

A THETA FUNCTION IDENTITY AND ITS IMPLICATIONS

ZHI-GUO LIU

ABSTRACT. In this paper we prove a general theta function identity with four parameters by employing the complex variable theory of elliptic functions. This identity plays a central role for the cubic theta function identities. We use this identity to re-derive some important identities of Hirschhorn, Garvan and Borwein about cubic theta functions. We also prove some other cubic theta function identities. A new representation for $\prod_{n=1}^{\infty}(1 - q^n)^{10}$ is given. The proofs are self-contained and elementary.

1. INTRODUCTION

Throughout this paper we will use q to denote $\exp(2\pi i\tau)$ with $\text{Im}(\tau) > 0$. We will use the familiar notation

$$(1.1) \quad (z; q)_{\infty} = \prod_{n=0}^{\infty} (1 - zq^n),$$

and sometimes write

$$(1.2) \quad (a, b, c, \dots; q)_{\infty} = (a; q)_{\infty} (b; q)_{\infty} (c; q)_{\infty} \dots.$$

It is easy to see that for any integer $m > 0$,

$$(1.3) \quad (z, zq, \dots, zq^{m-1}; q^m)_{\infty} = (z; q)_{\infty}.$$

If we define ω to be the primitive cube root of unity given by $\omega = \exp(\frac{2\pi i}{3})$, then, using the identity $(1 - x)(1 - x\omega)(1 - x\omega^2) = 1 - x^3$, we find that

$$(1.4) \quad (z, z\omega, z\omega^2; q)_{\infty} = (z^3; q^3)_{\infty}.$$

The well-known Jacobi triple product identity is

$$(1.5) \quad (q, z, q/z; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n$$

(see [1, pp. 21-22], [2, p. 35], [7], and [10]).

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Jacobi's theta function $\theta_1(z|\tau)$ is defined as

$$\begin{aligned} \theta_1(z|\tau) &= -iq^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{(2n+1)iz} \\ (1.6) \quad &= 2q^{1/8} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z \end{aligned}$$

(see, for example, [16, p. 463]).

Using the triple product identity, we can find the infinite product representation for $\theta_1(z|\tau)$, namely,

$$\begin{aligned} \theta_1(z|\tau) &= 2q^{1/8}(\sin z)(q, qe^{2iz}, qe^{-2iz}; q)_{\infty} \\ (1.7) \quad &= iq^{1/8}e^{-iz}(q, e^{2iz}, qe^{-2iz}; q)_{\infty} \end{aligned}$$

(see, for example, [16, p. 469]).

In [13], we use the complex variable theory of elliptic functions to establish a general theta function identity. We then derive some remarkable theta function identities related to the modular equations of degree 5; in particular, we give new proofs of the two fundamental identities satisfied by the Rogers-Ramanujan continued fraction. In this paper we set up the following general theta function identity with four parameters by employing the same method. This is a notable identity, which contains many interesting cubic theta function identities as special cases.

Theorem 1. *Suppose $f(z)$ is an entire function satisfying the functional equations*

$$(1.8) \quad f(z + \pi) = -f(z) \quad \text{and} \quad f(z + \pi\tau) = -q^{-3/2}e^{-6iz}f(z).$$

Then the following identity holds:

$$\begin{aligned} &\{f(z) - f(-z)\} \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau) \\ &= \{f(y) - f(-y)\} \theta_1(x|\tau)\theta_1(z|\tau)\theta_1(x+z|\tau)\theta_1(x-z|\tau) \\ (1.9) \quad &- \{f(x) - f(-x)\} \theta_1(y|\tau)\theta_1(z|\tau)\theta_1(y+z|\tau)\theta_1(y-z|\tau). \end{aligned}$$

The contents are organized as follows. In Section 2, we prove Theorem 1 using the classical theory of elliptic functions. In Section 3, the following theta function identity is first proved using Theorem 1.

