

DYNAMICAL SYSTEMS DISJOINT FROM ANY MINIMAL SYSTEM

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ABSTRACT. Furstenberg showed that if two topological systems (X, T) and (Y, S) are disjoint, then one of them, say (Y, S) , is minimal. When (Y, S) is nontrivial, we prove that (X, T) must have dense recurrent points, and there are countably many maximal transitive subsystems of (X, T) such that their union is dense and each of them is disjoint from (Y, S) . Showing that a weakly mixing system with dense periodic points is in \mathcal{M}^\perp , the collection of all systems disjoint from any minimal system, Furstenberg asked the question to characterize the systems in \mathcal{M}^\perp . We show that a weakly mixing system with dense regular minimal points is in \mathcal{M}^\perp , and each system in \mathcal{M}^\perp has dense minimal points and it is weakly mixing if it is transitive. Transitive systems in \mathcal{M}^\perp and having no periodic points are constructed. Moreover, we show that there is a distal system in \mathcal{M}^\perp .

Recently, Weiss showed that a system is weakly disjoint from all weakly mixing systems iff it is topologically ergodic. We construct an example which is weakly disjoint from all topologically ergodic systems and is not weakly mixing.

§1. INTRODUCTION

By a *topological dynamical system* (TDS for short) (X, T) we mean a compact metric space X with a continuous surjective map T from X to itself. Recall that (X, T) is *transitive* if for each pair of open (i.e., nonempty and open) subsets U and V , $N(U, V) = \{n \in \mathbb{Z}_+ : T^{-n}V \cap U \neq \emptyset\}$ is infinite. (X, T) is *weakly mixing* if $(X \times X, T \times T)$ is transitive. $x \in X$ is a *transitive point* if $\{T(x), T^2(x), \dots\}$ is dense in X . It is well known if (X, T) is transitive, then the set of transitive points is a dense G_δ set (denoted by $Tran_T$) and if $Tran_T = X$, we say that (X, T) is *minimal*. For a minimal system (X, T) each point of X is called a *minimal point*. The orbit of x , $orb(x, T)$, is the set $\{x, T(x), \dots\}$. The ω -limit set of x , $\omega(x, T)$, is the set $\bigcap_{n \geq 0} \text{cl}(\{T^i(x) : i \geq n\})$.

Let \mathcal{M} be the collection of all minimal systems. The notion of *disjointness* of two TDS was introduced in [F]. If (X, T) and (Y, S) are two TDS, we say $J \subset X \times Y$ is a *joining* of X and Y if J is a nonempty closed invariant set and is projected onto X and Y . If each joining is equal to $X \times Y$, we then say that (X, T) and (Y, S) are *disjoint* or $(X, T) \perp (Y, S)$ or $X \perp Y$. Furstenberg [F] showed that if two systems are disjoint, then one of them is minimal. Thus the most natural

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question is to characterize \mathcal{M}^\perp (**Question E** in [F]). It is known [F] that a weakly mixing system with dense periodic points is in \mathcal{M}^\perp . We show that a weakly mixing system with dense regular minimal points is in \mathcal{M}^\perp , and each system in \mathcal{M}^\perp has dense minimal points and it is weakly mixing if it is transitive. Transitive systems in \mathcal{M}^\perp and having no periodic points are constructed. Moreover, we show that if $(X, T) \perp (Y, S)$, and (Y, S) is nontrivial minimal, then (X, T) must have a dense set of recurrent points, and there are countably many maximal transitive subsystems of (X, T) such that their union is dense and each of them is disjoint from (Y, S) . Furthermore, we show that there is a distal system in \mathcal{M}^\perp , and if (X, T) is equicontinuous and in \mathcal{M}^\perp , then (X, T) is consisting of fixed points.

A TDS (X, T) is an *E-system* if it is transitive and there is an invariant measure μ with full support, i.e., $\text{supp}(\mu) = X$; (X, T) is an *M-system* if it is transitive and the set of minimal points is dense (for some special *M*-systems see [DY]); and (X, T) is *topologically ergodic* (TE, for short) if (X, T) is transitive and for each nonempty open subsets U, V of X , $N(U, V)$ is *syndetic*, i.e., with bounded gaps. It is known that a minimal system is an *E*-system, and an *E*-system is TE [GH2]. Two TDS are *weakly disjoint* if their product is transitive. Note that if both (X, T) and (Y, S) are transitive and $(X, T) \perp (Y, S)$, then they are weakly disjoint. Call a system

- *extremely scattering* if it is weakly disjoint from all TE systems,
- *strongly scattering* if it is weakly disjoint from all *E*-systems,
- *scattering* if it is weakly disjoint from all minimal systems, and
- *weakly scattering* if it is weakly disjoint from all minimal equicontinuous systems.

Note that weak mixing implies extreme scattering as for a weakly mixing system $N(U, V)$ is *thick*, i.e., it contains arbitrary long intervals of natural numbers [F]. Recently Weiss [We] showed that a system is weakly disjoint from all weakly mixing systems iff it is TE (for the recent development related to this result see [SY]). We construct an example which is extremely scattering and is not weakly mixing. Note that in [AG] the authors showed that weak mixing and scattering are different properties, and in [HY] we have showed that extreme scattering and strong scattering are different properties. It remains open if strong scattering, scattering and weak scattering are different properties.

For a TDS (X, T) with a metric d , we say (\tilde{X}, \tilde{T}) is the *natural extension* of (X, T) , if $\tilde{X} = \{(x_1, x_2, \dots) : T(x_{i+1}) = x_i, x_i \in X, i \in \mathbb{N}\}$, which is a subspace of the product space $\prod_{i=1}^{\infty} X$ with the compatible metric d_T defined by

$$d_T((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}.$$

Moreover, $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$ is the shift homeomorphism, i.e., $\tilde{T}(x_1, x_2, \dots) = (T(x_1), x_1, x_2, \dots)$. Let $\pi_i : \tilde{X} \rightarrow X$ be the projection to the i -th coordinate. To end the section we prove first that when considering disjointness we may assume each map is a homeomorphism, and then prove a disjoint theorem for distal systems.

Proposition 1.1. *Let (X, T) , (Y, S) and (Z, W) be TDS.*

- (1) *If $(X, T) \perp (Y, S)$ and (Z, W) is a factor of (X, T) , then $(Z, W) \perp (Y, S)$.*

- (2) $(X, T) \perp (Y, S)$ iff $(\tilde{X}, \tilde{T}) \perp (\tilde{Y}, \tilde{S})$, where (\tilde{X}, \tilde{T}) and (\tilde{Y}, \tilde{S}) are the natural extensions of (X, T) and (Y, S) respectively.
- (3) $\mathcal{M}^{\perp\perp} = \mathcal{M}$.

Proof. (1) Let $\pi : (X, T) \rightarrow (Z, W)$ be the factor map. Let J be a joining of (Z, W) and (Y, S) . Then $J' = \{(x, y) \in X \times Y : (\pi(x), y) \in J\}$ is a joining of (X, T) and (Y, S) . Since $(X, T) \perp (Y, S)$, $J' = X \times Y$. Moreover, $J = \pi \times id(J') = Z \times Y$, hence $(Z, W) \perp (Y, S)$.

(2) (\Rightarrow) For each $i \in \mathbb{N}$ let $\pi_i : \tilde{X} \rightarrow X$ and $\phi_i : \tilde{Y} \rightarrow Y$ be the projections to the i -th coordinates respectively. Then $(\tilde{X} \times \tilde{Y}, \tilde{T} \times \tilde{S})$ is a natural extension of $X \times Y$ and $\pi_i \times \phi_i$ is the projection to the i -th coordinate.

Let \tilde{J} be a joining of (\tilde{X}, \tilde{T}) and (\tilde{Y}, \tilde{S}) , and $J_i = \pi_i \times \phi_i(\tilde{J})$, $i \in \mathbb{N}$. Then J_i is a joining of (X, T) and (Y, S) . It follows that $J_i = X \times Y$ and consequently $\tilde{J} = \tilde{X} \times \tilde{Y}$. That is, $(\tilde{X}, \tilde{T}) \perp (\tilde{Y}, \tilde{S})$.

(\Leftarrow) It follows immediately from (1).

(3) It follows just because there exists some nonminimal system in \mathcal{M}^\perp , e.g., weakly mixing systems with dense periodic points and any system disjoint from a nonminimal system is minimal. □

Remark 1.2. In [AG], Akin and Glasner introduced the notion of residual property. Disjointness from all minimal systems is a residual property by Theorem 2.6 in [AG].

Let \mathcal{D} be the collection of all distal systems. The following theorem was proved in [P] and for completeness we include a proof.

Theorem 1.3. *A dynamical system is in \mathcal{D}^\perp iff it is minimal and weakly mixing.*

Proof. Assume that $(X, T) \perp \mathcal{D}$. As there are nonminimal distal systems, (X, T) is minimal. Let (Y, S) be the maximal equicontinuous factor of (X, T) . Then $(X, T) \perp (Y, S)$. This shows that $(Y, S) \perp (Y, S)$ and thus (Y, S) is trivial. Hence by [M] (X, T) is weakly mixing.

Now let (X, T) be minimal and weakly mixing, and (Y, S) be distal. It is known that (Y, S) is a union of minimal distal subsets. Let $J \subset X \times Y$ be a joining. Assume $x \in X$, $y \in Y$ and M_y is the orbit closure of y . It is easy to see that $J_y = orb((x, y), T \times S) \subset J$ and is a joining of (X, T) and (M_y, S) . It is known [F] that $(X, T) \perp (M_y, S)$. Thus $J_y = X \times M_y \subset J$ which implies $J = X \times Y$. □

§2. TRANSITIVE SYSTEMS DISJOINT FROM ANY MINIMAL SYSTEM

In [F] Furstenberg asked the question to characterize systems disjoint from any distal or minimal system. Theorem 1.3 gives a complete answer to the first part of the question. In Sections 2, 3 and 4 we provide a partial answer to the second part. We will give some necessary conditions and some sufficient conditions for membership in \mathcal{M}^\perp .

If (X, T) is a TDS and x is a minimal point of T , then for each neighborhood U of x , $N(x, U) = \{n \in \mathbb{Z}_+ : T^n(x) \in U\}$ is syndetic [GH2]. Note that a subset A of \mathbb{Z}_+ is *piecewise syndetic* if it is the intersection of a syndetic set with a thick set, and it is *thickly syndetic* if for each $n \in \mathbb{N}$ there is a syndetic subset $\{w_1^n, w_2^n, \dots\}$ of A such that $\{w_i^n, w_i^n + 1, \dots, w_i^n + n\} \subset A$ for each i . For further investigation we need the following lemma.

Lemma 2.1. *Let (X, T) be a transitive system and $x \in \text{Tran}_T$. Then*

- (1) *(X, T) is an M -system if and only if for each neighborhood U of x , $N(x, U)$ is piecewise syndetic.*
- (2) *Let K be a minimal set of (X, T) . Then (X, T) has only one minimal set iff for each neighborhood U of K , $N(x, U)$ is thickly syndetic.*

Proof. (1) If (X, T) is an M -system, it is clear that for each neighborhood U of x , $N(x, U)$ is piecewise syndetic, since for each minimal point $y \in U$, $N(y, U)$ is syndetic and there are $\{n_i\}$ such that $T^{n_i}(x) \rightarrow y$.

Now we assume that for each neighborhood U of x , $N(x, U)$ is piecewise syndetic. Let $\epsilon > 0$ with $\text{cl}(B_{\epsilon/2}(x)) \subset U$. Thus there are $p \in \mathbb{N}$ and $\{m_j^i : i \in \mathbb{N} \text{ and } 1 \leq j \leq i\} \subset N(x, B_{\epsilon/2}(x))$ such that $m_1^i < \dots < m_i^i$ and $m_{j+1}^i - m_j^i \leq p$ for $1 \leq j \leq i - 1$. Let y be a limit point of $\{T^{m_i^i}(x)\}$. Then it is clear that $N(y, U)$ is syndetic. Let M be a minimal set in the orbit closure of y under T . We claim that $M \cap \text{cl}(U) \neq \emptyset$. In fact, if $M \cap \text{cl}(U) = \emptyset$, then there is an open set $V \supset M$ with $U \cap V = \emptyset$. It is clear that $N(y, V)$ is thick. Thus $N(y, U)$ is not syndetic, a contradiction.

Since $M \cap \text{cl}(U) \neq \emptyset$, it follows by the transitivity of (X, T) that (X, T) is an M -system.

(2) Assume that (X, T) has only one minimal set K . For each neighborhood U of K , let $U_i \subset U$ be a neighborhood of K such that if $T^j(x) \in U_i$, then $T^{j+k}(x) \in U$ for each $1 \leq k \leq i$. As $N(x, U_i)$ is syndetic for each i , we have that $N(x, U)$ is thickly syndetic.

Conversely assume that T has a minimal set K_1 with $K_1 \cap K = \emptyset$. Then $N(x, V)$ is thick for each neighborhood V of K_1 disjoint from U . Then $N(x, U) \subset \mathbb{Z}_+ \setminus N(x, V)$ cannot be syndetic, a contradiction. \square

Definition. A subset A of \mathbb{Z}_+ is called an m -set if there exist a minimal system (Y, S) , $y \in Y$, and an open subset V of Y such that $A \supset N(y, V)$.

