A THETA FUNCTION IDENTITY AND ITS IMPLICATIONS

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Abstract. In this paper we prove a general theta function identity with four parameters by employing the complex variable theory of elliptic functions. This identity plays a central role for the cubic theta function identities. We use this identity to re-derive some important identities of Hirschhorn, Garvan and Borwein about cubic theta functions. We also prove some other cubic theta function identities. A new representation for $Q_{1}(n)$ is given.

The proofs are self-contained and elementary.

1. Introduction

Throughout this paper we will use $q$ to denote $\exp(2\pi i \tau)$ with $\text{Im}(\tau) > 0$. We will use the familiar notation

$$ (z; q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n), $$

and sometimes write

$$ (a, b, c, \cdots ; q)_\infty = (a; q)_\infty (b; q)_\infty (c; q)_\infty \cdots. $$

It is easy to see that for any integer $m > 0$,

$$ (z, z^3, \cdots, z^m ; q_\infty) = (z; q)_\infty. $$

If we define $\omega$ to be the primitive cube root of unity given by $\omega = \exp\left(\frac{2\pi i}{3}\right)$, then, using the identity $(1 - x)(1 - x\omega)(1 - x\omega^2) = 1 - x^3$, we find that

$$ (z, z\omega, z\omega^2 ; q)_\infty = (z^3 ; q^3)_\infty. $$

The well-known Jacobi triple product identity is

$$ (q, z, q/z ; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^n (n-1)/2 z^n $$

(see [1] pp. 21-22, [2] p. 35, [7], and [10]).
Jacobi’s theta function $\theta_1(z|\tau)$ is defined as

$$
\theta_1(z|\tau) = -iq^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{(2n+1)iz}
$$

(1.6)

and

$$
\theta_1(z|\tau) = 2q^{1/8} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z
$$

(see, for example, [16, p. 463]).

Using the triple product identity, we can find the infinite product representation for $\theta_1(z|\tau)$, namely,

$$
\theta_1(z|\tau) = 2q^{1/8} (\sin z)(q, q e^{2iz}, q e^{-2iz}; q)_{\infty}
$$

(1.7)

(see, for example, [16, p. 469]).

In [13], we use the complex variable theory of elliptic functions to establish a general theta function identity. We then derive some remarkable theta function identities related to the modular equations of degree 5; in particular, we give new proofs of the two fundamental identities satisfied by the Rogers-Ramanujan continued fraction. In this paper we set up the following general theta function identity with four parameters by employing the same method. This is a notable identity, which contains many interesting cubic theta function identities as special cases.

**Theorem 1.** Suppose $f(z)$ is an entire function satisfying the functional equations

$$
f(z + \pi) = -f(z) \quad \text{and} \quad f(z + \pi \tau) = -q^{-3/2} e^{-6iz} f(z).
$$

(1.8)

Then the following identity holds:

$$
\{ f(z) - f(-z) \} \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x + y|\tau) \theta_1(x - y|\tau)
$$

$$
= \{ f(y) - f(-y) \} \theta_1(x|\tau) \theta_1(z|\tau) \theta_1(x + z|\tau) \theta_1(x - z|\tau)
$$

(1.9)

$$
- \{ f(x) - f(-x) \} \theta_1(y|\tau) \theta_1(z|\tau) \theta_1(y + z|\tau) \theta_1(y - z|\tau).
$$

The contents are organized as follows. In Section 2, we prove Theorem 1 using the classical theory of elliptic functions. In Section 3, the following theta function identity is first proved using Theorem 1.

**Theorem 2.** Let $\theta_1(z|\tau)$ be the Jacobi theta function defined by [14]. Then for any $x, y, z$, we have

$$
\theta_1(y|\tau/3) \theta_1(x|\tau) \theta_1(z|\tau) \theta_1(x - z|\tau) \theta_1(x + z|\tau)
$$

$$
- \theta_1(x|\tau/3) \theta_1(y|\tau) \theta_1(z|\tau) \theta_1(y - z|\tau) \theta_1(y + z|\tau)
$$

(1.10)

$$
\theta_1(z|\tau/3) \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x - y|\tau) \theta_1(x + y|\tau).
$$

