A QUADRATIC APPROXIMATION TO THE SENDOV RADIUS NEAR THE UNIT CIRCLE

MICHAEL J. MILLER

ABSTRACT. Define $S(n, \beta)$ to be the set of complex polynomials of degree $n \geq 2$ with all roots in the unit disk and at least one root at $\beta$. For a polynomial $P$, define $|P|_\beta$ to be the distance between $\beta$ and the closest root of the derivative $P'$. Finally, define $r_n(\beta) = \sup\{|P|_\beta : P \in S(n, \beta)\}$. In this notation, a conjecture of Bl. Sendov claims that $r_n(\beta) \leq 1$.

In this paper we investigate Sendov's conjecture near the unit circle, by computing constants $C_1$ and $C_2$ (depending only on $n$) such that $r_n(\beta) \sim 1 + C_1(1 - |\beta|) + C_2(1 - |\beta|)^2$ for $|\beta|$ near 1. We also consider some consequences of this approximation, including a hint of where one might look for a counterexample to Sendov's conjecture.

1. Introduction

In 1962, Sendov conjectured that if a polynomial (with complex coefficients) has all its roots in the unit disk, then within one unit of each of its roots lies a root of its derivative. More than 50 papers have been published on this conjecture, but it has been verified in general only for polynomials of degree at most 8 [4].

Let $n \geq 2$ be an integer and let $\beta$ be a complex number of modulus at most 1. Define $S(n, \beta)$ to be the set of polynomials of degree $n$ with complex coefficients, all roots in the unit disk and at least one root at $\beta$. For a polynomial $P$, define $|P|_\beta$ to be the distance between $\beta$ and the closest root of the derivative $P'$. Finally, define $r_n(\beta) = \sup\{|P|_\beta : P \in S(n, \beta)\}$, and note that $r_n(\beta) \leq 2$ (since by the Gauss-Lucas Theorem [5, Theorem 6.1] all roots of each $P'$ are also in the unit disk, and so each $|P|_\beta \leq 2$). In this notation, Sendov's conjecture claims that $r_n(\beta) \leq 1$.

In estimating $r_n(\beta)$, we will assume without loss of generality (by rotation) that $0 \leq \beta \leq 1$. It is already known that $r_2(\beta) = (1 + \beta)/2$ and that

$$r_3(\beta) = \left[3\beta + (12 - 3\beta^2)^{1/2}\right]/6$$

[9, Theorem 2], that $r_n(0) = (1/n)^{(n-1)/n}$ [2, Lemma 4 and $p(z) = z^n - z$, that $r_n(1) = 1$ [10, Theorem 1], and that $r_n(\beta) \leq \min(1.08332, 1 + 0.72054/n)$ [1, Corollary 1 and equation (3)].
Since $r_n(1) = 1$, an obvious place to look for counterexamples to Sendov’s conjecture is in a neighborhood of $\beta = 1$. This has already been done in [7, Theorem 3] and [12], where a linear upper bound on $r_n(\beta)$ suffices to verify the Sendov conjecture if $\beta$ is sufficiently close to 1. Unfortunately, having only an upper bound leaves many interesting questions about the conjecture unanswered. In this paper we investigate Sendov’s conjecture much more thoroughly near $\beta = 1$, by providing a quadratic approximation to $r_n(\beta)$ with

**Theorem 1.** Let $n \geq 3$, let $k$ be the largest integer such that $k \leq (n+1)/3$ and let

\[
\begin{align*}
    u_1 &= \cos \frac{2\pi k}{n + 1}, \quad u_2 = \cos \frac{2\pi (k + 1)}{n + 1}, \\
    D_1 &= \frac{-2u_1u_2 - 1}{2(1 - u_1)(1 - u_2)}, \quad D_2 = \frac{-1}{2(1 - u_1)(1 - u_2)}, \\
    D_3 &= (-1 - 4D_1 - 3D_1^2 + 2D_2^2)/2, \\
    D_4 &= (3D_1 - 4D_2 + 3D_1^2 - 2D_1D_2 - 6D_2^2)/2, \\
    D_5 &= (2 + 4D_1 + 5D_2 + 2D_1^2 + 4D_1D_2 + 3D_2^2)/2, \\
    D_6 &= (2D_2 + 2D_1D_2 + 3D_2^2)/2 \quad \text{and} \\
    D &= D_3n + D_4 + D_5/n + D_6/n^2.
\end{align*}
\]

If $n = 3$ or $n = 5$, then let $\alpha = 3/2$; otherwise let $\alpha = 2$. If $n = 5$, then let $\Delta = 7/225$; otherwise let $\Delta = 0$. Then for $\beta$ sufficiently close to 1, we have

\[
    r_{n+1}(\beta) = 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}.
\]

Before proving this theorem, we will examine some of its consequences. Our first consequence improves on estimates in [7] and [12] (by providing a value for the coefficient of the linear term) with

**Corollary 2.** For all $n \geq 2$ we have $r_n(\beta) \leq 1 - (3/10)(1 - \beta) + \mathcal{O}(1 - \beta)^2$.

**Proof.** Recall that for $2 \leq n \leq 3$ we have formulas for $r_n(\beta)$, and so the result for those values of $n$ follows from the Taylor series of these formulas at $\beta = 1$. As we will show in part 6 of Lemma 8 the quantity $D_1 + D_2/n \leq -3/10$ for all $n \geq 3$, and so the rest of Corollary 2 follows from Theorem 1. \[\square\]

As we will show in part 6 of Lemma 8 at $n = 4$ we have $D_1 + D_2/n = -3/10$, so Corollary 2 provides the smallest possible linear upper bound for $r_n(\beta)$ that is independent of $n$.

A second consequence of Theorem 1 shows that the result of [7, Theorem 3] is the best possible (in the sense that 1/3 cannot be replaced by a larger number), with

**Corollary 3.** There exist constants $K_n > 0$ with $\lim_{n \to \infty} K_n = 1/3$ such that

\[
    r_{n+1}(\beta) = 1 - K_n(1 - \beta) + \mathcal{O}(1 - \beta)^2.
\]

**Proof.** Choose $K_n = -(D_1 + D_2/n)$ and note that by Theorem 1 we have $r_{n+1}(\beta) = 1 - K_n(1 - \beta) + \mathcal{O}(1 - \beta)^2$. As we shall see in parts 5 and 6 of Lemma 8 the quantity $D_1 + D_2/n$ is negative and tends to $-1/3$. \[\square\]
Recall that \( r_n(0) = (1/n)^{1/(n-1)} \). This quantity is increasing in \( n \), so it is tempting to conjecture that for all fixed \( \beta \) the quantity \( r_n(\beta) \) is increasing in \( n \). Indeed, the graphs in [6, figure 4.8] provide some evidence of this for \( n = 4, 6, 8, 10, \) and \( 12 \). Unfortunately, this conjecture is false, as is shown by

**Corollary 4.** For \( \beta \) sufficiently close to 1 we have \( r_6(\beta) < r_4(\beta) \).

**Proof.** By Theorem 4 and the constants we will compute at the beginning of section 2 we know that

\[
r_4(\beta) = 1 - (1/3)(1 - \beta) + \mathcal{O}(1 - \beta)^2
\]

and that

\[
r_6(\beta) = 1 - (11/30)(1 - \beta) + \mathcal{O}(1 - \beta)^2,
\]

and the conclusion follows.

Corollary 2 hints of the existence of a better-than-Sendov result, for near \( \beta = 1 \) it appears that \( r_6(\beta) \) is bounded above by a function that is independent of \( n \) and strictly less than one. Unfortunately, moving up to the quadratic approximation in Theorem 4 casts doubt upon such a result. To see this, note that as \( n \) goes to infinity, then \( k/(n+1) \) tends to 1/3, so \( u_1 \) and \( u_2 \) tend to \(-1/2\), so \( D_3 \) tends to \( 4/91 \) and \( D_4 \) tends to \(-9/10 \), and so \( D + \Delta \) tends to infinity. Indeed, for \( n \) sufficiently large one might expect \( r_{n+1}(\beta) > 1 \) roughly when \( D_1(1 - \beta) + (D_3n + D_4)(1 - \beta)^2 > 0 \), i.e. when \( \beta < 1 + D_1/(D_3n + D_4) \sim 1 - 27/(4n - 9) \), provided that this \( \beta \) is “sufficiently close to 1”. This is an intriguing possibility that is clearly worthy of further investigation.

We will verify Theorem 4 by proving the following three propositions:

**Proposition 5.** Assume the notation of Theorem 4. Then for all polynomials \( P \in S(n+1, \beta) \), we have

\[
|P|_{\beta} \leq 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1}.
\]

**Proposition 6.** There are polynomials \( P \in S(6, \beta) \) with

\[
|P|_{\beta} = 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{5/2}.
\]

**Proposition 7.** Assume the notation of Theorem 4. Then there are real polynomials \( P \in S(n+1, \beta) \) with

\[
|P|_{\beta} = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1}.
\]

From the definition of \( D \) in Theorem 4 and the constants we will compute at the beginning of section 2 we see that for \( n = 5 \) we have \( D_1 + D_2/n = -11/30 \) and \( D + \Delta = 29/450 \), so Propositions 5 and 6 together imply that Theorem 4 is true for \( n = 5 \). Note that for \( n \neq 5 \) we have \( \Delta = 0 \), so Propositions 5 and 6 taken together imply that Theorem 4 is true for \( n \neq 5 \).

In [8] it was proved that if \( n = 5 \) and if \( \beta \) is sufficiently close to 1, then maximal polynomials in \( S(n+1, \beta) \) (those for which \( |P|_{\beta} = r_{n+1}(\beta) \)) must be nonreal. Taken together, Theorem 4 and Proposition 6 provide strong evidence that this is true only for \( n = 5 \) (although it is conceivable that this could fail for higher-order approximations).
2. Preliminaries

We begin by computing some values (that we will subsequently need) for the constants that appear in Theorem 1, obtaining:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_1 + D_2/n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>-1/4</td>
<td>-1/4</td>
<td>-1/3</td>
</tr>
<tr>
<td>4</td>
<td>$-1 + \sqrt{5}$</td>
<td>$-1 - \sqrt{5}$</td>
<td>-1/5</td>
<td>-2/5</td>
<td>-3/10</td>
</tr>
<tr>
<td>5</td>
<td>$-1/2$</td>
<td>$-1$</td>
<td>-1/3</td>
<td>-1/6</td>
<td>-11/30</td>
</tr>
<tr>
<td>6</td>
<td>-0.225</td>
<td>-0.9010</td>
<td>-0.3014</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>-0.7071</td>
<td>-0.2929</td>
<td>-0.2929</td>
<td>-0.3347</td>
</tr>
<tr>
<td>9</td>
<td>-0.3090</td>
<td>-0.8090</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.1423</td>
<td>-0.6549</td>
<td>-0.3138</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We next establish some relationships between these constants with

**Lemma 8.** Assume the notation of Theorem 1. Then

1. $u_2 < -1/2 \leq u_1$, and $u_1 \leq 0$ for $n \neq 4$, and $u_2 > -1$ for $n \neq 3, 5$,
2. $u_1 + u_2 < 0$ and $u_1 u_2 > -1$,
3. $2nu_1 + n + 1 \geq 1$ and $2nu_2 + n + 1 < 0$,
4. $D_1 < 0$ and $D_2 < 0$,
5. $\lim_{n \to \infty} D_1 + D_2/n = -1/3$,
6. $-1 < D_1 + D_2/n \leq -3/10$, with equality only at $n = 4$, and
7. $1 + (1 + D_1 + D_2)(u_1 - 1) - D_2(2u_1^2 - 2) = 0$ for $i = 1$ and $i = 2$.