For a transitive system whether it is in \mathcal{M}^\perp can be checked through m -sets as the following theorem shows. For a minimal dynamical system (Y, S) , we define

$$\mathcal{F}_Y = \{A \subset \mathbb{Z}_+ : A \supset N(y, V) \text{ for some } y \in Y \text{ and open subset } V\} \text{ and}$$

$$k\mathcal{F}_Y = \{B \subset \mathbb{Z}_+ : B \cap A \neq \emptyset \text{ for each } A \in \mathcal{F}_Y\}.$$

Theorem 2.2. *Let (X, T) be a transitive system and $x \in \text{Trans}_T$. Then*

- (1) *$(X, T) \in \mathcal{M}^\perp$ iff $N(x, U) \cap A \neq \emptyset$ for any neighborhood U of x and any m -set A .*
- (2) *$(X, T) \perp (Y, S)$ iff for any open neighborhood U of x , one has $N(x, U) \in k\mathcal{F}_Y$.*

Proof. We show (1), and (2) is similar.

Let $(X, T) \in \mathcal{M}^\perp$. For any m -set A , there exist a minimal system (Y, S) , $y \in Y$, and an open subset V of Y such that $A \supset N(y, V)$.

Set $J = \text{cl}(\text{orb}((x, y), T \times S))$. Then J is a joining of (X, T) and (Y, S) . Since $(X, T) \perp (Y, S)$, $J = X \times Y$. Hence for any neighborhood U of x one has $N((x, y), U \times V) \neq \emptyset$, i.e., $N(x, U) \cap A \neq \emptyset$.

Conversely, let (Y, S) be a minimal system and J a joining of (X, T) and (Y, S) . It is clear that, there is $y \in Y$ with $(x, y) \in J$. For any neighborhood U of x and any open subset V of Y , one has $N(x, U) \cap N(y, V) \neq \emptyset$. This implies

$\text{cl}(\text{orb}((x, y), T \times S)) \cap (U \times V) \neq \emptyset$. Since $\text{cl}(\text{orb}((x, y), T \times S))$ is $T \times S$ -invariant and closed, one has $\text{cl}(\text{orb}((x, y), T \times S)) = X \times Y$. Thus $J = X \times Y$. \square

For any $S \subset \mathbb{Z}_+$ let 1_S be the indication function from \mathbb{Z}_+ to $\{0, 1\}$, i.e., $1_S(s) = 1$ if $s \in S$ and $1_S(s) = 0$ if $s \notin S$. Note that if $s = (s(0), s(1), \dots) \in \Sigma = \{0, 1\}^{\mathbb{Z}_+}$, then we use $s[n; m]$ to denote $(s(n), s(n + 1), \dots, s(m))$ whenever $n \leq m$. Let $\sigma : \Sigma \rightarrow \Sigma$ be the shift map and, for a finite word A , let $|A|$ stand for the length of A .

Let \mathcal{P} be the collection of all subsets of \mathbb{Z}_+ . A subset \mathcal{F} of \mathcal{P} is a *family*, if it is hereditary upwards. That is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. Any subset \mathcal{A} of \mathcal{P} can generate a family $[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$.

Clearly, all m -sets form a family, we call it *the family of m -sets*. Now, we have

Proposition 2.3. *The family of m -sets is the family generated by the sets whose indicator functions are the minimal points of (Σ, σ) .*

Proof. We denote the family generated by the sets whose indicator functions are the minimal points of (Σ, σ) by \mathcal{F}_m . Clearly, if 1_F is the indicator function of F , then $F = N(1_F, [1])$, where $[1] = \{s \in \Sigma : s(0) = 1\}$. Hence \mathcal{F}_m is contained in the family of m -sets.

On the other hand, let A be an m -set. Then there exist a minimal system (X, T) with metric d , $x \in X$ and an open subset V of X such that $A \supset N(x, V)$. It is easy to see that we can shrink V to an open subset V' whose boundary is disjoint from the orbit of x .

Then do the classical lifting trick, a la Glasner, Adler, etc. Let

$$Y = \{(z, t) \in X \times \Sigma : t(i) = 1 \text{ implies } T^i z \in \text{cl}(V') \\ \text{and } t(i) = 0 \text{ implies } T^i z \in \text{cl}(X \setminus V')\}.$$

Then Y is a $T \times \sigma$ -invariant closed subset of $X \times \Sigma$. Since the orbit of x doesn't meet the boundary of V' , there is a unique $t \in \Sigma$ such that $(x, t) \in Y$ and t is the indicator function of $N(x, V')$. Take a minimal subset J of $(Y, T \times \sigma)$ with $J \subset \text{cl}(\text{orb}((x, t), T \times \sigma))$ and let $\pi_X : J \rightarrow X$ be the projective map. Since (X, T) is minimal, $\pi_X(J) = X$. Hence $(x, t) \in J$. Projecting J to Σ we see that t is a minimal point. Hence $A \in \mathcal{F}_m$ as $A \supset N(x, V')$ and $t = 1_{N(x, V')}$. \square

The following theorem is crucial for this section.

Theorem 2.4. *Every thickly syndetic set contains an m -set.*

Proof. Let $F \subset \mathbb{Z}_+$ be a thickly syndetic subset. We will construct $y^n = 1_{F_n} \in \{0, 1\}^{\mathbb{Z}_+}$ such that $F_n \subset F$ and $y = \lim y^n = 1_A$ is a minimal point. Let $Y = \text{cl}(\text{orb}(y, \sigma))$ and $[1] = \{x \in Y : x(0) = 1\}$. As $A \subset F$ and $A = N(y, [1])$, the theorem follows.

To obtain y^n we construct a finite word A_n such that y^n begins with A_n , A_n appears in y^n syndetically and A_{n+1} begins with A_n . The reason we can do this is that $1^n = (1, \dots, 1)$ (n times) appears in 1_F syndetically for each $n \in \mathbb{N}$. More precisely we do as follows.

Step 1. Construct A_1 and $F_1 \subset F$ such that A_1 appears in $y^1 = 1_{F_1}$ with gaps bounded by l_1 and y^1 begins with A_1 .

Let $\min F = k_1 - 1$ and $A_1 = 1_F[0; k_1 - 1]$. Set $r_1 = k_1$. As F is thickly syndetic, 1^{r_1} appears in F at a syndetic set $W_1 = \{w_1^1, w_2^1, \dots\}$. Without loss of generality,

assume that $2r_1 \leq w_{j+1}^1 - w_j^1 \leq l_1$ and $2k_1 \leq w_1^1 \leq l_1$, where l_1 is some number in \mathbb{N} . Put $u_i^1 = w_i^1, i \in \mathbb{N}$. Choose $y^1 \in \{0, 1\}^{\mathbb{Z}^+}$ such that

$$y^1[0; k_1 - 1] = A_1, \quad y^1[u_i^1; u_i^1 + k_1 - 1] = A_1$$

and

$$y^1(j) = 0 \quad \text{if } j \in \mathbb{Z}_+ \setminus ([0; k_1 - 1] \cup \bigcup_{i=1}^{\infty} [u_i^1; u_i^1 + k_1 - 1]).$$

It is easy to see that A_1 appears in y^1 with gaps bounded by l_1 and $F_1 \subset F$, where $1_{F_1} = y^1$.

Step 2. Construct A_2 and $F_2 \subset F$ such that

- (1) A_2 has the form of $A_1V_1A_1$ and if $k_2 = |A_2|$, then $A_2 = y^1[0; k_2 - 1]$.
- (2) $y^2[0; k_2 - 1] = A_2$ and A_1, A_2 appear in y^2 syndetically with gaps bounded by l_1 and l_2 respectively.
- (3) $F_2 = \{i \in \mathbb{Z}_+ : y^2(i) = 1\} \subset F$.

Set $k_2 = u_1^1 + k_1$ and let $A_2 = y^1[0; k_2 - 1]$. Then A_2 has the form of $A_1V_1A_1$. Let $r_2 = 2l_1 + 2k_1 + k_2$. As F is thickly syndetic, 1^{r_2} appears in F at a syndetic set $W_2 = \{w_1^2, w_2^2, \dots\}$. Without loss of generality, assume that $2r_2 \leq w_{j+1}^2 - w_j^2 \leq l_2 - (l_1 + k_1)$ and $2k_2 \leq w_1^2 \leq l_2 - (l_1 + k_1)$, where l_2 is some number in \mathbb{N} .

To get y^2 we change y^1 at places $[w_i^2; w_i^2 + r_2 - 1]$ for each $i \in \mathbb{N}$. It is enough to show how this is done at $[w_1^2; w_1^2 + r_2 - 1]$.

Let k, j satisfy that $u_{k-1}^1 < w_1^2 \leq u_k^1$ and $u_j^1 + k_1 - 1 \leq w_1^2 + r_2 - 1 < u_{j+1}^1 + k_1 - 1$. Let l be the integer part of $(u_j^1 - 1 - u_k^1 - k_1 - k_2)/k_1$.

Put $u_1^2 = u_k^1 + k_1$. Let $y^2[u_1^2; u_1^2 + k_2 - 1] = A_2$ and $y^2[u_1^2 + k_2 + pk_1; u_1^2 + k_2 + (p + 1)k_1 - 1] = A_1$ for $p = 0, 1, \dots, l - 1$. That is, first we put A_2 at place u_1^2 and then we put as many A_1 as we can. We do the same at all places $[w_i^2; w_i^2 + r_2 - 1]$; we get $u_i^2 \in [w_i^2, w_i^2 + r_2 - 1]$ with $y^2[u_i^2; u_i^2 + k_2 - 1] = A_2, i = 1, 2, \dots$

In such a way we get y^2 . It is easy to see that y^1 and y^2 differ possibly at $[w_i^2; w_i^2 + r_2 - 1]$. Thus

$$F_2 = \{i \in \mathbb{Z}_+ : y^2(i) = 1\} \subset F_1 \cup \bigcup_{i=1}^{\infty} [w_i^2; w_i^2 + r_2 - 1].$$

At the same time A_1, A_2 appear in y^2 syndetically with gaps bounded by l_1 and l_2 respectively by the construction.

Step 3. Construct A_{m+1} and $F_{m+1} \subset F$ inductively such that

- (1) A_{m+1} has the form of $A_mV_mA_m$ and if $k_{m+1} = |A_{m+1}|$, then $A_{m+1} = y^m[0; k_{m+1} - 1]$.
- (2) $y^{m+1}[0; k_{m+1} - 1] = A_{m+1}$ and A_i appear in y^{m+1} syndetically with gaps bounded by l_i for each $1 \leq i \leq m + 1$.
- (3) $F_{m+1} = \{i \in \mathbb{Z}_+ : y^{m+1}(i) = 1\} \subset F$.

Set $k_{m+1} = u_1^m + k_m$ and let $A_{m+1} = y^m[0; k_{m+1} - 1]$. Then A_{m+1} has the form of $A_mV_mA_m$. Let $r_{m+1} = 2l_m + 2k_m + k_{m+1}$. As F is thickly syndetic, $1^{r_{m+1}}$ appears in F at a syndetic set $W_{m+1} = \{w_1^{m+1}, w_2^{m+1}, \dots\}$. Without loss of generality, assume that $2r_{m+1} \leq w_{j+1}^{m+1} - w_j^{m+1} \leq l_{m+1} - (l_m + k_m)$ and $2k_{m+1} \leq w_1^{m+1} \leq l_{m+1} - (l_m + k_m)$, where l_{m+1} is some number in \mathbb{N} .

To get y^{m+1} we change y^m at places $[w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1]$ for each $i \in \mathbb{N}$. It is enough to show how this is done at $[w_1^{m+1}; w_1^{m+1} + r_{m+1} - 1]$.

Let k, j satisfy that $u_{k-1}^m < w_1^{m+1} \leq u_k^m$ and $u_j^m + k_m - 1 \leq w_1^{m+1} + r_{m+1} - 1 < u_{j+1}^m + k_m - 1$.

Put $u_1^{m+1} = u_k^m + k_m$. Let $y^{m+1}[u_1^{m+1}; u_1^{m+1} + k_{m+1} - 1] = A_{m+1}$ and

$$y^{m+1}[u_1^{m+1}, u_j^m - 1] = A_{m+1}(A_m)^{p_m}(A_{m-1})^{p_{m-1}} \dots (A_1)^{p_1} B_{m+1},$$

where B_{m+1} is a word and p_1, \dots, p_m are natural numbers with

$$|B_{m+1}| < k_1,$$

$$|B_{m+1}| + k_1 p_1 < k_2, \text{ and}$$

$$|B_{m+1}| + k_1 p_1 + \dots + k_i p_i < k_{i+1} \text{ for each } 1 \leq i \leq m - 1.$$

That is, first we put A_{m+1} at place u_1^{m+1} and start from $u_1^{m+1} + k_{m+1}$ to u_j^m ; we put as many A_m as we can and then we put as many A_{m-1} as we can and so on. We do the same at all places $[w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1]$; we get $u_i^{m+1} \in [w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1]$ with $y^{m+1}[u_i^{m+1}; u_i^{m+1} + k_{m+1} - 1] = A_{m+1}$, $i = 1, 2, \dots$

In such a way we get y^{m+1} . It is easy to see that y^{m+1} and y^m differ possibly only at $[w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1]$, $i = 1, 2, \dots$. Thus

$$F_{m+1} = \{i \in \mathbb{Z}_+ : y^{m+1}(i) = 1\} \subset F_m \cup \bigcup_{i=1}^{\infty} [w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1].$$

At the same time A_i appears in y^{m+1} syndetically with gaps bounded by l_i for each $1 \leq i \leq m + 1$ by the construction.