Then we use Theorem 2 to derive a interesting theta function identity given in Theorem 3 and from which we establish a new representation for $(q; q)_\infty^{10}$. In Section 4, we derive some identities for the Hirschhorn-Garvan-Borwein two-variable cubic theta functions; our method is different from that of Hirschhorn, Garvan, and Borwein. In Section 5, we use Theorem 4 to prove the following remarkable theta function identities.
Theorem 3. Let \( \theta_1(x|\tau) \) be the Jacobi theta function defined as in (1.10), and let \( a(q) \) be the Ramanujan function (see [3]) defined by

\[
a(q) := 1 + 6 \sum_{n=0}^{\infty} \left( \frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}} \right).
\]

Then, we have

\[
\theta_1^3(x + \frac{\pi}{3}|\tau) + \theta_1^3(x - \frac{\pi}{3}|\tau) - \theta_1^3(x|\tau) = 3a(q)\theta_1(3x|3\tau)
\]

and

\[
\theta_1^3(x|3\tau) - q^{1/2}e^{2ix}\theta_1^3(x + \pi|x|3\tau) - q^{1/2}e^{-2ix}\theta_1^3(x - \pi|x|3\tau) = a(q)\theta_1(x|\tau).
\]

Equality (1.13) is [3, p.142, Entry 3]. In Section 6, we set up the following identities using Theorem 1.

Theorem 4. We have

\[
(q; q)^3_\infty \theta_1^3(x|\tau)\theta_1(3y|3\tau) - (q; q)^3_\infty \theta_1^3(y|\tau)\theta_1(3x|3\tau)
\]

\[
= 3q^{1/4}(q^3; q^3)^3_\infty \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x + y|\tau)
\]

and

\[
(q; q)^3_\infty \theta_1^3(x|\tau)\theta_\tau(y|\tau) - (q; q)^3_\infty \theta_1^3(y|\tau)\theta_1(x|\tau)
\]

\[
= q^{-1/12}(q^{1/3}; q^{1/3})^3_\infty \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau).
\]

2. The proof of Theorem 1

To prove the theorem, we require the following Lemma 1. Lemma 1 is a fundamental theorem of elliptic functions and can be found in [3, p. 22]. Recently, in [3] [11] [12] [13] [14] [15], we have used Lemma 1 to set up many important theta function identities.

Lemma 1. The sum of all the residues of an elliptic function vanishes in the period parallelogram.

Proof. Suppose that \( f(u) \) is the given function satisfying the functional equations (1.8). Then we consider the function

\[
g(u) = \frac{\theta_1(2u|\tau)f(u)}{\theta_1(u|\tau)\theta_1(u - x|\tau)\theta_1(u + x|\tau)\theta_1(u - y|\tau)\theta_1(u + y|\tau)\theta_1(u - z|\tau)\theta_1(u + z|\tau)}.
\]

Here we temporarily assume that \( 0 < x, y, z < \pi \) are three distinct parameters different from the zero points of \( \theta_1(2u|\tau)f(u) \). Using the functional equations

\[
\theta_1(z + \pi|\tau) = -\theta_1(z|\tau) \quad \text{and} \quad \theta_1(z + \pi|\tau) = -q^{-1/2}e^{-2iz}\theta_1(z|\tau),
\]

we can verify that \( g(u + \pi) = g(u) \) and \( g(u + \pi|x|\tau) = g(u) \). Hence \( g(u) \) is an elliptic function with periods \( \pi \) and \( x \). We readily find that \( x, \pi - x, y, \pi - y, z, \pi - z \) are its only poles and all its poles are simple poles. In this paper we will use \( \text{res}(g; \alpha) \) to denote the residue of \( g \) at \( \alpha \). Then Lemma 1 gives

\[
\text{res}(g; x) + \text{res}(g; \pi - x) + \text{res}(g; y) + \text{res}(g; \pi - y) + \text{res}(g; z) + \text{res}(g; \pi - z) = 0.
\]
In this paper, we will also use the prime to denote the partial derivative with respect to \( u \). Now, we have the following elementary calculation:

\[
\begin{align*}
\res(g; x) &= \lim_{u \to x} (u - x)g(u) \\
&= \lim_{u \to x} \theta_1(2u|\tau)f(u) \\
&= \frac{\theta_1(2u|\tau)f(u)}{\theta_1(u|\tau)\theta_1(u + x|\tau)\theta_1(u - y|\tau)\theta_1(u + y|\tau)\theta_1(u - z|\tau)\theta_1(u + z|\tau)} \\
&\quad \times \lim_{u \to x} \frac{u - x}{\theta_1(u - x|\tau)} \\
&= \frac{f(x)}{\theta_1'(0|\tau)\theta_1(x|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau)\theta_1(x - z|\tau)\theta_1(x + z|\tau)}.
\end{align*}
\]

(2.4)

In the same way we can show that

\[
\res(\pi - x) = \frac{-f(-x)}{\theta_1'(0|\tau)\theta_1(x|\tau)\theta_1(x - y|\tau)\theta_1(x + y|\tau)\theta_1(x - z|\tau)\theta_1(x + z|\tau)}.
\]

(2.5)

Noting that \( g(u) \) is symmetric in \( x, y, \) and \( z \), we interchange \( x \) and \( y \) in (2.4) and (2.5) to obtain

\[
\begin{align*}
\res(g; y) &= \frac{-f(y)}{\theta_1'(0|\tau)\theta_1(y|\tau)\theta_1(y - x|\tau)\theta_1(y + x|\tau)\theta_1(y - z|\tau)\theta_1(y + z|\tau)} \\
\res(g; y) &= \frac{f(-y)}{\theta_1'(0|\tau)\theta_1(y|\tau)\theta_1(y - x|\tau)\theta_1(y + x|\tau)\theta_1(y - z|\tau)\theta_1(y + z|\tau)}.
\end{align*}
\]

(2.6) and (2.7)

Similarly we find that

\[
\begin{align*}
\res(g; z) &= \frac{f(z)}{\theta_1'(0|\tau)\theta_1(z|\tau)\theta_1(z - x|\tau)\theta_1(z + x|\tau)\theta_1(z - y|\tau)\theta_1(z + y|\tau)} \\
\res(g; z) &= \frac{-f(-z)}{\theta_1'(0|\tau)\theta_1(z|\tau)\theta_1(z - x|\tau)\theta_1(z + x|\tau)\theta_1(z - y|\tau)\theta_1(z + y|\tau)}.
\end{align*}
\]

(2.8) and (2.9)

Substituting (2.4) and (2.5) into (2.3), after some simplification, we obtain (2.1). By analytic continuation, we know (2.1) holds for all \( x, y, \) and \( z \), and so this completes the proof of Theorem 1.

\section{A new representation for \((q;q)_\infty^{10}\)}

In this section we first give a proof of Theorem 2 using Theorem 1.

\textbf{Proof.} It is easy to verify that \( \theta_1(x|\frac{\pi}{3}) \) satisfies all the conditions of Theorem 2. By taking \( f(x) = \theta_1(x|\frac{\pi}{3}) \) in Theorem 1, we immediately obtain Theorem 2 and thus we complete the proof of Theorem 2.

Using the identity \((z, z\omega, z\omega^2; q)_\infty = (z^3; q)_\infty^3\), where \( \omega = \exp(\frac{2\pi i}{3}) \), and the infinite product representation for \( \theta_1(z|\tau) \), we readily find that

\[
\theta_1(3z|3\tau) = \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty^3}\theta_1(z|\tau)\theta_1(z + \frac{\pi}{3}|\tau)\theta_1(z - \frac{\pi}{3}|\tau).
\]

(3.1)

Appealing to the infinite product representation for \( \theta_1(z|\tau) \), we can also find that

\[
\theta_1\left(\frac{\pi}{3}|\tau\right) = \theta_1\left(\frac{2\pi}{3}|\tau\right) = \sqrt{3}q^{1/8}(q^3; q)_\infty^3.
\]

(3.2)
Taking \( z = \frac{x}{3} \) in Theorem 2 and then using (3.1) and (3.2) in the resulting equation, we obtain the following identity.

