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ELLIPTIC EQUATIONS WITH BMO COEFFICIENTS IN LIPSCHITZ DOMAINS

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Abstract. In this paper, we study inhomogeneous Dirichlet problems for elliptic equations in divergence form. Optimal regularity requirements on the coefficients and domains for the $W^{1,p}$ $(1 < p < \infty)$ estimates are obtained. The principal coefficients are supposed to be in the John-Nirenberg space with small BMO semi-norms. The domain is supposed to have Lipschitz boundary with small Lipschitz constant. These conditions for the $W^{1,p}$ theory do not just weaken the requirements on the coefficients; they also lead to a more general geometric condition on the domain.

1. Introduction

There has been much research activity (see, e.g., [1, 2, 5, 6, 10, 12, 13, 14, 16, 17, 23, 24, 25, 26, 28, 29, 30, 31, 33]) concerning the classical Calderón-Zygmmund estimates established in [11]. In this paper, we will employ the method used in [33] to investigate the minimal requirements on the coefficients and the smoothness requirement on the domains for the following Dirichlet problem:

$$\left\{ \begin{array}{rcll} -(a_{ij}u_{x_j})_{x_i} = -div(A\nabla u) & = & div\mathbf{f} = (f^i)_{x_i} & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^n$.

The general setting for this work will be the following: (i) approximation of (1.1) by small L^2-perturbation of constant coefficient equations, (ii) decay estimates on the size of distribution functions of the maximal function $\mathcal{M}(|\nabla u|^2)$, and (iii) the Vitali covering lemma.

Throughout this paper we assume that the matrix $A = \{a_{ij}\}$ of the coefficients is defined on $\mathbb{R}^n$, as follows from [1, 20]. The main assumption on A is that it is in the John-Nirenberg space (cf. [19]) of the functions of bounded mean oscillation with small BMO semi-norms. For this assumption, we need the following definition.

Definition 1.1 (Small BMO semi-norm assumption)**.** We say that the matrix A of coefficients is (δ, R)-vanishing if

$$\sup_{0<r\le R} \sup_{x\in\mathbb{R}^n} \frac{1}{|B_r|} \int_{B_r(x)} |A(y) - \overline{A}_{B_r(x)}|^2 dy \le \delta^2. \tag{1.2}$$

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We would like to point out that our assumption that A is (δ, R)-vanishing refines the assumption in other papers (see e.g. [2, 5, 6, 13, 14, 16, 17, 21, 23, 26, 29, 30, 31]) that A is in VMO (cf. [27]). Recall that the class VMO consists of functions with bounded mean oscillation whose integral oscillation over balls shrinking to a point converges uniformly to zero.

Our main assumption on the domain is that it is locally the graph of a Lipschitz continuous function with small Lipschitz constant. More precisely, we have the following definition.

Definition 1.2 (Lipschitz domain assumption)**.** We say that a domain Ω is (δ, R)-Lipschitz if for every $x_0 \in \partial\Omega$ and every $r \in (0, R]$, there exists a Lipschitz continuous function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap B_r(x_0) = \{x = (x_1, ..., x_{n-1}, x_n) = (x', x_n) \in B_r(x_0) : x_n > \gamma(x')\}$$

and

$$\sup_{x', y' \in T_r(x_0), x' \neq y'} \frac{|\gamma(x') - \gamma(y')|}{|x' - y'|} \leq \delta$$

in some coordinate system.

First we remark that γ in the definition above is Lipschitz continuous with small Lipschitz constant if and only if it is in $W^{1,\infty}$ with small $\|\nabla\gamma\|_{L^\infty}$ (see Theorem 4 of chapter 5 in [15]). For further discussions regarding work on Lipschitz domains we cite, for example, [3, 4, 18]. We would like to point out that our assumption that Ω is (δ, R)-Lipschitz weakens the assumption in the paper [16] that $\partial\Omega$ is in $C^{1,1}$ and the assumption in [2] that $\partial\Omega$ is in C^1. We remark that one might assume that R in both Definition 1.1 and Definition 1.2 is 1 by scaling the given equations, while δ is scaling-invariant. Throughout this paper we mean δ to be a small positive constant.

In this work we are interested in finding an a priori inequality like

$$\|\nabla u\|_{L^p(\Omega)} \leq C\|\mathbf{f}\|_{L^p(\Omega)} \qquad (1 < p < \infty) \tag{1.3}$$

for some constant C independent of u and $\mathbf{f}$ under the assumptions that A is (δ, R)-vanishing and Ω is (δ, R)-Lipschitz for $\delta > 0$ small.

In [11], Calderón and Zygmund proved that (1.3) holds true when $A = I_n$, and their results were extended by Morrey in [25], Simader in [28], and Campanato in [12] to the the case of uniformly continuous coefficients. In the case of discontinuous coefficients, N.G. Meyers in [24] provided an example in which an estimate like (1.3) in general does not hold, in particular if one considers the equation

$$-div(A\nabla u) = 0, \tag{1.4}$$

where

$$A(x, y) = \frac{1}{4(x^2 + y^2)} \begin{bmatrix} 4x^2 + y^2 & 4xy \\ 4xy & x^2 + 4y^2 \end{bmatrix}.$$

It is easy to see that $u(x, y) = \frac{x}{(x^2+y^2)^{1/4}}$ is a solution of equation (1.4) and $\nabla u \in L^p(B_1)$ for $p < 4$, but $\nabla u \notin L^p(B_1)$ for $p \geq 4$. In [16], Di Fazio considered the Dirichlet problem of (1.1) when $A \in VMO$ (thus possibly discontinuous) and $\partial\Omega \in C^{1,1}$. He proved the well-posedness of the problem in $W_0^{1,p}(\Omega)$, and showed that the estimate (1.3) is valid for $1 < p < \infty$. His technique is based on representation formulas in terms of singular integral operators and commutators (see, e.g., [13, 14]). In [2], P. Auscher and M. Qafsaoui weaken the assumptions in [16], allowing real

and complex nonsymmetric operators and C^1 boundary. They also considered the corresponding inhomogeneous Neumann problem. Their main tool is an appropriate representation for the Green function on the upper half space. In [10], Caffarelli and Peral used a method of application (see section 1 of [10]) to give an alternative proof of $W^{1,p}$ regularity in the case of a linear elliptic equation, which uses the general theory of singular integrals (see [1, 2, 5, 6, 11, 12, 13, 14, 16, 17, 21, 22, 23, 24, 25, 28, 29, 30, 31]). More precisely, they deduced the following:

Theorem 1.3 ([10])**.** *Let p be a real number with $p > 2$. Assume there exists a small $\delta = \delta(p) > 0$ such that*

$$\|A - I_n\|_\infty \leq \delta.$$

Then all solutions u in H^1 of $-div(A\nabla u) = 0$ in some bounded domain Ω of $\mathbb{R}^n$ satisfy $u \in W^{1,p}$.

Here we use the classical weak solution; namely, we have the following definition.

Definition 1.4. Let $1 < p,\ q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then a weak solution of (1.1) is a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega A\nabla u \nabla\varphi dx = -\int_\Omega \mathbf{f}\nabla\varphi dx,\ \forall\varphi \in W_0^{1,q}(\Omega).$$

Let us state the main result of this paper.

Theorem 1.5. *Let p be a real number with $1 < p < \infty$. Then there is a small $\delta = \delta(\Lambda, p, n, R) > 0$ such that for all A with uniformly elliptic A (see Definition 2.1) and (δ, R)-vanishing; for all Ω with Ω (δ, R)-Lipschitz; and for all $\mathbf{f}$ with $\mathbf{f} \in L^p(\Omega; \mathbb{R}^n)$, the Dirichlet problem* (1.1) *has a unique weak solution with the estimate*

$$\int_\Omega |\nabla u|^p dx \leq C \int_\Omega |\mathbf{f}|^p dx,$$

where the constant C is independent of u and $\mathbf{f}$.

Remark 1.6. We only consider the existence of our weak solution when $p > 2$. Uniqueness follows easily from the case $p = 2$. Then a duality argument ends the proof when $1 < p < 2$. We henceforth suppose that $p > 2$.

Our approach is very much influenced by [10, 33]. In [10], the Calderón-Zygmund decomposition was used. In this paper the Vitali covering lemma will be used, as it was in [8, 33]. Our basic tools in this approach are the Vitali covering lemma, the Hardy-Littlewood maximal function and the compactness method. In a forthcoming paper (see [7]), we extend the present results to parabolic equations.