**Proof.** From the definition of $k$ in Theorem 1 the relationship between $k$ and $n$ depends on the residue of $n$ modulo 3. For increasing values of $n$ in each of the three residue classes, the sequence $k/(n+1)$ increases to (or is equal to) $1/3$ and the sequence $(k+1)/(n+1)$ strictly decreases to $1/3$, so the values of $u_1$ decrease to (or are equal to) $-1/2$ and the values of $u_2$ strictly increase to $-1/2$. Since the values of $u_1$ decrease (or remain constant) in each residue class, and since $u_1 \leq 0$ for $n = 3, 5$ and 7, then $u_1 \leq 0$ for all $n \neq 4$. Since the values of $u_2$ strictly increase in each residue class, and since $u_2 > -1$ for $n = 4$ and $u_2 = -1$ for $n = 3$ and $n = 5$, then $u_2 > -1$ for $n \neq 3, 5$. This completes the proof of part 1 of the lemma.

For $n = 4$, we have $u_1 + u_2 = -1/2$ and $u_1 u_2 = -1/4$. For $n \neq 4$ we have from part 1 that $u_2 < u_1 \leq 0$, and part 2 of the lemma follows trivially.

Since $u_1 \geq -1/2$, then $2nu_1 + n + 1 \geq 1$. For $n = 3, 4$ and 5 we have $(k+1)/(n+1) \leq 1/2$. Since in each residue class this quotient strictly decreases to $1/3$, then for all $n \geq 3$ we have $2\pi(k+1)/(n+1) \in (2\pi/3, \pi]$. Now $\cos x \leq 1/2 - 3x/(2\pi)$ on this interval, and from the definition of $k$ in Theorem 1 we know that $k \geq (n - 1)/3$, so

$$u_2 = \cos \frac{2\pi(k+1)}{n+1} \leq \frac{1}{2} - \frac{3(k+1)}{n+1} \leq \frac{1}{2} - \frac{n+2}{n+1} = - \frac{n+1}{2n}$$

which completes the proof of part 3 of the lemma.

At $n = 4$, we have $D_1 = -1/5$ and $D_2 = -2/5$. For $n \neq 4$ we know from part 1 of Lemma 8 that $u_2 < u_1 \leq 0$ so from the definitions of $D_1$ and $D_2$ in Theorem 1 we see that $D_1 < 0$ and $D_2 < 0$. This completes the proof of part 4 of the lemma.

As $n$ tends to infinity, $u_1$ and $u_2$ tend to $-1/2$, so $D_1$ tends to $-1/3$ and $D_2$ is bounded. This completes the proof of part 5 of the lemma.
By part 2 of Lemma 8 we have \( u_1 + u_2 < 0 \) and \( u_1u_2 > -1 \). Since by part 4 of Lemma 8 we know that \( D_2 < 0 \), then
\[
D_1 + D_2/n > D_1 + D_2 = -\frac{1 + u_1u_2}{1 + u_1u_2 - (u_1 + u_2)} > -1.
\]
From part 1 of Lemma 8 we know that \( u_2 < -1/2 \leq u_1 \), so by computing the partial derivatives of \( D_1 \) we see that \( \partial D_1/\partial u_1 > 0 \) and \( \partial D_1/\partial u_2 \leq 0 \). Since in each residue class \( u_1 \) decreases to \(-1/2\) and \( u_2 \) increases to \(-1/2\), then in each residue class \( D_1 \) decreases to \(-1/3\). At \( n = 5, 6 \) and \( 10 \) we have \( D_1 < -3/10 \), and hence \( D_1 + D_2/n < D_1 < -3/10 \) for all \( n \geq 3 \) except possibly \( n = 3, 4 \) and \( 7 \). Checking the values of \( D_1 + D_2/n \) (computed at the beginning of section 2) for these exceptional values completes the proof of part 6 of the lemma.

Expressing \( D_1 \) and \( D_2 \) in terms of \( u_1 \) and \( u_2 \) and simplifying the result verifies part 7, and thus completes the proof of Lemma 8.

We now estimate the size of the coefficients of \( P' \) with

**Proposition 9.** Suppose that \( P \in S(n+1, \beta) \) with \( P' \) monic and \( |P'|_\beta \geq \beta \). Let \( P'(z) = \prod_{j=1}^{n}(z - \zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \). Then
1. each \( \Re[\zeta_j] = O(1 - \beta) \) and each \( \Im[\zeta_j] = O(1 - \beta)^{1/2} \);
2. each \( a_{n-k} = O(1 - \beta)^{k/2} \);
3. for \( k \) odd, each \( \Re[a_{n-k}] = O(1 - \beta)^{(k+1)/2} \), and
4. for \( k \) even, each \( \Im[a_{n-k}] = O(1 - \beta)^{(k+1)/2} \).

**Proof.** Parts 1–3 were proved in [8, Proposition 4]. Part 4 is proved similarly to part 3, by noting that each term of \( \Im[a_{n-k}] \) is a product of \( k \) of the \( \Re[\zeta_j] \)'s and \( \Im[\zeta_j] \)'s, and that for \( k \) even, each term has at least one \( \Re[\zeta_j] \), so from part 1 of Proposition 8 we have that \( \Im[a_{n-k}] = O(1 - \beta)^{(k+1)/2} \).

To have \( P \in S(n+1, \beta) \) requires that the moduli of the roots of \( P \) are all at most 1. We estimate these moduli with

**Proposition 10.** Assume the notation of Theorem 7. Let \( P \) be a polynomial with \( P'(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \) and \( P(\beta) = 0 \). Let \( z \neq \beta \) be a root of \( P \), let \( \omega \) be the \((n+1)\)th root of 1 that is closest to \( z \) and let \( R = (1 - \beta) + a_{n-1}(\omega^n - 1)/n + \cdots + a_0(\omega - 1) \).

1. For \( 0 < r \leq 1 \), if each \( a_k = O(1 - \beta)^r \), then \( |z|^2 = 1 - 2\Re[R] + O(1 - \beta)^{2r} \).
2. Suppose that
   \[
   a_{n-1} = n(1 + D_1 + D_2)(1 - \beta) + O(1 - \beta)^{\alpha},
   a_{n-2} = - (n - 1)D_2(1 - \beta) + O(1 - \beta)^{\alpha}, \text{ and }
   a_{n-k} = O(1 - \beta)^{\alpha} \quad \text{for} \quad k \geq 3
   \]
and define
   \[
   \Gamma_2 = 2(1 + D_1 + D_2)(D_1 - 2D_2 + nD_2) \quad \text{and } \quad \Gamma_1 = -\Gamma_2 + (-2 - 4D_1)n + (1 + 4D_1 - 4D_2).
   \]
   If \( \Re[\omega] = u_i \) for \( i = 1 \) or \( i = 2 \), then
   \[
   |z|^{2n+2} = 1 - 2(n + 1)\Re[R] + (n + 1)(\Gamma_1 + \Gamma_2u_i)(1 - \beta)^2 + O(1 - \beta)^{\alpha+1}.
   \]
Proof. Since \( \beta = 1 - (1 - \beta) \), then by the binomial theorem \( \beta^k = 1 - k(1 - \beta) + O(1 - \beta)^2 \). Since \( z \) is a root of \( P \) we have

\[
0 = P(z) = \int_{\beta}^{z} P'(t) \, dt = \frac{z^{n+1} - \beta^{n+1}}{n+1} + a_{n-1} \frac{z^n - \beta^n}{n} + \cdots + a_0 (z - \beta),
\]

and solving for \( z^{n+1} \) gives us

\[
z^{n+1} = \beta^{n+1} - (n+1) \left[ a_{n-1} \frac{z^n - \beta^n}{n} + \cdots + a_0 (z - \beta) \right].
\]

By hypothesis, as \( \beta \) goes to 1 the \( a_k \) all tend to 0 so the roots of \( P \) tend to the roots of \( z^{n+1} - 1 \), and so the \( \omega \) appearing in the hypotheses is well defined.

Now each \( \beta^k = 1 + O(1 - \beta) \), and by the hypothesis of part 1 each \( a_k = O(1 - \beta)^r \). Putting these estimates into equation (2.1), we see that \( z^{n+1} = 1 + O(1 - \beta)^r \). Then \( z = \omega + O(1 - \beta)^r \) and so \( (\omega^k - 1)/k = (\omega^k - 1)/k + O(1 - \beta)^r \).

Now note that each \( a_{n-k} = O(1 - \beta)^r \) and that each \( \beta^k = 1 - k(1 - \beta) + O(1 - \beta)^2 \). Substituting these estimates into equation (2.1) gives

\[
z^{n+1} = 1 - (n+1)(1 - \beta) - (n+1) \left[ a_{n-1} \frac{\omega^n - 1}{n} + \cdots + a_0 (\omega - 1) \right] + O(1 - \beta)^{2r}
\]

Note that \( R = O(1 - \beta)^r \) so

\[
(1 - R)^{n+1} = 1 - (n+1)R + O(1 - \beta)^{2r} = z^{n+1} + O(1 - \beta)^{2r},
\]

so \( z = \omega(1 - R) + O(1 - \beta)^{2r} \) and hence \( |z|^2 = z\bar{z} = 1 - 2R[R] + O(1 - \beta)^{2r} \). This finishes the proof of part 1.