In such a way for each $m \in \mathbb{N}$ we defined a finite word A_m . Let $y = \lim A_m = \lim y^m$. By the construction, A_m appears in y with gaps bounded by l_m for each $m \in \mathbb{N}$. That is, y is a minimal point for the shift. It is obvious that $y \neq (0, 0, \dots)$. Let $Y = \text{orb}(y, \sigma)$ and $U = \{x \in Y : y(0) = 1\}$. Then

$$\emptyset \neq N(y, U) = \bigcup_{i=1}^{\infty} \{i \in \mathbb{Z}_+ : A_n(i) = 1, 0 \leq i \leq k_n - 1\} \subset \bigcup_{i=1}^{\infty} F_n \subset F.$$

Thus F contains the m -set $N(y, U)$. □

Remark 2.5. There exists an m -set which is not thick. For example, let (X, T) be a nontrivial minimal system and $x \in X$. If U and V are disjoint open subsets of X , then $N(x, U)$ and $N(x, V)$ are disjoint m -sets. In particular, both are syndetic and so neither is thick.

Now we are ready to show the main result of the section.

Theorem 2.6. *Let (X, T) be a transitive TDS. If $(X, T) \perp \mathcal{M}$, then (X, T) is a weakly mixing M -system without nontrivial minimal factor.*

Proof. Let $x \in \text{Tran}_T$ and let U be a nonempty neighborhood x . By Theorem 2.2, $N(x, U) \cap A \neq \emptyset$ for any m -set A . This implies that $N(x, U)$ has nonempty intersection with any thickly syndetic set by Theorem 2.4. Thus $N(x, U)$ is piecewise syndetic. By Lemma 2.1 (X, T) is an M -system. Since a nontrivial system is never disjoint from itself, it follows from Proposition 1.1(1) that a system in \mathcal{M}^\perp has only trivial minimal factors. Hence (X, T) has no nontrivial minimal factor.

Since (X, T) is transitive, (X, T) is weakly disjoint from all minimal systems, i.e., (X, T) is scattering [BHM]. By [AG, Theorem 2.9(b)], (X, T) is weakly disjoint from all M -systems. Particularly, (X, T) is weakly mixing. □

Remark 2.7. The condition in Theorem 2.6 is not sufficient. For example, let (X, T) be a weakly mixing M -system with a fixed point p and let (Y, S) be a minimal strongly mixing system. Then $X \times Y$ is a weakly mixing M -system. It is clear that $X \times Y$ is not disjoint from Y . Collapsing $\{p\} \times Y$ to a point p' we get a system Z which is not disjoint from Y . Moreover, Z is a weakly mixing M -system without nontrivial minimal factor, as p' is a fixed point.

§3. A TRANSITIVE SYSTEM IN \mathcal{M}^\perp WITHOUT PERIODIC POINTS

For a transitive system we have obtained some necessary conditions for a system in \mathcal{M}^\perp . Since Furstenberg has shown that each weakly mixing system with a dense set of periodic points is in \mathcal{M}^\perp , it is natural to ask if this is a necessary condition. We will give a negative answer by constructing a transitive system in \mathcal{M}^\perp without periodic points. In the process to do this, we also obtain some sufficient condition. To start, we first show

Lemma 3.1. *Let $A \subset \mathbb{Z}_+$ be an m -set. Then there exists $r \in \mathbb{Z}_+$ such that $N_k(A, r) = \{i \in \mathbb{Z}_+ : ki + r \in A\}$ is an m -set and so is syndetic for any $k \in \mathbb{Z}_+$.*

Proof. Since A is an m -set, there exist a minimal system (X, T) , $x \in X$, and an open subset U of X such that $A \supset N(x, U)$. Take $r \in \mathbb{Z}_+$ and a neighborhood V of x such that $T^r V \subset U$. Then $N(x, U) \supset N(x, V) + r$. Note that x is also a minimal point of T^k for any $k \in \mathbb{Z}_+$ and thus $\{i \in \mathbb{Z}_+ : T^{ki}x \in V\}$ is an m -set and so is syndetic. Therefore, $N_k(A, r) \supset \{i \in \mathbb{Z}_+ : T^{ki}x \in V\}$ is an m -set and so is syndetic for any $k \in \mathbb{Z}_+$. \square

Now we give a natural generalization of dense periodic points.

Definition. *Let (X, T) be a TDS. We say (X, T) has dense small periodic sets, if for any open subset U of X there exist a nonempty closed $A \subset U$ and $k \in \mathbb{N}$ such that A is invariant for T^k .*

Concerning this notion we have

Proposition 3.2. *Let (X, T) be a TDS. Then (X, T) has dense small periodic sets if and only if for any open subset U of X there exist $p \in X$, $k \in \mathbb{N}$ such that $N_k(p, U) = \{i \in \mathbb{Z}_+ : T^{ki}p \in U\}$ is thick.*

Proof. Let (X, T) have dense small periodic sets. Then for any open subset U of X there exist a closed $A \subset U$ and a positive integer k such that A is invariant for T^k . Take any $p \in A$; then $N_k(p, U) = \mathbb{Z}_+$ is thick.

Conversely, assume for any open subset U of X there exist $p \in X$, $k \in \mathbb{N}$ such that $N_k(p, U)$ is thick. Let W be an open subset of X and V an open subset with $\text{cl}(V) \subset W$. Then there exist $p \in X$ and $k \in \mathbb{N}$ such that $N_k(p, V)$ is thick. Hence we can find $n_1 < n_2 < \dots$ and $y \in X$ such that $T^{kn_1}p, T^{k(n_1+1)}p, \dots, T^{k(n_1+l)}p \in V$ for any $l \in \mathbb{N}$ and $\lim_{l \rightarrow \infty} T^{kn_l}p = y$. Clearly, $\text{orb}(y, T^k) \subset \text{cl}(V)$. Let $A = \text{cl}(\text{orb}(y, T^k))$. Then $A \subset \text{cl}(V) \subset W$ and A is closed and invariant for T^k . This shows that (X, T) has dense small periodic sets. \square

A totally transitive system with dense small periodic sets can be characterized as follows.

Proposition 3.3. *Let (X, T) be a TDS. Then (X, T) is a totally transitive system having dense small periodic sets if and only if there exists a transitive point x such*

that for any neighborhood U of x we have property (\star) , i.e.,

- (\star) for any $r \in \mathbb{Z}_+$ there exists $k \in \mathbb{N}$ such that $N_k(x, U, r) = \{i \in \mathbb{Z}_+ : T^{ki+r}x \in U\}$ is thick.

When these conditions hold then for every $x \in \text{Tran}_T$ and every neighborhood U of x , condition (\star) holds.

Proof. Let (X, T) be a totally transitive system having dense small periodic sets and $x \in \text{Tran}_T$. Then $x \in \text{Tran}_{T^k}$ for all $k \in \mathbb{N}$; see [B].

For any neighborhood U of x , by Proposition 3.2, there exist $p \in X$, $k \in \mathbb{N}$ and $n_1 < n_2 < \dots$ such that $N(p, U) \supset \bigcup_{i=1}^\infty \{kn_i, k(n_i + 1), \dots, k(n_i + i)\}$. Thus for any natural number l we can find a neighborhood V_l of p such that when $y \in V_l$, one has $T^{kn_l}y, T^{k(n_l+1)}y, \dots, T^{k(n_l+l)}y \in U$. For any $r \in \mathbb{Z}_+$, since $T^r x$ is a transitive point of T^k , there exists $m \in \mathbb{Z}_+$ with $T^{mk}T^r x \in V_l$. Moreover, $N_k(x, U, r) \supset \{n_l + m + 1, n_l + m + 2, \dots, n_l + m + l\}$ for each $l \in \mathbb{N}$. This shows $N_k(x, U, r)$ is thick.

Conversely, assume there exists a transitive point x such that for any neighborhood U of x we have property (\star) . To show that (X, T) is a totally transitive system having dense small periodic sets, we need only to show that for any open subset W of X and $n \in \mathbb{N}$, $N_n(x, W) \neq \emptyset$ and there exist $k \in \mathbb{N}$ and $p \in X$ such that $N_k(p, W)$ is thick.

Now let W be a given open subset of X and $n \in \mathbb{N}$. Since x is a transitive point, there exists $s \in \mathbb{N}$ such that $T^s x \in W$. Take a small open neighborhood U of x with $T^s U \subset W$ and $r \in \mathbb{N}$ with $n|(r + s)$. For U and r , since we have property (\star) , there exists $k \in \mathbb{N}$ such that $N_k(x, U, r)$ is thick. Let $p = T^{r+s}x$, then $N_k(p, W) \supset N_k(x, W, r + s) \supset N_k(x, U, r)$ as $T^s U \subset W$. Therefore, $N_k(p, W)$ is thick.

Clearly, $N_k(p, W) \cap n\mathbb{N} \neq \emptyset$. For $m \in N_k(p, W) \cap n\mathbb{N}$, we have $T^{km+r+s}x \in W$. As $n|(km + r + s)$, one has $N_n(x, W) \neq \emptyset$. □

Theorem 3.4. *Let (X, T) be a transitive TDS with a transitive point x . If for any neighborhood U of x we have property (\star) , then (X, T) is in \mathcal{M}^\perp . That is, if (X, T) is a totally transitive TDS having dense small periodic sets, then $(X, T) \in \mathcal{M}^\perp$.*

Proof. By Theorem 2.2, it remains to show that $N(x, U) \cap A \neq \emptyset$ for any neighborhood U of x and any m -set A . By Lemma 3.1, there exists $r \in \mathbb{Z}_+$ such that $N_k(A, r) = \{i \in \mathbb{Z}_+ : ki + r \in A\}$ is syndetic for any $k \in \mathbb{Z}_+$. Since property (\star) holds, there exists $k_r \in \mathbb{N}$ such that $N_{k_r}(x, U, r) = \{i \in \mathbb{Z}_+ : T^{k_r i+r}x \in U\}$ is thick. Thus $N_{k_r}(x, U, r) \cap N_{k_r}(A, r) \neq \emptyset$. This implies that $N(x, U) \cap A \neq \emptyset$. □

A minimal point x is *regular* [GH2] if for each neighborhood V of x there is k such that $N(x, V) \supset k\mathbb{Z}_+$.

Clearly, a periodic point is a regular minimal point and a regular minimal point is a minimal point. In order to characterize the minimal system containing a regular minimal point, we need the notions of an almost one-to-one extension and an adding machine.

Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map. We say π is an *almost one-to-one extension*, if $\text{Inj}_\pi = \{x \in X : \pi^{-1}\pi(x) = \{x\}\}$ is dense in X .

Given $K = (k_1, k_2, \dots)$ with $k_i \geq 1$, we define $\Sigma_K = \prod_{i=1}^\infty \{0, 1, \dots, k_i - 1\}$, where $\{0, 1, \dots, k_i - 1\}$ and Σ_K are equipped with the discrete and the product topology respectively. If $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are two elements

of Σ_K , then their sum $x \oplus y = (z_1, z_2, \dots)$ is defined as follow. If $x_1 + y_1 < k_1$, then $z_1 = x_1 + y_1$; if $x_1 + y_1 \geq k_1$, then $z_1 = x_1 + y_1 - k_1$ and we carry 1 to the next position. The other terms z_2, \dots are successively determined in the same fashion. Let $T_K : \Sigma_K \rightarrow \Sigma_K$ be defined by $T_K(z) = z \oplus 1$ for each $z \in \Sigma_K$, where $1 = (1, 0, 0, \dots)$. It is known that T_K is a minimal map, which is called an *adding machine*. We note that if $\{i \in \mathbb{N} : k_i > 1\}$ is finite, then T_K is periodic and Σ_K is the unique periodic orbit of T_K . Now we have

Proposition 3.5. *Let (X, T) be a minimal system. Then (X, T) contains a regular minimal point if and only if it is an almost one-to-one extension of an adding machine.*

Proof. See the Appendix. □

The following is an immediate corollary of Theorem 3.4.

Corollary 3.6. *If (X, T) is a totally transitive TDS and has dense periodic points or dense regular minimal points, then $(X, T) \in \mathcal{M}^\perp$. Particularly, (X, T) is weakly mixing.*

We remark that a nontrivial adding machine or finite system is never totally transitive, and hence the systems described in Corollary 3.6 are never minimal. Moreover, it is known that a totally transitive system having dense periodic points is weakly mixing [B].

Using Theorem 3.4 we will construct a transitive TDS which is in \mathcal{M}^\perp and has no periodic point. Namely, we have

Example 3.7. There is a transitive TDS in \mathcal{M}^\perp and has no periodic point.

Proof. See the Appendix. □

§4. THE GENERAL CASE

Now we consider the question which system is disjoint from all minimal systems without the assumption of transitivity. We will show such a system must have a dense set of minimal points. First we prove that the set of recurrent points is dense. Note that for a dynamical system (X, T) , $R(T)$ is the set of all recurrent points of T .