Theorem 2. *Let $\theta_1(z|\tau)$ be the Jacobi theta function defined by (1.6). Then for any x, y , and z , we have*

$$\begin{aligned} &\theta_1(y|\frac{\tau}{3})\theta_1(x|\tau)\theta_1(z|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau) \\ &\quad - \theta_1(x|\frac{\tau}{3})\theta_1(y|\tau)\theta_1(z|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau) \\ (1.10) \quad &= \theta_1(z|\frac{\tau}{3})\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned}$$

Then we use Theorem 2 to derive a interesting theta function identity given in Theorem 5 and from which we establish a new representation for $(q; q)_{\infty}^{10}$. In Section 4, we derive some identities for the Hirschhorn-Garvan-Borwein two-variable cubic theta functions; our method is different from that of Hirschhorn, Garvan, and Borwein. In Section 5, we use Theorem 1 to prove the following remarkable theta function identities.

Theorem 3. Let $\theta_1(x|\tau)$ be the Jacobi theta function defined as in (1.6), and let $a(q)$ be the Ramanujan function (see [4]) defined by

$$(1.11) \quad a(q) := 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

Then, we have

$$(1.12) \quad \theta_1^3\left(x + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(x - \frac{\pi}{3}|\tau\right) - \theta_1^3(x|\tau) = 3a(q)\theta_1(3x|3\tau)$$

and

$$(1.13) \quad \theta_1^3(x|3\tau) - q^{1/2}e^{2ix}\theta_1^3(x + \pi\tau|3\tau) - q^{1/2}e^{-2ix}\theta_1^3(x - \pi\tau|3\tau) = a(q)\theta_1(x|\tau).$$

Equality (1.13) is [3, p.142, Entry 3]. In Section 6, we set up the following identities using Theorem 1.

Theorem 4. We have

$$(1.14) \quad \begin{aligned} & (q; q)_{\infty}^3 \theta_1^3(x|\tau)\theta_1(3y|3\tau) - (q; q)_{\infty}^3 \theta_1^3(y|\tau)\theta_1(3x|3\tau) \\ &= 3q^{1/4}(q^3; q^3)_{\infty}^3 \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau) \end{aligned}$$

and

$$(1.15) \quad \begin{aligned} & (q; q)_{\infty}^3 \theta_1^3(x|\tau)\theta_1\left(y\left|\frac{\tau}{3}\right.\right) - (q; q)_{\infty}^3 \theta_1^3(y|\tau)\theta_1\left(x\left|\frac{\tau}{3}\right.\right) \\ &= q^{-1/(12)}(q^{1/3}; q^{1/3})_{\infty}^3 \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau). \end{aligned}$$

2. THE PROOF OF THEOREM 1

To prove the theorem, we require the following Lemma 1. Lemma 1 is a fundamental theorem of elliptic functions and can be found in [6, p. 22]. Recently, in [9, 11, 12, 13, 14, 15], we have used Lemma 1 to set up many important theta function identities.

Lemma 1. *The sum of all the residues of an elliptic function vanishes in the period parallelogram.*

Proof. Suppose that $f(u)$ is the given function satisfying the functional equations (1.8). Then we consider the function

$$(2.1) \quad g(u) = \frac{\theta_1(2u|\tau)f(u)}{\theta_1(u|\tau)\theta_1(u - x|\tau)\theta_1(u + x|\tau)\theta_1(u - y|\tau)\theta_1(u + y|\tau)\theta_1(u - z|\tau)\theta_1(u + z|\tau)}.$$

Here we temporarily assume that $0 < x, y, z < \pi$ are three distinct parameters different from the zero points of $\theta_1(2u|\tau)f(u)$. Using the functional equations

$$(2.2) \quad \theta_1(z + \pi|\tau) = -\theta_1(z|\tau) \quad \text{and} \quad \theta_1(z + \pi\tau|\tau) = -q^{-1/2}e^{-2iz}\theta_1(z|\tau),$$

we can verify that $g(u + \pi) = g(u)$ and $g(u + \pi\tau) = g(u)$. Hence $g(u)$ is an elliptic function with periods π and $\pi\tau$. We readily find that $x, \pi - x, y, \pi - y, z, \pi - z$ are its only poles and all its poles are simple poles. In this paper we will use $\text{res}(g; \alpha)$ to denote the residue of g at α . Then Lemma 1 gives

$$(2.3) \quad \text{res}(g; x) + \text{res}(g; \pi - x) + \text{res}(g; y) + \text{res}(g; \pi - y) + \text{res}(g; z) + \text{res}(g; \pi - z) = 0.$$