The remaining sections are organized in the following way. In section 2, we give auxiliary notation, necessary function spaces, some definitions and some geometric analysis results. In section 3, we discuss the interior regularity for the gradient of the solutions. A global regularity is derived for the Dirichlet problem of (1.1) in section 4. Our optimal regularity condition on the domain is discussed in the last section 5.

2. Some preliminary facts from real analysis

2.1. Geometric notation.

(1) $\mathbb{R}^n = n$-dimensional real Euclidean space.
(2) $e_i = (0, ..., 1, ..., 0)$ is the i-th standard coordinate vector.

(3) A typical point in $\mathbb{R}^n$ is $x = (x', x_n)$.
(4) $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$.
(5) $B_r = \{y \in \mathbb{R}^n : |y| < r\}$ is an open ball on $\mathbb{R}^n$ with center 0 and radius $r > 0$, $B_r(x) = B_r + x$, $B_r^+ = B_r \cap R^n_+$, $B_r^+(x) = B_r^+ + x$, $T_r = B_r \cap \{x_n = 0\}$, and $T_r(x) = T_r + x$.
(6) $\Omega_r = \Omega \cap B_r$, $\Omega_r(x) = \Omega \cap B_r(x)$.
(7) ∂B_r^+ is the boundary of B_r^+, and $\partial_c B_r^+ = \partial B_r \cap \{x_n > 0\}$ is the curved part of ∂B_r^+.

2.2. **Matrix of coefficients.**

Definition 2.1. We say that A is uniformly elliptic if there exists a positive constant Λ such that

$$\Lambda^{-1}|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda|\xi|^2, \text{ a.e. } x \in \mathbb{R}^n, \ \forall \xi \in \mathbb{R}^n.$$

(1) We write $A = \{a_{ij}\}$ to mean an $n \times n$ matrix with (i,j)-th entry a_{ij}.
(2) $|A| = \sqrt{(A : A)} = \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ and $\|A\|_\infty = \sup_y |A(y)|$.
(3) A is supposed to be uniformly elliptic.
(4) A is supposed to be $A : (\delta, R)$-vanishing (see Definition 1.1).

Remark 2.2. In this work, A is not assumed to be symmetric.

2.3. **Notation for functions.**

(1) If $u : \Omega \to \mathbb{R}$, we write $u(x)$ $(x \in \Omega)$. If $\mathbf{f} : \Omega \to \mathbb{R}^n$, we write $\mathbf{f}(x) = (f^1(x), ..., f^n(x))$ $(x \in \Omega)$.
(2)
$$\overline{f}_{B_r} = \frac{1}{|B_r|}\int_{B_r} f(x)dx$$
is the average of f over B_r.

2.4. **Notation for derivatives.**

(1) $\nabla u = (u_{x_1}, ..., u_{x_n})$ is the gradient of u.
(2) $div\mathbf{f} = \sum_{i=1}^n (f^i)_{x_i} = (f^i)_{x_i}$ is the divergence of $\mathbf{f} = (f^1, f^2, \cdots, f^n)$.

2.5. **Notation for estimates.** We employ the letter C to denote a universal constant depending usually on the dimension, ellipticity, and the geometric quantities of $\partial\Omega$.

2.6. **Function spaces.**

(1) $C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) : u \text{ has compact support in } \Omega\}$.
(2) $L^p(\Omega) = \{u : \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{1/p} < \infty\}$ for $1 \le p < \infty$.
(3) $L^\infty(\Omega) = \{u : \|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega |u| < \infty\}$.
(4) Let u and v be two locally integral functions. Then we say that v is the i-th weak derivative of u if for any $\varphi \in C_0^\infty(\Omega)$,
$$\int_\Omega u\frac{\partial \varphi}{\partial x_i}dx = -\int_\Omega v\varphi dx.$$
We denote by $\frac{\partial u}{\partial x_i}$ the i-th weak derivative of u. Then we say that u is in the space $W^{1,p}(\Omega)$ if u has weak derivatives $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$ and $u \in L^p(\Omega)$.

$W^{1,p}(\Omega)$ is a Banach space under the norm

$$\left(\|u\|^p_{L^p(\Omega)} + \sum_i \|\frac{\partial u}{\partial x_i}\|^p_{L^p(\Omega)} \right)^{1/p}.$$

In the case $p = 2$, $H^1 = W^{1,2}$ is a Hilbert space. We say $u \in W^{1,p}_0(\Omega)$ if $Eu \in W^{1,p}(\mathbb{R}^n)$, where Eu is the zero extension of u to $\mathbb{R}^n$.

We end this subsection by introducing the following standard arguments of measure theory.

Lemma 2.3 ([9]). *Suppose that f is a nonnegative and measurable function in a bounded domain Ω. Let $\theta > 0$ and $m > 1$ be constants. Then for $0 < p < \infty$,*

$$f \in L^p(\Omega) \text{ if and only if } S = \sum_{k \geq 1} m^{kp} |\{x \in \Omega : f(x) > \theta m^k\}| < \infty \tag{2.1}$$

and

$$\frac{1}{C} S \leq \|f\|^p_{L^p(\Omega)} \leq C(|\Omega| + S),$$

where $C > 0$ is a constant depending only on θ, m, and p.

2.7. **Preliminary tools.** One of our main tools is the following Vitali covering lemma.

Lemma 2.4 ([32]). *Let E be a measurable set. Suppose that a class of balls B_α covers E:*

$$E \subset \bigcup_\alpha B_\alpha.$$

Suppose the radius of B_α is bounded from above. Then there exist a disjoint $\{B_{\alpha_i}\}_{i=1}^\infty \subset \{B_\alpha\}_\alpha$ such that

$$E \subset \bigcup_i 5B_{\alpha_i},$$

where $5B_{\alpha_i}$ is the ball with 5 times the radius of B_{α_i}. Consequently, we have

$$|E| \leq 5^n \sum_i |B_{\alpha_i}|.$$

We use the Hardy-Littlewood maximal function which controls the local behavior of a function.

Definition 2.5. Let f be a locally integrable function. Then

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy$$

is called the Hardy-Littlewood maximal function of f. We also use

$$\mathcal{M}_\Omega f = \mathcal{M}(\chi_\Omega f),$$

if f is not defined outside Ω. We will drop the index Ω if Ω is understood clearly in the context.

We know that the value of an L^1 function at a particular point does not make good sense in a qualitative way, even when the point is a Lebesque point. However, the Hardy-Littlewood maximal function at a point does, since it is invariant with respect to scaling. The basic theorem for the Hardy-Littlewood maximal function is the following.

Theorem 2.6 ([32]). (1) *$f \in L^p(\mathbb{R}^n)$ with $p > 1$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$. Moreover,*

$$\|\mathcal{M}f\|_{L^p} \le C\|f\|_{L^p}. \tag{2.2}$$

(2) *If $f \in L^1(\mathbb{R}^n)$, then*

$$|\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > t\}| \le \frac{C}{t}\int |f|dx. \tag{2.3}$$

Inequality (2.2) is called a strong p-p estimate and (2.3) is called a weak 1-1 estimate.

We will use the following version of the Vitali covering lemma for discussions of interior $W^{1,p}$ regularity.

Theorem 2.7 ([33]). *Assume that C and D are measurable sets, $C \subset D \subset B_1$, and that there exists an $\epsilon > 0$ such that*

$$|C| < \epsilon|B_1| \tag{2.4}$$

and

$$\forall x \in B_1,\ \forall r \in (0,1] \text{ with } |C \cap B_r(x)| \ge \epsilon|B_r(x)|,\ B_r(x) \cap B_1 \subset D. \tag{2.5}$$

Then

$$|C| \le 10^n\epsilon|D|.$$

For our global estimate, we introduce another version of the Vitali covering lemma.