**Theorem 5.** For any \( x, y \) we have the identity

\[
q^{1/12}(q; q)^2 \theta_1(x|3\tau) + q^{1/12}(q; q)^2 \theta_1(y|x\tau) - q^{1/12}(q; q)^2 \theta_1(y|x\tau) - q^{1/12}(q; q)^2 \theta_1(x|3\tau)
\]

(3.3) \[ \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x - y|\tau) \theta_1(x + y|\tau) \]

Differentiating both sides of (1.6) with respect to \( y \) and then setting \( z = 0 \), we readily find that

\[
\theta_1'(0|\tau) = q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2},
\]

(3.4) and by successive differentiations, we obtain

\[
\theta_1''(0|\tau) = -q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1)^3 q^{n(n+1)/2}.
\]

(3.5) Differentiation of (1.7) gives

\[
\theta_1'(0|\tau) = 2q^{1/8}(q; q)^3.
\]

Comparing (3.4) and (3.5), we obtain Jacobi’s identity

\[
(q; q)^3 = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2}.
\]

(3.7)

In the next theorem we will give a new representation for \((q; q)_{10}^{10} \). An entirely different proof of this new representation has been given in the paper [5]. Using this identity, the authors have also given a short proof of Ramanujan’s famous congruence \( p(11n + 6) \equiv 0 \text{ (mod 11)} \), where \( p(n) \) denotes the number of unrestricted partitions of the positive integer \( n \).

**Theorem 6.** We have

\[
32(q; q)_{10}^{10} = 9 \left( \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1)^3 q^{3n(n+1)/2} \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/6} \right)
\]

(3.8) \[ \left( \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{3n(n+1)/2} \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/6} \right). \]

(3.9)

**Proof.** Dividing both sides of (3.3) by \( y \) and then allowing \( y \to 0 \), we obtain

\[
3q^{1/12}(q; q)^2 \theta_1'(0|3\tau) \theta_1(x|\tau) - q^{1/12}(q; q)^2 \theta_1'(0|3\tau) \theta_1(3x|\tau) = \theta_1'(0|\tau) \theta_1^3(x|\tau).
\]

(3.10)

Using (3.11) in this equation, we obtain the following interesting identity:

\[
(q; q)_\infty \theta_1^3(x|\tau) = 3q^{1/3}(q; q)_\infty \theta_1(x|\tau) - (q^{1/3}; q^{1/3})_\infty \theta_1(3x|3\tau).
\]

(3.11)

Differentiating both sides of the identity with respect to \( x \) three times, and then setting \( x = 0 \), we obtain

\[
6(q; q)_\infty \theta_1'(0|\tau)^3 = 3q^{1/3}(q; q)_\infty \theta_1'(0|\tau)^3 - 27(q^{1/3}; q^{1/3})_\infty \theta_1''(0|3\tau).
\]

(3.12)
Using (3.6) in the left side of this equation, we find that

\begin{equation}
48q^{3/8}(q; q)^{10}_\infty = 3q^{1/3}(q^3; q^3)^3_\infty \theta''_1(0) \frac{T}{3} - 27(q^{1/3}; q^{1/3})^3_\infty \theta''(0)_{3\tau}.
\end{equation}

Using (3.5) and (3.7) in the right side of the equation, we arrive at (3.3). This completes the proof of Theorem 6. \qed

4. Some identities for the Hirschhorn-Garvan-Borwein two-variable cubic theta functions

Noting the fact that \( \theta_1(x - y|\tau) = -\theta_1(y - x|\tau) \), we can rewrite (3.3) as

\begin{equation}
q^{1/(12)}(q; q)^2_\infty \theta_1(y|\frac{T}{3}) \theta_1(3x|3\tau) - q^{1/(12)}(q; q)^2_\infty \theta_1(x|\frac{T}{3}) \theta_1(3y|3\tau)
\end{equation}

(4.1) = \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(y - x|\tau) \theta_1(x + y|\tau).

