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# ON THE SINGULAR SPECTRUM OF SCHRÖDINGER OPERATORS WITH DECAYING POTENTIAL

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Abstract. The relation between Hausdorff dimension of the singular spectrum of a Schrödinger operator and the decay of its potential has been extensively studied in many papers. In this work, we address similar questions from a different point of view. Our approach relies on the study of the so-called Krein systems. For Schrödinger operators, we show that some bounds on the singular spectrum, obtained recently by Remling and Christ-Kiselev, are optimal.

## Introduction

We consider a Schrödinger operator $L_q y = -y'' + qy$ on the positive half-line $\mathbb{R}_+$ with boundary condition $y'(0) + hy(0) = 0$. Assume that $q \in L^\infty(\mathbb{R}_+)$ is a real-valued function and $h \in \mathbb{R} \cup \{\infty\}$. Denote the spectral measure of the operator $L_q$ by $\rho$.

Recently, Remling [22] proved the following theorem.

**Theorem 0.1** ([22], [23]). *If $|q(x)| \le C(1+x)^{-\beta}$ with $1/2 < \beta \le 1$, then the support of the (possible) singular part of $\rho$ has Hausdorff dimension less than or equal to $2(1-\beta)$.*

Actually, the stronger result was obtained, that is, the set of all positive spectral parameters such that the transfer matrix is not bounded at infinity has Hausdorff dimension less than or equal to $2(1-\beta)$.

A result of the same nature was proved in [3]. We give a slightly weaker version here.

**Theorem 0.2** ([3]). *Suppose that $0 < \gamma \le 1$ and*

$$\int_0^\infty (1+s)^\gamma q^2(s)\, ds < \infty.$$

*Then the support of the (possible) singular part of $\rho$ has Hausdorff dimension less than or equal to $1-\gamma$.*

This theorem readily implies the following statement.

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**Theorem 0.3.** *Suppose that* $0 < \gamma \leq 1$ *and*

$$\int_x^\infty q^2(s)\,ds \leq \frac{C}{(1+x)^\gamma}.$$

*Then the support of the (possible) singular part of* $\rho$ *has Hausdorff dimension less than or equal to* $1-\gamma$.

However, the presence of a non-trivial singular continuous part of $\rho$ for some potentials was only guessed. More attention to the subject was attracted when Simon [28] asked the following question.

**Question 0.4.** *Do there exist potentials* $q$ *on* $\mathbb{R}_+$ *so that* $|q(x)| \leq C(1+x)^{-(1/2+\varepsilon)}$, $(\varepsilon > 0)$ *and the spectral measure of* $L_q$ *has a non-trivial singular continuous part?*

The first example of a potential from $L^2(\mathbb{R}_+)$ with $\rho$ having a singular continuous component for some $h$ was given by Denisov [6]. Deift-Killip [5] proved that for potentials from $L^2(\mathbb{R}_+)$, the essential support of the absolutely continuous part of the spectral measure is the whole positive half-line. Later, Kiselev [16] constructed Schrödinger operators with potentials decaying arbitrarily slower than $C(1+x)^{-1}$ and having an embedded singular continuous component.

In [6], the construction was carried out in the opposite direction. It started with a specific spectral measure. Then the analysis of the corresponding inverse spectral problem was used to establish the required properties of the potential.

One of the main results of the present paper is the following theorem.

**Theorem 0.5.** *For any* $0 < \gamma_0 < 1$ *and* $0 < \gamma < \gamma_0$, *there is a potential* $q$ *with the properties:*

i) *The support of the singular component of* $\rho$ *has Hausdorff dimension exactly equal to* $1-\gamma_0$.
ii) *The following estimate holds:*

$$\int_x^\infty q^2(s)ds \leq \frac{C}{(1+x)^\gamma}.$$

Thus, we see that the spectral measures of Schrödinger operators from Theorem 0.3 can indeed contain a singular continuous component. Furthermore, we show that the inequality in the above theorem is sharp. Results of a similar flavor for other differential systems can be found in Sections 2.1 and 3.1.

Our methods are essentially different from those of [3], [16], [22]. The approach of this paper is based on certain well-known results from the theory of orthogonal polynomials [13], [29], [30]. Therefore, we start with the continuous analogs of orthogonal polynomials, the solutions of the so-called Krein systems. For these systems, we study the questions discussed above. It turns out that the problem can be reduced to a certain problem of minimization. We analyze this minimization problem using elementary methods of complex analysis and approximation theory.

The plan of the paper is as follows. In Section 1, we introduce some notation and discuss well-known results. In Section 2, we study the Krein systems case. Results obtained for Krein systems are applied to Dirac and Schrödinger operators in Section 3. The theorems for orthogonal polynomials are in Section 4.

We conclude the introduction with some notation. Given a measure $\sigma$ on $\mathbb{R}$, $\sigma_s, \sigma_{ac}$ refer to its singular and absolutely continuous components, respectively. Lebesgue measure is denoted by $m$. The Hausdorff dimension of a Borel set $E \subset \mathbb{R}$

is denoted by $\dim_{\mathrm{H}} E$. The characteristic function of a set $K$ is denoted by $\chi_K$. Abbreviation "a.e." means "almost everywhere". As usual, $W^{m,2}(\mathbb{R}_+)$ stand for the standard Sobolev spaces on $\mathbb{R}_+$, $m$ being the smoothness index. For $f \in L^2(\mathbb{R})$, $\hat{f}$ is its Fourier transform. $C$ is a constant changing from one relation to another.