Theorem 2.8. *Assume that C and D are measurable sets with $C \subset D \subset B_1^+$, and that there exists an $\epsilon > 0$ such that*

$$|C| < \epsilon|B_1^+| \tag{2.6}$$

and

$$\text{for every } x \in B_1^+ \text{ with } |C \cap B_r(x)| \ge \epsilon|B_r|,\ B_r(x) \cap B_1^+ \subset D. \tag{2.7}$$

Then

$$|C| \le 2(10)^n\epsilon|D|.$$

Proof. In view of (2.6), for almost every $x \in B_1^+$, there exists a small $r_x > 0$ such that

$$|C \cap B_{r_x}(x)| = \epsilon|B_{r_x}| \text{ and } |C \cap B_r(x)| < \epsilon|B_r(x)|\ , \forall r \in (r_x, 1]. \tag{2.8}$$

Since $\{B_{r_x}(x) \cap C : x \in C\}$ is a covering of C with $r_x \le 1$, by the Vitali covering lemma, there exists a disjoint subcovering $\{B_{r_i}(x_i) \cap C : x_i \in C\}_{i=1}^\infty$ such that

$$C \subset \bigcup_i B_{5r_i}(x_i) \text{ and } |C| \le 5^n \sum |B_{r_i}|. \tag{2.9}$$

Then, by (2.8), we see that

$$|C \cap B_{5r_i}(x_i)| < \epsilon|B_{5r_i}| = 5^n\epsilon|B_{r_i}| = 5^n|C \cap B_{r_i}(x_i)|. \tag{2.10}$$

Now we will claim that

$$|B_{r_i}| \le 2^{n+1}|B_{r_i}(x_i) \cap B_1^+|. \tag{2.11}$$

We observe that for any fixed $r > 0$, $\inf\{|B_r(x) \cap B_1^+| : x \in B_1^+\} = |B_r(e_1) \cap B_1^+|$. Now since

$$B_r(e_1) \cap B_1^+ \supset B_{\frac{r}{2}}^+((1 - r/2)e_1),$$

we see that

$$|B_r(x) \cap B_1^+| \geq |B_r(e_1) \cap B_1^+| \geq |B_{\frac{r}{2}}^+| = 2^{-(n+1)}|B_r(x)|.$$

This estimate implies (2.11). Finally, by (2.10), (2.11) and (2.7), we get

$$\begin{aligned}
|C| &= \left|\bigcup_i (B_{5r_i}(x_i) \cap C)\right| \\
&\leq \sum_i |B_{5r_i}(x_i) \cap C| \\
&< \epsilon \sum_i |B_{5r_i}(x_i)| \\
&= 5^n \epsilon \sum_i |B_{r_i}(x_i)| \\
&\leq 2(10)^n \epsilon \sum_i |(B_{r_i}(x_i) \cap B_1^+)| \\
&= 2(10)^n \epsilon \left|\bigcup_i (B_{r_i}(x_i) \cap B_1^+)\right| \\
&\leq 2(10)^n \epsilon |D|,
\end{aligned}$$

which finishes the proof. □

3. Interior estimates

In this section we investigate the interior $W^{1,p}$ $(2 < p < \infty)$ estimates for the divergence form elliptic equation

$$-div(A(x)\nabla u) = div\mathbf{f} \tag{3.1}$$

in a bounded open set $\Omega \subset \mathbb{R}^n$. Our main assumption is that the matrix A of the coefficients is (δ, R)-vanishing. The main tool in this section is a version of the Vitali covering lemma (see Theorem 2.7). Let us state the main result of this section.

Theorem 3.1. *Let p be a real number with $2 < p < \infty$. There is a small $\delta = \delta(\Lambda, p, n, R) > 0$ so that for all A with A uniformly elliptic and (δ, R)-vanishing, and for all $\mathbf{f}$ with $\mathbf{f} \in L^p(B_6; \mathbb{R}^n)$, if u is a weak solution of the elliptic PDE* (3.1) *in B_6, then u belongs to $W^{1,p}(B_1)$ with the estimate*

$$\|\nabla u\|_{L^p(B_1)} \leq C\left(\|u\|_{L^p(B_6)} + \|\mathbf{f}\|_{L^p(B_6)}\right),$$

where the constant C is independent of u and $\mathbf{f}$.

Remark 3.2. We can change the ball B_6 in Theorem 3.1 to any ball B_R with $R > 1$.

Let us start with the following definition.

Definition 3.3. We say that $u \in H^1(B_R)$ $(R > 0)$ is a weak solution of (3.1) if

$$\int_{B_R} A\nabla u \nabla \varphi dx = -\int_{B_R} \mathbf{f}\nabla\varphi dx \text{ for any } \varphi \in H_0^1(B_R).$$

We need the following standard energy estimate.

Lemma 3.4. *Assume that u is a weak solution of (3.1) in B_2. Then*

$$\int_{B_2} \phi^2 |\nabla u|^2 dx \le C \left(\int_{B_2} \phi^2 |\mathbf{f}|^2 \, dx + \int_{B_2} |\nabla \phi|^2 |u|^2 dx \right), \ \forall \phi \in C_0^\infty(B_2).$$

Remark 3.5. We remark that Lemma 3.4 is used not just for the proof of interior $W^{1,\infty}$ regularity of our approximation solution v with constant coefficients, but also for the proof of Corollary 3.7.

Lemma 3.6. *For any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon) > 0$ such that for any weak solution u of* (3.1) *in B_4 with*

$$\frac{1}{|B_4|} \int_{B_4} |\nabla u|^2 dx \le 1, \ \frac{1}{|B_4|} \int_{B_4} \left(|\mathbf{f}|^2 + |A - \overline{A}_{B_4}|^2 \right) dx \le \delta^2, \tag{3.2}$$

there exists a weak solution v of

$$-div(\overline{A}_{B_4} \nabla v) = 0 \ in \ B_4$$

such that

$$\int_{B_4} |u - v|^2 dx \le \epsilon^2. \tag{3.3}$$

Proof. We prove this lemma by contradiction. If not, there exist $\epsilon_0 > 0$, $\{A_k\}_{k=1}^\infty$, $\{u_k\}_{k=1}^\infty$, and $\{\mathbf{f}_k\}_{k=1}^\infty$ such that u_k is a weak solution of

$$-div(A_k \nabla u_k) = div \mathbf{f}_k \tag{3.4}$$

in B_4 with

$$\frac{1}{|B_4|} \int_{B_4} |\nabla u_k|^2 dx \le 1, \ \frac{1}{|B_4|} \int_{B_4} \left(|\mathbf{f}_k|^2 + |A_k - \overline{A_k}_{B_4}|^2 \right) dx \le \frac{1}{k^2}. \tag{3.5}$$

But,

$$\int_{B_4} |u_k - v_k|^2 dx > \epsilon_0^2 \tag{3.6}$$

for any weak solution v_k of

$$-div(\overline{A_k}_{B_4} \nabla v_k) = 0 \text{ in } B_4. \tag{3.7}$$

By (3.5), $\{u_k - \overline{u_k}_{B_4}\}_{k=1}^\infty$ is bounded in $H^1(B_4)$, and so $\{u_k - \overline{u_k}_{B_4}\}$ has a subsequence, which we denote as $\{u_k - \overline{u_k}\}$, such that

$$u_k - \overline{u_k} \rightharpoonup u_0 \text{ in } H^1(B_4), \ u_k - \overline{u_k} \to u_0 \text{ in } L^2(B_4) \tag{3.8}$$

for some $u_0 \in H^1(B_4)$. Since $\left\{\overline{A_k}_{B_4}\right\}$ is bounded in L^∞, it has a subsequence, which we denote as $\left\{\overline{A_k}\right\}_{k=1}^\infty$, such that

$$\|\overline{A_k} - A_0\|_\infty \to 0 \text{ in } \mathbb{R} \text{ as } k \to \infty \tag{3.9}$$

for some constant matrix A_0. But then, by (3.5), we have

$$A_k \to A_0 \text{ in } L^2(B_4). \tag{3.10}$$

First we will show that u_0 itself is a weak solution of

$$-div(A_0 \nabla u_0) = 0 \text{ in } B_4. \tag{3.11}$$

To do this, fix $\varphi \in H_0^1(B_4)$. Then, by (3.4), we have

$$\int_{B_4} A_k \nabla u_k \nabla \varphi dx = \int_{B_4} \mathbf{f}_k \nabla \varphi dx. \tag{3.12}$$