Proposition 4.1. *Let (X, T) be a TDS.*

- (1) *If $X \perp Y$ with (Y, S) minimal and nontrivial, then $R(T)$ is a dense G_δ set of X .*
- (2) *Let T be invertible, V be an open set of X and $Y = cl(\bigcup_{i \in \mathbb{Z}} T^n(V))$. If $(X, T) \perp \mathcal{M}$, then $(Y, T) \perp \mathcal{M}$.*
- (3) *If there are transitive sub-systems (X_i, T) of (X, T) satisfying that $\bigcup_i X_i$ is dense in X and $(X_i, T) \perp \mathcal{M}$ for each $i \in \mathbb{N}$. Then $(X, T) \perp \mathcal{M}$.*
- (4) *Let (X, T) be an equicontinuous system. If $(X, T) \in \mathcal{M}^\perp$, then each point of X is a fixed point. Consequently, if (Y, S) is TDS with $(Y, S) \in \mathcal{M}^\perp$, then the maximal equicontinuous factor of (Y, S) is consisting of fixed points.*

Proof. (1) First we assume that (X, T) is a homeomorphism. Let $\Omega(T)$ be the set of nonwandering points of T . If $\Omega(T) \neq X$, then there is $x \in X \setminus \Omega(T)$. Thus there

is an open set U of X with $x \in U$ and $T^i(U) \cap T^j(U) = \emptyset$ whenever $i \neq j \in \mathbb{Z}$. Let $y \in Y$ and

$$J = cl\left(\bigcup_{n \in \mathbb{Z}} T^n(U) \times \{S^n(y)\}\right) \cup \left(X \setminus \bigcup_{i \in \mathbb{Z}} T^i(U)\right) \times Y.$$

It is easy to see that J is a joining between (X, T) and (Y, S) and thus $J = X \times Y$. As Y is not trivial, $Y \setminus \{y\}$ is nonempty and open. As $U \times (Y \setminus \{y\}) \subset X \times Y = J$, we have

$$U \times (Y \setminus \{y\}) \subset cl\left(\bigcup_{n \in \mathbb{Z}} T^n(U) \times \{S^n(y)\}\right).$$

As $U \times (Y \setminus \{y\})$ is nonempty and open in $X \times Y$, there is $n \in \mathbb{Z}$ such that $(U \times (Y \setminus \{y\})) \cap (T^n(U) \times \{S^n(y)\}) \neq \emptyset$. It is clear that $n \neq 0$. Thus there is $n \neq 0$ such that $U \cap T^n(U) \neq \emptyset$, a contradiction. Thus we have proved $\Omega(T) = X$. By [F] $R(T)$ is a dense G_δ set of X .

In the general case by Proposition 1.1 $(\tilde{X}, \tilde{T}) \perp (\tilde{Y}, \tilde{S})$. By the above argument we have $R(\tilde{T})$ is dense in \tilde{X} . It follows that $R(T)$ is dense in X . By [F] $R(T)$ is a dense G_δ set of X .

(2) If $(X, T) \perp \mathcal{M}$ and $Y = X$, then $(Y, T) \perp \mathcal{M}$.

Now consider the case that $Y \neq X$. Assume that $J \subset Y \times Z$ is a joining, where (Z, S) is a minimal system. Let $J' = J \cup cl(X \setminus Y) \times Z$.

As $(X, T) \perp \mathcal{M}$, we have that $J' = X \times Z$. This implies that $J \supset V \times Z$. As J is closed and invariant, we get that $J \supset (\bigcup_{n \in \mathbb{Z}_+} T^n(V)) \times Z$. Since $R(T)$ is dense in X , we have $T^{-n}(V) \subset cl(\bigcup_{m \in \mathbb{Z}_+} T^m(V))$ for each $n \in \mathbb{N}$. That is, $J = Y \times Z$ and hence $(Y, T) \perp \mathcal{M}$.

(3) Let (Y, S) be a minimal system and J is a joining between (X, T) and (Y, S) . For each $i \in \mathbb{N}$ let x_i be a transitive point of (X_i, T) . For each i there is y_i such that $(x_i, y_i) \in J$. Thus $J_i = cl(orb(x_i, y_i), T \times S) \subset J$. It is easy to see that J_i is a joining between (X_i, T) and (Y, S) . Thus $J_i = X_i \times Y$. This implies that $J = X \times Y$, i.e., $(X, T) \perp \mathcal{M}$.

(4) As (X, T) is equicontinuous, for each $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(T^n(x), T^n(y)) < \epsilon$ for each $n \in \mathbb{Z}$. Assume that there is $x \in X$ with $T(x) \neq x$. Let $\epsilon < d(x, T(x))/4$ and δ be the number corresponding to ϵ with $\delta < \epsilon$. Moreover, let $U = B(x, \delta)$ and $V = B(T(x), \delta)$.

Set $X_1 = cl(\bigcup_{i \in \mathbb{Z}} T^i(U))$ and $Y_1 = orb(x, T)$. As (Y_1, T) is minimal, $(X, T) \perp (Y_1, T)$. Then $X_1 \perp Y_1$ by (2). Let $J = cl(\bigcup_{i \in \mathbb{Z}} (T^i(U) \times \{T^i x\}))$. Then J is a joining of X_1 and Y_1 . Thus, $J = X_1 \times Y_1$. This implies that there is n with

$$T^n(U) \times \{T^n(x)\} \cap U \times (V \cap Y_1) \neq \emptyset.$$

That is, $T^n(U) \cap U \neq \emptyset$ and $T^n(x) \in V \cap Y_1$, a contradiction. □

Now we proceed to show that if $(X, T) \in \mathcal{M}^\perp$, then the set of minimal points is dense in X . To do this we need

Definition. A sequence $\{F_n\}_1^\infty$ of thickly syndetic subsets of \mathbb{Z}_+ is uniform, if for any $l \in \mathbb{N}$ there exists $\phi(l)$ such that for every $n \in \mathbb{N}$, we can find a subset $\{w_1^n(l), w_2^n(l), w_3^n(l) \cdots\}$ of \mathbb{Z}_+ with $w_1^n(l) \leq \phi(l)$, $w_{i+1}^n(l) - w_i^n(l) \leq \phi(l)$ and $\{w_i^n(l), w_i^n(l) + 1, \dots, w_i^n(l) + l - 1\} \subset F_n$, $i \in \mathbb{N}$.

Strengthening Theorem 2.3 we have

Theorem 4.2. *Let $\{F'_n\}_{n=1}^{+\infty}$ be a uniform sequence of thickly syndetic subsets of \mathbb{Z}_+ . Then there is a minimal system $(Y, \sigma) \subset (\{0, 1\}^{\mathbb{Z}_+}, \sigma)$ such that for each $n \in \mathbb{N}$ there exists $y_n \in Y$ with $N(y_n, [1]) \subset F'_n$, where $[1] = \{y \in Y : y(0) = 1\}$.*

Proof. We will modify the construction in Theorem 2.3. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a map such that for each $i \in \mathbb{N}$, $f^{-1}(i)$ is infinite. Let $F_n = F'_{f(n)}$ for each $n \in \mathbb{N}$. Then $\{F_n\}$ is a uniform sequence of thickly syndetic sets.

First we modify the construction of y^1 in step 1 in Theorem 2.3 to obtain y_n^1 for each $n \in \mathbb{N}$, namely we construct A_1 such that A_1 appears in $y_n^1 = 1_{F_n^1}$ with gaps bounded by l_1 and $F_n^1 \subset F_n$.

Take $k_1 \in \mathbb{N}$ and $A_1 = 0 \dots 01$ with $|A_1| = k_1$. Set $r_1 = k_1$. As $\{F_n\}$ is a uniform sequence of thickly syndetic sets, 1^{r_1} appears in F_n at a syndetic set $\{w_1^n(r_1), w_2^n(r_1), \dots\}$. Without loss of generality, assume that $w_1^n(r_1) \leq l_1 - k_1$ and $2r_1 \leq w_{j+1}^n - w_j^n \leq l_1$, where l_1 is some number in \mathbb{N} . Choose $y_n^1 \in \{0, 1\}^{\mathbb{Z}_+}$ such that

$$y_n^1[w_i^n(r_1); w_i^n(r_1) + k_1 - 1] = A_1 \text{ and } y_n^1(j) = 0 \text{ if } j \in \mathbb{Z}_+ \setminus (\bigcup_{i=1}^{\infty} [w_i^n(r_1); w_i^n(r_1) + k_1 - 1]).$$

It is easy to see that for each $n \in \mathbb{N}$, A_1 appears in y_n^1 with gaps bounded by l_1 and $F_n^1 \subset F_n$, where $1_{F_n^1} = y_n^1$.

Let $A_2 = A_1 y_1^1[0; a_1] A_1$ with $a_1 > \phi(k_1)$ and $y_1^1[0; a_1] = A'_1 A_1$. It is easy to see that $A_2[k_1; k_1 + a_1] \subset F_1$.

Set $k_2 = |A_2|$. Now we modify the construction of step 2 in Theorem 2.3 as we did (for step 1) to obtain y_n^2 (as 1^{k_2} appears in F_n syndetically). Then A_1, A_2 appear in y_n^2 syndetically with gaps bounded by $2l_1$ and l_2 respectively. Set $A_3 = A_2 y_2^2[0; a_2] A_2$ with $a_2 > \phi(k_2)$ and $y_2^2[0; a_2] = A'_2 A_2$. It is easy to see that $A_3[k_2, k_2 + a_2] \subset F_2$.

By the same arguments, we get y_n^i and we set $A_{i+1} = A_i y_i^i[0; a_i] A_i$ with $a_i > \phi(k_i)$ and $y_i^i[0; a_i] = A'_i A_i$. We have $A_{i+1}[k_i, k_i + a_i] \subset F_i$. Moreover, A_1, \dots, A_i appear in y_n^i syndetically with gaps bounded by $2l_1, \dots, 2l_{i-1}, l_i$ respectively for each n .

Let $y = \lim A_i$. Then y is a minimal point. For each n let y_n be a limit point of $\{A_{i+1}[k_i, k_i + a_i]\}_{f(i)=n}$. Then we have $N(y_n, [1]) \subset F'_n$. □

With the help of Theorem 4.2 we now show

Theorem 4.3. *Let (X, T) be a dynamical system. If (X, T) is in \mathcal{M}^\perp , then (X, T) has a dense set of minimal points.*

Proof. First assume that T is a homeomorphism and that the set of minimal points M is not dense in X . Then there are an open set U of X and an open neighborhood V of $cl(M)$ with $U \cap V = \emptyset$.

By Theorem 4.1(1), $R(T)$ is dense in U . Then we may take recurrent points $\{x_n\}_{n=1}^{+\infty} \subset U$ such that $\{x_n : n \in \mathbb{N}\}$ is dense in U . Let $F_n = N(x_n, U^c) \supset N(x_n, V)$, $n = 1, 2, \dots$. We claim that

$$\{F_n\}_{n=1}^{+\infty} \text{ is a uniform sequence of thickly syndetic subsets of } \mathbb{Z}_+.$$

Proof of claim. For any $l > 0$, there exists a neighborhood $W_l \subset V$ of $cl(M)$ such that $T^k(W_l) \subset V$ for each $1 \leq k \leq l$. As $\bigcup_{i \in \mathbb{Z}_+} T^{-i} W_l = X$ and X is compact, there exists $\phi(l) > 0$ such that $\bigcup_{0 \leq i \leq \phi(l)} T^{-i} W_l = X$.

For each $n \in \mathbb{N}$, let $N(x_n, W_l) = \{w_1^n, w_2^n, \dots\}$. It is easy to see that $w_1^n \leq \phi(l)$, $w_{i+1}^n - w_i^n \leq \phi(l)$ and $\{w_i^n, w_i^n + 1, \dots, w_i^n + l - 1\} \subset N(x_n, V)$. As $F_n \supset N(x_n, V)$, this shows that $\{F_n\}_{n=1}^{+\infty}$ is a uniform sequence of thickly syndetic subsets of \mathbb{Z}_+ .

By the above claim and Theorem 4.2, there is a minimal system $(Y, \sigma) \subset (\{0, 1\}^{\mathbb{Z}_+}, \sigma)$ such that for each $n \in \mathbb{N}$ there exists $y_n \in Y$ with $N(y_n, [1]) \subset F_n$, where $[1] = \{y \in Y : y(0) = 1\}$. Let

$$J = \text{cl}\left(\bigcup_{i=0}^{+\infty} \bigcup_{n=1}^{+\infty} (T \times \sigma)^i(x_n, y_n)\right) \cup \left(X \setminus \bigcup_{i=-\infty}^{+\infty} T^i U\right) \times Y.$$

As $\{x_n : n \in \mathbb{N}\}$ is dense in U and each x_n is recurrent, we have

$$\bigcup_{i=-\infty}^{+\infty} T^i U \subset \text{cl}\left(\bigcup_{i=0}^{+\infty} \bigcup_{n=1}^{+\infty} T^i x_n\right).$$

Thus J is a joining of X and Y , and hence $J = X \times Y$.

It is easy to see that $\text{cl}\left(\bigcup_{i=0}^{+\infty} \bigcup_{n=1}^{+\infty} (T \times \sigma)^i(x_n, y_n)\right) \supset U \times [1]$. As $U \times [1]$ is open, we have

$$\left(\bigcup_{i=0}^{+\infty} \bigcup_{n=1}^{+\infty} (T \times \sigma)^i(x_n, y_n)\right) \cap (U \times [1]) \neq \emptyset.$$

Therefore, there exist i and n with $(T^i x_n, \sigma^i y_n) \in U \times [1]$, i.e., $N(x_n, U) \cap N(y_n, [1]) \neq \emptyset$, which contradicts the fact that $N(y_n, [1]) \subset F_n = N(x_n, U^c)$.