Using the second equation of (1.7) in the above equation and then replacing \( e^{2ix} \) by \( x \) and \( e^{2iy} \) by \( y \) in the resulting equation, we obtain the following infinite product identity:

**Theorem 7.** We have

\begin{equation}
(q; q)^2_\infty (q^{1/3}; q^{1/3})_\infty (q^3; q^3; q^3)_\infty
- yx^{-1}(q; q)^2_\infty (q^{1/3}; q^{1/3}; y; q^{1/3})_\infty (q^3; q^3; q^3)_\infty
\end{equation}

(4.2) = (q, x, q/x; q)_\infty (q, y, q/y; q)_\infty (q, y/x, qx/y; q)_\infty (q, xy, q/xy; q)_\infty.

Appealing to the Jacobi triple product identity, we have

\begin{equation}
(q; q)^2_\infty (q^{1/3}; q^{1/3}; q^{1/3})_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n-1)/2} y^n
- yx^{-1}(q; q)^2_\infty (q^3; q^3; q^3)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n-1)/2} y^n
\end{equation}

(4.3) = (q, x, q/x; q)_\infty \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n-1)/2} y^n \right)

\times \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n-1)/2} y^n \right).

It is easy to see that the coefficient of \( y \) on the right side of the above equation is

\begin{equation}
-(q, x, q/x; q)_\infty \sum_{m, n=-\infty}^{\infty} q^{m^2 + mn + n^2 - m - n} x^n y^n
\end{equation}

and the coefficient of \( y \) on the left side is

\begin{equation}
-x^{-1}(q; q)^2_\infty (q^3; q^3; q^3)_\infty.
\end{equation}

By equating the above two quantities, we arrive at

\begin{equation}
\sum_{m, n=-\infty}^{\infty} q^{m^2 + mn + n^2 - m - n} x^n y^n
= (q; q)_\infty (q^3; q^3)_\infty (1 + x + x^{-1}) \frac{(q^3 x^3; q^3; q^3)_\infty}{(qx, q/x; q)_\infty}.
\end{equation}
Making the changes of indices \(m \to -m\) and \(n \to -n\), we obtain

\[
\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n} = (q; q)_\infty (q^3; q^3)_\infty (1 + x + x^{-1}) \frac{(q^3 x^3, q^3/x^3; q^3)_\infty}{(qx, qx/x; q)_\infty}.
\]

(4.7)

This is the same as in [8, p. 675, Equation (1.23)].

Equating the terms independent of \(y\) in (4.3), we find that

\[
\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n} = q^{1/3} (q; q)_\infty (q^3; q^3)_\infty (1 + x + x^{-1}) \frac{(q^3 x^3, q^3/x^3; q^3)_\infty}{(qx, qx/x; q)_\infty} + (q; q)_\infty (q^{1/3}; q^{1/3})_\infty \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_\infty}{(qx, qx/x; q)_\infty}.
\]

(4.8)

Combining the above two equations, we find that

\[
(q; q)_\infty (q^{1/3}; q^{1/3})_\infty \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_\infty}{(qx, qx/x; q)_\infty} = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n} - q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n}.
\]

(4.9)

Let \(\omega\) be the primitive cube root of unity given by \(\omega = \exp(\frac{2\pi i}{3})\). Then we readily find that the right side of (4.10) is equal to

\[
\sum_{m,n=-\infty}^{\infty} q^{(m^2+mn+n^2)/3} x^{n} \omega^{m-n}
\]

(see [8, p. 678] for the details). Therefore, we have

\[
\sum_{m,n=-\infty}^{\infty} q^{(m^2+mn+n^2)/3} x^{n} \omega^{m-n} = (q; q)_\infty (q^{1/3}; q^{1/3})_\infty \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_\infty}{(qx, qx/x; q)_\infty}.
\]

(4.11)

Writing \(q\) as \(q^3\), we obtain

\[
\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{n} \omega^{m-n} = (q; q)_\infty (q^3; q^3)_\infty \frac{(qx, qx^{-1}; q)_\infty}{(q^3 x, q^3/x; q^3)_\infty}.
\]

(4.12)

This is the same as in [8, p. 675, equation (1.22)].
If we denote
\begin{equation}
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\end{equation}
\begin{align}
a(q, x) &= \sum_{m, n = -\infty}^{\infty} q^{m^2 + mn + n^2} x^{m-n}, \\
b(q, x) &= \sum_{m, n = -\infty}^{\infty} q^{m^2 + mn + n^2} x^{n} \omega^{m-n}, \\
c(q, x) &= q^{1/3} \sum_{m, n = -\infty}^{\infty} q^{m^2 + mn + m + n} x^{m-n},
\end{align}
then (4.8) can be written as
\begin{equation}
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\end{equation}
\begin{equation}
a(q, x) = c(q, x) + b(q, x),
\end{equation}
from which one can easily derive the following important identity of Hirschhorn, Garvan, and Borwein (see [8, p.682] for the details).