## 1. Preliminaries

1.1. In this subsection, we briefly discuss some simple properties of polynomials that are orthogonal on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. A detailed presentation of the subject can be found in [13], [29], [30]. The unit disk in the complex plane is denoted by $\mathbb{D}$.

Let $\sigma$ be a probability measure on $\mathbb{T}$. Let $\{\varphi_n\}$ be polynomials, orthonormal with respect to $\sigma$, that is, $\int_{\mathbb{T}} \varphi_n \overline{\varphi_m}\, d\sigma = \delta_{nm}$, $\delta_{nm}$ being the Kronecker symbol. We also consider monic orthogonal polynomials $\{\psi_n\}$, that is,

$$\int_{\mathbb{T}} \psi_n(t)\overline{\psi_m}(t)\, d\sigma(t) = k_n \delta_{nm},$$

where $\psi_n(z) = z^n + \cdots$, and $k_n = ||\psi_n||_\sigma^2$. These polynomials can be explicitly computed. Consider momenta of $\sigma$, given by $c_k = \int_{\mathbb{T}} e^{ikt}\, d\sigma(t)$. Define the matrix

$$M_n = \begin{bmatrix} c_0 & c_1 & \dots & c_n \\ \bar{c}_1 & c_0 & \dots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}_n & \bar{c}_{n-1} & \dots & c_0 \end{bmatrix}$$

and let $\Delta_n = \det M_n$. It is easy to see that

$$\psi_n(z) = \sum_{k=0}^{n} (-1)^k \frac{\Delta_{n+1,n+1-k}}{\Delta_{n-1}} z^{n-k}, \tag{1.1}$$

where $\Delta_{n+1,n+1-k}$ denotes the determinant of $M_n$ with dropped $(n+1)$-th row and $(n+1-k)$-th column. The sequence $\{\psi_n\}$ generates the set $\{a_n\}, a_n \in \mathbb{D}$, of Verblunsky coefficients by means of the relations

$$\begin{cases} \psi_{n+1}(z) &= z\psi_n(z) - \bar{a}_n \psi_n^*(z), \\ \psi_{n+1}^*(z) &= \psi_n^*(z) - a_n z \psi_n(z), \end{cases} \tag{1.2}$$

where $\psi_0(z) = \psi_0^*(z) = 1$ and $\psi_n^*(z) = z^n \overline{\psi_n(1/\overline{z})}$. Vice versa, given a sequence $\{a_n\}, a_n \in \mathbb{D}, n = 0, 1, \ldots$, we can define $\sigma$ and the orthogonal polynomials $\{\psi_n\}$ uniquely; see [13] and [15, Ch. 5] for details.

We say that $\sigma$ is a Szegő measure if $\log \sigma'_{ac} \in L^1(\mathbb{T})$. The following theorem is classical.

**Theorem 1.1** ([13], [30]). *The following assertions are equivalent:*

i) *$\sigma$ is a Szegő measure.*
ii) *The series $\sum_{k=0}^{\infty} |\varphi_n(z)|^2$ converges for at least one (and hence, for all) $z \in \mathbb{D}$.*
iii) *There exists a subsequence $\{\varphi_{n_k}^*\}$ bounded for at least one (and hence, for all) $z \in \mathbb{D}$.*
iv) *The sequence $\{a_k\}$ lies in $l^2$.*

*In the above case, the limit $\pi(z) = \lim_{n\to\infty} \varphi_n^*(z),\ z \in \mathbb{D}$, exists, and $\pi^{-1} \in H^2(\mathbb{D})$.*

Furthermore, $2\pi\sigma'_{ac}(\theta) = |\pi(e^{i\theta})|^{-2}$, $\pi$ is an outer function, and

$$\sum_{k=n}^{\infty} |a_k|^2 \le C \inf_{p\in\mathcal{P}_n} ||\pi_0 - p||_\sigma^2; \tag{1.3}$$

see [13, Ch. 2]. In this formula, $\mathcal{P}_n$ is the space of polynomials of degree less than or equal to $n$, and $\pi_0 = \chi_{\mathbb{T}\setminus E}\pi$, where $E = \text{supp } \sigma_s$.

1.2. In this subsection, we introduce the so-called Krein systems and briefly discuss their properties. A modern presentation of the topic is in [8]; see also [2], [18], [19], [24] in this connection.

By a Krein system (a K-system), we mean a system of differential equations

$$\begin{cases} P'(r,\lambda) &= i\lambda P(r,\lambda) - \overline{A(r)}P_*(r,\lambda), \\ P'_*(r,\lambda) &= -A(r)P(r,\lambda), \end{cases} \tag{1.4}$$

with boundary conditions $P(0,\lambda) = P_*(0,\lambda) = 1$. We suppose that $A \in C(\mathbb{R}_+)$, $\lambda \in \mathbb{C},\ r \in \mathbb{R}_+$.