In this paper, we will also use the prime to denote the partial derivative with respect to u . Now, we have the following elementary calculation:

$$\begin{aligned}
 \text{res}(g; x) &= \lim_{u \rightarrow x} (u - x)g(u) \\
 &= \lim_{u \rightarrow x} \frac{\theta_1(2u|\tau)f(u)}{\theta_1(u|\tau)\theta_1(u + x|\tau)\theta_1(u - y|\tau)\theta_1(u + y|\tau)\theta_1(u - z|\tau)\theta_1(u + z|\tau)} \\
 &\quad \times \lim_{u \rightarrow x} \frac{u - x}{\theta_1(u - x|\tau)} \\
 (2.4) \quad &= \frac{f(x)}{\theta_1'(0|\tau)\theta_1(x|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau)\theta_1(x - z|\tau)\theta_1(x + z|\tau)}.
 \end{aligned}$$

In the same way we can show that

$$(2.5) \quad \text{res}(g; \pi - x) = \frac{-f(-x)}{\theta_1'(0|\tau)\theta_1(x|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau)\theta_1(x - z|\tau)\theta_1(x + z|\tau)}.$$

Noting that $g(u)$ is symmetric in x, y , and z , we interchange x and y in (2.4) and (2.5) to obtain

$$(2.6) \quad \text{res}(g; y) = \frac{-f(y)}{\theta_1'(0|\tau)\theta_1(y|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau)\theta_1(y - z|\tau)\theta_1(y + z|\tau)}$$

and

$$(2.7) \quad \text{res}(g; \pi - y) = \frac{f(-y)}{\theta_1'(0|\tau)\theta_1(y|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau)\theta_1(y - z|\tau)\theta_1(y + z|\tau)}.$$

Similarly we find that

$$(2.8) \quad \text{res}(g; z) = \frac{f(z)}{\theta_1'(0|\tau)\theta_1(z|\tau)\theta_1(x - z|\tau)\theta_1(x + z|\tau)\theta_1(y - z|\tau)\theta_1(y + z|\tau)}$$

and

$$(2.9) \quad \text{res}(g; \pi - z) = \frac{-f(-z)}{\theta_1'(0|\tau)\theta_1(z|\tau)\theta_1(x - z|\tau)\theta_1(x + z|\tau)\theta_1(y - z|\tau)\theta_1(y + z|\tau)}.$$

Substituting (2.4)-(2.9) into (2.3), after some simplification, we obtain (2.1). By analytic continuation, we know (2.1) holds for all x, y , and z , and so this completes the proof of Theorem 1. □

3. A NEW REPRESENTATION FOR $(q; q)_\infty^{10}$

In this section we first give a proof of Theorem 2 using Theorem 1.

Proof. It is easy to verify that $\theta_1(x|\frac{\tau}{3})$ satisfies all the conditions of Theorem 1. By taking $f(x) = \theta_1(x|\frac{\tau}{3})$ in Theorem 1, we immediately obtain Theorem 2; and thus we complete the proof of Theorem 2. □

Using the identity $(z, z\omega, z\omega^2; q)_\infty = (z^3; q^3)_\infty$, where $\omega = \exp(\frac{2\pi i}{3})$, and the infinite product representation for $\theta_1(z|\tau)$, we readily find that

$$(3.1) \quad \theta_1(3z|3\tau) = -\frac{(q^3; q^3)_\infty}{(q; q)_\infty^3} \theta_1(z|\tau)\theta_1(z + \frac{\pi}{3}|\tau)\theta_1(z - \frac{\pi}{3}|\tau).$$

Appealing to the infinite product representation for $\theta_1(z|\tau)$, we can also find that

$$(3.2) \quad \theta_1(\frac{\pi}{3}|\tau) = \theta_1(\frac{2\pi}{3}|\tau) = \sqrt{3}q^{1/8}(q^3; q^3)_\infty.$$

Taking $z = \frac{\pi}{3}$ in Theorem 2 and then using (3.1) and (3.2) in the resulting equation, we obtain the following identity.