From the hypotheses of part 2, we know that \( \Re[\omega] = u_i \) for \( i = 1 \) or \( i = 2 \). Suppose for the moment that \( \Re[\omega] = u_1 \) and write \( \omega = u_1 + iv_1 \). Since \( \omega^{n+1} = 1 \), then \( |\omega| = 1 \), so \( \omega^n = \overline{\omega} \) and \( \Re[\omega^2] = 2u_1^2 - 1 \). Let \( A = [-(1 + D_1 + D_2) + 2D_2 u_1]v_1 \).

From part 7 of Lemma [8] we see that

\[
\Re[1 + (1 + D_1 + D_2)(\overline{\omega} - 1) - D_2(\overline{\omega}^2 - 1)] = 0
\]

and so using the estimates of the \( a_{n-k} \)'s given in the hypotheses of part 2, we get

\[
R = (1 - \beta) + a_{n-1} \frac{\overline{\omega} - 1}{n} + a_{n-2} \frac{\overline{\omega}^2 - 1}{n-1} + \cdots + a_0 (\omega - 1)
\]

\[
= \left[ 1 + (1 + D_1 + D_2)(\overline{\omega} - 1) - D_2(\overline{\omega}^2 - 1) \right] (1 - \beta) + O(1 - \beta)^\alpha
\]

\[
= iA(1 - \beta) + O(1 - \beta)^\alpha.
\]

The hypotheses of part 2 imply that each \( a_k = O(1 - \beta) \), so from the proof of part 1 with \( r = 1 \) we have \( z = \omega(1 - R) + O(1 - \beta)^2 = \omega[1 - iA(1 - \beta)] + O(1 - \beta)^\alpha \)

and so

\[
(\omega^k - 1)/k = (\omega^k - 1)/k + (1 - iA\omega^k)(1 - \beta) + O(1 - \beta)^\alpha.
\]

Let \( G = n/2 - n(1 + D_1 + D_2)(1 - iA\overline{\omega}) + (n - 1)D_2(1 - iA\overline{\omega}^2) \). Then from equation (2.1) and the estimates of the \( a_k \)'s given in the hypotheses of part 2 we
get

\[ z^{n+1} = 1 - (n+1)(1-\beta) + \left(\frac{n+1}{2}\right)(1-\beta)^2 \]

\[ - (n+1) \left[ a_{n-1} \left( \frac{\omega^n-1}{n} + (1-i\omega^n)(1-\beta) \right) \right. \]

\[ + a_{n-2} \left( \frac{\omega^{n-1}-1}{n-1} + (1-i\omega^{n-1})(1-\beta) \right) \]

\[ + a_{n-3} \frac{\omega^{n-2}-1}{n-2} + \cdots + a_0(\omega-1) \] + O(1-\beta)^{n+1}

Then since \( R = iA(1-\beta) + O(1-\beta)^n \) we have

\[ |z|^{2n+2} = z^{n+1} \]

\[ = 1 - 2(n+1)R|R| + (n+1) \left[ 2R[G] + (n+1)A^2 \right] (1-\beta)^2 + O(1-\beta)^{n+1}. \]

Thus to complete the proof of part 2 of Proposition \[10\] for the case \( \text{Re}[\omega] = u_1 \) we need only verify that \( 2R[G] + (n+1)A^2 = \Gamma_1 + \Gamma_2 u_1 \).

Let \( D_0 = 1 + D_1 + D_2 \), so from the definition of \( A \) we see that

\[ A = (-D_0 + 2D_2 u_1)v_1. \]

Note that \( \text{Re}[\omega] = \Im[\omega] \). Then from the definition of \( G \) we have

\[ \text{Re}[G] = n/2 - nD_0(1-Av_1) + (n-1)D_2(1-2Au_1v_1) \]

\[ = n/2 - nD_0 + (n-1)D_2 - A[n(-D_0v_1 + 2D_2u_1v_1) - 2D_2u_1v_1] \]

\[ = (-n/2 - nD_1 - D_2) - nA^2 + 2AD_2u_1v_1 \]

so

\[ 2\text{Re}[G] + (n+1)A^2 = (-n - 2nD_1 - 2D_2) + (-n + 1)A^2 + 4AD_2u_1v_1. \]

Now

\[ 2D_2u_1^2 = \frac{-u_1^2}{(1-u_1)(1-u_2)} = D_0u_1 + (D_2-D_1), \]

so

\[ Av_1 = (-D_0 + 2D_2u_1)(1 - u_1) \]

\[ = -D_0 + 2D_2u_1 - u_1(-D_0u_1 + 2D_2u_1) \]

\[ = -D_0 + (D_1 + D_2)u_1. \]

Using these two equalities, we see that

\[ A^2 = (-D_0 + 2D_2u_1)[-D_0 + (D_1 + D_2)u_1] \]

\[ = D_0^2 + (-D_0D_1 - 3D_0D_2)u_1 + (D_1 + D_2)(2D_2u_1^2) \]

\[ = D_0^2 - D_1^2 + D_2^2 - 2D_0D_2u_1 \]

and

\[ 2AD_2u_1v_1 = 2D_2u_1[-D_0 + (D_1 + D_2)u_1] \]

\[ = -2D_0D_2u_1 + (D_1 + D_2)[D_0u_1 + (D_2 - D_1)] \]

\[ = D_0(D_1 - D_2)u_1 + (D_2^2 - D_1^2). \]
Thus from equation (2.2) we have
\[ 2\Re(G) + (n + 1)A^2 = (-n - 2nD_1 - 2D_2) + (-n + 1)(D_0^2 - D_1^2 + D_2^2 - 2D_0D_2u_1) \\
+ 2[D_0(D_1 - D_2)u_1 + (D_2^2 - D_1^2)] \\
= (-1 - 2D_1 - D_0^2 + D_1^2 - D_2^2)n + (-2D_2 + D_0^2 - 3D_1^2 + 3D_2^2) \\
+ 2D_0u_1(D_1 - 2D_2 + nD_2) \\
= \Gamma_1 + \Gamma_2u_1. \]

This finishes the proof of part 2 of Proposition 10 for the case \( \Re[\omega] = u_1 \). Since \( D_1 \) and \( D_2 \) are symmetric in \( u_1 \) and \( u_2 \), swapping \( u_1 \) and \( u_2 \) in this proof verifies part 2 of Proposition 10 for the remaining case \( \Re[\omega] = u_2 \), and thus completes the proof of Proposition 10. \( \square \)

Finally, consider the linear transformation \( T \) which takes functions to real numbers via
\[ T(f) = \frac{(2nu_1 + n + 1)f(u_2) - (2nu_2 + n + 1)f(u_1)}{2(u_1 - u_2)}. \]
Recall that by Lemma 8 we have \( u_1 - u_2 > 0, 2nu_1 + n + 1 > 0 \) and \( 2nu_2 + n + 1 < 0 \), so \( T/n \) is a weighted average. This implies that \( T \) preserves inequalities, in the sense that if \( f(u_1) \leq g(u_1) \) and \( f(u_2) \leq g(u_2) \), then \( T(f) \leq T(g) \).

In the process of analyzing several inequalities, we will need the following values of the transformation \( T \):
\[ T(1) = n, \]
\[ T(1 + 2n) = n - 1, \]
\[ T(1 + 2n + 2u) = \frac{[n + 2 + 2(n + 1)(u_1 + u_2) + 4nu_1u_2]}{D_2} = \frac{n + 1 + D_1 + 3nD_1 + 3D_2}{D_2}, \]
\[ T\left(\frac{1}{1 - u}\right) = n + nD_1 + D_2, \]
\[ T\left(\frac{u}{1 - u}\right) = nD_1 + D_2. \]

We will also use the results of

Lemma 11. For the linear transformation \( T \) defined in equation (2.3) we have
1. \( T(1 + 4u + 4u^2)/(n - 2) < 1/2 \) for \( n \neq 3, 4 \) and 6, and
2. \( T(8u^2 + 8u^3) \geq 0 \) for all \( n \).

Proof. From the formula for \( T(1 + 4u + 4u^2) \) in (2.4) and from part 3 of Lemma 8 we have
\[ \partial T(1 + 4u + 4u^2)/\partial u_1 = -2(2nu_2 + n + 1) > 0 \]
\[ \partial T(1 + 4u + 4u^2)/\partial u_2 = -2(2nu_1 + n + 1) < 0. \]
Recall from the proof of Lemma 8 that for each residue class of \( n \) modulo 3 the values of \( u_1 \) decrease and the values of \( u_2 \) increase, so the signs of the partial derivatives above imply that in each residue class the values of \( T(1 + 4u + 4u^2) \) decrease. Since \( 1 + 4u + 4u^2 = (1 + 2u)^2 \geq 0 \) and since \( T \) preserves inequalities, then \( T(1 + 4u + 4u^2) \geq 0 \), so the values of \( T(1 + 4u + 4u^2)/(n - 2) \) also decrease.
in each residue class. Using the formula for \( T(1 + 4u + 4u^2) \) in (2.4) and the values of the \( u_i \) computed at the beginning of section 2, we calculate the values of \( T(1 + 4u + 4u^2)/(n - 2) \) at \( n = 5, 7 \) and 9, getting respectively 1/3, 0.4627 and 0.3372. Since they are all less than 1/2, this proves part 1 of Lemma 11.

Since by definition \( u_i \geq -1 \), then \( 8u_i^2 + 8u_i^3 = 8u_i^2(1 + u_i) \geq 0 \) for both \( i = 1 \) and \( i = 2 \), and so part 2 of Lemma 11 follows from our observation that \( T \) preserves inequalities.

Finally, we will deal with polynomials that are “almost” in \( S(n, \beta) \) using

**Lemma 12.** Suppose that \( P \) is a polynomial of degree \( n \) with all roots in \( \{ z : |z| \leq 1 + O(1 - \beta)^\gamma \} \), one root at \( \beta \), and all other roots bounded away from \( \beta \). Then there is a polynomial \( Q \in S(n, \beta) \) such that \( |Q|_\beta = |P|_\beta + O(1 - \beta)^\gamma \).

**Proof.** If \( P \in S(n, \beta) \), then we may take \( Q = P \). If not, then at least one root of \( P \) has modulus greater than 1. In this case, let

\[
c = \max \left\{ \frac{|z|^2 - 1}{|z|^2} : z \text{ is a root of } P \text{ and } |z| > 1 \right\}.
\]

Since by hypothesis \( |z - \beta| \) is bounded away from 0 and \( |z| \leq 1 + O(1 - \beta)^\gamma \), then \( 0 < c \leq O(1 - \beta)^\gamma \). In particular, for \( \beta \) sufficiently close to 1 we have \( 0 < c < 1 \).