It turns out that a K-system defines a unique positive measure $\sigma$ on $\mathbb{R}$, $\int_{\mathbb{R}}(1+\lambda^2)^{-1}d\sigma(\lambda)d\lambda < \infty$, with the following characteristic property. Introduce the Fourier transform $\mathcal{F} : L^2(\mathbb{R}_+) \to L^2(\sigma)$ by the formula

$$(\mathcal{F}f)(\lambda) = \int_0^\infty f(r)P(r,\lambda)\,dr.$$

The inverse Fourier transform $\mathcal{F}^{-1} : L^2(\sigma) \to L^2(\mathbb{R}_+)$ is given by the relation

$$(\mathcal{F}^{-1}g)(r) = \int_{\mathbb{R}} g(\lambda)\overline{P(r,\lambda)}\,d\sigma(\lambda),$$

and we have $\mathcal{F}^{-1}\mathcal{F}f = f$ for any $f \in L^2(\mathbb{R}_+)$; see [8], [18], [26]. The above integrals should be understood in the $L^2$ sense.

**Theorem 1.2** ([18], [26])**.** *For any $f \in L^2(\mathbb{R}_+)$, the Parseval equality holds:*

$$||\mathcal{F}f||_\sigma^2 = \int_{\mathbb{R}} |\mathcal{F}f(\lambda)|^2\,d\sigma(\lambda) = \int_0^\infty |f(s)|^2\,ds.$$

Notice that $\mathcal{F}$ is not necessarily a unitary map.

It turns out that the functions $P(r,.)$ from (1.4) and monic orthogonal polynomials $\psi_n$ from (1.2) have a lot in common.

We describe the solution of the inverse spectral problem for a class of K-systems to be used later. The general construction can be found in [8], [18], [24]. Let $\sigma_0 = m/(2\pi)$ and let $\sigma$ be a positive measure on $\mathbb{R}$. Assume that $\text{supp}\,(\sigma - \sigma_0)$ is compact. Introduce

$$H(t) = \int_{\mathbb{R}} e^{i\lambda t}\,d(\sigma - \sigma_0)(\lambda).$$

The function $H \in C^\infty(\mathbb{R})$ gives rise to an integral equation for the "resolvent" kernel $\Gamma_r$,

$$\Gamma_r(t,\tau) + \int_0^r H(t-s)\Gamma_r(s,\tau)\,ds = H(t-\tau). \tag{1.5}$$

Then, the following lemma holds.

**Lemma 1.3.** *The functions*

$$
\begin{aligned}
P(r,\lambda) &= e^{i\lambda r}\left(1-\int_0^r \Gamma_r(s,0)e^{-i\lambda s}\,ds\right),\\
P_*(r,\lambda) &= 1-\int_0^r \Gamma_r(0,s)e^{i\lambda s}\,ds
\end{aligned}
\tag{1.6}
$$

*are solutions to the K-system* (1.4) *with* $A(r)=\Gamma_r(0,r)$.

*Sketch of the proof.* The lemma is classical [18]. We quote the main steps of its proof for reader's convenience only. Relation (1.5) readily yields that, for $t,\tau\in[0,r]$,

$$
\begin{aligned}
\Gamma_r(t,\tau) &= \overline{\Gamma_r(\tau,t)}, & (\Gamma_r(t,\tau))'_r &= -\Gamma_r(t,r)\Gamma_r(r,\tau),\\
\Gamma_r(t,\tau) &= \Gamma_r(r-\tau,r-t). & &
\end{aligned}
$$

The first equality is immediate, since $H(t)=\overline{H(-t)}, t\in[0,r]$. The second relation is obtained by differentiating (1.5) with respect to $r$ [14, Ch. 4, Sect. 7], [19, Sects. 1.4, 1.5]. The last identity is just the change of variable $s_1=r-s$. Now, we take $P_*(r,\lambda)$, defined by (1.6), and compute its derivative

$$
\begin{aligned}
(P_*(r,\lambda))'_r &= -\Gamma_r(0,r)e^{i\lambda r}+\int_0^r \Gamma_r(0,r)\Gamma_r(r,s)e^{i\lambda s}\,ds\\
&= -\Gamma_r(0,r)\left(e^{i\lambda r}-\int_0^r \Gamma_r(r-s,0)e^{i\lambda s}\,ds\right)\\
&= -A(r)P(r,\lambda),
\end{aligned}
$$

which is precisely the second equation in (1.6). The first equation (1.6) is deduced similarly. □

It is instructive to compare formulas (1.4) and (1.6) to (1.2) and (1.1), respectively. Notice also that if a measure $\sigma$ is even, then the corresponding function $H$ is real-valued, and so are $\Gamma_r$ and $A$.

Since $H\in C^\infty(\mathbb{R}_+)$, the Fredholm formula for the resolvents $\Gamma_r(t,\tau)$ yields $A\in C^\infty(\mathbb{R}_+)$. We also have a weakened version of Theorem 1.1.

**Theorem 1.4** ([7], [18], [25], [31])**.** *The following assertions are equivalent:*

i) *$\sigma$ is a Szegő-type measure on $\mathbb{R}$, that is, $(1+\lambda^2)^{-1}\log\sigma'_{ac}\in L^1(\mathbb{R})$.*
ii) *The integral $\int_0^\infty |P(r,\lambda)|^2\,dr$ converges for at least one (and hence, for all) $\lambda\in\mathbb{C}_+$.*
iii) *$\liminf_{r\to\infty}|P_*(r,\lambda)|$ is finite for at least one (and hence, for all) $\lambda\in\mathbb{C}_+$.*

*In the above cases, there exists a limit $\Pi(\lambda)=\lim_{r_n\to\infty}P_*(r_n,\lambda)$, $\lambda\in\mathbb{C}_+$, and $[(\lambda+i)\Pi]^{-1}\in H^2(\mathbb{C}_+)$.*

*Remark* 1.5. If $A\in L^2(\mathbb{R}_+)$, then i)–iii) are satisfied, and the limit $\Pi(\lambda)$ is independent of the choice of $r_n$. The function $\Pi$ is outer and $2\pi\sigma'_{ac}(\lambda)=|\Pi(\lambda+i0)|^{-2}$; see [25].

