoted by \diamond .

Take a point $x_0 \in W^u(m_0)$ such that $x_0 \neq m_0$. Let $D_\alpha \subset W^u(m_0) - m_0$ be the family of disks (α ranges over the interval $[0; 1]$) with the following properties:

- $x_0 \in D_0$, $\varphi(x_0) \notin D_0$, $\varphi(x_0) \in D_1$, $D_0 \cap \varphi(D_0) = \emptyset$.
- $D_{\alpha_1} \subset D_{\alpha_2}$ whenever $\alpha_1 < \alpha_2$.
- The boundary ∂D_α varies continuously with respect to α .

Such a family exists, because $x_0 \neq \varphi(x_0)$ and $W^u(m_0)$ is homeomorphic to \mathbb{R}^2 .

Step 7.1. *There exists an index $\alpha_0 \in (0; 1)$ such that*

$$\emptyset \neq D_{\alpha_0} \cap \varphi(D_{\alpha_0}) = \partial D_{\alpha_0} \cap \partial(\varphi(D_{\alpha_0})).$$

Proof of Step 7.1. If $\alpha_1 < \alpha_2$, then $\varphi(D_{\alpha_1}) \subset \varphi(D_{\alpha_2})$, because φ is a homeomorphism. Since $D_0 \cap \varphi(D_0) = \emptyset$, we have that $D_\alpha \cap \varphi(D_\alpha) = \emptyset$ for α sufficiently small. On the other hand, $D_{\alpha_1} \cap \varphi(D_{\alpha_1}) \neq \emptyset$, because $\varphi(x_0) \in D_1$. Since ∂D_α varies continuously with respect to α , $\partial\varphi(D_\alpha) = \varphi(\partial D_\alpha)$ varies continuously with respect to α as well. Hence there exists $\alpha_0 \in (0; 1)$ such that the disks D_{α_0} , $\varphi(D_{\alpha_0})$ have a non-empty intersection at points of the boundaries only. \diamond

Step 7.2. *There exists a curve $C \subset W^u(m_0) - m_0$, with no self-intersections, such that $C \cap \varphi(C) = \{x_0, \varphi(x_0)\}$, where $x_0, \varphi(x_0)$ are the endpoints of C .*

Proof of Step 7.2. According to step 7.1, there is a point $x_1 \in \partial D_{\alpha_0} \cap \varphi(\partial D_{\alpha_0})$ for some $\alpha_0 \in (0, 1)$. Since $\varphi^2 = id$ and m_0 is the only fixed point of φ , it follows that $\varphi(x_1) \in \partial D_{\alpha_0} \cap \varphi(\partial D_{\alpha_0})$ and $\varphi(x_1) \neq x_1$. There is the path $C_{01} \subset D_{\alpha_0}$ with no self-intersections such that C_{01} connects the points x_0, x_1 (because D_{α_0} is a disk). Due to step 7.1, the interior of D_{α_0} and the image of this interior under φ have no intersections. Therefore, $\varphi(C_{01}) \subset \varphi(D_{\alpha_0})$ is a path with no self-intersections. Moreover, $\varphi(C_{01})$ connects $\varphi(x_0)$ and $\varphi(x_1)$, and $C_{01} \cap \varphi(C_{01}) = \emptyset$. Obviously, the curve $\varphi(C_{01})$ does not divide the disk $\varphi(D_{\alpha_0})$. Hence there is a path C_{12} with no self-intersections that connects the points $\varphi(x_0)$ and x_1 . Moreover, we can assume that $\varphi(C_{01}) \cap C_{12} = \varphi(x_0)$. By definition, put $C = C_{01} \cup C_{12}$. Since $\varphi^2 = id$, it follows that $C \cap \varphi(C) = \{x_0, \varphi(x_0)\}$. \diamond

Step 7.3. *The set $M_c = \bigcup_{x \in C} (x, y)_\emptyset^s$ is homeomorphic to the open Möbius band. Moreover, any arc $(x, y)_\emptyset^s$ is transversal (in the topological sense) to the middle line of M_c .*