Let \( Q \) be the polynomial with roots \( \{ z - c(z - \beta) : z \text{ is a root of } P \} \). Since the mapping \( z \mapsto z - c(z - \beta) \) is a contraction of the plane that leaves \( \beta \) fixed and moves all roots of \( P \) (and hence \( P' \)) toward \( \beta \) by at most \( O(1 - \beta)^\gamma \), then \( Q(\beta) = 0 \) and \( |Q|_\beta = |P|_\beta + O(1 - \beta)^\gamma \). Thus we need only show that all roots of \( Q \) are in the unit disk.

Note that for \( t \) real the image of the mapping \( t \mapsto z - t(z - \beta) \) is a line, with \( t = 0 \) mapping to \( z \), and \( t = 1 \) mapping to \( \beta \), and \( t = (|z|^2 - 1)/|z - \beta|^2 \) mapping to

\[
z - \frac{|z|^2 - 1}{|z|^2}(z - \beta) = z - \frac{z\bar{z} - 1}{\bar{z} - \beta} = \frac{1 - \beta z}{\bar{z} - \beta}.
\]

If \( z \) is in the unit disk, then the images of every \( t \) between 0 and 1 lie on the line between \( z \) and \( \beta \), hence in the unit disk. If \( z \) is not in the unit disk, then \( |(1 - \beta z)/(z - \beta)| < 1 \) and so the images of every \( t \) between \( (|z|^2 - 1)/|z - \beta|^2 \) and 1 lie on the line between \( (1 - \beta z)/(\bar{z} - \beta) \) and \( \beta \), hence in the unit disk. Thus for every root \( z \) of \( P \), the image of \( z \) lies in the unit disk, so all roots of \( Q \) are in the unit disk and so \( Q \in S(n, \beta) \). This completes the proof of Lemma 12. \( \square \)

3. **Proof of Proposition 5**

Take any \( P \in S(n + 1, \beta) \), assume without loss of generality that \( P' \) is monic, and write \( P'(z) = \prod_{j=1}^{n}(z - \zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \).

If \( |P|_\beta \leq 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2 \), then Proposition 4 is trivially true. Thus we may assume without loss of generality that

\[
(3.1) \quad |P|_\beta \geq 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2.
\]

From part 6 of Lemma 8 we have that \( D_1 + D_2/n > -1 \), and so inequality (3.1) implies that \( |P|_\beta \geq \beta \) as long as \( \beta \) is sufficiently close to 1. Note that \( P \) thus satisfies all the hypotheses of Proposition 8.
We begin by estimating some relationships between the coefficients of \( P' \) with

**Lemma 13.** Suppose that \( \Im[a_{n-1}] = \mathcal{O}(1 - \beta)^{3/2} \) and that each
\[
|\zeta_j - \beta| = 1 + (D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2.
\]

Then
\[
\begin{align*}
1. & \quad \Im[a_{n-2}] = -(3/2)\Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2} \quad \text{and} \\
2. & \quad \Re[a_{n-3}] + 2\Re[a_{n-4}] = (n - 2)(1 + D_1 + D_2/n)(1 - \beta)\Re[a_{n-2}] + \mathcal{O}(1 - \beta)^3.
\end{align*}
\]

**Proof.** Let each \( \zeta_j = x_j + iy_j \) and note that by Proposition \ref{prop:1} we have \( x_j = \mathcal{O}(1 - \beta) \) and \( y_j = \mathcal{O}(1 - \beta)^{1/2} \). Note that by hypothesis, \( \sum_i y_i = -\Im[a_{n-1}] = \mathcal{O}(1 - \beta)^{3/2} \) and that each
\[
(\beta - x_j)^2 + y_j^2 = |\beta - \zeta_j|^2 = 1 + 2(D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2,
\]
so solving for \( x_j \) gives us
\[(3.2)
\[
\begin{align*}
x_j &= y_j^2/2 - (1 + D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2.
\end{align*}
\]

Note that \( \Im[a_{n-3}] = -\sum_{i<j<k} \Im[\zeta_i\zeta_j\zeta_k] = \sum_{i<j<k} y_i y_j y_k + \mathcal{O}(1 - \beta)^{5/2} \), so
\[
\begin{align*}
\mathcal{O}(1 - \beta)^{5/2} &= \sum_i y_i \sum_{i<j} y_i y_j = \sum_{i\neq j} y_i^2 y_j + 3 \sum_{i<j<k} y_i y_j y_k \\
&= \sum_{i\neq j} y_i^2 y_j + 3\Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2}
\end{align*}
\]
and so \( \sum_{i\neq j} y_i y_j = -3\Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2} \). Then using equation \ref{eq:3.2} we have
\[
\begin{align*}
\Im[a_{n-2}] &= \sum_{i<j} \Im[\zeta_i\zeta_j] = \sum_{i\neq j} x_i y_j \\
&= (1/2) \sum_{i\neq j} y_i^2 y_j - (1 + D_1 + D_2/n)(1 - \beta) \sum_{i\neq j} y_j + \mathcal{O}(1 - \beta)^{5/2} \\
&= -(3/2)\Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2},
\end{align*}
\]
which completes the proof of part 1 of Lemma \ref{lem:13}.

Let \( S \) be the set of triples \((i, j, k)\) of distinct integers from 1 to \( n \) with \( j < k \). Note that \( \Re[a_{n-2}] = \sum_{i<j} \Re[\zeta_i\zeta_j] = -\sum_{i<j} y_i y_j + \mathcal{O}(1 - \beta)^2 \) and \( \Re[a_{n-3}] = -\sum_{i<j<k} \Re[\zeta_i\zeta_j\zeta_k] = \sum_{S} x_i y_j y_k + \mathcal{O}(1 - \beta)^3 \). Furthermore,
\[
\begin{align*}
\mathcal{O}(1 - \beta)^3 &= \sum_i y_i \sum_{j<k} y_j y_k y_l = \sum_S y_i^2 y_j y_k y_l + 4 \sum_{i<j<k<l} y_i y_j y_k y_l,
\end{align*}
\]
so
\[
\begin{align*}
\Re[a_{n-4}] &= \sum_{i<j<k<l} \Re[\zeta_i\zeta_j\zeta_k\zeta_l] = \sum_{i<j<k<l} y_i y_j y_k y_l + \mathcal{O}(1 - \beta)^3 \\
&= (-1/4) \sum_S y_i^2 y_j y_k + \mathcal{O}(1 - \beta)^3.
\end{align*}
\]
Then using equation (3.2) we have

\[ \Re[a_{n-3}] + 2\Re[a_{n-4}] = \sum_{S} (x_i - \frac{y_i^2}{2}) y_j y_k + \mathcal{O}(1 - \beta)^3 \]

\[ = -(1 + D_1 + D_2/n)(1 - \beta)(n - 2) \sum_{j<k} y_j y_k + \mathcal{O}(1 - \beta)^3 \]

\[ = (n - 2)(1 + D_1 + D_2/n)(1 - \beta)\Re[a_{n-2}] + \mathcal{O}(1 - \beta)^3, \]

which completes the proof of Lemma 13.

We now establish a lower bound on \( \Re[a_{n-4}] \) with

**Lemma 14.** Suppose that

\[ \Im[a_{n-1}] = \mathcal{O}(1 - \beta)^\alpha, \]

\[ \Re[a_{n-2}] = -(n - 1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^\alpha, \text{ and} \]

\[ \Im[a_{n-3}] = \mathcal{O}(1 - \beta)^\alpha. \]

If \( n = 5 \), then define \( \delta = -1/15 \); otherwise define \( \delta = 0 \). Then

\[ \Re[a_{n-4}] \geq \delta(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1}. \]

**Proof.** Let each \( \zeta_j = x_j + iy_j \) and recall by Proposition 19 that \( x_j = \mathcal{O}(1 - \beta) \) and \( y_j = \mathcal{O}(1 - \beta)^{1/2} \). Let \( F(y) = \prod_{i=1}^{n} (y + y_i) = y^n + b_{n-1}y^{n-1} + \cdots + b_0 \). Note that

\[ \Re[a_{n-4}] = \sum_{i<j<k<l} \Re[\zeta_i \zeta_j \zeta_k \zeta_l] = \sum_{i<j<k<l} y_i y_j y_k y_l + \mathcal{O}(1 - \beta)^3 \]

and that by hypothesis

\[ b_{n-1} = \sum_i y_i = \sum_i \Im[\zeta_i] = -\Im[a_{n-1}] = \mathcal{O}(1 - \beta)^\alpha, \]

\[ b_{n-2} = \sum_{i<j} y_i y_j = -\sum_{i<j} \Re[\zeta_i \zeta_j] + \mathcal{O}(1 - \beta)^2 = -\Re[a_{n-2}] + \mathcal{O}(1 - \beta)^2 \]

\[ = (n - 1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^\alpha, \text{ and} \]

\[ b_{n-3} = \sum_{i<j<k} y_i y_j y_k = -\sum_{i<j<k} \Im[\zeta_i \zeta_j \zeta_k] + \mathcal{O}(1 - \beta)^{5/2} = \Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2} \]

\[ = \mathcal{O}(1 - \beta)^\alpha. \]

Let

\[ f(y) = F^{(n-4)}(y) \]

\[ = \frac{n!}{24} y^4 + \frac{(n - 1)!}{6} b_{n-1} y^3 + \frac{(n - 2)!}{2} b_{n-2} y^2 + (n - 3)! b_{n-3} y + (n - 4)! b_{n-4}. \]

Now by definition \( F \) has all real roots, hence by Rolle’s Theorem (from elementary calculus) so does \( f \). Then the “reverse” of \( f \) defined by \( y^4 f(1/y) = (n - 4)! b_{n-4} y^4 + \cdots + n!/24 \) has all real roots, so by Rolle’s theorem so does the reverse’s second derivative

\[ 12(n - 4)! b_{n-4} y^2 + 6(n - 3)! b_{n-3} y + (n - 2)! b_{n-2}. \]
Since this quadratic has all real roots, then its discriminant is nonnegative, so
\[ 6(n - 3)!b_{n-3}^2 - 48(n - 2)!(n - 4)!b_{n-2}b_{n-4} \geq 0. \]
Using our estimates of the \( b_{n-k} \)'s (including \( b_{n-4} = O(1 - \beta)^2 \)), this implies that
\[ -D_2(1 - \beta)b_{n-4} \geq O(1 - \beta)^{2a} \]
and so \( b_{n-4} \geq O(1 - \beta)^{2a-1} \). Now for \( n \neq 3, 5 \) we have \( \alpha = 2 \) and so \( \Re[a_{n-4}] = b_{n-4} + O(1 - \beta)^3 \geq O(1 - \beta)^3 \), which finishes the proof of Lemma 14 for these values of \( n \).