In the general case we pass to the natural extension and observe that minimal points are mapped to minimal points by factor maps. □

For a dynamical system (X, T) , a transitive subsystem (Y, T) is *maximal* if it is maximal among all transitive subsystems by the inclusion. We have

Theorem 4.4. *Let (X, T) be a TDS and (Y, T) be a transitive subsystem. Then there is a MTSS containing (Y, T) . Consequently, if $R(T) = X$, then the union of all MTSS is dense in X .*

Proof. Let \mathcal{A} be the collection of all transitive subsystems containing Y . By Hausdorff maximal principal (see, for example, [K, p. 32]) there is a maximal nest \mathcal{A}' containing Y . It is easy to see that $\text{cl}\left(\bigcup_{Z \in \mathcal{A}'} Z\right)$ is transitive and maximal. □

Inspired by the above theorem and Theorem 4.1 one may have

Conjecture. *If $(X, T) \perp \mathcal{M}$, then there are countably many MTSS such that their union is dense in X and each of them is in \mathcal{M}^\perp .*

Unfortunately the conjecture is not true in general. However, we will show that it is true in a certain sense, i.e., if $(X, T) \perp (Y, S)$ with Y nontrivial minimal, then there are countably many MTSS such that their union is dense in X and each of them is disjoint from Y . We also give an example to show that this is the best situation we can expect, namely there are distal systems in \mathcal{M}^\perp .

Theorem 4.5. *Let (X, T) be a TDS and (Y, S) be nontrivial minimal system. Then $(X, T) \perp (Y, S)$ if and only if there exist countably many MTSS such that their union is dense in X and each of them is disjoint from Y .*

Proof. The sufficiency is easy and it remains to show the necessity. First we assume that T and S are homeomorphisms.

Let $(X, T) \perp (Y, S)$. For any open subsets V of Y and $W \subset X$, put

$$X(U, V) = \{x \in X : N_{\mathbb{Z}}(x, U) \cap N_{\mathbb{Z}}(y, V) \neq \emptyset \text{ for any } y \in Y\},$$

$$\tilde{X}(U, V) = (X \setminus \text{cl}(\bigcup_{n \in \mathbb{Z}} T^n U)) \cup X(U, V),$$

where $N_{\mathbb{Z}}(x, U) = \{n \in \mathbb{Z} : T^n x \in U\}$ and $N_{\mathbb{Z}}(y, V) = \{n \in \mathbb{Z} : S^n y \in V\}$.

As $N_{\mathbb{Z}}(Tx, U) = N_{\mathbb{Z}}(x, U) - 1$ and $N_{\mathbb{Z}}(Sy, V) = N_{\mathbb{Z}}(y, V) - 1$, it is not difficult to see that $TX(W, V) = X(W, V)$. We have

Claim. For any open set U of X and open set V of Y , the invariant set $\tilde{X}(U, V)$ is a dense open subset of X .

Proof of Claim. First, we show that $\tilde{X}(U, V)$ is an open subset of X . Since $X \setminus \text{cl}(\bigcup_{n \in \mathbb{Z}} T^n U)$ is open, it remains to show that $X(U, V)$ is open. Let $x_n \notin X(U, V)$ and $\lim_{n \rightarrow \infty} x_n = x$. Then for each $n \in \mathbb{N}$ there exists $y_n \in Y$ such that $N_{\mathbb{Z}}(x_n, U) \cap N_{\mathbb{Z}}(y_n, V) = \emptyset$, i.e., $(x_n, y_n) \notin \bigcup_{m \in \mathbb{Z}} (T \times S)^m U \times V$. Without loss of generality, assume $\lim_{n \rightarrow \infty} y_n = y$. Since $\bigcup_{m \in \mathbb{Z}} (T \times S)^m U \times V$ is open, one has $(x, y) \notin \bigcup_{m \in \mathbb{Z}} (T \times S)^m U \times V$, i.e., $N_{\mathbb{Z}}(x, U) \cap N_{\mathbb{Z}}(y, V) = \emptyset$. Hence $x \notin X(U, V)$. This shows that $X \setminus X(U, V)$ is a closed set, i.e., $X(U, V)$ is an open set.

Next we show that $\tilde{X}(U, V)$ is dense in X . Let $W \subset U$ be an open subset of X . If $X(W, V) \cap W = \emptyset$, then for each $x \in W$ there exists $y(x) \in Y$ such that $N_{\mathbb{Z}}(x, W) \cap N_{\mathbb{Z}}(y(x), V) = \emptyset$.

Let

$$J = (X \setminus (\bigcup_{n \in \mathbb{Z}} T^n W)) \times Y \cup \text{cl}(\bigcup_{n \in \mathbb{Z}} \bigcup_{x \in W} (T^n \times S^n)(x, y(x))).$$

Clearly, J is a joining of X and Y . As $N_{\mathbb{Z}}(x, W) \cap N_{\mathbb{Z}}(y(x), V) = \emptyset$ for each $x \in W$, one has

$$\text{cl}(\bigcup_{n \in \mathbb{Z}} \bigcup_{x \in W} (T^n \times S^n)(x, y(x))) \cap W \times V = \emptyset.$$

Therefore, $J \cap W \times V = \emptyset$. Hence $J \neq X \times Y$ which contradicts $(X, T) \perp (Y, S)$. This shows that $X(W, V) \cap W \neq \emptyset$ and thus $X(U, V) \cap W \neq \emptyset$. As W is arbitrary and $X(U, V)$ is T -invariant, one has $\text{cl}(X(U, V)) \supset \text{cl}(\bigcup_{n \in \mathbb{Z}} T^n U)$. Therefore, $\tilde{X}(U, V)$ is dense in X . This ends the proof of claim. \square

Let $\{V_n\}_{n=1}^{+\infty}$ and $\{U_m\}_{m=1}^{+\infty}$ be basis of Y and X respectively. Put

$$(4.5.1) \quad R_X = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \tilde{X}(U_m, V_j).$$

As $\tilde{X}(U_m, V_j)$ is a dense open invariant subset of X for $m, j \in \mathbb{N}$ by the claim, R_X is a dense G_{δ} set of X . Now we use a terminology given by Auslander: A set $A \subset X$ has the *capturing property* if $\omega(p, T) \cap A \neq \emptyset$ implies $p \in A$. It is easy to see that any open invariant subset of X has the capturing property and any intersection of capturing sets is capturing. In particular, R_X has the capturing property.

For each $x \in R_X \cap R(T)$, set $X_x = \text{cl}(\text{orb}(x, T))$. Then (X_x, T) is a transitive subsystem. Hence by Theorem 4.4 there exists $x' \in R(T)$ such that $(X_{x'}, T)$ is a

maximal transitive subsystem and $X_{x'} \supset X_x$, where $X_{x'} = \text{cl}(\text{orb}(x', T))$. Since R_X has the capturing property and $x \in X_{x'} \cap R_X$, one gets $x' \in R_X$.

Now, we show that $(X_{x'}, T) \perp (Y, S)$. Let J be a joining of $X_{x'}$ and Y , and $J' = \bigcap_{n \in \mathbb{Z}} (T \times S)^n J$. Then J' is a joining with $T \times S(J') = J'$. Take $y \in Y$ such that $(x', y) \in J'$.

For any open neighborhood U of x' and open set V of Y , there exist U_m, V_j such that $x \in U_m \subset U$ and $V_j \subset V$. By (4.5.1), $x' \in X(U_m, V_j)$. In particular, $N_{\mathbb{Z}}(x', U_m) \cap N_{\mathbb{Z}}(y, V_j) \neq \emptyset$. As $T \times S(J') = J'$ and $(x', y) \in J'$, $J' \cap U_m \times V_j \neq \emptyset$. Thus $J' \supset \{x'\} \times Y$ and hence $J' = X_{x'} \times Y$. Therefore, $(X_{x'}, T) \perp (Y, S)$.

By Proposition 4.1(1), $R(T)$ is a dense G_δ set of X . Moreover, $R_X \cap R(T)$ is dense in X . As $\bigcup_{x \in R_X \cap R(T)} X_{x'}$ is dense in X , we can find countably many MTSS such that their union is dense in X and each of them is disjoint from Y .

In the general case we pass to the natural extension. Let \tilde{X} and \tilde{Y} be natural extensions of X and Y respectively. Then by Proposition 1.1, $\tilde{X} \perp \tilde{Y}$. Thus there exist countably many MTSS \tilde{X}_i of \tilde{X} such that their union is dense in \tilde{X} and each of them is disjoint from \tilde{Y} . Let $\pi : \tilde{X} \rightarrow X$ be the projection to the first coordinate. Then $\pi(\tilde{X}_i)$ are MTSS of X and their union is dense in X . It is clear that $\tilde{X}_i \perp \tilde{Y}$. □

Using Theorem 4.5, we get easily

Corollary 4.6. *Let (X, T) be a dynamical system. Then $(X, T) \perp \mathcal{M}$ iff for any minimal system (Y, S) there exist countably many MTSS (depending on Y) such that their union is dense in X and each of them is disjoint from Y .*

To finish the section we now construct the example we promised. We need

Definition. Let (X, T) be a TDS and f a complex-valued continuous function on X which is not identically 0. We say that f is an *eigenfunction* for T if there exists $\lambda \in \mathbb{C}$ such that $f(Tx) = \lambda f(x)$ for any $x \in X$. We then call λ the *eigenvalue* for T corresponding to the eigenfunction f . We denote the set of eigenvalues of T by $\text{Eig}(T)$.

We will give an equivalence condition for a distal system in \mathcal{M}^\perp . To do so we need Lemma 4.7 and 4.8.

Lemma 4.7 ([Wa]). *Let (X, T) be a transitive TDS and T a homeomorphism. Then $\text{Eig}(T)$ forms a countable subgroup of K , where K is the unit circle in complex plane.*

If $T : K \rightarrow K$ is defined by $T(z) = e^{i2\pi\theta}z$, where θ is irrational, then $\text{Eig}(T) = \{e^{i2\pi n\theta} : n \in \mathbb{Z}\}$.

Lemma 4.8. *Let (X, T) and (Y, S) be minimal TDS.*

- (1) *If (X, T) is distal and (Y_{eq}, S) is the maximal equicontinuous factor of (Y, S) , then $(X, T) \perp (Y, S)$ if and only if $(X, T) \perp (Y_{\text{eq}}, S)$.*
- (2) *If (X, T) is equicontinuous, then $(X, T) \perp (Y, S)$ iff X and Y have no nontrivial common factor.*
- (3) *If (X, T) is equicontinuous and $\text{Eig}(T) \cap \text{Eig}(S) = \{1\}$, then $(X, T) \perp (Y, S)$.*

Proof. For the proofs of (1) and (2), see [A]. Now we show (3). By (2), it remains to show that X and Y have no nontrivial common factor.

Assume that X and Y have nontrivial common factor (Z, H) . Then (Z, H) is a minimal equicontinuous system, and there exists $\lambda \in \text{Eig}(H) \setminus \{1\}$. As $\text{Eig}(H) \subset \text{Eig}(T) \cap \text{Eig}(S)$, $\lambda \in \text{Eig}(T) \cap \text{Eig}(S)$, a contradiction. \square

Theorem 4.9. *Let (X, T) be a distal system. Then $(X, T) \perp \mathcal{M}$ iff (X, T) is disjoint from any minimal equicontinuous system.*

Proof. It remains to show that if (X, T) is disjoint from any minimal equicontinuous system, then $(X, T) \perp \mathcal{M}$. Let (Y, S) be a minimal system and let (Y_{eq}, S) be its maximal equicontinuous factor. If Y_{eq} is trivial, then (Y, S) is minimal and weakly mixing. By Theorem 1.3, $(X, T) \perp (Y, S)$.

Now, assume that Y_{eq} is nontrivial. Since $(X, T) \perp (Y_{\text{eq}}, S)$, by Corollary 4.6 there exist transitive subsystems (X_i, T) of (X, T) satisfying $\bigcup_i X_i$ is dense in X and $(X_i, T) \perp (Y_{\text{eq}}, S)$ for each i . As (X_i, T) is distal and transitive, (X_i, T) is minimal and distal. By Lemma 4.8(1), one has $(X_i, T) \perp (Y, S)$. Hence $(X, T) \perp (Y, S)$. This implies that $(X, T) \perp \mathcal{M}$. \square

Finally, we can construct the following example.

Example 4.10. Let $X = [0, 1] \times K$ and define $T : X \rightarrow X$ by $T(x, z) = (x, e^{i2\pi x} z)$. Then (X, T) is distal and $(X, T) \perp \mathcal{M}$.

Proof. Clearly, (X, T) is distal. By Theorem 4.9 it remains to show (X, T) is disjoint from any minimal equicontinuous system. Let (Y, S) be a minimal equicontinuous system. Then $\text{Eig}(S)$ is a countable set of K . Set $E_X = \{x \in [0, 1] \setminus \mathbb{Q} : e^{i2\pi nx} \notin \text{Eig}(S) \text{ for each } n \in \mathbb{Z} \setminus \{0\}\}$. As $\text{Eig}(S)$ is countable, it is easy to see that E_X is dense in $[0, 1]$.

For each $x \in E_X$, let $X_x = \{x\} \times K$. X_x can be considered as an irrational rotation of K . By Lemma 4.7, $\text{Eig}(X_x, T) = \{e^{i2\pi nx} : n \in \mathbb{Z}\}$. From the definition of E_X , $\{e^{i2\pi nx} : n \in \mathbb{Z}\} \cap \text{Eig}(S) = \{1\}$. By Lemma 4.8 (3), $(X_x, T) \perp (Y, S)$. This shows that (X, T) is disjoint from any minimal equicontinuous system by Proposition 4.1(3). Thus $(X, T) \perp \mathcal{M}$. \square

§5. AN EXTREMELY SCATTERING, NONWEAKLY MIXING EXAMPLE

In this section we will construct an extremely scattering, nonweakly mixing example. To do this we need some lemmas to check when a system is not weakly mixing and is extremely scattering. Roughly speaking, the reason such an example exists is that not all syndetic sets can be realized by a dynamical system (Lemma 5.2).