Proof of Step 7.3. Let us consider the open rectangle $P = \bigcup_{x \in C - x_0} (x, y)_\emptyset^s$. Then M_c is obtained by adding to P the arc $(x_0, \varphi(x_0))_\emptyset^s$. The relation $\varphi^2 = id$ implies that this addition can be represented as attaching two opposite sides of the closed rectangle $clos P$ by an orientation-reversing homeomorphism. Hence, M_c is homeomorphic to the open Möbius band. By construction, any arc $(x, y)_\emptyset^s$ is transversal (in the topological sense) to the middle line of the Möbius band. \diamond

By step 7.2, the union $C \cup \varphi(C) = c_0$ is a simple closed curve in $W^u(m_0)$. Therefore, c_0 bounds on $W^u(m_0)$ the open disk D_c .

Step 7.4. $m_0 \in D_c$.

Proof of Step 7.4. By construction, $\varphi(c_0) = c_0$. Since φ is a homeomorphism, it follows that $\varphi(D_c) = D_c$ and $\varphi(c_0 \cup D_c) = c_0 \cup D_c$, because the exterior of c_0 on $W^u(m_0)$ is homeomorphic to an open annulus. Hence, φ has a fixed point in $c_0 \cup D_c$. Since m_0 is the unique fixed point of φ , $m_0 \in D_c$. \diamond

Step 7.5. *There exists an open subset $C^3 \subset M^3$ that is homeomorphic to the product $M_b \times (0, 1)$, where M_b is the open Möbius band.*

Proof of Step 7.5. We keep the notation of the steps above. Take an open simple arc $l_0 \subset D_c$ such that $l_0 \cap \varphi(l_0) = \emptyset$ and one of endpoints is x_0 . Because $\varphi(x_0) \neq x_0$ and φ is a homeomorphism, such an arc exists. Since $\varphi(C) \cap C$ consists of only two points x_0 and $\varphi(x_0)$, it follows that there is an open strip $P_1 \subset D_c$, bounded by the curve C and arcs l_0 and $\varphi(l_0)$, such that $\varphi(P_1) \cap P_1 = \emptyset$ and $m_0 \notin \varphi(P_1) \cup P_1$. Therefore, $\varphi(P_1)$ is the open strip bounded by the curve $\varphi(C)$ and the arcs l_0 and $\varphi(l_0)$. As a consequence, the union $P = \varphi(l_0) \cup l_0 \cup \varphi(P_1) \cup P_1$ is an open strip in D_c . Moreover, one component of the boundary ∂P coincides with the boundary $\partial D_c = \varphi(C) \cup C$ of D_c . Let C_1 be the side of $P_1 \subset D_c$ which is opposite the side $C \stackrel{\text{def}}{=} C_0$. Note that the other pair of opposite sides l_0 and $\varphi(l_0)$ is invariant under φ . Therefore there exists a splitting of the closed rectangle $\varphi(l_0) \cup l_0 \cup P_1 \cup C_0 \cup C_1 = \text{clos } P_1$ into disjoint arcs C_α (α ranges over the interval $[0, 1]$) such that endpoints of each arc map onto each other by φ . Moreover, without loss of generality we can assume that the arcs C_α , $0 \leq \alpha \leq 1$, form a topological foliation on $\text{clos } P_1$. It follows from the equality $\varphi(P_1) \cap P_1 = \emptyset$ that, given any $\alpha \in [0, 1]$, $C_\alpha \cup \varphi(C_\alpha)$ is a closed simple curve. As a consequence, the family $C_\alpha \cup \varphi(C_\alpha)$, $\alpha \in [0, 1]$, forms the topological foliation on P with leaves $C_\alpha \cup \varphi(C_\alpha)$ that are closed simple curves invariant under φ . Therefore, $C_\alpha \cup \varphi(C_\alpha)$ is the boundary of the set $M_\alpha = \bigcup_{x \in C_\alpha \cup \varphi(C_\alpha)} (x, \varphi(x))_\emptyset^s$, which is homeomorphic to the open Möbius band M_b , step 7.3. Since stable manifolds are pairwise disjoint, the M_α are pairwise disjoint as well.