**Theorem 8.** We have
\begin{equation}
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\end{equation}
\begin{equation}
a^3(q, x) = c^3(q, x) + b^2(q, 1)b(q, x^3).
\end{equation}

5. The proof of Theorem 8

Using the identity \((z, zq, zq^2; q^3)_{\infty} = (z; q)_{\infty}\) and the infinite product representation for \(\theta_1(z|\tau)\), we readily find that
\begin{equation}
\theta_1(z|\tau) = \frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}^3} \theta_1(z|\tau) \theta_1(z + \frac{\pi}{3}|\tau) \theta_1(z - \frac{\pi}{3}|\tau),
\end{equation}
and we can also find that
\begin{equation}
\theta_1(\frac{2\pi}{3}|\tau) = i q^{-5/(24)}(q^{1/3}; q^{1/3})_{\infty} \quad \text{and} \quad \theta_1(\frac{5\pi}{3}|\tau) = i q^{5/(24)}(q^{1/3}; q^{1/3})_{\infty}.
\end{equation}
We recall the definition of the Ramanujan function
\begin{equation}
a(q) := 1 + 6 \sum_{n=0}^{\infty} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).
\end{equation}
Using logarithmic differentiation on (1.7), we obtain
\begin{equation}
\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} = -i - 2i \sum_{n=0}^{\infty} \frac{q^n e^{2iz}}{1 - q^n e^{2iz}} + 2i \sum_{n=1}^{\infty} \frac{q^n e^{-2iz}}{1 - q^n e^{-2iz}}.
\end{equation}
Comparing the above two equations, we infer that
\begin{equation}
a(q) = -2 + 3i \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)}.
\end{equation}
It is well-known that the trigonometric series expansion for the logarithmic derivative of \(\theta_1(z|\tau)\) is
\begin{equation}
\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin 2nz
\end{equation}
(see [16] p.489).
Thus, we have

\begin{equation}
\frac{\theta_1'(x)}{\theta_1(x)} = \frac{\pi}{3} a(q).
\end{equation}

Now we are ready to prove Theorem 3.

**Proof.** Differentiating both sides of (1.9) with respect to $z$ and then setting $z = 0$, we obtain

\begin{equation}
\theta_1'(0) \{ f(y) - f(-y) \} \theta_1^3(x|\tau) - \theta_1'(0) \{ f(x) - f(-x) \} \theta_1^3(y|\tau) = 2f'(0)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau) \theta_1(x+y|\tau).
\end{equation}

Inserting $\theta_1'(0) = 2q^{1/8}(q; q)_\infty^3$ in the above equation, we have

\begin{equation}
q^{1/8}(q; q)_\infty^3 \{ f(y) - f(-y) \} \theta_1^3(x|\tau) - q^{1/8}(q; q)_\infty^3 \{ f(x) - f(-x) \} \theta_1^3(y|\tau) = f'(0)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau) \theta_1(x+y|\tau).
\end{equation}

It is easy to verify that $\theta_1'(x+\frac{\pi}{3}|\tau)$ satisfies (1.8), so we can take $f(x) = \theta_1^3(x+\frac{\pi}{3}|\tau)$ in (5.9). By (3.2) and (5.7) and a direct computation, we have

\begin{equation}
f'(0) = 3\theta_1^3(0|\tau) \left( \frac{\theta_1'}{\theta_1} \right) \left( \frac{\pi}{3} \right) = 9q^{3/8}(q^3; q^3)_\infty^3 a(q);
\end{equation}

and thus we have

\begin{align}
q^{1/8}(q; q)_\infty^3 \left\{ \theta_1^3(y + \frac{\pi}{3}|\tau) + \theta_1^3(y - \frac{\pi}{3}|\tau) \right\} \theta_1^3(x|\tau) - q^{1/8}(q; q)_\infty^3 \left\{ \theta_1^3(x + \frac{\pi}{3}|\tau) + \theta_1^3(x - \frac{\pi}{3}|\tau) \right\} \theta_1^3(y|\tau) = 9q^{3/8}(q^3; q^3)_\infty^3 a(q)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau) \theta_1(x+y|\tau).
\end{align}