1.3. In this subsection, we discuss some basic properties of Hardy spaces in the upper half-plane $\mathbb{C}_+$. A complete information on the topic can be found, for instance, in [12], [17]. We also prove several auxiliary lemmas we will use later.

The space $H^p(\mathbb{C}_+),\ 1 \le p < \infty$, is a space of analytic functions on $\mathbb{C}_+$ with the property

$$||f||_p^p = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p \, dx < \infty.$$

The space $H^\infty(\mathbb{C}_+)$ is a space of uniformly bounded analytic functions on $\mathbb{C}_+$. It is well known that functions from $H^p(\mathbb{C}_+), 1 \le p \le \infty$, have boundary values a.e. on $\mathbb{R}$. In particular, for $f \in H^p(\mathbb{C}_+), 1 \le p < \infty$,

$$||f||_p^p = \int_{\mathbb{R}} |f(x)|^p \, dx.$$

The spaces $H^p(\mathbb{C}_+)$ possess the so-called reproducing kernels. Namely, let $k_{z_0}(z) = 1/(-2\pi i(z - \bar{z}_0)),\ z_0 \in \mathbb{C}_+$. It is obvious that $k_{z_0} \in H^p(\mathbb{C}_+), 1 < p \le \infty$, and, for any $f \in H^q(\mathbb{C}_+), 1/p + 1/q = 1$,

$$f(z_0) = (f, k_{z_0}) = \int_{\mathbb{R}} f(x)\overline{k_{z_0}(x)} \, dx = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(x)}{x - z_0} \, dx,$$

$(.,.)$ being the standard duality between $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$.

Let $w \in L^\infty(\mathbb{R}_+),\ w \ge 0$ be such that $w^{-1}$ is locally summable and $w(x) = 1$ outside an interval $[-a_0, a_0]$ for a fixed $a_0 \ge 0$. In particular, we have $\log w \in L^1(\mathbb{R}_+)$. We consider an outer function $g \in H^\infty(\mathbb{C}_+)$ with the property $|g|^2 = w$ a.e. on $\mathbb{R}$, or, equivalently,

$$g(z) = \exp\left( \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{x - z} \log w(x) \, dx \right).$$

We begin with two elementary lemmas on integrals depending on a parameter $s \in \mathbb{R}$.

**Lemma 1.6.** *Let $w$ be a function described above. Then, for $y \ge 1$,*

$$\int_{\mathbb{R}} |k_{iy}(x+s) - k_{iy}(x)|^2 \frac{dx}{w(x)} \le C \frac{s^2}{y^3}.$$

*Proof.* First, we compute an integral using properties of the kernels $k_{iy}$,

$$\begin{aligned} \int_{\mathbb{R}} |k_{iy}(x+s) - k_{iy}(x)|^2 \, dx &= (k_{-s+iy} - k_{iy}, k_{-s+iy} - k_{iy}) \\ &= (k_{-s+iy} - k_{iy})(-s+iy) - (k_{-s+iy} - k_{iy})(iy). \end{aligned}$$

A simple computation shows that the last expression equals $s^2/(2\pi\, y(s^2 + 4y^2))$. Notice also that

$$|k_{iy}(s+x) - k_{iy}(x)|^2 = \frac{1}{4\pi^2} \frac{s^2}{((x+s)^2 + y^2)(x^2 + y^2)}.$$

Consequently,

$$\begin{aligned} \int_{\mathbb{R}} |k_{iy}(x+s) - k_{iy}(x)|^2 \frac{dx}{w(x)} &= \int_{-a_0}^{a_0} |k_{iy}(x+s) - k_{iy}(x)|^2 \left( \frac{1}{w(x)} - 1 \right) dx \\ &+ \int_{\mathbb{R}} |k_{iy}(x+s) - k_{iy}(x)|^2 \, dx \\ &\le C \left( \frac{s^2}{y^4} + \frac{s^2}{y(s^2 + 4y^2)} \right) \le C \frac{s^2}{y^3}. \end{aligned}$$

The lemma is proved. □

*Remark* 1.7. If $w$ is an even function, $g(z)$ is real for $z \in i\mathbb{R}_+$.

We put $f = g(\cdot + s)/g$ for a fixed $s \in \mathbb{R}$.

**Lemma 1.8.** *Let functions $g$ and $f$ be as above. Then, for $y \geq 1$*

i) $$|f(iy) - 1| \leq C\,\frac{|s|}{y^2},$$

ii) $$\int_{\mathbb{R}} |k_{iy}(x)|^2 |f(x) - 1|^2\,dx \leq \textstyle\int_{\mathbb{R}} |k_{iy}(x)|^2 ||f(x)|^2 - 1|\,dx + \frac{1}{2\pi y}|\operatorname{Re}(1 - f(iy))|.$$

*Proof.* The function $f$ admits the following representation:

$$f(z) = \exp\left(\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\frac{1}{x-(s+z)} - \frac{1}{x-z}\right)\log w(x)\,dx\right).$$

We have

$$|\log f(iy)| \leq \frac{1}{2\pi}\int_{-a_0}^{a_0}\left|\frac{s\,\log w(x)}{(x-s-iy)(x-iy)}\right|dx \leq C\,\frac{|s|}{y^2}.$$

Using an obvious estimate $|e^z - 1| \leq C|z|, |z| \leq 1$, we obtain the first claim of the lemma.