Lemma 14 is trivially true for \( n = 3 \), since then \( \Re[a_{n-4}] = 0 \geq O(1 - \beta)^{5/2} \).

Finally, for \( n = 5 \) we have that
\[ f(y) = 5y^4 + 4b_{n-1}y^3 + 3b_{n-2}y^2 + 2b_{n-3}y + b_{n-4} \]
has all real roots, hence by Rolle's theorem so does its derivative
\[ f'(y) = 20y^3 + 12b_{n-1}y^2 + 6b_{n-2}y + 2b_{n-3}. \]

A classical result (see e.g. [11, p. 289]) states that if a cubic polynomial \( ax^3 + bx^2 + cx + d \) has all real roots, then its discriminant is nonnegative, so
\[ 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 \geq 0. \]
Applying this to \( f'(y) \), we have
\[ -4[20][6b_{n-2}]^3 - 27[20]^2[2b_{n-3}]^2 \geq O(1 - \beta)^4, \]
which implies that \( 2b_{n-2}^3 + 5b_{n-3} \leq O(1 - \beta)^4 \). Since for \( n = 5 \) we have \( D_2 = -1/6 \), then by hypothesis \( b_{n-2} = (-2/3)(1 - \beta) + O(1 - \beta)^{3/2} \), and so
\[ b_{n-3}^2 \leq (-2/5)b_{n-2}^2 + O(1 - \beta)^4 \]
\[ = (16/135)(1 - \beta)^3 + O(1 - \beta)^{7/2}. \]

We also have that the first derivative of the reverse of \( f \)
\[ 4b_{n-4}y^3 + 6b_{n-3}y^2 + 6b_{n-2}y + 4b_{n-1} \]
has all real roots, so applying our classical result gives
\[ [6b_{n-3}]^2[6b_{n-2}]^2 - 4[4b_{n-4}][6b_{n-2}]^3 \geq O(1 - \beta)^6. \]
Dividing this by \( 144b_{n-2}^4 \) and recalling that \( b_{n-2} = (-2/3)(1 - \beta) + O(1 - \beta)^{3/2} \) yields
\[ 9b_{n-3}^2 + 16(1 - \beta)b_{n-4} \geq O(1 - \beta)^{7/2}. \]
Combining these two inequalities implies that for \( n = 5 \) we have
\[ \Re[a_{n-4}] = b_{n-4} + O(1 - \beta)^3 \]
\[ \geq -9b_{n-3}^2 + O(1 - \beta)^{5/2} \]
\[ \geq (-1/15)(1 - \beta)^2 + O(1 - \beta)^{5/2}. \]
This completes the proof of Lemma 14. \( \square \)

We now begin the proof of Proposition 9. Our first step will be to show that
\( |P|_3 \leq 1 + (D_1 + D_2/n)(1 - \beta) + O(1 - \beta)^2 \). Recall that \( P \) satisfies the hypotheses of Proposition 9, so each \( a_{n-k} = O(1 - \beta)^{k/2} \). Let \( \omega \neq 1 \) be any \( (n + 1) \)st root of 1 and let \( z \) be the root of \( P \) (so \( |z| \leq 1 \)) closest to \( \omega \). Then in Proposition 10 we have
\[ R = (1 - \beta) + a_{n-1}(\omega^n - 1)/n + \cdots + a_0(\omega - 1) \]
\[ = a_{n-1}(\omega^n - 1)/n + O(1 - \beta) \]
and so by part 1 of Proposition [10] with \( r = 1/2 \), we have
\[
|z|^2 = 1 - 2\Re[a_{n-1}(\omega^n - 1)/n] + O(1 - \beta).
\]

Since \( |z| \leq 1 \) and \( \omega^n = \overline{\omega} \), this implies that \( \Re[a_{n-1}(\overline{\omega} - 1)] \geq O(1 - \beta) \). Expanding the product and noting that by Proposition [9] we have \( \Re[a_{n-1}] = O(1 - \beta) \), we get that \( \Im[a_{n-1}][\omega] \geq O(1 - \beta) \). Choosing \( \omega \) non-real and repeating this argument with \( \overline{\omega} \) substituted for \( \omega \) provides that \( \Im[a_{n-1}][\overline{\omega}] \geq O(1 - \beta) \) and so \( \Im[a_{n-1}] = O(1 - \beta) \). Thus we have \( a_{n-1} = O(1 - \beta) \).

Recall that each \( a_{n-k} = O(1 - \beta)^k/2 \), so we now know that each \( a_{n-k} = O(1 - \beta) \). Since \( \omega^{n-k} = \overline{\omega}^{k+1} \), by part 1 of Proposition [10] with \( r = 1 \) we have
\[
|z|^2 = 1 - 2\Re\left((1 - \beta) + a_{n-1}\frac{\bar{\omega} - 1}{n} + a_{n-2}\frac{\bar{\omega}^2 - 1}{n - 1} + a_{n-3}\frac{\bar{\omega}^3 - 1}{n - 2}\right) + O(1 - \beta)^2.
\]

Since \( |z| \leq 1 \) this implies that
\[
(3.3) \quad -\Re\left[a_{n-1}\frac{\bar{\omega} - 1}{n} + a_{n-2}\frac{\bar{\omega}^2 - 1}{n - 1} + a_{n-3}\frac{\bar{\omega}^3 - 1}{n - 2}\right] \leq (1 - \beta) + O(1 - \beta)^2.
\]

Averaging the expressions obtained by substituting \( \omega \) and \( \overline{\omega} \) into inequality (3.3) and noting that by Proposition [9] we have \( \Re[a_{n-3}] = O(1 - \beta)^2 \) we get
\[
(3.4) \quad \Re[a_{n-1}]\Re\left[\frac{1 - \omega}{n}\right] + \Re[a_{n-2}]\Re\left[\frac{1 - \omega^2}{n - 1}\right] \leq (1 - \beta) + O(1 - \beta)^2.
\]

Let \( u = \Re[\omega] \). Note that since \( |\omega| = 1 \), then \( \Re[\omega^2] = 2u^2 - 1 \), so dividing inequality (3.4) by \( 1 - u \), we get
\[
(3.5) \quad \frac{\Re[a_{n-1}]}{n} + \frac{\Re[a_{n-2}]}{n - 1}(2 + 2u) \leq \frac{1 - \beta}{1 - u} + O(1 - \beta)^2
\]
for each \( \omega \neq 1 \). In particular, inequality (3.5) holds for \( u = u_1 \) and \( u = u_2 \) as defined in Theorem 1.

Applying the linear transformation \( T \) defined in equation (2.3) to inequality (3.5), and using the values computed in (2.4), we see that
\[
(3.6) \quad \Re[a_{n-1}] + \Re[a_{n-2}] \leq (n + nD_1 + D_2)(1 - \beta) + O(1 - \beta)^2.
\]

Recall that \( P(z) = \prod_{j=1}^{n}(z - \zeta_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \), that each \( a_{n-k} = O(1 - \beta) \) and that \( \Re[a_{n-3}] = O(1 - \beta)^2 \). Then
\[
(3.7) \quad |P|_{2n}^2 = (\min_j |\beta - \zeta_j|)^{2n} \leq \prod_{j=1}^{n} |\beta - \zeta_j|^2 = |P'(\beta)|^2
\]
\[
= |P'(\beta)T'(\beta)| = \beta^{2n} + 2\Re[a_{n-1}]\beta^{2n-1} + 2\Re[a_{n-2}]\beta^{2n-2} + O(1 - \beta)^2
\]
\[
= 1 - 2n(1 - \beta) + 2\Re[a_{n-1}] + 2\Re[a_{n-2}] + O(1 - \beta)^2
\]
\[
= [1 - (1 - \beta) + (\Re[a_{n-1}] + \Re[a_{n-2}])/n]^{2n} + O(1 - \beta)^2
\]
and so using inequalities (3.7) and then (3.6) we have
\[
(3.8) \quad |P|_\beta \leq 1 - (1 - \beta) + (\Re[a_{n-1}] + \Re[a_{n-2}])/n + O(1 - \beta)^2
\]
\[
\leq 1 + (D_1 + D_2/n)(1 - \beta) + O(1 - \beta)^2.
\]

This completes our first step.
showing that $a_{n-1} = n(1 + D_1 + D_2)(1 - \beta) + \mathcal{O}(1 - \beta)\alpha$, $a_{n-2} = -(n-1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)\alpha$, and $a_{n-k} = \mathcal{O}(1 - \beta)\alpha$ for $k \geq 3$.

Combining inequalities (3.1) and (3.8), we see that $|P|_{\beta} = 1 + (D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2$.

Since equation (3.8) is thus an equality, then so are equations (3.7) and (3.6), and thus equation (3.5) for $u = u_i$ and equations (3.4) and (3.3) for $\Re[\omega] = u_j$.

Combining inequalities (3.1) and (3.8), we see that $\Re[a_{n-1}] = n(1 + D_1 + D_2)(1 - \beta) + \mathcal{O}(1 - \beta)^2$

$$= n(1 + D_1 + D_2)(1 - \beta) + \mathcal{O}(1 - \beta)^2$$

and $\Re[a_{n-2}] = -\frac{n-1}{2(1-u_1)(1-u_2)}(1 - \beta) + \mathcal{O}(1 - \beta)^2$

$$= -(n-1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^2.$$  

Note that from Proposition 11 we have that $\Re[a_{n-k}] = \mathcal{O}(1 - \beta)^2$ for $k \geq 3$, so we now have the correct real parts for our second step. Thus we need only show that each $\Im[a_{n-k}] = \mathcal{O}(1 - \beta)\alpha$.

Recalling the definitions of $u_1$ and $u_2$ in Theorem 11 we can choose $\omega_1$ and $\omega_2$ to be $(n + 1)$st roots of 1 so that $\Re[\omega_i] = u_i$. For $\omega = \omega_1$, expanding the products in equality (3.3) and cancelling those terms of equality (3.4) gives us

$$\left|\frac{3[a_{n-1}]}{n}\Re[\omega] + \frac{3[a_{n-2}]}{n-1}\Re[\omega^2] + \frac{3[a_{n-3}]}{n-2}\Re[\omega^3]\right| = \mathcal{O}(1 - \beta)^2.$$  

Consider the case $i = 1$. Since $|\omega_1| = 1$ and since by part 1 of Lemma 8 we have $-1/2 \leq u_1 < 1$, then $\Re[\omega_1] \neq 0$. Now by Proposition 7 $\Im[a_{n-k}] = \mathcal{O}(1 - \beta)^{3/2}$ for $k \geq 2$, so equation (3.11) implies that $\Im[a_{n-1}] = \mathcal{O}(1 - \beta)^{3/2}$. If $n = 3$ or $n = 5$, then by definition $\alpha = 3/2$, so this completes our second step for those two values of $n$.