Theorem 5. *For any x, y we have the identity*

$$(3.3) \quad q^{1/(12)}(q; q)_{\infty}^2 \theta_1(x|\frac{\tau}{3})\theta_1(3y|3\tau) - q^{1/(12)}(q; q)_{\infty}^2 \theta_1(y|\frac{\tau}{3})\theta_1(3x|3\tau) = \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau).$$

Differentiating both sides of (1.6) with respect to z and then setting $z = 0$, we readily find that

$$(3.4) \quad \theta_1'(0|\tau) = q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2},$$

and by successive differentiations, we obtain

$$(3.5) \quad \theta_1'''(0|\tau) = -q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1)^3 q^{n(n+1)/2}.$$

Differentiation of (1.7) gives

$$(3.6) \quad \theta_1'(0|\tau) = 2q^{1/8}(q; q)_{\infty}^3.$$

Comparing (3.4) and (3.6), we obtain Jacobi's identity

$$(3.7) \quad (q; q)_{\infty}^3 = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2}.$$

In the next theorem we will give a new representation for $(q; q)_{\infty}^{10}$. An entirely different proof of this new representation has been given in the paper [5]. Using this identity, the authors have also given a short proof of Ramanujan's famous congruence $p(11n+6) \equiv 0 \pmod{11}$, where $p(n)$ denotes the number of unrestricted partitions of the positive integer n .

Theorem 6. *We have*

$$(3.8) \quad 32(q; q)_{\infty}^{10} = 9 \left(\sum_{n=-\infty}^{\infty} (-1)^n (2n + 1)^3 q^{3n(n+1)/2} \right) \left(\sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/6} \right) - \left(\sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{3n(n+1)/2} \right) \left(\sum_{n=-\infty}^{\infty} (-1)^n (2n + 1)^3 q^{n(n+1)/6} \right).$$

(3.9)

Proof. Dividing both sides of (3.3) by y and then allowing $y \rightarrow 0$, we obtain

$$(3.10) \quad 3q^{1/(12)}(q; q)_{\infty}^2 \theta_1'(0|3\tau)\theta_1(x|\frac{\tau}{3}) - q^{1/(12)}(q; q)_{\infty}^2 \theta_1'(0|\frac{\tau}{3})\theta_1(3x|3\tau) = \theta_1'(0|\tau)\theta_1^3(x|\tau).$$

Using (3.6) in this equation, we obtain the following interesting identity:

$$(3.11) \quad (q; q)_{\infty} \theta_1^3(x|\tau) = 3q^{1/3}(q^3; q^3)_{\infty}^3 \theta_1(x|\frac{\tau}{3}) - (q^{1/3}; q^{1/3})_{\infty}^3 \theta_1(3x|3\tau).$$

Differentiating both sides of the identity with respect to x three times, and then setting $x = 0$, we obtain

$$(3.12) \quad 6(q; q)_{\infty} \theta_1'(0|\tau)^3 = 3q^{1/3}(q^3; q^3)_{\infty}^3 \theta_1'''(0|\frac{\tau}{3}) - 27(q^{1/3}; q^{1/3})_{\infty}^3 \theta_1'''(0|3\tau).$$

Using (3.6) in the left side of this equation, we find that

$$(3.13) \quad 48q^{3/8}(q; q)_\infty^{10} = 3q^{1/3}(q^3; q^3)_\infty^3 \theta_1'''(0|\frac{\tau}{3}) - 27(q^{1/3}; q^{1/3})_\infty^3 \theta_1'''(0|3\tau).$$

Using (3.5) and (3.7) in the right side of the equation, we arrive at (3.9). This completes the proof of Theorem 6. \square

4. SOME IDENTITIES FOR THE HIRSCHHORN-GARVAN-BORWEIN TWO-VARIABLE CUBIC THETA FUNCTIONS

Noting the fact that $\theta_1(x - y|\tau) = -\theta_1(y - x|\tau)$, we can rewrite (3.3) as

$$(4.1) \quad \begin{aligned} & q^{1/(12)}(q; q)_\infty^2 \theta_1(y|\frac{\tau}{3}) \theta_1(3x|3\tau) - q^{1/(12)}(q; q)_\infty^2 \theta_1(x|\frac{\tau}{3}) \theta_1(3y|3\tau) \\ &= \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(y - x|\tau) \theta_1(x + y|\tau). \end{aligned}$$

Using the second equation of (1.7) in the above equation and then replacing e^{2ix} by x and e^{2iy} by y in the resulting equation, we obtain the following infinite product identity:

Theorem 7. *We have*

$$(4.2) \quad \begin{aligned} & (q; q)_\infty^2 (q^{1/3}, x, q^{1/3}/x; q^{1/3})_\infty (q^3, y^3, q^3/y^3; q^3)_\infty \\ & - yx^{-1}(q; q)_\infty^2 (q^{1/3}, y, q^{1/3}/y; q^{1/3})_\infty (q^3, x^3, q^3/x^3; q^3)_\infty \\ &= (q, x, q/x; q)_\infty (q, y, q/y; q)_\infty (q, y/x, qx/y; q)_\infty (q, xy, q/xy; q)_\infty. \end{aligned}$$

Appealing to the Jacobi triple product identity, we have

$$(4.3) \quad \begin{aligned} & (q; q)_\infty^2 (q^{1/3}, x, q^{1/3}/x; q^{1/3})_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n-1)/2} y^{3n} \\ & - yx^{-1}(q; q)_\infty^2 (q^3, x^3, q^3/x^3; q^3)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/6} y^n \\ &= (q, x, q/x; q)_\infty \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} y^n \right) \\ & \times \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} y^n x^{-n} \right) \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} y^n x^n \right). \end{aligned}$$

It is easy to see that the coefficient of y on the right side of the above equation is

$$(4.4) \quad -(q, x, q/x; q)_\infty \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2-m-n} x^{n-m}$$

and the coefficient of y on the left side is

$$(4.5) \quad -x^{-1}(q; q)_\infty^2 (q^3, x^3, q^3/x^3; q^3)_\infty.$$

By equating the above two quantities, we arrive at

$$(4.6) \quad \begin{aligned} & \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2-m-n} x^{n-m} \\ &= (q; q)_\infty (q^3; q^3)_\infty (1 + x + x^{-1}) \frac{(q^3 x^3, q^3/x^3; q^3)_\infty}{(qx, q/x; q)_\infty}. \end{aligned}$$

Making the changes of indices $m \rightarrow -m$ and $n \rightarrow -n$, we obtain

$$(4.7) \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n} = (q; q)_{\infty} (q^3; q^3)_{\infty} (1+x+x^{-1}) \frac{(q^3 x^3, q^3/x^3; q^3)_{\infty}}{(qx, q/x; q)_{\infty}}.$$

This is the same as in [8, p.675, Equation (1.23)].

Equating the terms independent of y in (4.3), we find that

$$(4.8) \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n} = q^{1/3} (q; q)_{\infty} (q^3; q^3)_{\infty} (1+x+x^{-1}) \frac{(q^3 x^3, q^3/x^3; q^3)_{\infty}}{(qx, q/x; q)_{\infty}} + (q; q)_{\infty} (q^{1/3}; q^{1/3})_{\infty} \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_{\infty}}{(qx, q/x; q)_{\infty}}.$$

Combining the above two equations, we find that

$$(4.9) \quad (q; q)_{\infty} (q^{1/3}; q^{1/3})_{\infty} \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_{\infty}}{(qx, q/x; q)_{\infty}} = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n} - q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n}.$$

Let ω be the primitive cube root of unity given by $\omega = \exp(\frac{2\pi i}{3})$. Then we readily find that the right side of (4.9) is equal to

$$(4.10) \quad \sum_{m,n=-\infty}^{\infty} q^{(m^2+mn+n^2)/3} x^n \omega^{m-n}$$

(see [8, p. 678] for the details). Therefore, we have

$$(4.11) \quad \sum_{m,n=-\infty}^{\infty} q^{(m^2+mn+n^2)/3} x^n \omega^{m-n} = (q; q)_{\infty} (q^{1/3}; q^{1/3})_{\infty} \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_{\infty}}{(qx, q/x; q)_{\infty}}.$$

Writing q as q^3 , we obtain

$$(4.12) \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^n \omega^{m-n} = (q; q)_{\infty} (q^3; q^3)_{\infty} \frac{(qx, qx^{-1}; q)_{\infty}}{(q^3 x, q^3/x; q^3)_{\infty}}.$$

This is the same as in [8, p. 675, equation (1.22)].