The proof of ii) relies on the properties of the reproducing kernels. Observing that $k_{iy}f \in H^2(\mathbb{C}_+)$, we get

$$\begin{aligned}\int_{\mathbb{R}} |k_{iy}(x)|^2 |f(x) - 1|^2\,dx &= (k_{iy}(f-1), k_{iy}(f-1)) \\ &= (k_{iy}f, k_{iy}f) - 2\operatorname{Re}(k_{iy}f, k_{iy}) + (k_{iy}, k_{iy}).\end{aligned}$$

It is plain that $(k_{iy}f, k_{iy}) = f(iy)/(4\pi y)$ and $(k_{iy}, k_{iy}) = 1/(4\pi y)$. Hence,

$$\begin{aligned}\int_{\mathbb{R}} |k_{iy}(x)|^2 |f(x) - 1|^2\,dx &= (k_{iy}f, k_{iy}f) - \frac{1}{2\pi y}\operatorname{Re} f(iy) + \frac{1}{4\pi y} \\ &= \left\{(k_{iy}f, k_{iy}f) - \frac{1}{4\pi y}\right\} + \frac{1}{2\pi y}\operatorname{Re}(1 - f(iy)).\end{aligned}$$

To finish the proof, we notice that

$$(k_{iy}f, k_{iy}f) - \frac{1}{4\pi y} \leq \int_{\mathbb{R}} |k_{iy}(x)|^2 ||f(x)|^2 - 1|\,dx.$$

□

1.4. We will need an approximation argument for the subsequent construction; see [1], [21] in this connection. Take $f \in L^\infty(\mathbb{R})$ and let

$$(f)_r(x) = r\int_{\mathbb{R}} K(r(x-s))f(s)\,ds, \tag{1.7}$$

where $K(x) = \dfrac{12}{\pi}\left\{\dfrac{\sin(x/2)}{x}\right\}^4$ and $r \geq 0$.

**Lemma 1.9.** *Let $f \in L^2(d\sigma) \cap L^\infty(\mathbb{R})$. Then*

$$||(f)_r - f||_\sigma^2 \leq \frac{C}{r^3}\int_{\mathbb{R}}\left\{\frac{\sin rs/2}{s}\right\}^4 ||f(\cdot + s) - f||_\sigma^2\,ds,$$

*where $||\cdot||_\sigma$ refers to the norm in $L^2(d\sigma)$.*

*Proof.* It is clear that

$$(f)_r(x) - f(x) = \frac{12}{\pi r^3} \int_{\mathbb{R}} \left\{ \frac{\sin rs/2}{s} \right\}^4 (f(x+s) - f(x))\, ds.$$

Applying the Hölder inequality, we get

$$|(f)_r(x) - f(x)|^2 \le \frac{C}{r^3} \int_{\mathbb{R}} \left\{ \frac{\sin rs/2}{s} \right\}^4 |f(x+s) - f(x)|^2\, ds.$$

Integration in $x$ with respect to $\sigma$ concludes the proof. □

At last, we denote the Paley-Wiener space of entire functions of exponential type $r$ by $\boldsymbol{\mathcal{F}}_r$, that is,

$$\boldsymbol{\mathcal{F}}_r = \left\{ F : F(\lambda) = \int_0^r e^{i\lambda s} f(s)\, ds,\ f \in L^2[0,r] \right\}.$$

Observe that if $f \in H^2(\mathbb{C}_+)$, then $(f)_r \in \boldsymbol{\mathcal{F}}_{2r}$. Indeed, $\hat{f}$ is supported on $\mathbb{R}_+$, $\widehat{K(rx)}$ is supported on $[-2r, 2r]$ (see [1, Sect. 71]), and, consequently, $\widehat{(f)_r} = \widehat{K(rx)}\hat{f}$ lives on $[0, 2r]$.

## 2. Krein systems

2.1. Suppose that the measure $\sigma$ of a K-system satisfies the Szegő-type condition (see Theorem 1.4). Our goal is to understand how the properties of the singular and absolutely continuous parts of $\sigma$ and their mutual location influence the properties of the coefficient $A$. We distinguish between two different cases. In the first case, $\sigma_{ac}$ and $\sigma_s$ are well agreed. In the second case, a "good" $\sigma_{ac}$ and a singular component $\sigma_s$ are chosen more or less independently. By "good" we mean that the density $\sigma'_{ac}$ is infinitely smooth, bounded above and bounded below from zero.

In the first case, we have the following theorem.

**Theorem 2.1.** *Let $0 < \gamma_0 < 1$. For any $0 < \gamma < \gamma_0$, there exists a K-system with the properties:*

i) $\dim_{\mathrm{H}} \operatorname{supp} \sigma_s = 1 - \gamma_0$.

ii) *The corresponding real-valued coefficient $A$ satisfies the inequality*

$$\int_r^\infty A^2(s)\, ds \le \frac{C}{(1+r)^\gamma}.$$

Notice that the bound for the above integral matches perfectly the estimate for Schrödinger operators (Theorem 0.3).