Assume then without loss of generality that $n \neq 3, 5$. Again by part 1 of Lemma 8 we have $-1 < u_2 < u_1 < 1$ so $\Re[\omega_1] \neq 0$. Thus we may divide equation (3.10) by $\Im[\omega_1]$ to obtain

$$\left|\frac{3[a_{n-1}]}{n}\Re[\omega] + \frac{3[a_{n-2}]}{n-1}(2u_i) + \frac{3[a_{n-3}]}{n-2}(4u_i^2 - 1)\right| = \mathcal{O}(1 - \beta)^2.$$  

Now subtracting equality (3.11) with $i = 2$ from equality (3.11) with $i = 1$ and dividing by $2(u_1 - u_2)$ produces

$$\left|\frac{3[a_{n-2}]}{n-1} + \frac{3[a_{n-3}]}{n-2}2(u_1 + u_2)\right| = \mathcal{O}(1 - \beta)^2.$$  

Since equation (3.7) is an equality, we have each $|\beta - \zeta_j| = |P|_{\beta} + \mathcal{O}(1 - \beta)^2$. Recall that $\Im[a_{n-1}] = \mathcal{O}(1 - \beta)^{3/2}$ and that $|P|_{\beta} = 1 + (D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2$.
\( \mathcal{O}(1 - \beta)^2 \). Then by part 1 of Lemma 13 we have \( \Im[a_{n-2}] = (-3/2) \Im[a_{n-3}] + \mathcal{O}(1 - \beta)^{5/2} \), so substituting into (3.12) we have

\[
\Im[a_{n-3}] \left[ \frac{-3}{n-1} + \frac{2(u_1 + u_2)}{n-2} \right] = \mathcal{O}(1 - \beta)^2.
\]

Now by part 2 of Lemma 3 we have \( u_1 + u_2 < 0 \) so the quantity in brackets is non-zero. Then \( \Im[a_{n-3}] = \mathcal{O}(1 - \beta)^2 \), and so solving back in equations (3.12) and (3.11) we find that \( \Im[a_{n-k}] = \mathcal{O}(1 - \beta)^2 \) for all \( k \leq 3 \). Note that by Proposition 9 we have \( a_{n-k} = \mathcal{O}(1 - \beta)^2 \) for all \( k \geq 4 \), and so \( \Im[a_{n-k}] = \mathcal{O}(1 - \beta)^2 \) for all \( k \).

Since \( n \neq 3, 5 \), then by definition \( \alpha = 2 \), and so this finishes the proof of our second step.

We will now finish the proof of Proposition 5. Consider only those roots \( z \) of \( P \) such that the nearest \( \omega \) has \( \Re[\omega] = u_i \). In our second step, we verified the hypotheses of part 2 of Proposition 10 so we have

\[
|z|^2 = 1 - 2(n + 1) \Re[R] + (n + 1)(\Gamma_1 + \Gamma_2 u_i)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}.
\]

Since \( |z| \leq 1 \), this implies that

\[
-\Re[R] \leq -\frac{\Gamma_1 + \Gamma_2 u_i}{2}(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}
\]

and so from the definition of \( R \) in Proposition 10 we have

\[
-\Re \left[ a_{n-1} \frac{\overline{w} - 1}{n} + a_{n-2} \frac{\overline{w}^2 - 1}{n-1} + \cdots + a_0 (\omega - 1) \right] \leq (1 - \beta) - \frac{\Gamma_1 + \Gamma_2 u_i}{2}(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}.
\]

Since \( \Re(\overline{w}) = u_i \), this inequality is also valid when \( \omega \) is replaced by \( \overline{w} \). Note that by Proposition 9 we have \( \Re[a_{n-k}] = \mathcal{O}(1 - \beta)^3 \) for \( k \geq 5 \), so averaging these two inequalities gives us

\[
(3.13) \quad \frac{\Re[a_{n-1}]}{n} \Re[1 - \omega] + \cdots + \frac{\Re[a_{n-4}]}{n-3} \Re[1 - \omega^4] \leq (1 - \beta) - \frac{\Gamma_1 + \Gamma_2 u_i}{2}(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1}.
\]

Note that since \( |\omega| = 1 \), then \( \Re[\omega^2] = 2u_i^2 - 1 \), \( \Re[\omega^3] = 4u_i^3 - 3u_i \) and \( \Re[\omega^4] = 8u_i^4 - 8u_i^2 + 1 \). Dividing inequality (3.13) by \( 1 - u_i \), we get

\[
\frac{\Re[a_{n-1}]}{n} + \frac{\Re[a_{n-2}]}{n-1} (2 + 2u_i) + \frac{\Re[a_{n-3}]}{n-2} (1 + 4u_i + 4u_i^2) + \frac{\Re[a_{n-4}]}{n-3} (8u_i^2 + 8u_i^3) \leq \frac{1 - \beta}{1 - u_i} - \frac{(\Gamma_1 + \Gamma_2 u_i)(1 - \beta)^2}{2(1 - u_i)} + \mathcal{O}(1 - \beta)^{\alpha + 1}.
\]

Applying to this the linear transformation \( T \) defined in (2.3) and using the values computed in (2.4), we get an inequality of the form

\[
\Re[a_{n-1}] + c_3 \Re[a_{n-2}] + c_4 \Re[a_{n-3}] + c_4 \Re[a_{n-4}] \leq (n + nD_1 + D_2)(1 - \beta)
\]

\[
- \left[ (\Gamma_1/2)(n + nD_1 + D_2) + (\Gamma_2/2)(nD_1 + D_2) \right] (1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha + 1},
\]

where \( c_3 = T(1 + 4u + 4u^2)/(n - 2) \) and \( c_4 = T(8u^2 + 8u^3)/(n - 3) \).
Define
\[
Q = (-\Gamma_1/2)(n + nD_1 + D_2) - (\Gamma_2/2)(nD_1 + D_2) - (n - 1)(n - 2)(1 - c_3)D_2(1 + D_1 + D_2/n).
\]

Recall from our second step that for all \( n \) we have that \( \Re[a_{n-1}] = \mathcal{O}(1 - \beta)^{3/2} \), and that \( \Re[a_{n-2}] = - (n - 1)D_2(1 - \beta) + \mathcal{O}(1 - \beta)^2 \), and that each \( |\zeta_j - \beta| = 1 + (D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^2 \). Then by part 2 of Lemma 13 we have \( \Re[a_{n-1}] + 2\Re[a_{n-4}] = -(n - 1)(n - 2)D_2(1 + D_1 + D_2/n)(1 - \beta) + \mathcal{O}(1 - \beta)^3 \).

Adding 1 - \( c_3 \) times this to inequality (3.14) gives us
\[
(3.16) \quad \Re[a_{n-1}] + \Re[a_{n-2}] + \Re[a_{n-3}] + (2 - 2c_3 + c_4)\Re[a_{n-4}] \\
\leq (n + nD_1 + D_2)(1 - \beta) + Q(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1}.
\]

Note that Lemma 11 implies that \( c_3 < 1/2 \) for \( n \neq 3, 4, \) and 6 and that \( c_4 \geq 0 \) for all \( n \). Using the definition of \( T \) in (4.3), we calculate that for \( n = 4 \) we have \( c_3 = 3/2 \) and \( c_4 = 4 \), and for \( n = 6 \) we have \( c_3 = 0.729 \) and \( c_4 = 0.972 \). Thus for all \( n \geq 4 \) we have 1 - 2\( c_3 \) and \( c_4 > 0 \). Note also that by our second step and Lemma 13 we have \( \Re[a_{n-4}] \geq \delta(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1} \). Since \( \delta = 0 \) except when \( n = 5 \), and for \( n = 5 \) we calculate \( c_3 = 1/3 \) and \( c_4 = 2 \), then
\[
-(1 - 2c_3 + c_4)\Re[a_{n-4}] \leq -(1 - 2c_3 + c_4)\delta(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1} \\
= -(7\delta/3)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1}.
\]

Adding this to equation (3.16) gives us
\[
(3.17) \quad \Re[a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}] \\
\leq (n + nD_1 + D_2)(1 - \beta) + (Q - 7\delta/3)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1}.
\]

Let
\[
Q_1 = -n(1 - \beta) + a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5} \quad \text{and} \quad Q_2 = n(n - 1)(1 - \beta)^2/2 - [(n - 1)a_{n-1} + (n - 2)a_{n-2}](1 - \beta).
\]

Recall from our first step that each \( a_{n-k} = \mathcal{O}(1 - \beta) \) so \( Q_1 = \mathcal{O}(1 - \beta) \) and \( Q_2 = \mathcal{O}(1 - \beta)^2 \).

Now from our second step we know that \( a_{n-k} = \mathcal{O}(1 - \beta)^{\alpha} \) for \( k \geq 3 \), and from Proposition 9 we know that \( a_{n-k} = \mathcal{O}(1 - \beta)^{3} \) for \( k \geq 6 \), so
\[
P'(\beta) = \beta^n + a_{n-1}\beta^{n-1} + \cdots + a_0 \\
= 1 - n(1 - \beta) + \frac{n(n - 1)}{2}(1 - \beta)^2 + a_{n-1}[1 - (n - 1)(1 - \beta)] \\
+ a_{n-2}[1 - (n - 2)(1 - \beta)] + a_{n-3} + a_{n-4} + a_{n-5} + \mathcal{O}(1 - \beta)^{\alpha+1} \\
= 1 + Q_1 + Q_2 + \mathcal{O}(1 - \beta)^{\alpha+1}.
\]

Then \( |P'(\beta)|^2 = P'(\beta)\overline{P'(\beta)} = 1 + 2\Re(Q_1) + 2\Re(Q_2) + |Q_1|^2 + \mathcal{O}(1 - \beta)^{\alpha+1} \). Note from our second step that each \( \Im[a_{n-k}] = \mathcal{O}(1 - \beta)^{\alpha} \) so \( \Im(Q_1) = \mathcal{O}(1 - \beta)^{\alpha} \). Then \( (1 + \Re(Q_1) + \Re(Q_2))^2 = |P'(\beta)|^2 + \mathcal{O}(1 - \beta)^{\alpha+1} \) and so \( |P'(\beta)| = 1 + \Re(Q_1) + \Re(Q_2) + \mathcal{O}(1 - \beta)^{\alpha+1} \).
\(O(1 - \beta)^{\alpha+1}\). Substituting the values of \(Q_1\) and \(Q_2\) and using the results of our second step gives us

\[
|P'(\beta)| = 1 - n(1 - \beta) + \Re[a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}]
\]