The theorem on the continuous dependence of invariant manifolds on initial conditions implies that the sets M_α vary continuously with respect to α . Since the strip P is homeomorphic to the product $S^1 \times (0, 1)$, it follows that $C^3 = \bigcup_{x \in P} (x, \varphi(x))_\emptyset^s = \bigcup_{\alpha \in (0, 1)} M_\alpha$ is homeomorphic to the product $M_b \times (0, 1)$. \diamond

It is well known that the Möbius band is a non-orientable surface. Hence, C^3 is a non-orientable 3-manifold. As a consequence, M^3 is a non-orientable 3-manifold as well [24]. This concludes the proof. \square

Theorem 7.3. *Let $f : M^3 \rightarrow M^3$ be a structurally stable diffeomorphism of a closed 3-manifold M^3 . Then the spectral decomposition of f contains neither codimension one non-orientable expanding attractors nor codimension one non-orientable contracting repellers. In particular, f has no codimension one Plykin attractors.*

Proof. Assume the converse. Without loss of generality we can assume that f has a codimension one non-orientable expanding attractor Ω . According to Theorem 7.2, M^3 is non-orientable. Let \tilde{M}^3 be the orientable manifold such that $p : \tilde{M}^3 \rightarrow M^3$ is a double covering for M^3 . Then there exists a diffeomorphism $\tilde{f} : \tilde{M}^3 \rightarrow \tilde{M}^3$ which is a lift of f , i.e., $f \circ p = p \circ \tilde{f}$. It is easy to see that \tilde{f} is an A -diffeomorphism. Since p is a local diffeomorphism, \tilde{f} satisfies the strong transversality condition as well. By Robinson's theorem [47], \tilde{f} is a structurally stable diffeomorphism. The

preimage $p^{-1}(\Omega)$ contains a codimension one attractor $\tilde{\Omega}$, because Ω is an attractor. Since Ω is locally homeomorphic to the product of a Cantor set and \mathbb{R}^2 , so is $\tilde{\Omega}$. It follows that $\tilde{\Omega}$ is an expanding attractor of codimension one.

Recall that \tilde{M}^3 is orientable. Due to Theorem 7.2, $\tilde{\Omega}$ is orientable and has no 1-bunches. By Corollary 1.3, $\tilde{\Omega}$ has 2-bunches only.

Since Ω is non-orientable, it follows that there is an open component $R \subset M^3 - \Omega$ such that the accessible boundary of R contains the 1-bunch $W^u(m_0) \subset \Omega$, where m_0 is a boundary periodic point. Without loss of generality we can assume that m_0 is a fixed point of f . Let $\tilde{m}_1 \in \tilde{\Omega}$ be the lift under p of the point m_0 . Because p is a double covering, \tilde{m}_1 is a boundary periodic point of $\tilde{\Omega}$. Since every bunch of $\tilde{\Omega}$ is a 2-bunch, it follows that there is a boundary periodic point $\tilde{m}_2 \in \tilde{\Omega}$, $\tilde{m}_2 \neq \tilde{m}_1$, associated with \tilde{m}_1 .

By Corollary 7.2, there are $k + 1 \geq 1$ repelling periodic points $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{k+1}$ and $k \geq 0$ saddle periodic points P_1, \dots, P_k of the index 2 such that the following conditions hold:

- $a_{12} = \{\tilde{m}_1\} \cup W_\emptyset^s(\tilde{m}_1) \cup \tilde{\alpha}_1 \cup W^s(P_1) \cup P_1 \cup \tilde{\alpha}_2 \cup \dots \cup P_k \cup \tilde{\alpha}_{k+1} \cup W_\emptyset^s(\tilde{m}_2) \cup \{\tilde{m}_2\}$ is a compact arc with no self-intersections.
- On the arc a_{12} , the repelling periodic points $\tilde{\alpha}_i$ alternate with the saddle periodic points P_i .