Since $\nabla u_k \rightharpoonup \nabla u_0$ and $A_k \to A_0$ in $L^2(B_4)$, $A_k \nabla u_k \rightharpoonup A_0 \nabla u_0$ in $L^2(B_4)$. Then letting $k \to \infty$ in (3.12), we have

$$\int_{B_4} A_0 \nabla u_0 \nabla \varphi dx = 0,$$

and this is (3.11). Note that

$$-div(\overline{A_k}\nabla u_0) = -div\left((\overline{A_k} - A_0)\nabla u_0\right) - div(A_0 \nabla u_0) = -div\left((\overline{A_k} - A_0)\nabla u_0\right)$$

in B_4. Let h_k be the weak solution of

$$\begin{cases} -div(\overline{A_k}\nabla h_k) &= -div\left((\overline{A_k} - A_0)\nabla u_0\right) \quad \text{in} \quad B_4, \\ h_k &= 0 \quad \text{on} \quad \partial B_4. \end{cases} \tag{3.13}$$

Then $u_0 - h_k$ is a weak solution of

$$-div\left(\overline{A_k}\nabla(u_0 - h_k)\right) = 0 \text{ in } B_4. \tag{3.14}$$

Furthermore, by (3.13)

$$\begin{aligned} \|h_k\|_{L^2(B_4)} \le C\|\nabla h_k\|_{L^2(B_4)} &\le C\|(\overline{A_k} - A_0)\nabla u_0\|_{L^2(B_4)} \\ &\le C\|\overline{A_k} - A_0\|_\infty \|\nabla u_0\|_{L^2(B_4)} \\ &\le C\|\overline{A_k} - A_0\|_\infty, \end{aligned}$$

and so we have

$$\begin{aligned} \|u_k - (u_0 - h_k)\|_{L^2(B_4)} &\le \|u_k - u_0\|_{L^2(B_4)} + \|h_k\|_{L^2(B_4)} \\ &\le \|u_k - u_0\|_{L^2(B_4)} + C\|\overline{A_k} - A_0\|_\infty. \end{aligned}$$

This estimate, (3.8) and (3.9) imply that

$$\|u_k - (u_0 - h_k)\|_{L^2(B_4)} \to 0 \text{ as } k \to \infty.$$

But this is a contradiction to (3.6) by (3.14). □

Corollary 3.7. *For any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon) > 0$ such that for any weak solution u of* (3.1) *in B_4 with*

$$\frac{1}{|B_4|}\int_{B_4} |\nabla u|^2 dx \le 1, \ \frac{1}{|B_4|}\int_{B_4} \left(|\mathbf{f}|^2 \ + |A - \overline{A}_{B_4}|^2\right) dx \le \delta^2, \tag{3.15}$$

there exists a weak solution v of

$$-div(\overline{A}_{B_4}\nabla v) = 0$$

in B_4 such that

$$\int_{B_2} |\nabla(u - v)|^2 dx \le \epsilon^2.$$

Proof. In view of Lemma 3.6 and (3.15), for any $\eta > 0$, there are a small $\delta = \delta(\eta) > 0$ and a corresponding weak solution v of

$$-div\left(\overline{A}_{B_4}\nabla v\right) = 0 \text{ in } B_4$$

such that

$$\int_{B_4} |u - v|^2 dx \le \eta^2 \tag{3.16}$$

provided that

$$\frac{1}{|B_4|}\int_{B_4}\left(|\mathbf{f}|^2 \ +|A-\overline{A}_{B_4}|^2\right)dx \le \delta^2.$$

First we see that $u-v$ is a weak solution of

$$-div(A\nabla(u-v)) = div(\mathbf{f}+(A-\overline{A}_{B_4})\nabla v) \text{ in } B_4. \tag{3.17}$$

In view of (3.17) and Lemma 3.4, we get the following estimate:

$$\begin{aligned}
\int_{B_2}|\nabla(u-v)|^2dx &\le C\left(\int_{B_3}|\mathbf{f}+(A-\overline{A}_{B_4})\nabla v|^2dx+\int_{B_3}|u-v|^2dx\right)\\
&\le C\left(\int_{B_3}|\mathbf{f}|^2dx+\int_{B_3}|(A-\overline{A}_{B_4})\nabla v|^2dx+\int_{B_3}|u-v|^2dx\right)\\
&\le C\left(\int_{B_3}|\mathbf{f}|^2dx+\int_{B_3}|A-\overline{A}_{B_4}|^2dx+\int_{B_3}|u-v|^2dx\right)\\
&\le C\left(\int_{B_4}\left(|\mathbf{f}|^2+|A-\overline{A}_{B_4}|^2\right)dx+\int_{B_4}|u-v|^2dx\right).
\end{aligned}$$

Here we used the interior $W^{1,\infty}$ regularity of $v\in H^1(B_4)$. This estimate and (3.16) imply finally

$$\begin{aligned}
\int_{B_2}|\nabla(u-v)|^2dx &\le C\left(\int_{B_4}\left(|\mathbf{f}|^2+|A-\overline{A}_{B_4}|^2\right)dx+\int_{B_4}|u-v|^2dx\right)\\
&\le C\left(|B_4|\delta^2+\eta^2\right)\\
&= \epsilon^2,
\end{aligned}$$

by taking η and δ satisfying the last identity above. This completes our proof. □

We use a local estimate of solutions of $-div(A\nabla u)=div\mathbf{f}$ by comparing with a weak solution of the approximation equation $-div(\overline{A}\nabla v)=0$, where $\overline{A}$ is a local average of A.

Lemma 3.8. *There is a constant N_1 so that for any $\epsilon>0$ there exists a small $\delta=\delta(\epsilon)>0$, and if u is a weak solution of*

$$-div(A\nabla u)=div\mathbf{f}$$

in $\Omega\supset B_6$ with

$$B_1\cap\{x\in\Omega:\mathcal{M}(|\nabla u|)^2(x)\le 1\}\cap\{x\in\Omega:\mathcal{M}(|\mathbf{f}|^2)(x)\le\delta^2\}\neq\emptyset \tag{3.18}$$

and A $(\delta,6)$-vanishing, then

$$|\{x\in\Omega:\mathcal{M}(|\nabla u|)^2(x)>N_1^2\}\cap B_1|<\epsilon|B_1|. \tag{3.19}$$

Proof. By (3.18), there exists a point $x_0\in B_1$ such that

$$\frac{1}{|B_r|}\int_{B_r(x_0)\cap\Omega}|\nabla u|^2dx\le 1,\ \frac{1}{|B_r|}\int_{B_r(x_0)\cap\Omega}|\mathbf{f}|^2dx\le\delta^2,\ \forall r>0. \tag{3.20}$$

Since $B_4(0)\subset B_5(x_0)$, by (3.20) we have

$$\frac{1}{|B_4|}\int_{B_4}|\mathbf{f}|^2dx\le\frac{|B_5|}{|B_4|}\frac{1}{|B_5|}\int_{B_5(x_0)}|\mathbf{f}|^2dx\le\left(\frac{5}{4}\right)^n\delta^2. \tag{3.21}$$

Similarly, we find that

$$\frac{1}{|B_4|}\int_{B_4}|\nabla u|^2dx \le \left(\frac{5}{4}\right)^n. \tag{3.22}$$

In view of (3.22) and (3.21), and from the assumption that A is $(\delta,6)$-vanishing, we can apply Corollary 3.7 with u replaced by $\left(\frac{4}{5}\right)^n u$ and $\mathbf{f}$ replaced by $\left(\frac{4}{5}\right)^n \mathbf{f}$, to find that for any $\eta>0$, there exists a small $\delta(\eta)$ and a corresponding weak solution v of

$$-div\left(\overline{A}_{B_4}\nabla v\right)=0$$

in B_4 such that

$$\int_{B_2}|\nabla(u-v)|^2dx \le \eta^2 \tag{3.23}$$

provided that

$$\frac{1}{|B_4|}\int_{B_4}\left(|\mathbf{f}|^2+|A-\overline{A}_{B_4}|^2\right)dx \le \delta^2.$$

Now we can use the interior $W^{1,\infty}$ regularity of v to see that there is a constant N_0 such that

$$\|\nabla v\|^2_{L^\infty(B_3)} \le N_0^2. \tag{3.24}$$

We claim that

$$\{x:\mathcal{M}(|\nabla u|)^2>N_1^2\}\cap B_1 \subset \{x:\mathcal{M}_{B_4}(|\nabla(u-v)|)^2>N_0^2\}\cap B_1, \tag{3.25}$$

where $N_1^2=\sup\left\{2^n,4N_0^2\right\}$. To see this, suppose that

$$x_1\in\{x:\mathcal{M}_{B_4}(|\nabla(u-v)|)^2\le N_0^2\}\cap B_1. \tag{3.26}$$

For $r\le 2$, $B_r(x_1)\subset B_3$, and by (3.26) and (3.24), we have

$$\frac{1}{|B_r|}\int_{B_r(x_1)}|\nabla u|^2dx \le \frac{2}{|B_r|}\int_{B_r(x_1)}(|\nabla(u-v)|^2+|\nabla v|^2)dx \le 4N_0^2. \tag{3.27}$$