(3.18)

\[+ (n - 1)[n/2 - n(1 + D_1 + D_2) + (n - 2)D_2](1 - \beta)^2 + O(1 - \beta)^{\alpha+1}.
\]

Using the first line of inequality (3.7), then inequalities (3.18) and (3.17), we have

\[
|P|_\beta^n \leq |P'(\beta)|
\]

(3.19)

\[\leq 1 + (nD_1 + D_2)(1 - \beta)
\]

and so comparing this with the definition of \(D\) in Theorem 1, we see that

\[(n - 2)(1 - c_3D_2) = (1 + 3D_1 + D_2)n + (1 + D_1 + D_2).
\]

Substituting these values into equation (3.19) and collecting like powers of \(n\), we conclude that

\[
Q = \left[1 - D_1 - D_1^2 + D_2^2\right]n^2 + \left[\frac{1}{2}D_1 + nD_1 + D_1^2 - 3D_2^2\right] + \left[1 + 2D_1 + \frac{1}{2}D_2 + D_1^2 + D_1D_2 + 2D_2^2\right] + [D_2 + D_1D_2 + D_2^2]/n
\]

and so comparing this with the definition of \(D\) in Theorem 1, we see that

\[
Q - (n - 1)(n/2 + nD_1 + 2D_2) = nD_1 + n(n - 1)/(D_1 + D_2/n^2).
\]

Substituting this into inequality (3.19), we have

\[
|P|_\beta^n \leq 1 + (nD_1 + D_2)(1 - \beta)
\]

\[+ nD_1 + n(n - 1)/(D_1 + D_2/n^2) - 7\delta/3] (1 - \beta)^2 + O(1 - \beta)^{\alpha+1}
\]

\[= \left[1 + (D_1 + D_2/n)(1 - \beta) + \left(D - \frac{7\delta}{3n}\right) (1 - \beta)^2\right] + O(1 - \beta)^{\alpha+1}.
\]
Note that (from the definitions of $\delta$ in Lemma 11 and $\Delta$ in Theorem 1) for all $n$ we have $\Delta = -7\delta/(3n)$, and so

$$|P|_{\beta} \leq 1 + (D_1 + D_2/n)(1 - \beta) + (D + \Delta)(1 - \beta)^2 + O(1 - \beta)^{\alpha + 1}.$$ 

This completes the proof of Proposition 5.

4. Proof of Proposition 6

This proof parallels the proof of [8, Theorem 2]. We begin by letting

$$u = -\frac{i\sqrt{15}}{15}(1 - \beta)^{1/2} - \frac{6}{10}(1 - \beta) + \frac{i\sqrt{15}}{300}(1 - \beta)^{3/2} - \frac{33}{600}(1 - \beta)^2$$

and

$$v = \frac{4i\sqrt{15}}{15}(1 - \beta)^{1/2} - \frac{1}{10}(1 - \beta) + \frac{46i\sqrt{15}}{300}(1 - \beta)^{3/2} + \frac{532}{600}(1 - \beta)^2.$$ 

Let $P'(z) = (z - u)^4(z - v)$ and let $P(z) = \int_{\beta}^{z} P'(t) \, dt$. Note that $u - \beta = -1 + u + (1 - \beta)$ so

$$|u - \beta|^2 = \left[ -1 + (4/10)(1 - \beta) - (33/600)(1 - \beta)^2 \right]^2 + \left[ (-\sqrt{15}/15)(1 - \beta)^{1/2} + (\sqrt{15}/300)(1 - \beta)^{3/2} \right]^2 = 1 - (11/15)(1 - \beta) + (79/300)(1 - \beta)^2 + O(1 - \beta)^3$$

and $v - \beta = -1 + v + (1 - \beta)$ so

$$|v - \beta|^2 = \left[ -1 + (9/10)(1 - \beta) + (532/600)(1 - \beta)^2 \right]^2 + \left[ (4\sqrt{15}/15)(1 - \beta)^{1/2} + (46\sqrt{15}/300)(1 - \beta)^{3/2} \right]^2 = 1 - (11/15)(1 - \beta) + (79/300)(1 - \beta)^2 + O(1 - \beta)^3.$$ 

Now

$$\left[ 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 \right]^2 = 1 - (11/15)(1 - \beta) + (79/300)(1 - \beta)^2 + O(1 - \beta)^3,$$

and so we have

$$|P|_{\beta} = \min\{|u - \beta|, |v - \beta|\} = 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 + O(1 - \beta)^3.$$ 

By definition $P$ is of degree 6 and $P(\beta) = 0$. Thus to verify that $P \in S(6, \beta)$ we need only show that all the roots of $P$ remain in the closed unit disk when $\beta$ is sufficiently close to 1. Now

$$u^2 = (-1/15)(1 - \beta) + (2i\sqrt{15}/25)(1 - \beta)^{3/2} + O(1 - \beta)^2,$$

$$u^3 = (i\sqrt{15}/225)(1 - \beta)^{3/2} + O(1 - \beta)^2,$$

and

$$u^4 = O(1 - \beta)^2,$$
so writing $P'(z) = z^5 + a_4 z^4 + \cdots + a_0$, we calculate that 
\[ a_4 = -(4u + v) = (5/2)(1 - \beta) - (i\sqrt{15}/6)(1 - \beta)^{3/2} - (2/3)(1 - \beta)^2 \]
\[ a_3 = u(6u + 4v) = (2/3)(1 - \beta) - (2i\sqrt{15}/15)(1 - \beta)^{3/2} + 3(1 - \beta)^2 + O(1 - \beta)^{5/2} \]
\[ a_2 = -(4u + 6v) = (4i\sqrt{15}/45)(1 - \beta)^{3/2} + (7/5)(1 - \beta)^2 + O(1 - \beta)^{5/2} \]
\[ a_1 = u^3(u + 4v) = (-1/15)(1 - \beta)^2 + O(1 - \beta)^{5/2} \]
\[ a_0 = -u^4v = O(1 - \beta)^{5/2}. \]

Recall from the values computed at the beginning of section 2 that for $n = 5$ we have $a = 3/2, u_1 = -1/2, u_2 = -1, D_1 = -1/3$ and $D_2 = -1/6$. Note that in part 2 of Proposition 10 the values of the $a_k$’s computed above satisfy the hypotheses, and that $\Gamma_2 = -5/6$ and $\Gamma_1 = -13/6$.

Let us apply part 2 of Proposition 10 to the case $\omega = -1$. Note that $\Re[\omega] = u_2$ and $\Gamma_1 + \Gamma_2 u_2 = -4/3$. Since $\omega = -1$ we have 
\[ R = (1 - \beta) - (2/5)a_4 - (2/3)a_2 - 2a_0, \]
and so 
\[ \Re[R] = \Re[((1 - \beta) - (2/5)[(5/2)(1 - \beta) - (2/3)(1 - \beta)^2] - (2/3)(7/5)(1 - \beta)^2 + O(1 - \beta)^{5/2} = (-2/3)(1 - \beta)^2 + O(1 - \beta)^{5/2}. \]

Thus by part 2 of Proposition 10 we have 
\[ |z|^2 = 1 - (2/3)(1 - \beta)^2 + 6(-4/3)(1 - \beta)^2 + O(1 - \beta)^{5/2} = 1 + O(1 - \beta)^{5/2}, \]
and so $|z| = 1 + O(1 - \beta)^{5/2}$.

Let us now apply part 2 of Proposition 10 to the case $\omega = (1/2)(-1 \pm i\sqrt{3})$. Note that $\Re[\omega] = u_1$ and $\Gamma_1 + \Gamma_2 u_1 = -7/4$. Now 
\[ R = (1 - \beta) + (a_4/10)(-3 \pm i\sqrt{3}) + (a_3/8)(-3 \pm i\sqrt{3}) \]
\[ + (a_1/4)(-3 \pm i\sqrt{3}) + (a_0/2)(-3 \pm i\sqrt{3}) \]
so 
\[ \Re[R] = (1 - \beta) - (3/10)[(5/2)(1 - \beta) - (2/3)(1 - \beta)^2] \pm (\sqrt{3}/10)(-\sqrt{15}/6)(1 - \beta)^{3/2} - (3/8)\left[(2/3)(1 - \beta) + 3(1 - \beta)^2\right] \pm (\sqrt{3}/8)(-2\sqrt{15}/15)(1 - \beta)^{3/2} - (3/4)(-1/15)(1 - \beta)^2 + O(1 - \beta)^{5/2} \]
\[ = (-7/8)(1 - \beta)^2 + O(1 - \beta)^{5/2}. \]

Thus by part 2 of Proposition 10 we have 
\[ |z|^2 = 1 - 12(-2/3)(1 - \beta)^2 + 6(-4/3)(1 - \beta)^2 + O(1 - \beta)^{5/2} = 1 + O(1 - \beta)^{5/2}, \]
so $|z| = 1 + O(1 - \beta)^{5/2}$. 

Finally, let us apply part 1 of Proposition 11 with $r = 1$ to the case $\omega = (1/2)(1 \pm i\sqrt{3})$. Note that

$$ R = (1 - \beta) + (a_4/10)(-1 \mp i\sqrt{3}) + (a_3/8)(-3 \mp i\sqrt{3}) + O(1 - \beta)^{3/2} $$

so

$$ \Re[R] = (1 - \beta) + (-1/10)(5/2)(1 - \beta) + (-3/8)(2/3)(1 - \beta) + O(1 - \beta)^{3/2} $$

$$ = (1/2)(1 - \beta) + O(1 - \beta)^{3/2}. $$

Thus by part 1 of Proposition 11 we have $|z|^2 = 1 - (1 - \beta) + O(1 - \beta)^{3/2}$ and so $|z| = 1 - (1/2)(1 - \beta) + O(1 - \beta)^{3/2}$.

At this stage, we know that $|P_\beta| = 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 + O(1 - \beta)^3$ and that if $\beta$ is sufficiently close to 1, then all roots $z$ of $P$ have $|z| \leq 1 + O(1 - \beta)^{3/2}$. Since the roots of $P$ approach the roots of $z^6 - 1$, then the non-\(\beta\) roots of $P$ are bounded away from $\beta$. Thus by Lemma 12 there is a polynomial $Q \in S(6, \beta)$ with $|Q|_\beta = 1 - (11/30)(1 - \beta) + (29/450)(1 - \beta)^2 + O(1 - \beta)^{3/2}$. This completes the proof of Proposition 11.