If we denote

$$(4.13) \quad a(q, x) = \sum_{m, n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n},$$

$$(4.14) \quad b(q, x) = \sum_{m, n=-\infty}^{\infty} q^{m^2+mn+n^2} x^n \omega^{m-n},$$

$$(4.15) \quad c(q, x) = q^{1/3} \sum_{m, n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n},$$

then (4.8) can be written as

$$(4.16) \quad a(q, x) = c(q, x) + b(q^{1/3}, x),$$

from which one can easily derive the following important identity of Hirschhorn, Garvan, and Borwein (see [8, p.682] for the details).

Theorem 8. *We have*

$$(4.17) \quad a^3(q, x) = c^3(q, x) + b^2(q, 1)b(q, x^3).$$

5. THE PROOF OF THEOREM 3

Using the identity $(z, zq, zq^2; q^3)_{\infty} = (z; q)_{\infty}$ and the infinite product representation for $\theta_1(z|\tau)$, we readily find that

$$(5.1) \quad \theta_1\left(z\left|\frac{\tau}{3}\right.\right) = \frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}^3} \theta_1(z|\tau) \theta_1\left(z + \frac{\pi\tau}{3}|\tau\right) \theta_1\left(z - \frac{\pi\tau}{3}|\tau\right),$$

and we can also find that

$$(5.2) \quad \theta_1\left(\frac{\pi\tau}{3}|\tau\right) = iq^{-1/(24)}(q^{1/3}; q^{1/3})_{\infty} \quad \text{and} \quad \theta_1\left(\frac{2\pi\tau}{3}|\tau\right) = iq^{-5/(24)}(q^{1/3}; q^{1/3})_{\infty}.$$

We recall the definition of the Ramanujan function

$$(5.3) \quad a(q) := 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

Using logarithmic differentiation on (1.7), we obtain

$$(5.4) \quad \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} = -i - 2i \sum_{n=0}^{\infty} \frac{q^n e^{2iz}}{1 - q^n e^{2iz}} + 2i \sum_{n=1}^{\infty} \frac{q^n e^{-2iz}}{1 - q^n e^{-2iz}}.$$

Comparing the above two equations, we infer that

$$(5.5) \quad a(q) = -2 + 3i \frac{\theta_1'}{\theta_1}(\pi\tau|3\tau).$$

It is well-known that the trigonometric series expansion for the logarithmic derivative of $\theta_1(z|\tau)$ is

$$(5.6) \quad \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin 2nz$$

(see [16, p.489]).

Thus, we have

$$(5.7) \quad \frac{\theta'_1}{\theta_1}\left(\frac{\pi}{3}|\tau\right) = \frac{1}{\sqrt{3}}a(q).$$

Now we are ready to prove Theorem 3.

Proof. Differentiating both sides of (1.9) with respect to z and then setting $z = 0$, we obtain

$$(5.8) \quad \begin{aligned} & \theta'_1(0|\tau) \{f(y) - f(-y)\} \theta_1^3(x|\tau) - \theta'_1(0|\tau) \{f(x) - f(-x)\} \theta_1^3(y|\tau) \\ & = 2f'(0)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned}$$

Inserting $\theta'_1(0|\tau) = 2q^{1/8}(q; q)_\infty^3$ in the above equation, we have

$$(5.9) \quad \begin{aligned} & q^{1/8}(q; q)_\infty^3 \{f(y) - f(-y)\} \theta_1^3(x|\tau) - q^{1/8}(q; q)_\infty^3 \{f(x) - f(-x)\} \theta_1^3(y|\tau) \\ & = f'(0)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned}$$

It is easy to verify that $\theta_1^3(x + \frac{\pi}{3}|\tau)$ satisfies (1.8), so we can take $f(x) = \theta_1^3(x + \frac{\pi}{3}|\tau)$ in (5.9). By (3.2) and (5.7) and a direct computation, we have

$$(5.10) \quad f'(0) = 3\theta_1^3\left(\frac{\pi}{3}|\tau\right) \left(\frac{\theta'_1}{\theta_1}\right)\left(\frac{\pi}{3}|\tau\right) = 9q^{3/8}(q^3; q^3)_\infty^3 a(q);$$

and thus we have

$$(5.11) \quad \begin{aligned} & q^{1/8}(q; q)_\infty^3 \left\{ \theta_1^3\left(y + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(y - \frac{\pi}{3}|\tau\right) \right\} \theta_1^3(x|\tau) \\ & - q^{1/8}(q; q)_\infty^3 \left\{ \theta_1^3\left(x + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(x - \frac{\pi}{3}|\tau\right) \right\} \theta_1^3(y|\tau) \\ & = 9q^{3/8}(q^3; q^3)_\infty^3 a(q)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned}$$