By Lemma 2.1, if $\tilde{x}_n \rightarrow \tilde{m}_1$ as $n \rightarrow \infty$, then $\tilde{y}_n \rightarrow \tilde{m}_2$, where $\tilde{x}_n \in W^u(\tilde{m}_1)$, $\tilde{y}_n \in W^s(\tilde{m}_2)$, and $(\tilde{x}_n, \tilde{y}_n)^s = (\tilde{x}_n, \tilde{y}_n)_\emptyset^s$. Since $(\tilde{x}_n, \tilde{y}_n)^s \cup \tilde{\Omega} = \emptyset$ and $p(\tilde{\Omega}) = \Omega$,

$$p((\tilde{x}_n, \tilde{y}_n)_\emptyset^s) = (p(\tilde{x}_n), p(\tilde{y}_n))_\emptyset^s.$$

The inclusion $p(\tilde{x}_n) \in W^u(m_0)$ implies $p(\tilde{y}_n) \in W^u(m_0)$, because $W^u(m_0)$ is a 1-bunch. Since p is a continuous map, $p(\tilde{x}_n) \rightarrow p(\tilde{m}_1) = m_0$ and $p(\tilde{y}_n) \rightarrow p(\tilde{m}_2)$. From the inclusion $p(\tilde{y}_n) \in W^u(m_0)$ and Lemma 2.1, it follows that $p(\tilde{y}_n) \rightarrow m_0$. Hence, $p(\tilde{m}_2) = m_0$.

The relation $p \circ f = \tilde{f} \circ p$ implies that p maps the invariant manifolds of \tilde{f} onto the invariant manifolds of f . Then

$$p(W_\emptyset^s(\tilde{m}_1)) = p(W_\emptyset^s(\tilde{m}_2)), \quad p(\tilde{\alpha}_1) = p(\tilde{\alpha}_{k+1}),$$

$$p(W^s(P_1)) = p(W^s(P_k)), \quad p(P_1) = p(P_k), \quad p(\tilde{\alpha}_2) = p(\tilde{\alpha}_k), \dots$$

Since the repelling periodic points $\tilde{\alpha}_i$ alternate with the saddle periodic points P_i on the arc a_{12} , we see that the number of periodic points on a_{12} is odd, because the endpoints are of saddle type. As a consequence, there is either a periodic point $\tilde{\alpha}_i$ with $p(W^s(P_{i-1})) = p(W^s(P_i))$ or a periodic point P_i with $p(W_1^s(P_i)) = p(W_2^s(P_i))$, where $W_1^s(P_i), W_2^s(P_i)$ are different components of $W^s(P_i) - P_i$. In both cases, there is a point ($\tilde{\alpha}_i$ or P_i) at which p is not a local homeomorphism. This contradiction concludes the proof. □

8. TOPOLOGICAL CLASSIFICATION

After Theorem 7.1, it is natural to classify structurally stable diffeomorphisms with expanding orientable attractors of codimension one on the n -torus T^n , $n \geq 3$. Let $f : T^n \rightarrow T^n$ be such a diffeomorphism. Denote by $f_* : H_1(T^n, \mathbb{R}^n) \rightarrow H_1(T^n, \mathbb{R}^n)$ the automorphism of the first homology group $H_1(T^n, \mathbb{R}^n) \simeq \mathbb{R}^n$ induced by f . The automorphism f_* is *hyperbolic* if f_* has no eigenvalues of absolute value 1. The following theorem was proved in [19], thm. 1, with no assumption on the structural stability.

Theorem 8.1. *Suppose $f : T^n \rightarrow T^n$ is a structurally stable diffeomorphism having an orientable expanding attractor Ω of codimension one. Then f_* is hyperbolic.*

Following [12], let us call a diffeomorphism $A : M^n \rightarrow M^n$ a π_1 -diffeomorphism if, given any homeomorphism $g : K \rightarrow K$ of a compact CW complex and any map $h : K \rightarrow M^n$ such that $A_*h_* = h_*g_*$, there is a unique base-point-preserving map $h' : K \rightarrow M^n$, homotopic to h , such that $A \circ h' = h' \circ g$.