5. Proof of Proposition 7

Let $b_1 = 1 + D_1 + D_2/n$, let $b_2 = (n-1)D_2$, and let $z_0 = -b_1(1 - \beta) - D(1 - \beta)^2$. Then $z_0 - \beta = -1 + (1 - b_1)(1 - \beta) - D(1 - \beta)^2$, and (for $\beta$ near 1) this is real and negative so $|z_0 - \beta| = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2$.

Now let $x$ be a real constant, depending only on $n$ (and to be determined later), and let

$$ q(z) = z^2 + [(b_2 + 2b_1)(1 - \beta) - 2x(1 - \beta)^2]z $$
\[+ \left[ -b_2(1 - \beta) + (b_1^2 + b_2 + 2D + 2x)(1 - \beta)^2 \right].$$

Now by part 4 of Lemma 8 we have $D_2 < 0$ and so $b_2 < 0$. Since the discriminant of $q(z)$ is $4b_2(1 - \beta) + O(1 - \beta)^2$, then (for $\beta$ near 1) the roots of $q$ are complex conjugates. If we denote these roots by $z_1$ and $\overline{z}_1$, then by writing $\beta = 1 - (1 - \beta)$ we have

$$ |z_1 - \beta|^2 = (z_1 - \beta)(\overline{z}_1 - \beta) = q(\beta) $$
$$ = 1 + (2b_1 - 2)(1 - \beta) + (1 - 2b_1 + b_1^2 + 2D)(1 - \beta)^2 + O(1 - \beta)^3 $$
$$ = [1 + (b_1 - 1)(1 - \beta) + D(1 - \beta)^2]^2 + O(1 - \beta)^3, $$

so $|z_1 - \beta| = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + O(1 - \beta)^3$.

Let $P'(z) = (z - z_0)^{n-2}q(z)$ and $P(z) = \int_{\beta}^{P'} P'(t) dt$, so

$$ |P|_\beta = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + O(1 - \beta)^3. $$

Now $z_0 = O(1 - \beta)$, so

$$ (z - z_0)^{n-2} = z^{n-2} - (n - 2)z_0 z^{n-3} + \binom{n-2}{2} z_0^2 z^{n-4} + O(1 - \beta)^3 $$
$$ = z^{n-2} + (n - 2) \left[ b_1 (1 - \beta) + D(1 - \beta)^2 \right] z^{n-3} $$
$$ + \binom{n-2}{2} b_1^2 (1 - \beta)^2 z^{n-4} + O(1 - \beta)^3. $$
Then letting $t_1 = (n^2 - n)b_1^2 / 2 + (n - 2)b_1b_2 + b_2$ we have

$$P'(z) = (z - z_0)^{n-2} q(z)$$

(5.1)

$$= z^n + [(n b_1 + b_2)(1 - \beta) + (n D - 2D - 2x)(1 - \beta^3)]z^{n-1} + [-b_2(1 - \beta) + (t_1 + 2D + 2x)(1 - \beta^2)]z^{n-2} - (n - 2)b_1b_2(1 - \beta^2)z^{n-3} + \mathcal{O}(1 - \beta^3).$$

Note that by its definition, $P$ is a polynomial of degree $n + 1$ and $P(\beta) = 0$. Thus to show that $P \in S(n + 1, \beta)$ it will suffice to show that all roots of $P$ remain in the unit disk when $\beta$ is sufficiently close to 1.

Let $\omega \neq 1$ be an $(n + 1)$th root of 1, let $u = \Re[\omega]$ and note that since $|\omega| = 1$, then $\Re[\omega^2] = 2u^2 - 1$, $\Re[\omega^3] = 4u^3 - 3u$, and $\omega^{n-k} = \omega^{k+1}$. Substituting the coefficients of equation (5.1) into the formula for $R$ in Proposition 10 we have

$$R = (1 - \beta) + (nb_1 + b_2)(1 - \beta)(\omega - 1)/n$$

$$- b_2(1 - \beta)(\omega^2 - 1)/(n - 1) + \mathcal{O}(1 - \beta^2).$$

Substituting the values of $b_1$ and $b_2$ into this formula, we see by part 1 of Proposition 10 with $r = 1$ that

$$|z|^2 = 1 - 2(1 - \beta)[1 + (1 + D_1 + D_2)(u - 1) - D_2(2u^2 - 2)] + \mathcal{O}(1 - \beta^2).$$

Recall from part 4 of Lemma 8 that $D_2 < 0$, so the quantity in square brackets is quadratic in $u$ with positive leading coefficient. By elementary calculus, its minimum (over all real numbers) occurs when $1 + D_1 + D_2 - 4D_2u = 0$, which happens when $u = (1 + D_1 + D_2)/(4D_2) = (u_1 + u_2)/2$, which is between $u_1$ and $u_2$. Now $u_1$ and $u_2$ are (by definition) the real parts of adjacent $(n + 1)$th roots of 1, so there are no possible values of $u$ between $u_1$ and $u_2$, so the minimum (over all possible values of $u$) must occur at either $u_1$ or $u_2$. From part 7 of Lemma 8 we see that at these values the quantity in square brackets is 0, and so the minimum value of the quantity in square brackets is 0. Thus for $\Re[\omega] \neq u_i$ the quantity in square brackets is positive, so for these values of $\omega$ and for $\beta$ sufficiently close to 1 we have $|z| < 1$, and so these roots remain in the unit disk.

Thus we need only concern ourselves with the case $\Re[\omega] = u_i$. In this case, by part 2 of Proposition 11 we have

$$|z|^{2n+2} = 1 - 2(n + 1)\Re[R] + (n + 1)(\Gamma_1 + \Gamma_2u_i)(1 - \beta)^2 + \mathcal{O}(1 - \beta)^{\alpha+1}. $$

To get $P \in S(n + 1, \beta)$ we will seek a value of $x$ so that $|z| = 1 + \mathcal{O}(1 - \beta)^{\alpha+1}$, so we will need

(5.2) $$\Re[R] - (1/2)(\Gamma_1 + \Gamma_2u_i)(1 - \beta)^2 = \mathcal{O}(1 - \beta)^{\alpha+1}$$

for both $i = 1$ and $i = 2$.

Substituting the coefficients of equation (5.1) into the formula for $R$ in Proposition 11 we have

$$R = (1 - \beta) + [(n b_1 + b_2)(1 - \beta) + (n D - 2D - 2x)(1 - \beta^2)](\omega - 1)/n$$

(5.3) $$+ [-b_2(1 - \beta) + (t_1 + 2D + 2x)(1 - \beta^2)](\omega^2 - 1)/(n - 1)$$

$$- (n - 2)b_1b_2(1 - \beta^2)(\omega^3 - 1)/(n - 2) + \mathcal{O}(1 - \beta^3).$$
Thus by Lemma 12, there is a real polynomial $Q$ and such that the coefficient of $(1-\beta)$ in $\mathbb{R}[R]$ is zero, so to satisfy equation (5.2) we need only find a value of $x$ such that the coefficient of $(1-\beta)^2$ in equation (5.2) is 0. We divide this coefficient by $u_i - 1$ and denote the result by $Z_i$, so

\begin{equation}
Z_i = \frac{(nD - 2D - 2x)(u_i - 1) + (t_1 + 2D + 2x)(2u_i^2 - 2)}{(n - 1)}\left[(1 - \beta)^2 + O(1 - \beta)^3\right].
\end{equation}

Note that the coefficient of $x$ in $Z_i$ is $-2/n + (4u_i + 4)/(n - 1)$, which is non-zero by part 3 of Lemma 8, so each equation $Z_i = 0$ has a solution for $x$. To show that these solutions are identical, we will show that $Z_1$ and $Z_2$ (considered as linear expressions in the variable $x$) are scalar multiples of each other.

To see this, we eliminate $x$ by applying the transformation $T$ defined in equation (2.3). Since in equation (3.14) we defined $c_3 = T(1 + 4u + 4u^2)/(n - 2)$, then from equations (2.4) we see that

\begin{equation}
T(Z_i) = nD + t_1 - (n - 1)(n - 2)c_3D_2(1 + D_1 + D_2/n) + (\Gamma_1/2)(n + nD_1 + D_2) + (\Gamma_2/2)(nD_1 + D_2).
\end{equation}

Comparing this to the value of $Q$ defined in equation (5.6), we see that

\begin{equation}
T(Z_i) = nD + t_1 - Q - (n - 1)(n - 2)D_2(1 + D_1 + D_2/n).
\end{equation}

Note that by equation (3.21) we have

\begin{equation}
Q = nD + \frac{n(n - 1)}{2}(D_1 + D_2/n)^2 + (n - 1)(n/2 + nD_1 + 2D_2).
\end{equation}

Substituting the values of $b_1$ and $b_2$ into our definition of $t_1$ gives us

\begin{equation}
t_1 = (n - 1)\left[\frac{(n/2)(1 + D_1 + D_2/n)^2 + (n - 2)D_2(1 + D_1 + D_2/n)}{2}\right]
\end{equation}

and so $Q - t_1 = nD - (n - 1)(n - 2)D_2(1 + D_1 + D_2/n)$. Substituting this into equation (5.6) gives us $T(Z_i) = 0$. Since $T(Z_i)$ is a linear combination of $Z_1$ and $Z_2$, this implies that $Z_1$ and $Z_2$ (considered as polynomials in $x$) are scalar multiples of one another, and so there is a single value of $x$ that satisfies equation (5.2) for both $i = 1$ and $i = 2$. Using this value of $x$, we have now constructed a real polynomial $P$ with

\begin{equation}|P|_{\beta} = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + O(1 - \beta)^3
\end{equation}

and such that all roots $z$ of $P$ have $|z| \leq 1 + O(1 - \beta)^{n+1}$. Since the roots of $P$ approach the roots of $z^{n+1} - 1$, then the non-$\beta$ roots of $P$ are bounded away from $\beta$. Thus by Lemma 12, there is a real polynomial $Q \in S(n + 1, \beta)$ with

\begin{equation}|Q|_{\beta} = 1 + (D_1 + D_2/n)(1 - \beta) + D(1 - \beta)^2 + O(1 - \beta)^{n+1}.
\end{equation}

This finishes the proof of Proposition 7.
A QUADRATIC APPROXIMATION TO THE SENDOV RADIUS

REFERENCES


Department of Mathematics, Le Moyne College, Syracuse, New York 13214
E-mail address: millermj@mail.lemoyne.edu

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use