As expected, the bound in the second case is worse.

**Theorem 2.2.** *Let $0 < \gamma_0 < 1$. For any $0 < \gamma < \gamma_0$, there exists a K-system with the properties:*

i) $\dim_{\mathrm{H}} \operatorname{supp} \sigma_s = 1 - \gamma_0$, *the density $\sigma'_{ac} \in C^\infty(\mathbb{R})$ is even and, in particular,*

$$\sigma'_{ac}(x) = 1/(2\pi) \begin{cases} 1/2, & |x| \le 1, \\ 1, & |x| > 2. \end{cases}$$

ii) *The real-valued coefficient $A$ satisfies the inequality*

$$\int_r^\infty A^2(s)\, ds \le \frac{C}{(1+r)^{\gamma/2}}. \tag{2.1}$$

*Remark* 2.3. If $\dim_{\mathrm{H}} \operatorname{supp} \sigma_s \to 0$, we get $A \in L^p(\mathbb{R}_+)$ with any $p > 4/3$.

2.2. The proofs of both theorems rely on several lemmas which are proved in this subsection.

**Lemma 2.4.** *If $A \in W^{1,2}(\mathbb{R}_+)$ is a real-valued coefficient of a K-system, then*

$$\lim_{y\to+\infty} y^2 \int_r^\infty P^2(s,iy)\,ds = \int_r^\infty A^2(s)\,ds \tag{2.2}$$

*for $r \geq 0$.*

*Proof.* Take the K-system (1.4) with $\lambda = iy$ and introduce $Q(r) = e^{yr}P(r,iy)$. Clearly, $Q$ is a solution of

$$\begin{cases} Q' &= -Ae^{yr}P_*, \\ P_*' &= -Ae^{-yr}Q \end{cases}$$

with boundary conditions $Q(0) = P_*(0) = 1$. We have

$$Q = 1 - \int_0^r A(s)e^{ys}P_*(s)\,ds, \tag{2.3}$$

$$P_* = 1 - \int_0^r A(s)e^{-ys}\left(1 - \int_0^s A(\xi)e^{y\xi}P_*(\xi)\,d\xi\right)ds. \tag{2.4}$$

Plug relation (2.4) in (2.3) and express $P$ through $Q$. This gives

$$\begin{aligned} P &= e^{-yr} - e^{-yr}\int_0^r A(s)e^{ys}\Big[1 - \int_0^s A(\xi)e^{-y\xi}\,d\xi \\ &+ \int_0^s A(\xi)e^{-y\xi}\int_0^\xi A(\eta)e^{y\eta}P_*(\eta)\,d\eta d\xi\Big]\,ds = I_1 - I_2 + I_3 - I_4. \end{aligned} \tag{2.5}$$

Now, we estimate integrals $\int_r^\infty I_i^2\,ds$. It is clear that

$$\int_r^\infty I_1^2\,ds \leq 2\frac{e^{-2yr}}{y}.$$

Furthermore,

$$I_2 = \frac{1}{y}\left(A(r) - A(0)e^{-yr} - e^{-yr}\int_0^r A'(s)e^{ys}\,ds\right) = \frac{A(r)}{y} + I_{21} + I_{22},$$

and, obviously, $||I_{21}||_2 \leq C/(y\sqrt{y})$. By the Young inequality for convolutions, $||I_{22}||_2 \leq C/y^2$ for large $y$. For $I_3$ from (2.5), we obtain

$$|I_3| \leq \frac{e^{-yr}}{\sqrt{y}}||A||_2\int_0^r |A(s)|e^{ys}\,ds.$$

Consequently, $||I_3||_2 \leq C/(y\sqrt{y})$. Equality (2.4) implies that $|P_*(r,iy)| \leq C$ uniformly in $r \geq 0$ and $y \geq 1$; see [7]. Hence, we get $||I_4||_2 \leq C/y^2$.

To compute the left-hand side of (2.2), we represent $|P(s,iy)|^2$ with the help of (2.5). The above estimates yield the claim of the lemma. □

In the lemma below, we use notations introduced in Section 1.3. We also let $E = \operatorname{supp}\sigma_s$ and $\Pi_0 = \chi_{\mathbb{R}\setminus E}k_{iy}\Pi$, where $\Pi$ is the function from Theorem 1.4.

**Lemma 2.5.** *Assume $A \in L^2(\mathbb{R}_+)$. Then*

$$\inf_{F\in\mathcal{F}_r} ||\Pi_0 - F||_{2,\sigma}^2 = \frac{1}{4\pi^2|\Pi(iy)|^2}\int_r^\infty |P(s,iy)|^2\,ds. \tag{2.6}$$

*Proof.* We begin with computation of $\mathcal{F}^{-1}\Pi_0$. Notice that

$$\left(\mathcal{F}^{-1}\Pi_0\right)(s) = \int_{\mathbb{R}} \Pi_0(\lambda)\overline{P(s,\lambda)}\, d\sigma(\lambda) = \frac{1}{2\pi}\overline{\left(\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{P(s,\lambda)}{\Pi(\lambda)}\frac{1}{\lambda - iy}\, d\lambda\right)}.$$