Let $f : T^n \rightarrow T^n$ be a structurally stable diffeomorphism having an orientable expanding attractor Ω of codimension one. Due to Theorem 8.1, f_* is hyperbolic. Take an algebraic automorphism $A(f) : T^n \rightarrow T^n$ such that $f_* = A(f)_*$. By Proposition 2.1 in [12], $A(f)_*$ is a π_1 -diffeomorphism. Then there exists a map $h : T^n \rightarrow T^n$, homotopic to the identity, such that $h \circ f = A(f) \circ h$. By definition, put

$$P(f, h) = \{x \in T^n \mid h^{-1}(x) \text{ contains at least two points}\}.$$

Lemma 8.1. *The map h is continuous and satisfies the following conditions:*

- $h(\Omega) = T^n$.
- If $\{p_i, q_i\}_{i=1}^k$ are the pairs of associated boundary periodic points of f , then $h(p_i) = h(q_i)$. Moreover, $h(p_i) = h(q_i)$ is a periodic point of $A(f)$ for each $i = 1, \dots, k$.
- $P(f, h) = \bigcup_{i=1}^k W^u(p_i) \cup W^u(q_i)$.
- Suppose K_i is a component of $M^n - \Omega$; then $h(K_i)$ is the unstable manifold $W^u(h(p_i)) = W^u(h(q_i))$ of $A(f)$, where p_i, q_i are associated boundary periodic points such that $\delta(K_i) = W^u(p_i) \cup W^u(q_i)$. Moreover, $h(K_i \cup \delta(K_i)) = W^u(h(p_i))$.
- If $\check{\Omega} \subset \Omega$ is a union of unstable manifolds that do not contain boundary periodic points, then the restriction $h|_{\check{\Omega}}$ is a homeomorphism onto its image $h(\check{\Omega})$.

Proof. The proof is similar to that of Theorem 2 in [19] and Lemma 4 in [20]. We omit the details. \square

The relation $h \circ f = A(f) \circ h$ and Lemma 8.1 imply that $A(f)$ is a codimension one Anosov diffeomorphism.

Given any pair (p, q) of associated boundary periodic points, we assign the number $n(p, q) \in \mathbb{N}$ of repelling periodic points that are inside the characteristic sphere S_{pq} corresponding to the 2-bunch $B = W^u(p) \cup W^u(q)$. This number is well defined, because it does not depend on the choice of a characteristic sphere. Obviously, $n(p, q) = n(f^m(p), f^m(q))$ for any $m \in \mathbb{Z}$. Therefore we can assign the number $n(p, q) \stackrel{\text{def}}{=} n(O(p, q))$ to the pair of orbits $O(p), O(q)$ of the points p, q .

Let $\{O(p_i, q_i)\}_{i=1}^k$ be the pairs of orbits of associated boundary points, and let $\{n(O(p_i, q_i))\}_{i=1}^k$ be the set of corresponding natural numbers defined above. To each periodic orbit $h(O(p_i)) = h(O(q_i))$ of $A(f)$, we assign the number $n(O(p_i, q_i))$. The collection $\{h(O(p_i)), n(O(p_i, q_i))\}_{i=1}^k$, denoted by $\mathcal{D}(f, h)$, is called the *data set* of f ,

$$\mathcal{D}(f, h) = \{h(O(p_i)), n(O(p_i, q_i))\}_{i=1}^k.$$

Let A be an arbitrary codimension one hyperbolic automorphism of T^n , and let $\{O_j\}_{j=1}^r$ be any finite family of periodic orbits of A . To any orbit O_j , let us assign an arbitrary natural number $n_j \in \mathbb{N}$. The collection $\{O_j, n_j\}_{j=1}^r$ is called an

admissible data set of the automorphism A . Note that by Theorem 7.1, a data set of f is admissible whenever f is structurally stable.