Taking $y = \frac{\pi}{3}$ in the above equation and noting that $\theta_1(\frac{\pi}{3}|\tau) = \theta_1(\frac{2\pi}{3}|\tau) = \sqrt{3}q^{1/8}(q^3; q^3)_\infty$, we find that

$$(5.12) \quad \begin{aligned} & \theta_1^3(x|\tau) - \theta_1^3\left(x + \frac{\pi}{3}|\tau\right) - \theta_1^3\left(x - \frac{\pi}{3}|\tau\right) \\ & = 3a(q)\frac{(q^3; q^3)_\infty}{(q; q)_\infty^3}\theta_1(x|\tau)\theta_1\left(x + \frac{\pi}{3}|\tau\right)\theta_1\left(x - \frac{\pi}{3}|\tau\right). \end{aligned}$$

From (3.1), we have

$$(5.13) \quad \theta_1(3x|3\tau) = -\frac{(q^3; q^3)_\infty}{(q; q)_\infty^3}\theta_1(x|\tau)\theta_1\left(x + \frac{\pi}{3}|\tau\right)\theta_1\left(x - \frac{\pi}{3}|\tau\right).$$

We substitute the above equation into the right side of (5.12) to obtain (1.12).

Proceeding through the same steps as before, by taking $f(x) = e^{2ix}\theta_1^3(x + \frac{\pi\tau}{3}|\tau)$ in (5.9) and then setting $y = \frac{\pi\tau}{3}$ and appealing to (5.1), (5.2), and (5.5), we can find that

$$(5.14) \quad \theta_1^3(x|\tau) - q^{1/6}e^{2ix}\theta_1^3\left(x + \frac{\pi\tau}{3}|\tau\right) - q^{1/6}e^{-2ix}\theta_1^3\left(x - \frac{\pi\tau}{3}|\tau\right) = a(q^{1/3})\theta_1\left(x|\frac{\tau}{3}\right).$$

Replacing q by q^3 , we obtain (1.13). Thus, we complete the proof of Theorem 3. \square

6. THE PROOF OF THEOREM 4

Proof. Taking $z = \frac{\pi}{3}$ in Theorem 1 and appealing to (3.1) and (3.2), we obtain

$$\begin{aligned}
 & \sqrt{3}q^{1/8}(q; q)_{\infty}^3 \{f(x) - f(-x)\} \theta_1(3y|3\tau) \\
 &= \sqrt{3}q^{1/8}(q; q)_{\infty}^3 \{f(y) - f(-y)\} \theta_1(3x|3\tau) \\
 (6.1) \quad &+ \left\{ f\left(\frac{\pi}{3}\right) - f\left(-\frac{\pi}{3}\right) \right\} \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau).
 \end{aligned}$$

It is easy to see that $\theta_1^3(x|\tau)$ satisfies all the conditions of Theorem 1. We choose $f(x) = \theta_1^3(x|\tau)$ and then use (3.2) in the resulting equation to obtain (1.14).

Taking $z = \frac{\pi\tau}{3}$ in Theorem 1 and appealing to (5.1) and (5.2), we obtain

$$\begin{aligned}
 & iq^{-1/(24)}(q; q)_{\infty}^3 \{f(x) - f(-x)\} \theta_1\left(y\left|\frac{\tau}{3}\right.\right) \\
 &= iq^{-1/(24)}(q; q)_{\infty}^3 \{f(y) - f(-y)\} \theta_1\left(x\left|\frac{\tau}{3}\right.\right) \\
 (6.2) \quad &- \left\{ f\left(\frac{\pi\tau}{3}\right) - f\left(-\frac{\pi\tau}{3}\right) \right\} \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau).
 \end{aligned}$$

Choosing $f(x) = \theta_1^3(x|\tau)$ and then using (5.2) in the resulting equation, we obtain (1.15). This completes the proof of Theorem 4. \square

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DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200062, PEOPLE'S REPUBLIC OF CHINA

E-mail address: `zgliu@math.ecnu.edu.cn`, `liuzg18@hotmail.com`