Taking $y = \frac{\pi}{3}$ in the above equation and noting that $\theta_1(\frac{\pi}{3}|\tau) = \theta_1(\frac{5\pi}{3}|\tau) = \sqrt{3}q^{1/8}(q^3; q^3)_\infty^3$, we find that

\begin{equation}
\theta_1^3(x|\tau) - \theta_1^3(x + \frac{\pi}{3}|\tau) - \theta_1^3(x - \frac{\pi}{3}|\tau) = 3a(q)\left( \frac{q^3; q^3}_\infty \right) \theta_1(x|\tau)\theta_1(x + \frac{\pi}{3}|\tau)\theta_1(x - \frac{\pi}{3}|\tau).
\end{equation}

From (3.4), we have

\begin{equation}
\theta_1(3x|3\tau) = \frac{(q^3; q^3)_\infty^3}{(q^3; q^3)_\infty^3} \theta_1(x|\tau)\theta_1(x + \frac{\pi}{3}|\tau)\theta_1(x - \frac{\pi}{3}|\tau).
\end{equation}

We substitute the above equation into the right side of (5.12) to obtain (1.12).

Proceeding through the same steps as before, by taking $f(x) = e^{2ix}\theta_1^3(x + \frac{\pi}{3}|\tau)$ in (5.9) and then setting $y = \frac{\pi}{3}$ and appealing to (5.1), (5.2), and (5.5), we can find that

\begin{equation}
\theta_1^3(x|\tau) - q^{1/6}e^{2ix}\theta_1^3(x + \frac{\pi}{3}|\tau) - q^{1/6}e^{-2ix}\theta_1^3(x - \frac{\pi}{3}|\tau) = a(q^{1/3})\theta_1(x|\frac{\tau}{3}).
\end{equation}

Replacing $q$ by $q^3$, we obtain (1.13). Thus, we complete the proof of Theorem 3. □
6. The proof of Theorem 4

Proof. Taking $z = \frac{\pi}{3}$ in Theorem 1 and appealing to (3.1) and (3.2), we obtain

$$\sqrt{3q}^{1/8}(q; q)_\infty^3 \{ f(x) - f(-x) \} \theta_1(3y/3\tau)$$

$$= \sqrt{3q}^{1/8}(q; q)_\infty^3 \{ f(y) - f(-y) \} \theta_1(3x/3\tau)$$

(6.1)

$$+ \left\{ f\left(\frac{\pi}{3}\right) - f\left(-\frac{\pi}{3}\right) \right\} \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x + y|\tau)\theta_1(x - y|\tau).$$

It is easy to see that $\theta_1^3(x|\tau)$ satisfies all the conditions of Theorem 1. We choose $f(x) = \theta_1^3(x|\tau)$ and then use (3.2) in the resulting equation to obtain (1.14).

Taking $z = \frac{\pi}{3}$ in Theorem 1 and appealing to (5.1) and (5.2), we obtain

$$iq^{-1/(24)}(q; q)_\infty^3 \{ f(x) - f(-x) \} \theta_1(y/3)$$

$$= iq^{-1/(24)}(q; q)_\infty^3 \{ f(y) - f(-y) \} \theta_1(x/3)$$

(6.2)

$$- \left\{ f\left(\frac{\pi}{3}\right) - f\left(-\frac{\pi}{3}\right) \right\} \theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x + y|\tau)\theta_1(x - y|\tau).$$

Choosing $f(x) = \theta_1^2(x|\tau)$ and then using (5.2) in the resulting equation, we obtain (1.15). This completes the proof of Theorem 4. \qed

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References


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