For $r>2$, $x_0\in B_r(x_1)\subset B_{2r}(x_0)$, and by (3.20), we have

$$\frac{1}{|B_r|}\int_{B_r(x_1)\cap\Omega}|\nabla u|^2dx \le \frac{2^n}{|B_{2r}|}\int_{B_{2r}(x_0)\cap\Omega}|\nabla u|^2dx \le 2^n. \tag{3.28}$$

Then (3.27) and (3.28) imply

$$x_1\in\{x:\mathcal{M}(|\nabla u|)^2\le N_1^2\}\cap B_1. \tag{3.29}$$

Thus assertion (3.25) follows from (3.26) and (3.29). By (3.25), weak 1-1 estimates (see Theorem 2.6), and (3.23), we obtain

$$\begin{aligned}
|\{x:\mathcal{M}(|\nabla u|)^2(x)>N_1^2\}\cap B_1| &\le |\{x:\mathcal{M}_{B_4}(|\nabla(u-v)|)^2(x)>N_0^2\}\cap B_1| \\
&< \frac{C}{N_0^2}\int_{B_2}|\nabla(u-v)|^2dx \\
&\le \frac{C}{N_0^2}\eta^2 \\
&= \epsilon|B_1|,
\end{aligned}$$

by taking η and δ satisfying the last identity above. This finishes our proof. □

From Lemma 3.8 and the scaling argument, we get the following lemma, which will be used later for our proofs.

Lemma 3.9. *Let B be a ball in $\mathbb{R}^n$. Assume that u is a weak solution of* (3.1) *in a domain $\Omega \supset 6B$. If $|\{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B| \geq \epsilon |B|$, then*

$$B \subset \{x \in \Omega : \mathcal{M}(|\nabla u|)^2(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2\}.$$

Now take N_1, ϵ, and the corresponding $\delta > 0$ given by Lemma 3.9.

Corollary 3.10. *Suppose that u is a weak solution of* (3.1) *in a domain $\Omega \supset B_6$. Assume that*

$$|\{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\}| < \epsilon |B_1|.$$

Let k be a positive integer and set $\epsilon_1 = 10^n \epsilon$. Then we have

$$\begin{aligned} |\{x \in B_1 : \mathcal{M}(|\nabla u|^2)(x) > N_1^{2k}\}| \quad \leq \quad & \sum_{i=1}^{k} \epsilon_1^i |\{x \in B_1 : \mathcal{M}(|\mathbf{f}|^2) > \delta^2 N_1^{2(k-i)}\}| \\ & + \epsilon_1^k |\{x \in B_1 : \mathcal{M}(|\nabla u|^2) > 1\}|. \end{aligned}$$

Proof. We will prove this corollary by induction on k. The case $k = 1$ comes from Lemma 3.9 and Theorem 2.7 on

$$\begin{aligned} C &:= \{x \in B_1 : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\}, \\ D &:= \{x \in B_1 : \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2\} \cup \{x \in B_1 : \mathcal{M}(|\nabla u|^2)(x) > 1\}. \end{aligned}$$

Assume now that the conclusion is valid for some positive integer k. Let us define $u_1 = u/N_1$ and $\mathbf{f}_1 = \mathbf{f}/N_1$. Then u_1 is a weak solution of

$$-div(A\nabla u_1) = div\mathbf{f}_1$$

in $\Omega \supset B_6$, and the following property holds:

$$|\{x \in \Omega : \mathcal{M}(|\nabla u_1|^2)(x) > N_1^2\}| < \epsilon |B_1|.$$

Then by the induction assumption, we have

$$\begin{aligned} & |\{x \in B_1 : \mathcal{M}(|\nabla u|^2)(x) > N_1^{2(k+1)}\}| \\ = \quad & |\{x \in B_1 : \mathcal{M}(|\nabla u_1|^2)(x) > N_1^{2k}\}| \\ \leq \quad & \sum_{i=1}^{k} \epsilon_1^i |\{x \in B_1 : \mathcal{M}(|\mathbf{f}_1|^2)(x) > \delta^2 N_1^{2(k-i)}\}| \\ & + \epsilon_1^k |\{x \in B_1 : \mathcal{M}(|\nabla u_1|^2)(x) > 1\}| \\ = \quad & \sum_{i=1}^{k+1} \epsilon_1^i |\{x \in B_1 : \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2 N_1^{2(k+1-i)}\}| \\ & + \epsilon_1^{k+1} |\{x \in B_1 : \mathcal{M}(|\nabla u|^2)(x) > 1\}|. \end{aligned}$$

Thus, the conclusion is valid for $k+1$, which completes the proof. □

We now prove Theorem 3.1. Corollary 3.10 is the primary technical tool.

Proof. Without loss of generality, we assume that

$$\|\mathbf{f}\|_{L^p(B_6)} \text{ is small enough} \tag{3.30}$$

and

$$|\{x \in B_6 : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_1| < \epsilon |B_1| \tag{3.31}$$

by multiplying the PDE (3.1) by a small constant depending on $\|\mathbf{f}\|_{L^2(B_6)}$ and $\|\nabla u\|_{L^2(B_6)}$. Since $\mathbf{f} \in L^p(B_6)$, we have $\mathcal{M}(|\mathbf{f}|^2) \in L^{p/2}(B_6)$ by strong p-p estimates

(see Theorem 2.6). In view of Lemma 2.3, there is a constant C, depending only on δ, p and N_1, such that

$$\sum_{k=0}^{\infty} N_1^{pk} |\{x \in B_1 : \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2 N_1^{2k}\}| \leq C \|\mathcal{M}(|\mathbf{f}|^2)\|_{L^{p/2}(B_6)}^{p/2}.$$

Then this estimate, strong p-p estimates, and (3.30) imply

$$\sum_{k=0}^{\infty} N_1^{pk} |\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2 N_1^{2k} \cap B_1\}| \leq 1. \tag{3.32}$$

Given $p > 2$, we will claim that $\mathcal{M}(|\nabla u|^2) \in L^{p/2}$ by using Lemma 2.3 when $f = \mathcal{M}(|\nabla u|^2)$ and $m = N_1^2$. Let us compute

$$\begin{aligned}
& \sum_{k=0}^{\infty} N_1^{pk} \left|\left\{x \in B_1 : \mathcal{M}(|\nabla u|^2)(x) > N_1^{2k}\right\}\right| \\
\leq \; & \sum_{k=1}^{\infty} N_1^{pk} \left(\sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in B_1 : \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2 N_1^{2(k-i)}\right\}\right| \right. \\
& \left. + \epsilon_1^k \left|\left\{x \in B_1 : \mathcal{M}(|\nabla u|^2)(x) > 1\right\}\right| \right) \\
= \; & \sum_{i=1}^{\infty} (N_1^p \epsilon_1)^i \left(\sum_{k=i}^{\infty} N_1^{p(k-i)} \left|\left\{x \in B_1 : \; \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2 N_1^{2(k-i)}\right\}\right| \right) \\
& + \sum_{k=1}^{\infty} (N_1^p \epsilon_1)^k \left|\left\{x \in B_1 : \mathcal{M}(|\nabla u|^2)(x) > 1\right\}\right| \\
\leq \; & C \sum_{k=1}^{\infty} (N_1^p \epsilon_1)^k \\
< \; & +\infty,
\end{aligned}$$

where we have used Corollary 3.10, (3.32) and took ϵ_1 so that $N_1^p \epsilon_1 < 1$. Then, this estimate and Lemma 2.3 imply $\mathcal{M}(|\nabla u|^2) \in L^{p/2}(B_6)$. Therefore, $\nabla u \in L^p(B_6)$. □

Remark 3.11. In the proof above we can take $N_1^p \epsilon_1 < 1$, since N_1 is a universal constant depending on the dimension and ellipticity. So we can choose ϵ, and the corresponding ϵ_1. This is the same case in the proof of Theorem 4.10.

Remark 3.12. Now that we have the interior L^p estimate for the gradient of u in a ball B_1, we can get the interior estimate by standard scaling and covering arguments.