Recall that for a dynamical system (X, T) , $x \in X$, and a pair of nonempty subsets U, V of X , $N(x, U) = \{n \in \mathbb{Z}_+ : T^n(x) \in U\}$ and $N(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}V \neq \emptyset\}$. The following lemma will be used in the construction and also can be viewed as another characterization of weak mixing.

Lemma 5.1. *Let (X, T) be a transitive TDS. If for any open subset U of X there exists $s = s_U \in \mathbb{Z}_+$ such that $s, s + 1 \in N(U, U)$, then (X, T) is weakly mixing.*

Proof. It is known that (X, T) is weakly mixing iff it is transitive and $N(U, U)$ is thick for any open U . Thus it remains to show $N(U, U)$ is thick for any open subset U of X . First we have

Claim. If $N(U, U)$ contains consecutive natural numbers of length k , then $N(U, U)$ contains consecutive natural numbers of length $k + 1$.

Proof of Claim. Let $a, a + 1, \dots, a + k - 1 \in N(U, U)$ with $a \in \mathbb{N}$. Then $U \cap T^{-(a+i)}U \neq \emptyset, 0 \leq i \leq k - 1$. Since (X, T) is transitive, there exist $t_0, t_1, \dots, t_{k-1} \in \mathbb{N}$ (by induction) such that

$$\bigcap_{i=0}^{k-1} T^{-t_i}(U \cap T^{-(a+i)}U) \neq \emptyset.$$

Put $D = \bigcap_{i=0}^{k-1} T^{-t_i}(U \cap T^{-(a+i)}U)$. By the assumption there is $s \in \mathbb{Z}_+$ such that $s, s + 1 \in N(D, D)$, i.e., $D \cap T^{-s}D \neq \emptyset$ and $D \cap T^{-(s+1)}D \neq \emptyset$. Clearly,

$$\begin{aligned} (U \cap T^{-(a+i)}U) \cap T^{-s}(U \cap T^{-(a+i)}U) &\neq \emptyset, \\ (U \cap T^{-(a+i)}U) \cap T^{-(s+1)}(U \cap T^{-(a+i)}U) &\neq \emptyset, \end{aligned}$$

for $i = 0, 1, \dots, k - 1$.

Particularly, $U \cap T^{-(s+a+i)}U \neq \emptyset$ and $U \cap T^{-(s+1+a+i)}U \neq \emptyset$ for $0 \leq i \leq k - 1$. Hence, one has $s + a, s + a + 1, \dots, s + a + k - 1, s + a + k \in N(U, U)$. This finishes the proof of the claim.

By the claim, $N(U, U)$ is thick for any open subset U of X . □

Lemma 5.2. *Let (X, T) be a topologically ergodic system. Then for any open subset U of X and $k \in \mathbb{N}$, $N_k(U) = \{i \in \mathbb{Z}_+ : 2^k i \in N(U, U)\}$ is syndetic.*

Proof. First, we consider the case when $k = 1$. If $N(U, U)$ doesn't contain odd numbers, clearly $N_1(U)$ is syndetic. Assume now there exists an odd number $a_1 \in N(U, U)$, that is, $U \cap T^{-a_1}U \neq \emptyset$. Put $D_1 = U \cap T^{-a_1}U$. Take any $m \in N(D_1, D_1)$, then $D_1 \cap T^{-m}D_1 \neq \emptyset$. This implies $U \cap T^{-m}U \cap T^{-(m+a_1)}U \neq \emptyset$. Thus, $\{m, a_1 + m\} \subset N(U, U)$. Since a_1 is odd and $N(D_1, D_1)$ is syndetic, it turns out that $N_1(U)$ is syndetic.

Assume that for any $1 \leq k \leq l$, $N_k(U)$ is syndetic for any open subset U of X . We shall show that $N_{l+1}(U)$ is syndetic for any open subset U of X .

If $N_l(U)$ doesn't contain odd numbers, then $N_{l+1}(U)$ is syndetic. Assume now there is an odd number $a_l \in N_l(U)$, i.e., $D_l = U \cap T^{-2^l a_l}U \neq \emptyset$. Similarly, one has $2^l m, 2^l(m + a_l) \in N(U, U)$ for each $m \in N_l(D_l)$. This implies that $m, m + a_l \in N_l(U)$ for each $m \in N_l(D_l)$. Since a_l is odd and $N_l(D_l)$ is syndetic (by the inductive assumption), $N_{l+1}(U)$ is syndetic. □

Lemma 5.3. *Let (X, T) be a transitive system with a transitive point x . If for any neighborhood U of x we have property (**):*

$$(**) \text{ for any } r \in \mathbb{Z}_+ \text{ there exists } k_r \in \mathbb{N} \text{ such that } N_{k_r}(U, r) = \{n \in \mathbb{N} : 2^{k_r} n - r \in N(U, U)\} \text{ is thick,}$$

then (X, T) is extremely scattering.

Proof. First let (Y, S) be invertible and topologically ergodic. Let U_1, U_2 be any open subsets of X and let V_1, V_2 be any open subsets of Y . Then by the definition

$$N(U_1 \times V_1, U_2 \times V_2) = \{n \in \mathbb{Z}_+ : (T \times S)^{-n}(U_2 \times V_2) \cap (U_1 \times V_1) \neq \emptyset\}.$$

As x is a transitive point, there are n_0, k such that $U = T^{-(n_0+k)}(U_2) \cap T^{-k}(U_1)$ is a nonempty neighborhood of x . Thus,

$$\begin{aligned} &N(U_1 \times V_1, U_2 \times V_2) \\ &= \{n \in \mathbb{Z}_+ : (T \times S)^{-(n+k)}(U_2 \times V_2) \cap (T \times S)^{-k}(U_1 \times V_1) \neq \emptyset\} \\ &= \{n \in \mathbb{Z}_+ : (T^{-(n+k)}U_2 \cap T^{-k}U_1) \times (S^{-(n+k)}V_2 \cap S^{-k}V_1) \neq \emptyset\} \\ &\supset n_0 + \{m \in \mathbb{Z}_+ : (T^{-m}U \cap U) \times (S^{-m}S^{-(n_0+k)}V_2 \cap S^{-k}V_1) \neq \emptyset\}. \end{aligned}$$

Since (Y, S) is transitive and S is a homeomorphism, there exist an open subset $V \in S^{-k}(V_1)$ and $r \in \mathbb{N}$ such that $S^{-r}V \subset S^{-(n_0+k)}(V_2)$. Thus

$$N(U_1 \times V_1, U_2 \times V_2) \supset n_0 + N(U \times V, U \times S^{-r}V).$$

By property (**) there exists k_r such that $N_{k_r}(U, r)$ is thick. By Lemma 5.2, one knows that $N_{k_r}(V)$ is syndetic.

Thus $N_{k_r}(U, r) \cap N_{k_r}(V) \neq \emptyset$. Let $m \in N_{k_r}(U, r) \cap N_{k_r}(V)$. Then $2^{k_r}m - r \in N(U, U)$ and $2^{k_r}m \in N(V, V)$, and thus we have

$$2^{k_r}m - r \in N(V, S^{-r}V) \cap N(U, U) = N(U \times V, U \times S^{-r}V).$$

Hence $n_0 + 2^{k_r}m - r \in N(U_1 \times V_1, U_2 \times V_2)$.

Now assume that (Y, S) is a topologically ergodic system and (Y_1, S_1) is the nature extension (Y, T) . Then (Y_1, S_1) is invertible and topologically ergodic. Hence by what we have proved, $(X \times Y_1, T \times S_1)$ is transitive. As $(X \times Y, T \times S)$ is a factor of $(X \times Y_1, T \times S_1)$, $(X \times Y, T \times S)$ is transitive, i.e., (X, T) is extremely scattering. This ends the proof. \square

With the above preparation we now start to construct the example we promised.

Theorem 5.4. *There is an extremely scattering, not weakly mixing system.*

Proof. We will construct the system in a one sided shift on two symbols (Σ, S) and the system is the closure of the orbit of a recurrent point $x = (x_0, x_1, \dots) \in \Sigma$. To do this, we construct inductively a sequence of finite words C_i such that C_{i+1} begins with C_i and x is just the limit of C_i .

Define $F : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{Z}_+$ by $F(l) = (\phi(l), \varphi(l))$ such that $\phi(l+1) \leq l$ and for any $(n, r) \in \mathbb{N} \times \mathbb{Z}_+$ there exist infinitely many $j \in \mathbb{N}$ with $F(j) = (n, r)$. To begin with we let

$$C_0 = (0), \quad C_1 = (0, 0, 1, 0, 0) = (x_0, x_1, \dots, x_4) \quad \text{and} \quad k_1 = 5.$$

Set

$$W_1 = \{2\} = \{i : x_i = 1, i \leq k_1 - 1\} \quad \text{and} \quad B_1 = W_1 - W_1 = \{0\}.$$

where $A - B = \{a - b \geq 0 : a \in A, b \in B\}$. Inductively we construct C_l . If k_l is the length of C_l , then we define

$$(5.1) \quad W_l = \{i : x_i = 1, 0 \leq i \leq k_l - 1\} \quad \text{and} \quad B_l = W_l - W_l.$$

Moreover, B_l satisfies

- (1)_l $1 \notin B_l$ and there is no $s \in \mathbb{Z}_+$ such that $\{s, s+1\} \subset B_l$.
- (2)_l C_l begins with C_{l-1} .

For $l = 1$, $(1)_l$ and $(2)_l$ are satisfied. Assume that we have constructed C_n for $1 \leq n \leq l$ with $(1)_n$ and $(2)_n$. We build C_{l+1} as follows.

Assume that $p_{l,1}, p_{l,2}, \dots, p_{l,l+1}, q_l$ are positive integers to be defined later and set

$$(5.2) \quad q_{l,i} = 2^{k_{\phi(l+1)}+1}(q_l + i) - k_{\phi(l+1)} - \varphi(l + 1), \text{ for } i = 1, 2, \dots, l + 1.$$

Let

$$(5.3) \quad C_{l+1} = C_l 0^{p_{l,1}} C_{\phi(l+1)} 0^{q_{l,1}} C_{\phi(l+1)} 0^{p_{l,2}} C_{\phi(l+1)} 0^{q_{l,2}} C_{\phi(l+1)} 0^{p_{l,3}} \dots 0^{p_{l,l+1}} C_{\phi(l+1)} 0^{q_{l,l+1}} C_{\phi(l+1)}.$$

Set

$$(5.4) \quad a_1^l = k_l + p_{l,1}, \quad a_{2i-1}^l = k_{\phi(l+1)} + p_{l,i} \text{ and } a_{2j}^l = q_{l,j} + k_{\phi(l+1)},$$

where $i = 2, 3, \dots, l + 1$ and $j = 1, 2, \dots, l + 1$.

By (5.1) and (5.4)

$$(5.5) \quad W_{l+1} = W_l \cup \bigcup_{k=1}^{2(l+1)} (W_{\phi(l+1)} + (a_1^l + a_2^l + \dots + a_k^l)).$$

As $W_{\phi(l+1)} \subset W_l$, we know that $W_l - W_{\phi(l+1)}, W_{\phi(l+1)} - W_l \subset W_l - W_l = B_l$. Thus from (5.1) and (5.5) we have

$$(5.6) \quad B_{l+1} \subset B_l \cup \bigcup_{k=1}^{2(l+1)} ((a_1^l + a_2^l + \dots + a_k^l) \pm B_l) \cup \bigcup_{1 \leq i < j \leq 2(l+1)} ((a_{i+1}^l + a_{i+2}^l + \dots + a_j^l) \pm B_{\phi(l+1)}).$$

We can take $p_{l,1}, p_{l,2}, \dots, p_{l,l+1}, q_l$ such that B_{l+1} satisfies $(1)_{l+1}$. We do this at the end of the proof.

Let $x = \lim_l C_l$ and X be the orbit closure of x under the shift S . We now prove that (X, S) is extremely scattering and not weakly mixing.

Let $U = \{y \in X : y_0 = 1\}$. Then

$$(5.7) \quad N(x, U) = \bigcup_{l=1}^{+\infty} W_l \text{ and } N(U, U) = \bigcup_{l=1}^{+\infty} B_l.$$

As $B_1 \subset B_2 \subset B_3 \subset \dots$ and $(1)_l$ is satisfied by all l we know that $N(U, U)$ is not thick, and consequently that (X, S) is not weakly mixing.

We now check that (X, S) satisfies property $(**)$ in Lemma 5.3. As x is recurrent, (X, S) is transitive. For each neighborhood V of x there is $n \in \mathbb{N}$ such that $V_n = [C_n] \subset V$. Let $r \in \mathbb{Z}_+$ and set $k = k_n + 1$. By the definition of F , there exists infinitely many $l \in \mathbb{N}$ such that

$$(5.8) \quad F(l + 1) = (\phi(l + 1), \varphi(l + 1)) = (n, r).$$

By the construction of C_{l+1} (5.3) and (5.8), it is easy to see

$$N(x, V_n) \supset \{a_1^l, a_1^l + a_2^l, \dots, a_1^l + a_2^l + \dots + a_{2(l+1)}^l\}.$$

By (5.4), (5.2) and (5.8) we have

$$a_{2i}^l = q_{l,i} + k_{\phi(l+1)} = 2^{k_n+1}(q_l + i) - \varphi(l + 1) = 2^k(q_l + i) - r.$$

Thus,

$$\begin{aligned} N(V, V) \supset N(V_n, V_n) &= N(x, V_n) - N(x, V_n) \supset \{a_2^l, a_4^l, \dots, a_{2(l+1)}^l\} \\ &= \{2^k(q_l + 1) - r, 2^k(q_l + 2) - r, \dots, 2^k(q_l + l + 1) - r\}. \end{aligned}$$

Hence $N_k(V, r) = \{n \in \mathbb{N} : 2^k n - r \in N(V, V)\} \supset \{q_l + 1, q_l + 2, \dots, q_l + (l + 1)\}$. This shows that $N_k(V, r)$ is thick. Hence (X, S) is extremely scattering by Lemma 5.3.