Taking into account the assumptions of the lemma and relations (1.6), we see that the last integral can be calculated by the Cauchy formula. Hence

$$(\mathcal{F}^{-1}\Pi_0)(s) = \frac{1}{2\pi}\overline{\left(\frac{P(s,iy)}{\Pi(iy)}\right)}. \tag{2.7}$$

Recall that [18]

$$|P_*(r,\lambda)|^2 - |P(r,\lambda)|^2 = 2\,\mathrm{Im}\,\lambda \int_0^r |P(s,\lambda)|^2 ds,$$

and $\lim_{r\to+\infty} P(r,\lambda) = 0,\ \lim_{r\to+\infty} P_*(r,\lambda) = \Pi(\lambda)$, for a fixed $\lambda \in \mathbb{C}_+$. Hence

$$\int_0^\infty |P(s,iy)|^2\, ds = \frac{1}{2y}|\Pi(iy)|^2,$$

or, as follows from (2.7),

$$\int_0^\infty |\mathcal{F}^{-1}\Pi_0|^2\, ds = \frac{1}{4\pi^2|\Pi(iy)|^2}\int_0^\infty |P(s,iy)|^2\, ds = \frac{1}{8\pi^2 y}.$$

On the other hand, we have $2\pi\sigma'_{ac}(\lambda) = |\Pi(\lambda + i0)|^{-2}$ by Remark 1.5, and a direct computation shows that

$$\int_{\mathbb{R}} |\Pi_0(\lambda)|^2\, d\sigma(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} |k_{iy}|^2\, d\lambda = \frac{1}{8\pi^2 y}.$$

Therefore,

$$\int_{\mathbb{R}} |\Pi_0(\lambda)|^2 d\sigma(\lambda) = \int_0^\infty |\mathcal{F}^{-1}\Pi_0|^2\, ds,$$

and we see that $\Pi_0$ belongs to the range of $\mathcal{F}$.

Pick a function $f_r \in L^2[0,r]$ and extend it to $[r,+\infty)$ by zero. It follows that

$$\mathcal{F}^{-1}(\Pi_0 - \mathcal{F}f_r)(s) = \frac{1}{2\pi\overline{\Pi(iy)}}\left(\overline{P(s,iy)} - 2\pi\overline{\Pi(iy)}f_r(s)\right).$$

Using the Parseval equality (see Theorem 1.2) and observing that $\mathcal{F}f_r \in \boldsymbol{\mathcal{F}}_r$, we conclude the proof. □

2.3. For a given $0 < \beta < 1$, we construct a non-negative function $w$ with certain special properties. This function gives rise to a measure $\sigma$. The measure, in turn, generates the K-system appearing in Theorem 2.1. The theorem is proved in the second part of the subsection.

Let $E^0 = [-1,1]$. At the first step, we set $E^1 = E^0 \backslash J_{00}$, where $J_{00}$ is the open middle interval of $E^0$, and $|J_{00}| = \beta|E^0|$. At the $(n+1)$-th step, we represent $E^n$ as $E^n = \bigcup_{k=1}^{2^n} I_{nk}$, $|I_{nk}| = 2((1-\beta)/2)^n$. Similarly, we define $E^{n+1} = \bigcup_{k=1}^{2^n}(I_{nk}\backslash J_{nk})$, where $J_{nk}$ are open middle intervals of $I_{nk}$, and $|J_{nk}| = \beta|I_{nk}|$, etc.

Consider $E_\beta = \bigcap_{k=0}^\infty E^k$. The set $E_\beta$ is the usual Cantor set of Hausdorff dimension

$$\dim_{\mathrm{H}} E_\beta = \frac{\log 2}{\log 2 - \log(1-\beta)}. \tag{2.8}$$

For any $M > 0$, we define the function $w = w_\beta$ as follows:

$$w(x) = \begin{cases} \min\{1, M|x+1|^\gamma\}, & x \le -1, \\ 0, & x \in E_\beta, \\ M\min\{|x-a|^\gamma, |x-b|^\gamma\}, & x \in J_{nk} = (a,b), \\ \min\{1, M|x-1|^\gamma\}, & x \ge 1, \end{cases} \tag{2.9}$$

where

$$0 < \gamma < \gamma_0 = \frac{-\log(1-\beta)}{\log 2 - \log(1-\beta)}. \tag{2.10}$$

Notice that $\gamma_0 + \dim_{\mathrm{H}} E_\beta = 1$. By definition, $w$ is even and lies in $\mathrm{Lip}_\gamma(\mathbb{R})$. It will be very important that $w = 0$ on $E_\beta$ and $w = 1$ outside a fixed interval.