4. Elliptic equations in Lipschitz domains

In this section we make some observations on the minimal regularity requirements on the coefficients and the domain of (1.1), and prove a global $W^{1,p}$ $(1 < p < \infty)$ estimate under the assumptions that the matrix of coefficients is (δ, R)-vanishing and the domain is (δ, R)-Lipschitz. Let us start with the following definition.

Definition 4.1. We say that $u \in H_0^1(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega A\nabla u \nabla\varphi dx = -\int_\Omega \mathbf{f}\nabla\varphi dx,\ \forall\varphi \in H_0^1(\Omega). \tag{4.1}$$

We localize our interest on a weak solution of

$$\begin{cases} -div(A\nabla u) &= div\mathbf{f} \quad \text{in} \quad B_R^+, \\ u &= 0 \quad \text{on} \quad T_R. \end{cases} \tag{4.2}$$

We study local estimates of solutions to (4.2) by comparison with solutions of

$$\begin{cases} -div\left(\overline{A}_{B_R^+}\nabla v\right) &= 0 \quad \text{in} \quad B_R^+, \\ v &= 0 \quad \text{on} \quad T_R, \end{cases} \tag{4.3}$$

with the following definition.

Definition 4.2. (1) We say that $u \in H^1(B_R^+)$ is a weak solution of (4.2) in B_R^+ if

$$\int_{B_R^+} A\nabla u\nabla\varphi dx = -\int_{B_R^+} \mathbf{f}\nabla\varphi dx,\ \forall\varphi \in H_0^1(B_R^+),$$

and its zero extension is in $H^1(B_R)$.

(2) We say that $v \in H^1(B_R^+)$ is a weak solution of (4.3) in B_R^+ if

$$\int_{B_R^+} \overline{A}_{B_R^+}\nabla v\nabla\varphi dx = 0,\ \forall\varphi \in H_0^1(B_R^+),$$

and its zero extension is in $H^1(B_R)$.

We need the following energy estimate for the proof of Corollary 4.5.

Lemma 4.3. *Assume that $u \in H^1(B_1^+)$ is a weak solution of (4.2) in B_1^+. Then*

$$\int_{B_1^+} \phi^2|\nabla u|^2 dx \le C\left(\int_{B_1^+} \phi^2|\mathbf{f}|^2 dx + \int_{B_1^+} |\nabla\phi|^2|u|^2 dx\right),\ \forall\phi \in C_0^\infty(B_1).$$

Proof. The proof comes from taking $\phi^2 u$ as a competitor in Definition 4.2. □

The following lemma is a so-called compact argument. It is also called the indirect method or blow-up method.

Lemma 4.4. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that for any weak solution u of (4.2) in B_4^+ with*

$$\frac{1}{|B_4|}\int_{B_4^+} |\nabla u|^2 dx \le 1,\ \frac{1}{|B_4|}\int_{B_4^+} \left(|\mathbf{f}|^2 + |A - \overline{A}_{B_4^+}|^2\right) dx \le \delta^2, \tag{4.4}$$

there exists a weak solution v of (4.3) in B_4^+ such that

$$\int_{B_4^+} |u - v|^2 dx \le \epsilon^2. \tag{4.5}$$

Proof. We prove it by contradiction. If not, there exist $\epsilon_0 > 0$, $\{A_k\}_{k=1}^\infty$, $\{u_k\}_{k=1}^\infty$, and $\{\mathbf{f}_k\}_{k=1}^\infty$ such that u_k is a weak solution of

$$\begin{cases} -div(A_k\nabla u_k) &= div\mathbf{f}_k \quad \text{in} \quad B_4^+, \\ u_k &= 0 \quad \text{on} \quad T_4, \end{cases} \tag{4.6}$$

with

$$\frac{1}{|B_4|}\int_{B_4^+}|\nabla u_k|^2dx \le 1,\ \frac{1}{|B_4|}\int_{B_4^+}\left(|\mathbf{f}_k|^2+|A_k-\overline{A_k}_{B_4^+}|^2\right)dx \le \frac{1}{k^2}. \tag{4.7}$$

But,

$$\int_{B_4^+}|u_k-v|^2dx > \epsilon_0^2 \tag{4.8}$$

for any weak solution v of (4.3) in B_4^+.

By (4.7) and from the fact that $u_k = 0$ on T_4, $\{u_k\}_{k=1}^\infty$ is bounded in $H^1(B_4^+)$. Consequently, there exists a subsequence, which we still denote as $\{u_k\}$, and u_0 in $H^1(B_4^+)$ with $u_0 = 0$ on T_4 such that

$$u_k \rightharpoonup u_0 \text{ in } H^1(B_4^+) \text{ and } u_k \to u_0 \text{ in } L^2(B_4^+). \tag{4.9}$$

Since $\left\{\overline{A_k}_{B_4^+}\right\}_{k=1}^\infty$ is bounded in L^∞, it has a subsequence, which we denote as $\{\overline{A_k}\}$, such that

$$\|\overline{A_k}-A_0\|_\infty \to 0 \text{ as } k\to\infty \tag{4.10}$$

for some constant matrix A_0. But then, by (4.10) and (4.7), we have

$$A_k \to A_0 \text{ in } L^2(B_4^+). \tag{4.11}$$

Next we will show that u_0 is a weak solution of

$$\begin{cases} -div(A_0\nabla u_0) &= 0 \quad \text{in} \quad B_4^+, \\ u_0 &= 0 \quad \text{on} \quad T_4. \end{cases} \tag{4.12}$$

To do this, fix any $\varphi \in H_0^1(B_4^+)$. Then, by (4.6), we have

$$\int_{B_4^+}A_k\nabla u_k\nabla\varphi dx = -\int_{B_4^+}\mathbf{f}_k\nabla\varphi dx. \tag{4.13}$$

Since $\nabla u_k \rightharpoonup \nabla u_0$ in $L^2(B_4^+)$ and $A_k \to A_0$ in $L^2(B_4^+)$, we have

$$A_k\nabla u_k \rightharpoonup A_0\nabla u_0 \text{ in } L^2(B_4^+) \text{ as } k\to\infty. \tag{4.14}$$

Let $k\to\infty$ in (4.13) and note (4.14). We find that

$$\int_{B_4^+}A_0\nabla u_0\nabla\varphi dx = 0. \tag{4.15}$$

The claim (4.12) comes from (4.15) and the fact that $u_0 = 0$ on T_4.

Finally, we get a contradiction to (4.8) by taking $v = u_0$ and k large enough. □

Corollary 4.5. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that for any weak solution u of (4.2) in B_4^+ with*

$$\frac{1}{|B_4|}\int_{B_4^+}|\nabla u|^2dx \le 1,\ \frac{1}{|B_4|}\int_{B_4^+}\left(|\mathbf{f}|^2+|A-\overline{A}_{B_4^+}|^2\right)dx \le \delta^2, \tag{4.16}$$

there exists a weak solution v of (4.3) in B_4^+ such that

$$\int_{B_2^+}|\nabla(u-v)|^2dx \le \epsilon^2. \tag{4.17}$$

Proof. In view of (4.16) and Lemma 4.4, for any $\eta > 0$, there exist a small $\delta = \delta(\epsilon)$ and a corresponding weak solution v of (4.3) in B_4^+ such that

$$\int_{B_4^+} |u - v|^2 dx \le \eta^2 \tag{4.18}$$

provided that

$$\frac{1}{|B_4|} \int_{B_4^+} \left(|\mathbf{f}|^2 + |A - \overline{A}_{B_4^+}|^2 \right) dx \le \delta^2.$$

First, we observe that $u - v$ is a weak solution of

$$-div(A\nabla(u - v)) = div(\mathbf{f} + (A - \overline{A}_{B_4^+})\nabla v) \tag{4.19}$$

in B_4^+ with $u - v = 0$ on T_4. According to Lemma 4.3, we get

$$\begin{aligned} \int_{B_2^+} |\nabla(u - v)|^2 dx &\le C \left(\int_{B_3^+} |\mathbf{f} + (A - \overline{A}_{B_4^+})\nabla v|^2 dx + \int_{B_3^+} |u - v|^2 \, dx \right) \\ &\le C \left(\int_{B_4^+} \left(|\mathbf{f}|^2 + |A - \overline{A}_{B_4^+}|^2 \right) dx + \int_{B_4^+} |u - v|^2 dx \right). \end{aligned}$$