To finish the proof we must choose $p_{l,1}, p_{l,2}, \dots, p_{l,l+1}, q_l$ such that B_{l+1} satisfies (1) _{$l+1$} . There are many ways to do this, for example, we take

$$q_l = k_l + \varphi(l + 1) \text{ (by (5.2) and (5.4) } a_{2j}^l \text{ is determined by } q_l),$$

$$p_{l,2} = \sum_{j=1}^{l+1} a_{2j}^l + 2k_l + 2 \text{ and inductively we take}$$

$$p_{l,i} \geq \sum_{j=2}^{i-1} a_{2j-1}^l + \sum_{j=1}^{l+1} a_{2j}^l + 2k_l + 2 \text{ for } 3 \leq i \leq l + 1. \text{ Finally, we take}$$

$$(5.9) \quad p_{l,1} \geq \sum_{i=2}^{l+1} a_{2i-1}^l + \sum_{j=1}^{l+1} a_{2j}^l + 2k_l + 2.$$

The reason we take $p_{l,1}, p_{l,2}, \dots, p_{l,l+1}, q_l$ in the above form is that we want the following conditions to hold.

- (i) if $1 \leq j \neq k \leq 2(l + 1)$, then the four intervals

$$[(a_1^l + \dots + a_j^l) \pm B_l] \quad \text{and} \quad [(a_1^l + \dots + a_k^l) \pm B_l]$$

are disjoint with the gaps ≥ 2 (see (5.6)) as $\max B_l \leq k_l$, where $[A]$ is the convex hull of a finite set A of \mathbb{Z}_+ in \mathbb{Z}_+ .

- (ii) if $1 \leq i < j \leq 2(l + 1)$ and $1 \leq k \leq 2(l + 1)$, then the four intervals

$$[(a_1^l + \dots + a_k^l) \pm B_l] \quad \text{and} \quad [(a_{i+1}^l + \dots + a_j^l) \pm B_{\phi(l+1)}]$$

are disjoint with the gaps ≥ 2 (see (5.6)) as $\max B_{\phi(l+1)}, \max B_l \leq k_l$.

- (iii) if $1 \leq i_1 < j_1 \leq 2(l + 1)$, $1 \leq i_2 < j_2 \leq 2(l + 1)$ and $\{i_1, j_1\} \neq \{i_2, j_2\}$, then the four intervals

$$[(a_{i_1+1}^l + \dots + a_{j_1}^l) \pm B_{\phi(l+1)}] \quad \text{and} \quad [(a_{i_2+1}^l + \dots + a_{j_2}^l) \pm B_{\phi(l+1)}]$$

are disjoint with the gaps ≥ 2 (see (5.6)) as $\max B_{\phi(l+1)} \leq k_{\phi(l+1)}$.

These in turn follow because (5.10), (5.11) and (5.12) below hold.

By choosing $p_{l,1}, p_{l,2}, \dots, p_{l,l+1}, q_l$, (5.2) and (5.4) we know $a_i^l \geq 2k_l + 2$, $i = 1, 2, \dots, 2(l + 1)$. Moreover, we have if $1 \leq j \neq k \leq 2(l + 1)$, then

$$(5.10) \quad |(a_1^l + \dots + a_j^l) - (a_1^l + \dots + a_k^l)| \geq 2k_l + 2.$$

By (5.4) and (5.9) we have $a_1^l - (a_2^l + a_3^l + \dots + a_{2(l+1)}^l) \geq 2k_l + 2$. Hence if $1 \leq i < j \leq 2(l + 1)$ and $1 \leq k \leq 2(l + 1)$, then

$$(5.11) \quad |(a_1^l + \dots + a_k^l) - (a_{i+1}^l + \dots + a_j^l)| \geq 2k_l + 2.$$

Now we claim that if $1 \leq i_1 < j_1 \leq 2(l + 1)$, $1 \leq i_2 < j_2 \leq 2(l + 1)$ and $\{i_1, j_1\} \neq \{i_2, j_2\}$, then

$$(5.12) \quad |(a_{i_1+1}^l + \dots + a_{j_1}^l) - (a_{i_2+1}^l + \dots + a_{j_2}^l)| \geq 2k_{\phi(l+1)} + 2.$$

Proof of the claim. Set $I_1 = \{i_1 + 1, \dots, j_1\}$, $I_2 = \{i_2 + 1, \dots, j_2\}$ and $I = (I_1 \Delta I_2) \cap \{3, 5, \dots, 2l + 1\}$.

Case 1: $I \neq \emptyset$.

Let $2m + 1 = \max I$. Then

$$\begin{aligned} & |(a_{i_1+1}^l + \dots + a_{j_1}^l) - (a_{i_2+1}^l + \dots + a_{j_2}^l)| \\ & \geq a_{2m+1}^l - \left(\sum_{i=1}^{l+1} a_{2i}^l + \sum_{j=1}^{m-1} a_{2j+1}^l \right) \quad (\text{by (5.9) and (5.4)}) \\ & \geq 2k_l + 2 \geq 2k_{\phi(l+1)} + 2. \end{aligned}$$

Case 2: $I = \emptyset$.

If $I_1 \cap \{3, 5, \dots, 2l + 1\} = \emptyset$, then $I_1 = \{i_1\}$, $I_2 = \{i_2\}$ and $i_1 \neq i_2$ are even.

$$|a_{i_1}^l - a_{i_2}^l| = |2^{k_{\phi(l+1)}+1} \frac{(i_2 - i_1)}{2}| \geq 2k_{\phi(l+1)} + 2.$$

If $I_1 \cap \{3, 5, \dots, 2l + 1\} = \{2j + 1, 2j + 3, \dots, 2s + 1\}$, then $I_1, I_2 \in \{\{2j, 2j + 1, \dots, 2s + 1\}, \{2j + 1, \dots, 2s + 1\}, \{2j, \dots, 2s + 1, 2(s + 1)\}, \{2j + 1, \dots, 2s + 1, 2(s + 1)\}\}$. Then

$$\begin{aligned} & |(a_{i_1+1}^l + \dots + a_{j_1}^l) - (a_{i_2+1}^l + \dots + a_{j_2}^l)| \\ & = a_{2j}^l \text{ or } |a_{2(s+1)}^l - a_{2j}^l| \text{ or } a_{2(s+1)}^l \text{ or } |a_{2j}^l + a_{2(s+1)}^l| \\ & \geq 2k_{\phi(l+1)} + 2. \end{aligned}$$

This ends the proof of the claim.

Now we check that B_{l+1} satisfies $(1)_{l+1}$.

Assume that there is s such that $\{s, s - 1\} \subset B_{l+1}$. By $(1)_l$, $s, s - 1$ do not belong to B_l simultaneously. As $\max B_{\phi(l+1)}, \max B_l \leq k_l$ and $a_i^l \geq 2k_l + 2$ ($1 \leq i \leq 2(l + 1)$), for $1 \leq k \leq 2(l + 1)$, $1 \leq i < j \leq 2(l + 1)$

$\min\{(a_1^l + \dots + a_j^l) \pm B_l\}, \min\{(a_{i+1}^l + \dots + a_j^l) \pm B_{\phi(l+1)}\} \geq 2k_l + 2 \geq \max B_l + 2$, we have (see (5.6))

$$\begin{aligned} \{s, s - 1\} \subset & \bigcup_{k=1}^{2(l+1)} ((a_1^l + a_2^l + \dots + a_k^l) \pm B_l) \\ & \cup \bigcup_{1 \leq i < j \leq 2(l+1)} ((a_{i+1}^l + a_{i+2}^l + \dots + a_j^l) \pm B_{\phi(l+1)}). \end{aligned}$$

By (5.10), (5.11), (5.12) and $B_{\phi(l+1)} \subset B_l$ there must exist $1 \leq i_s \leq j_s \leq 2(l + 1)$ such that

$$\{s, s - 1\} \subset (a_{i_s}^l + a_{i_s+1}^l + \dots + a_{j_s}^l) \pm B_l.$$

From $(1)_l$ there is no s with $\{s, s - 1\} \subset B_l$. Thus

$$(5.13) \quad s \in (a_{i_s}^l + \dots + a_{j_s}^l) + B_l \text{ and } s - 1 \in (a_{i_s}^l + \dots + a_{j_s}^l) - B_l.$$

Since

$$\min\{(a_{i_s}^l + \dots + a_{j_s}^l) + B_l\} = \max\{(a_{i_s}^l + \dots + a_{j_s}^l) - B_l\} = a_{i_s}^l + a_{i_s+1}^l + \dots + a_{j_s}^l,$$

(5.13) is impossible as $1 \notin B_l$. That is, there is no s with $\{s, s - 1\} \subset B_{l+1}$. Hence $1 \notin B_{l+1}$ since $0 \in B_{l+1}$. \square

§6. APPENDIX

In this Appendix we give the proofs of Proposition 3.5 and Example 3.7.

Proof of Proposition 3.5. Let $\pi : (X, T) \rightarrow (\Sigma_K, T_K)$ be an almost one-to-one extension, where $K = (k_1, k_2, \dots)$. Note that for any $w = (w_1, w_2, \dots) \in \Sigma_K$ and an open neighborhood $[w_1, w_2, \dots, w_l] = \{t \in \Sigma_K : t_i = w_i, i = 1, 2, \dots, l\}$ of w , one has

$$N(w, [w_1, w_2, \dots, w_l]) = (k_1 \cdot k_2 \cdots k_l)\mathbb{Z}_+.$$

Hence each point of Σ_K is a regular minimal point.

Now take $x \in X$ with $\pi^{-1}\pi(x) = \{x\}$. For any open neighborhood U of x , there exists an open neighborhood V of $\pi(x)$ such that $\pi^{-1}V \subset U$. Since $\pi(x)$ is a regular minimal point, we can find $k \in \mathbb{N}$ such that $N(\pi(x), V) \supset k\mathbb{Z}_+$. Therefore, $N(x, U) \supset N(x, \pi^{-1}V) = N(\pi(x), V) \supset k\mathbb{Z}_+$. This shows that x is a regular minimal point. Moreover, each point in Inj_π is a regular minimal point.

Conversely, let $x \in X$ be a regular minimal point. We have

Claim. For each open neighborhood U of x , there exist a clopen (closed and open) neighborhood $A \subset U$ of x and $l \geq 1$ such that $\{A, TA, \dots, T^{l-1}A\}$ is a clopen partition of X and $T^lA = A$.

Proof of the Claim. Take an open neighborhood V of x with $cl(V) \subset U$. Since x is a regular minimal point, there exists $k \in \mathbb{N}$ such that $N(x, V) \supset k\mathbb{Z}_+$. Thus $\omega(x, T^k) \subset cl(V) \subset U$. Let $A = \omega(x, T^k)$. Then $x \in A \subset U$ and (T^iA, T^k) is minimal, $0 \leq i \leq k - 1$. If $T^iA \neq T^jA$, then $T^iA \cap T^jA = \emptyset$. As $T^kA = \omega(T^kx, T^k) = \omega(x, T^k) = A$, there exists $0 < l \leq k$ such that $A, TA, \dots, T^{l-1}A$ are pairwise disjoint closed subsets of X and $T^lA = A$. Since $\bigcup_{i=0}^{l-1} T^iA = \bigcup_{i=0}^{k-1} T^iA = \omega(x, T) = X$, $\{A, TA, \dots, T^{l-1}A\}$ is a clopen partition of X . This finishes the proof of the claim.

By the above claim, it is easy to construct inductively a sequence of clopen neighborhood A_i of x and $l_i \geq 1$ such that $A_1 \supset A_2 \supset \dots, \bigcap_{i=1}^\infty A_i = \{x\}$ and $\{A_i, TA_i, \dots, T^{l_i-1}A_i\}, i \in \mathbb{N}$, is a clopen partition of X . Since $A_i \subset A_{i+1}, l_i | l_{i+1}$ for each $i \in \mathbb{N}$.

Take $l_0 = 1$ and $k_i = \frac{l_i}{l_{i-1}}, i \in \mathbb{N}$. Let $K = (k_1, k_2, \dots)$. Now we define $h : X \rightarrow \Sigma_k$ by

$$h(x)_i = \left[\frac{j_i}{l_{i-1}} \right], \text{ where } 0 \leq j_i \leq l_i - 1 \text{ with } x \in T^{j_i}A_i, \text{ for each } i \in \mathbb{N}.$$

Now we show that $h : (X, T) \rightarrow (\Sigma_K, T_K)$ is an almost one-to-one extension.