Suppose $\{O_j^1, n_j^1\}_{j=1}^{r_1}$ and $\{O_j^2, n_j^2\}_{j=1}^{r_2}$ are admissible data sets of codimension one hyperbolic automorphisms A_1 and A_2 respectively. These data sets are called *equivalent* if $r_1 = r_2 \stackrel{\text{def}}{=} r$ and there is an affine transformation $\psi : T^n \rightarrow T^n$ (i.e., the composition of an automorphism and translation, $\psi = Ax + \xi$) such that

$$\psi \circ A_1 = A_2 \circ \psi, \quad \psi\left(\bigcup_{j=1}^r O_j^1\right) = \bigcup_{j=1}^r O_j^2, \quad n(\psi(O_j)) = n(O_j), 1 \leq j \leq r.$$

Theorem 8.2. *Suppose $f_1, f_2 : T^n \rightarrow T^n$ are structurally stable diffeomorphisms having orientable expanding attractors of codimension one Ω_1 and Ω_2 respectively. Then f_1 and f_2 are conjugate if and only if the data sets $\mathcal{D}(f_1, h_1)$ and $\mathcal{D}(f_2, h_2)$ are equivalent, where $h_i : T^n \rightarrow T^n$ ($i = 1, 2$) are continuous maps homotopic to the identity and such that $h_i \circ f_i = A_i \circ h_i$, $(f_i)_* = (A_i)_*$.*

Proof. The proof is similar to the proof of Theorem 3 of [19] and Theorem 2 of [20], where the conjugacy was proved in the basins of the codimension one expanding attractors. Note that in those theorems, there was no assumption on structural stability of the diffeomorphisms f_1, f_2 (see also Theorem 2.5 of [45] and Theorem 4 of [63]). The proof is omitted. \square

Theorem 8.3. *Let A be a codimension one hyperbolic automorphism of T^n such that the stable manifolds of A are one-dimensional. Given an admissible data set $\{O_j, n_j\}_{j=1}^r$ of A , there is a structurally stable diffeomorphism $f : T^n \rightarrow T^n$ having an orientable expanding attractor of codimension one and such that $\mathcal{D}(f, h) = \{O_j, n_j\}_{j=1}^r$, where $h : T^n \rightarrow T^n$ is a continuous map homotopic to the identity with $h \circ f = A \circ h$ and $f_* = A_*$.*

Proof. Actually, f is constructed by the ‘surgery operation’ described by Smale [53] and Williams [59] to get a DA -diffeomorphism with a one-dimensional basic set on the 2-torus. The neat construction is in [31] and [48] (for the 2-torus), and [13] for the 3-torus, and [45] (for any n -torus, $n \geq 2$). According to Theorem 2.3 in [45], there is a structurally stable DA -diffeomorphism $f' : T^n \rightarrow T^n$ having a unique expanding orientable codimension one attractor Ω' and the data set $\{O_j, 1\}_{j=1}^r$. By a slight modification of f' inside the characteristic spheres of Ω' , one can get the desired diffeomorphism f with the data set $\{O_j, n_j\}_{j=1}^r$ and the expanding orientable codimension one attractor $\Omega = \Omega'$. \square

Theorems 8.2 and 7.3 imply the following corollary.

Corollary 8.1. *Suppose $f_1, f_2 : T^3 \rightarrow T^3$ are structurally stable diffeomorphisms having expanding attractors of codimension one Ω_1 and Ω_2 respectively. Then f_1 and f_2 are conjugate if and only if the data sets $\mathcal{D}(f_1, h_1)$ and $\mathcal{D}(f_2, h_2)$ are equivalent, where $h_i : T^3 \rightarrow T^3$ ($i = 1, 2$) are continuous maps homotopic to the identity and such that $h_i \circ f_i = A_i \circ h_i$ and $(f_i)_* = (A_i)_*$.*

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