Here we used the interior $W^{1,\infty}$ regularity of v. This estimate and (4.18) imply finally

$$\begin{aligned} \int_{B_2^+} |\nabla(u - v)|^2 dx &\le C \left(\int_{B_4^+} \left(|\mathbf{f}|^2 + |A - \overline{A}_{B_4^+}|^2 \right) dx + \int_{B_4^+} |u - v|^2 dx \right) \\ &\le C \left(|B_4|\delta^2 + \eta^2 \right) \\ &\le \epsilon^2, \end{aligned}$$

by taking η and δ satisfying the last inequality above. This completes our proof. □

Lemma 4.6. *There is a constant $N_1 > 0$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ with A uniformly elliptic and $(\delta, 6)$-vanishing. If $u \in H_0^1(\Omega)$ is a weak solution of (1.1) in $\Omega \supset B_6^+$ with $\partial\Omega \supset T_6$ and*

$$B_1^+ \cap \{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) \le 1\} \cap \{x \in \Omega : \mathcal{M}(|\mathbf{f}|^2)(x) \le \delta^2\} \ne \emptyset, \tag{4.20}$$

then

$$|\{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_1^+| < \epsilon |B_1^+|. \tag{4.21}$$

Proof. By (4.20), there exists a point $x_0 \in B_1^+$ such that for all $r > 0$,

$$\frac{1}{|B_r|} \int_{B_r^+(x_0)\cap\Omega} |\nabla u|^2 dx \le 1, \ \frac{1}{|B_r|} \int_{B_r^+(x_0)\cap\Omega} |\mathbf{f}|^2 dx \le \delta^2. \tag{4.22}$$

Since $B_4^+(0) \subset B_5^+(x_0)$ and by (4.22), we have

$$\frac{1}{|B_4|} \int_{B_4^+(0)} |\mathbf{f}|^2 dx \le \frac{|B_5|}{|B_4|} \frac{1}{|B_5|} \int_{B_5^+(x_0)} |\mathbf{f}|^2 dx \le \left(\frac{5}{4}\right)^n \delta^2. \tag{4.23}$$

Similarly, we see that

$$\frac{1}{|B_4|} \int_{B_4^+(0)} |\nabla u|^2 dx \le \left(\frac{5}{4}\right)^n. \tag{4.24}$$

In view of (4.23), (4.24), and from the assumption that A is $(\delta, 6)$-vanishing, we can employ Corollary 4.5 if we replace u and $\mathbf{f}$ by $\left(\frac{4}{5}\right)^n u$ and $\left(\frac{4}{5}\right)^n \mathbf{f}$, respectively,

to find that for any $\eta > 0$, there exist a small $\delta = \delta(\epsilon)$ and a weak solution v of (4.3) such that

$$\int_{B_2^+} |\nabla(u-v)|^2 dx \le \eta^2 \tag{4.25}$$

provided that

$$\frac{1}{|B_4|}\int_{B_4^+} \left(|\mathbf{f}|^2 + |A - \overline{A}_{B_4^+}|^2\right) dx \le \delta^2.$$

Now we can use the local $W^{1,\infty}$ estimates for v to see that there is a constant N_0 such that

$$\|\nabla v\|^2_{L^\infty(B_3^+)} \le N_0^2. \tag{4.26}$$

Now we set $N_1^2 := \max\left\{4N_0^2,\ 2^n\right\}$, and we claim that

$$\{x \in B_1^+ : \mathcal{M}(|\nabla u|^2) > N_1^2\} \subset \{x \in B_1^+ : \mathcal{M}_{B_4^+}(|\nabla(u-v)|^2) > N_0^2\}. \tag{4.27}$$

To see this, suppose that

$$x_1 \in \{x \in B_1^+ : \mathcal{M}_{B_4^+}(|\nabla(u-v)|^2)(x) \le N_0^2\}. \tag{4.28}$$

For $r \le 2$, $B_r^+(x_1) \subset B_3^+$, and by (4.28) and (4.26), we have

$$\frac{1}{|B_r|}\int_{B_r^+(x_1)\cap\Omega} |\nabla u|^2 dx \le \frac{2}{|B_r|}\int_{B_r^+(x_1)\cap\Omega} (|\nabla(u-v)|^2 + |\nabla v|^2) dx \le 4N_0^2.$$

For $r > 2$, $x_0 \in B_r^+(x_1) \subset B_{2r}^+(x_0)$, and by (4.22), we get

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r^+(x_1)\cap\Omega} |\nabla u|^2 dx &\le \frac{1}{|B_r|}\int_{B_{2r}^+(x_0)\cap\Omega} |\nabla u|^2 dx \\ &= \frac{2^n}{|B_{2r}|}\int_{B_{2r}^+(x_0)\cap\Omega} |\nabla u|^2 dx \le 2^n.\end{aligned}$$

This says that

$$x_1 \in \{x \in B_1^+ : \mathcal{M}(|\nabla u|^2)(x) \le N_1^2\}; \tag{4.29}$$

whence assertion (4.27) follows from (4.28) and (4.29). Now by (4.27), weak 1-1 estimates and (4.25), we have

$$\begin{aligned} & |\{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_1^+| \\ \le\ & |\{x \in \Omega : \mathcal{M}_{B_4^+}(|\nabla(u-v)|^2)(x) > N_0^2\} \cap B_1^+| \\ <\ & \frac{C}{N_0^2}\int_{B_2^+} |\nabla(u-v)|^2 dx \\ \le\ & \frac{C}{N_0^2}\eta^2 \\ =\ & \epsilon|B_1^+|, \end{aligned}$$

by taking η and δ satisfying the last identity above, which completes our proof. □

We can now deduce the following corollary immediately from the previous lemma and a scaling argument.

Corollary 4.7. *There is a constant $N_1 > 0$ so that for any $\epsilon,\ r > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ with A uniformly elliptic and $(\delta, 6r)$-vanishing. If $u \in H_0^1(\Omega)$ is a weak solution of (1.1) in $\Omega \supset B_{6r}^+$ with $\partial\Omega \supset T_{6r}$ and*

$$B_r^+ \cap \{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) \le 1\} \cap \{x \in \Omega : \mathcal{M}(|\mathbf{f}|^2)(x) \le \delta^2\} \neq \emptyset,$$

then

$$|\{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_r^+| < \epsilon |B_r^+|.$$

For the following lemma only, we denote by B_ρ ($\rho > 0$) a ball centered at a point of Ω or $\partial\Omega$ with radius ρ. We also denote by B_ρ^+ the intersection B_ρ and $\{x_n > 0\}$.

Lemma 4.8. *There is a constant $N_1 > 0$ so that for any $1 \ge \epsilon,\ r > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ with A uniformly elliptic and $(\delta, 96)$-vanishing. If $u \in H_0^1(\Omega)$ is a weak solution of (1.1) in $\Omega \supset B_{96}^+$ with $\partial\Omega \supset T_{96}$, and if*

$$|\{x \in B_1^+ : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_r| \ge \epsilon |B_r|, \tag{4.30}$$

then

$$B_r \cap \Omega \subset \{x \in B_1^+ : \mathcal{M}(|\nabla u|^2) > 1\} \cup \{x \in B_1^+ : \mathcal{M}(|\mathbf{f}|^2) > \delta^2\}. \tag{4.31}$$

Proof. We argue by contradiction. If B_r satisfies (4.30) and the conclusion (4.31) is false, then there exists $x_0 \in B_r \cap \Omega$ such that

$$\frac{1}{|B_\rho|}\int_{B_\rho(x_0)\cap\Omega} |\nabla u|^2 dx \le 1,\ \frac{1}{|B_\rho|}\int_{B_\rho(x_0)\cap\Omega} |\mathbf{f}|^2 dx \le \delta^2,\ \forall \rho > 0.$$

If $B_{6r} \cap \{x_n = 0\} = \emptyset$, this is an interior estimate (see Lemma 3.9). So suppose that $(x', 0) \in B_{6r} \cap \{x_n = 0\}$. Now observe that $B_{6r}^+ \subset B_{8r}^+(x_0) \subset B_{16r}^+(x', 0)$, to see that

$$\Omega \supset B_{96}^+ \supset B_{96r}^+(x', 0) \supset B_{16r}^+(x', 0) \supset B_r \cap B_1^+.$$

Now we apply Corollary 4.7 to the ball $B_{16r}^+(x', 0)$ with ϵ replaced by $\frac{\epsilon}{16^n}$, to obtain

$$\begin{aligned} & |\{x \in B_1^+ : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_r| \\ \le\ & |\{x \in B_1^+ : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_{16r}^+(x', 0)| \\ <\ & \frac{\epsilon}{16^n}|B_{16r}^+| \\ =\ & \epsilon |B_r^+|, \end{aligned}$$

which is a contradiction to (4.30). □

Now take N_1, ϵ and the corresponding δ given by Lemma 4.8.