First, we note that $x \in \bigcap_{i=1}^\infty T^{\sum_{j=1}^i l_{j-1} h(x)_j} A_i$. Hence

$$Tx \in \bigcap_{i=1}^\infty T^{\sum_{j=1}^i l_{j-1} h(x)_j + 1} A_i = \bigcap_{i=1}^\infty T^{\sum_{j=1}^i l_{j-1} (T_k h(x))_j} A_i.$$

This shows that $h(Tx) = T_K h(x)$. Second, for any $w = (w_1, w_2, \dots) \in \Sigma_K$, since $T^{\sum_{j=1}^{i+1} l_{j-1} w_j} A_{i+1} \subset T^{\sum_{j=1}^i l_{j-1} w_j} A_i, i \in \mathbb{N}$, one has $\bigcap_{i=1}^\infty T^{\sum_{j=1}^i l_{j-1} w_j} A_i \neq \emptyset$. For any $z \in \bigcap_{i=1}^\infty T^{\sum_{j=1}^i l_{j-1} w_j} A_i, h(z) = w$. Hence h is surjective. Third, since for each cylinder $[w_1, w_2, \dots, w_l] \subset \Sigma_K, h^{-1}([w_1, w_2, \dots, w_l]) = \bigcap_{i=1}^l T^{\sum_{j=1}^i l_{j-1} w_j} A_i$ is an open set of X, h is continuous. By the above discussion, we have seen that $h : (X, T) \rightarrow (\Sigma_K, T_K)$ is a factor map.

Finally, since for each $j \in \mathbb{Z}_+$, $\bigcap_{i=1}^{+\infty} T^j A_i = \{T^j x\}$, i.e., $h^{-1}(h(T^j x)) = \{T^j x\}$ and $orb(x, T)$ is dense in X , one knows that Inj_h is dense in X . Hence h is an almost one-to-one extension. \square

Now we present the construction. Note that for a word $A = (a(1), \dots, a(n))$ and a sequence y , $A < y$ means that A appears in y .

Proof of Example 3.7. We will construct a recurrent point $x \in \{0, 1, 2\}^{\mathbb{Z}_+}$ such that $X = cl(orb(x, \sigma))$ is the system which we need. Let $y = (12212112 \dots) \in \{1, 2\}^{\mathbb{Z}_+}$ be the Morse sequence. It is known [GH1] that Morse sequence y has the following property: $w^3 \not< y$ for any finite word w .

Define $F : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{Z}_+$ by $F(s) = (\phi(s), \varphi(s))$ such that $\phi(s) \leq s$, and for any $(k, r) \in \mathbb{N} \times \mathbb{Z}_+$ there exist infinitely many $j \in \mathbb{N}$ with $F(j) = (k, r)$.

We use induction to construct a recurrent point $x = (x_0, x_1, \dots) \in \{0, 1, 2\}^{\mathbb{Z}_+}$. To do this, we construct inductively a sequence of finite words A_i such that A_{i+1} begins with A_i and x is the limit of A_i . To begin with we let $A_1 = 0$ and $n_1 = |A_1| = 1$. Assume that we have constructed A_l for $1 \leq l \leq k$. Set $n_l = |A_l|$. For $l = k + 1$, put

$$\begin{aligned} A_{k+1} = & A_k y [0; m_k - 1] A_{\phi(k)} y [m_k; m_k + 9n_{\phi(k)} - 1] \dots \\ & A_{\phi(k)} y [m_k + 9in_{\phi(k)}; m_k + 9(i + 1)n_{\phi(k)} - 1] \dots \\ & A_{\phi(k)} y [m_k + 9(k - 1)n_{\phi(k)}; m_k + 9kn_{\phi(k)} - 1] A_{\phi(k)}, \end{aligned}$$

where $m_k = 10t_k n_{\phi(k)} - n_k + \varphi(k)$ and t_k is large enough with $m_k \geq \frac{8}{9}n_{k+1}$ (note that $n_{k+1} = n_k + m_k + (10k + 1)n_{\phi(k)}$).

Let $x = \lim_{k \rightarrow +\infty} A_k$ and $X = cl(orb(x, \sigma))$. By the construction of A_k , it is easy to see that for any given $(k, r) \in \mathbb{N} \times \mathbb{Z}_+$ and s with $F(s) = (k, r)$,

$$A_{s+1} = A_s y [0; m_s - 1] A_k y [m_s; m_s + 9n_k - 1] A_k y [m_s + 9n_k, m_s + 18n_k - 1] \dots A_k.$$

As $n_s + m_s + 10in_k = n_s + 10t_s n_k - n_s + r + 10in_k = 10n_k(t_s + i) + r$ and $N_k(x, U, r) = \{n \in \mathbb{Z}_+ : T^{kn+r}(x) \in U\}$, we have

$$N_{10n_k}(x, [A_k], r) \supset \bigcup_{s \text{ with } F(s)=(k,r)} \{t_s, t_s + 1, \dots, t_s + s - 1\}.$$

Therefore, $(X, \sigma) \in \mathcal{M}^\perp$ by Theorem 3.4.

Now, we show that (X, σ) has no periodic point. First, we have

Claim 1: For every $k \in \mathbb{N}$, one has

- (1)_k There is no finite word w such that $t = |w| \leq \frac{n_k}{3}$ and $w^3 = A_k[0; 3t - 1]$.
- (2)_k There is no finite word w such that $t = |w| \leq \frac{n_k}{3}$ and $w^3 = A_k[n_k - 3t; n_k - 1]$.

Proof of Claim 1. (1)₁ is obvious. Assume that (1)_k holds for $1 \leq k \leq l$ and (1)_k fails for $k = l + 1$. Then there exists a finite word w such that $t = |w| \leq \frac{n_k}{3}$ and $w^3 = A_k[0; 3t - 1]$. By (1)_l, $n_{l+1} \geq 3t > n_l$.

Since $m_l \geq \frac{8}{9}n_{l+1}$, one has $2t < n_l + m_l$. As $w(0) = A_k(0) = 0, 0 \not< y$ and $A_k(2t) = w(0) = 0$, one has $2t < n_l$. Moreover, $l \geq 2$ since $t \geq 1$ and $n_1 = 1$.

Since $m_{l-1} + n_{l-1} > \frac{n_l}{2} > t > \frac{n_l}{3} > n_{l-1}$ and $A_l[n_{l-1}; n_{l-1} + m_{l-1} - 1] = y[0; m_{l-1} - 1]$, one has $0 = w(0) = A_l(t) < y$, a contraction, as $0 \not< y$.

(2)₁ is obvious. Assume that (2)_k holds for $1 \leq k \leq l$ and fails for $k = l + 1$. Then there exists a finite word w such that $t = |w| \leq \frac{n_k}{3}$ and $w^3 = A_k[n_k - 3t; n_k - 1]$. By (2)_{φ(l)}, $n_{l+1} \geq 3t > n_{φ(l)}$. There are two cases for t .

Case 1: $t < n_{φ(l)}$. Since $A_k(n_k - 2t - 1) = w(t - 1) = A_k(n_k - 1) = 0$ and $0 \not\prec y$, one has $2t < n_{φ(l)}$. This shows that $\frac{n_{φ(l)}}{2} > t > \frac{n_{φ(l)}}{3}$. Moreover, one has $φ(l) \geq 2$ and

$$n_{φ(l)-1} \leq \frac{n_{φ(l)}}{2} - 1 < n_{φ(l)} - t - 1 < \frac{2n_{φ(l)}}{3} - 1 \leq n_{φ(l)-1} + m_{φ(l)-1}.$$

Thus $A_{n_{φ(l)}}(n_{φ(l)} - t - 1) < y[0; m_{φ(l)-1} - 1]$, a contradiction, as $A_{n_{φ(l)}}(n_{φ(l)} - t - 1) = w(t - 1) = 0$ and $0 \not\prec y$.

Case 2: $t \geq n_{φ(l)}$. Since $w[t - n_{φ(l)}; t - 1] = A_{φ(l)}, A_{φ(l)}(0) = A_{φ(l)}(n_{φ(l)} - 1) = 0$ and $0 \not\prec y$, there exists $i \in \mathbb{Z}_+$ such that

$$w = y[m_l + 9in_{φ(l)}; m_l + 9(i + 1)n_{φ(l)} - 1]A_{φ(l)} \cdots A_{φ(l)}y[m_l + 9(l - 1)n_{φ(l)}; m_l + 9ln_{φ(l)} - 1]A_{φ(l)}.$$

It follows by the above equality that $n_k - t \geq n_l + m_l$. Hence $t \leq n_k - (n_l + m_l) \leq \frac{1}{9}n_k \leq \frac{1}{8}m_l$. Moreover, $n_k - 3t > n_l$. Noting that $A_k[n_k - 3t; n_k - 1] = w^3$ and $w[9n_{φ(l)}; 10n_{φ(l)} - 1] = A_{φ(l)}$, one has $A_k[n_k - 3t + 9n_{φ(l)}; n_k - 3t + 10n_{φ(l)} - 1] = A_{φ(l)}$.

As $A_{φ(l)}(0) = A_{φ(l)}(n_{φ(l)} - 1) = 0$ and $0 \not\prec y$, one gets $n_k - 3t + 9n_{φ(l)} \geq n_l + m_l$. Let $u = y[m_l + 9in_{φ(l)}; m_l + 9ln_{φ(l)} - 1]$. If $n_k - 3t \geq n_l + m_l$, then

$$\begin{aligned} w^3 &= A_k[n_k - 3t; n_k - 1] \\ &= y[m_l + 9jn_{φ(l)}; m_l + 9(j + 1)n_{φ(l)} - 1]A_{φ(l)} \cdots \\ &\quad A_{φ(l)}y[m_l + 9(l - 1)n_{φ(l)}; m_l + 9ln_{φ(l)} - 1]A_{φ(l)}, \end{aligned}$$

where $j = l - 3(l - i + 1) + 1$. Hence $u^3 = y[m_l + 9jn_{φ(l)}; m_l + 9ln_{φ(l)} - 1] < y$, a contradiction.

If $n_k - 3t < n_l + m_l$, then

$$\begin{aligned} w^3 &= A_k[n_k - 3t; n_k - 1] \\ &= y[m_l - 9n_{φ(l)}; m_l - 1]A_{φ(l)}y[m_l; m_l + 9n_{φ(l)} - 1]A_{φ(l)} \cdots \\ &\quad A_{φ(l)}y[m_l + 9(l - 1)n_{φ(l)}; m_l + 9ln_{φ(l)} - 1]A_{φ(l)}. \end{aligned}$$

Hence $u^3 = y[m_l - 9n_{φ(l)}; m_l + 9ln_{φ(l)} - 1] < y$, a contradiction. This ends the proof of claim 1.

Second, we have

Claim 2: There is no finite word w with $w^8 < x$.

Proof of Claim 2. Assume that there exists a finite word w with $w^8 < x$. Let k be the least natural number with $w^8 < A_k$. Clearly, $k \geq 2$. Set $|w| = t$. Then there is $0 \leq l \leq n_k - 8t$ with $A_k[l; l + 8t - 1] = w^8$. By the choosing of k , one knows $w^8 \not\prec A_{k-1}$. Therefore, $l + 8t > n_{k-1}$.

Let $w = w(0)w(1) \cdots w(t - 1)$ and define

$$P^j w = w(j)w(j + 1) \cdots w(t - 1)w(0) \cdots w(j - 1), \quad j = 1, 2, \dots, t - 1.$$

Since 0 appears in both endpoint of A_m for each $m \in \mathbb{N}$, $0 \not\prec y$ and $w^3 \not\prec y$, one has $w, Pw, P^2w, \dots, P^{t-1}w \not\prec y$. There are two cases for l .

Case 1: $l < n_{k-1}$. Since $m_{k-1} \geq \frac{8}{9}n_k$, $l + 6t < n_{k-1} + m_{k-1}$. By $(1)_{k-1}$ of claim 1, $l + 3t \geq n_{k-1}$. This shows that $n_{k-1} \leq l + 3t < l + 6t < n_{k-1} + m_{k-1}$. Therefore, $w^3 < A_k[n_{k-1}; n_{k-1} + m_{k-1} - 1] = y[0; m_{k-1} - 1]$, a contradiction.

Case 2: $l \geq n_{k-1}$. Since $w \not\prec y$, $l \geq n_{k-1} + m_{k-1} - t$. If $t < n_{\phi(k-1)}$. By Claim 1 and $w, Pw, P^2w, \dots, P^{t-1}w \not\prec y$, one has $w^8 < A_{\phi(k-1)}$, a contradiction since $\phi(k-1) < k$. Thus $t \geq n_{\phi(k-1)}$. Since $w \not\prec y$, there exists $0 \leq m \leq k-1$ such that $l < s_k \leq l + 2t$, where $s_k = n_{k-1} + m_{k-1} + 10mn_{\phi(k-1)}$. Set $v = A_k[s_k; s_k + t - 1]$. Then $v^6 = A_k[s_k; s_k + 6t - 1]$.

Since $v[0; n_{\phi(k-1)} - 1] = A_{\phi(k-1)}$, $A_{\phi(k-1)}(0) = A_{\phi(k-1)}(n_{\phi(k-1)} - 1) = 0$ and $0 \not\prec y$, we can find $n \in \mathbb{Z}_+$ such that

$$v = A_{\phi(k-1)}y[m_{k-1} + 9mn_{\phi(k-1)}; m_{k-1} + 9(m+1)n_{\phi(k-1)} - 1] \cdots \\ A_{\phi(k-1)}y[m_{k-1} + 9(m+n)n_{\phi(k-1)}; m_l + 9(m+n+1)n_{\phi(k-1)} - 1].$$

Put $u = y[m_{k-1} + 9mn_{\phi(k-1)}; m_{k-1} + 9(m+n+1)n_{\phi(k-1)} - 1]$. Since $v^6 = A_k[s_k; s_k + 6t - 1]$, one has $u^6 < y$ which contradicts $u^3 \not\prec y$. This finishes the proof of Claim 2.

Thus, (X, σ) has no periodic point by Claim 2. \square

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