Corollary 4.9. *Suppose that $u \in H_0^1(\Omega)$ is a weak solution of (1.1) in $\Omega \supset B_{96}^+$ with $\partial\Omega \supset T_{96}$, with A uniformly elliptic and $(\delta, 96)$-vanishing. Assume that*

$$|\{x \in \Omega : \mathcal{M}(|\nabla u|^2) > N_1^2\} \cap B_1^+| < \epsilon |B_1^+|. \tag{4.32}$$

Let k be a positive integer and $\epsilon_1 = 2(10)^n\epsilon$. Then, we have

$$\begin{aligned} |\{x \in B_1^+ : \mathcal{M}(|\nabla u|^2) > N_1^{2k}\}| \ \le\ & \sum_{i=1}^{k} \epsilon_1^i |\{x \in B_1^+ : \mathcal{M}(|\mathbf{f}|^2) > \delta^2 N_1^{2(k-i)}\}| \\ & + \epsilon_1^k |\{x \in B_1^+ : \mathcal{M}(|\nabla u|^2)(x) > 1\}|. \end{aligned}$$

Proof. We argue by induction on k, the case $k = 1$ being from Lemma 4.8 and Theorem 2.8 on

$$C := \{x \in B_1^+ : \mathcal{M}(|\nabla u|^2) > N_1^2\},$$
$$D := \{x \in B_1^+ : \mathcal{M}(|\mathbf{f}|^2) > \delta^2\} \cup \{x \in B_1^+ : \mathcal{M}(|\nabla u|^2)(x) > 1\}.$$

Assume now that the conclusion is valid for some positive integer $k \geq 2$. Let us define $u_1 = \frac{u}{N_1}$ and $\mathbf{f}_1 = \frac{\mathbf{f}}{N_1}$. Then, u_1 is a weak solution of

$$\begin{cases} -div(A\nabla u_1) &= div\mathbf{f}_1 \quad \text{in} \quad \Omega, \\ u_1 &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

and satisfies

$$|\{x \in \Omega : \mathcal{M}(|\nabla u_1|^2)(x) > N_1^2\} \cap B_1^+| < \epsilon |B_1^+|.$$

Then, by the induction hypothesis and an easy computation, we have

$$\begin{aligned} & |\{x \in B_1^+ : \mathcal{M}(|\nabla u|^2)(x) > N_1^{2(k+1)}\}| \\ = \; & |\{x \in B_1^+ : \mathcal{M}(|\nabla_1 u|^2)(x) > N_1^{2k}\}| \\ \leq \; & \sum_{i=1}^{k} \epsilon_1^i |\{x \in B_1^+ : \mathcal{M}(|\mathbf{f}_1|^2)(x) > \delta^2 N_1^{2(k-i)}\}| \\ & + \epsilon_1^k |\{x \in B_1^+ : \mathcal{M}(|\nabla u_1|^2)(x) > 1\}| \\ = \; & \sum_{i=1}^{k+1} \epsilon_1^i |\{x \in B_1^+ : \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2 N_1^{2(k+1-i)}\}| \\ & + \epsilon_1^{k+1} |\{x \in B_1^+ : \mathcal{M}(|\nabla u|^2)(x) > 1\}|. \end{aligned}$$

This estimate in turn completes the induction on k. □

Using this iterative lemma, we can give a simple and elementary proof for the global $W^{1,p}$ regularity of the solution of a divergence form elliptic equation with Dirichlet boundary condition.

Theorem 4.10. *Let p be a real number with $2 < p < \infty$. There is a small $\delta = \delta(p, n, \Lambda) > 0$ so that, for all uniformly elliptic and $(\delta, 96)$-vanishing A, and for all $\mathbf{f} \in L^p(\Omega; \mathbb{R}^n)$, if $u \in H_0^1(\Omega)$ is a weak solution of the elliptic PDE* (1.1) *in $\Omega \supset B_{96}^+$ with $\partial\Omega \supset T_{96}$, then u belongs to $W^{1,p}(B_1^+)$ with the estimate*

$$\int_{B_1^+} |\nabla u|^p dx \leq C \int_{B_6^+} (|u|^p + |\mathbf{f}|^p) dx,$$

where the constant C is independent of u and $\mathbf{f}$.

Proof. As in the proof of Theorem 3.1, we assume that

$$|\{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_1^+| < \epsilon |B_1^+|$$

and

$$\sum_{k=0}^{\infty} N_1^{pk} |\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^2)(x) > \delta^2 N_1^{2k}\} \cap B_1^+| \leq 1. \tag{4.33}$$

Then, in view of Corollary 4.9, from analytic computations and (4.33), there exists a constant C depending on δ, N_1, and p such that

$$\sum_{k=0}^{\infty} N_1^{pk} |\{x \in \Omega : \mathcal{M}(|\nabla u|^2)(x) > N_1^{2k}\} \cap B_1^+| \leq C \sum_{k=0}^{\infty} (N_1^p \epsilon_1)^k < +\infty$$

for ϵ selected so small that $N_1^p \epsilon_1 < 1$. Then Lemma 2.3 leads to

$$\nabla u \in L^p(B_1^+),$$

which finishes our proof. □

Remark 4.11. We can change the number 6 in Theorem 4.10 to any number greater than 1 by a scaling argument, so the number 96 also can be changed to any number greater than 1.

5. Global $W^{1,p}$ regularity in Lipschitz domains with small Lipschitz constants

5.1. **Flattening argument.** Our next goal is to show why we need the assumption that the boundary of the domain Ω is a local graph and has a small Lipschitz constant. We choose any point $x_0 \in \partial\Omega$. For our purpose assume that

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x_n > \gamma(x')\}$$

for some $r > 0$ and some $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ with $Lip(\gamma)$ small. Now define

$$\begin{aligned} y_i &= x_i = \Phi^i(x) \ (1 \leq i \leq n-1), \\ y_n &= x_n - \gamma(x') = \Phi^n(x), \end{aligned}$$

and write

$$y = \Phi(x).$$

Set $\Phi := \Psi^{-1}$ so that $\Phi \circ \Psi = \Psi \circ \Phi = Identity$ and

$$x = \Psi(y).$$

Choose $s > 0$ so small that B_s^+ lies in $\Phi(\Omega \cap B_r(x_0))$, and define $v(y) = u(\Psi(y))$ for all $y \in B_s^+$. If u is a weak solution of the PDE

$$-div(A\nabla u) = div\mathbf{f} \text{ in } \Omega,$$

then v is a weak solution of

$$-div(A_1 \nabla v) = div\mathbf{f}_1 \text{ in } B_s^+,$$

where $\mathbf{f}_1(y) = \mathbf{f}(\Psi(y))$ and

$$A_1(y) = [\nabla\Phi(\Psi(y))|^T \cdot A(\Psi(y)) \cdot [\nabla\Phi(\Psi(y))]. \tag{5.1}$$

A simple computation gives us

$$\nabla\Phi = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ . & . & ... & \\ -\gamma_{x_1} & -\gamma_{x_2} & ... & 1 \end{bmatrix}.$$

Let us compute $[A_1]_{BMO}$, assuming that $[A]_{BMO}$ is small enough. After an easy computation, we have

$$\|\nabla\Phi \cdot \nabla\Phi\|_\infty = n + \|\nabla\gamma\|_\infty^2. \tag{5.2}$$

Then in view of (5.1) and (5.2), we see that

$$[A_1]_{BMO} \leq C([A]_{BMO} + Lip(\gamma)),$$

and so $[A_1]_{BMO}$ is small provided that $[A]_{BMO} + Lip(\gamma)$ is small, which is our optimal regularity requirement in this paper.

5.2. **Proof of Theorem 1.5.** We are finally set to give our proof of Theorem 1.5.

Proof. Now that we have the boundary L^p $(2 < p < \infty)$ estimates for the gradient of u in B_1^+ in Theorem 4.10, we can get the proof by standard scaling, covering and flattening arguments along with the interior estimate and a duality argument. □

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