Furthermore, for the chosen $\gamma$, we have $w^{-1} \in L^1[-1,1]$, and, consequently, $\dfrac{\log w(x)}{1+x^2} \in L^1(\mathbb{R})$. Indeed, denoting by $J_n$ an arbitrary interval $J_{nk}$ (they are of the same length), we see that

$$\begin{aligned} \int_{-1}^{1} w^{-1}\,dx &\le \frac{C}{M}\sum_{n=0}^{\infty} 2^n \int_0^{|J_n|/2} x^{-\gamma}\,dx \\ &\le \frac{C}{M}\sum_{n=0}^{\infty} 2^n \left(\frac{1-\beta}{2}\right)^{n(1-\gamma)} < \infty, \end{aligned} \tag{2.11}$$

since $2((1-\beta)/2)^{1-\gamma} < 1$ under condition (2.10). Moreover, we can fix $M$ large enough to ensure that $w = 1$ outside $[-2,2]$ and

$$\frac{1}{2\pi}\int_{-2}^{2} \frac{dx}{w} < \frac{4}{2\pi}. \tag{2.12}$$

*Proof of Theorem 2.1.* Let $0 < \gamma_0 < 1$ and $0 < \gamma < \gamma_0$ be parameters from the assumptions of the theorem. Pick an auxiliary parameter $0 < \beta < 1$ with the property $\gamma_0 + \dim_{\mathrm{H}} E_\beta = 1$ (see (2.8), (2.10)). Let $w$ be a function defined by relations (2.9) and (2.12). We set $d\sigma = 1/(2\pi w)dx + d\sigma_s$, where $\sigma_s$ is an even finite singular continuous measure on $E_\beta$ with the property

$$\int_{-2}^{2} d\sigma(x) = \frac{1}{2\pi}\int_{-2}^{2}\frac{dx}{w(x)} + d\sigma_s(x) = \frac{4}{2\pi}.$$

The measure $\sigma$ defines a Krein system with a real-valued coefficient $A$ through the solution of the inverse spectral problem (see Section 1.2). Moreover, $A \in W^{1,2}(\mathbb{R}_+)$ [6, Sect. 2]. Since $\log w \in L^1(\mathbb{R})$ and $w \in L^\infty(\mathbb{R})$, there is an outer function $\Pi \in H^\infty(\mathbb{C}_+)$ with the property $w = |\Pi|^2$. We put $\Pi_0 = \chi_{\mathbb{R}\setminus E_\beta} k_{iy}\Pi \in H^2(\mathbb{C}_+)$.

Lemmas 2.4 and 2.5 show that

$$\begin{aligned} \int_r^\infty A^2(s)\,ds &= \lim_{y\to+\infty} y^2 \int_r^\infty P^2(s,iy)\,ds \le C \lim_{y\to+\infty} y^2 \inf_{F\in\mathcal{F}_r} ||\Pi_0 - F||_\sigma^2 \\ &\le C \lim_{y\to+\infty} y^2 ||\Pi_0 - (\Pi_0)_{r/2}||_\sigma^2. \end{aligned} \tag{2.13}$$

Recalling Lemma 1.9, we obtain

$$||\Pi_0 - (\Pi_0)_{r/2}||_\sigma^2 \le \frac{C}{r^3}\int_{\mathbb{R}} \left\{\frac{\sin rs/4}{s}\right\}^4 ||\Pi_0(.+s) - \Pi_0||_\sigma^2\,ds. \tag{2.14}$$

The main part of the proof consists in estimating the norm, arising in the above integral, for $y \geq 1$ and $s \in \mathbb{R}$. We have

$$||\Pi_0(.+s) - \Pi_0||^2_\sigma = ||\Pi_0(.+s) - \Pi_0||^2_{\sigma_s} + ||\Pi_0(.+s) - \Pi_0||^2_{\sigma_{ac}}. \tag{2.15}$$

Since supp $\sigma_s = E_\beta \subset [-1,1]$, $w = |\Pi|^2 = 0$ on $E_\beta$, and $w \in \mathrm{Lip}_\gamma(\mathbb{R})$, we get for the first norm

$$\begin{aligned}\int_{\mathbb{R}} |\Pi_0(x+s) - \Pi_0(x)|^2 \, d\sigma_s(x) &= \int_{-1}^{1} |\Pi_0(x+s)|^2 \, d\sigma_s(x) \\ &\leq C \min\{|s|^\gamma, 1\} \int_{-1}^{1} |k_{iy}(x+s)|^2 d\sigma_s(x) \leq C \, \frac{\min\{|s|^\gamma, 1\}}{y^2}.\end{aligned}$$

As for the second norm in (2.15), we see

$$\begin{aligned}||\Pi_0(.+s) - \Pi_0||^2_{\sigma_{ac}} &= ||(k_{iy}\Pi)(.+s) - k_{iy}\Pi||^2_{\sigma_{ac}} \\ &\leq 2\left(||(k_{iy}(.+s) - k_{iy})\Pi(.+s)||^2_{\sigma_{ac}} + ||k_{iy}(\Pi(.+s) - \Pi)||^2_{\sigma_{ac}}\right)\end{aligned} \tag{2.16}$$

Recall that $d\sigma_{ac} = 1/(2\pi w)dx$, $w = |\Pi|^2$, and $w \in L^\infty(\mathbb{R})$. Hence,

$$\int_{\mathbb{R}} |k_{iy}(x+s) - k_{iy}(x)|^2 |\Pi(x+s)|^2 \, \frac{dx}{|\Pi(x)|^2} \leq C \int_{\mathbb{R}} |k_{iy}(x+s) - k_{iy}(x)|^2 \, \frac{dx}{|\Pi(x)|^2}.$$

Lemma 1.6 shows that the last integral is less than or equal to $C\, s^2/y^3$. Setting $f = \Pi(\cdot + s)/\Pi$, we rewrite the second term in (2.16) as

$$\int_{\mathbb{R}} |k_{iy}(x)|^2 |\Pi(x+s) - \Pi(x)|^2 \, \frac{dx}{|\Pi(x)|^2} = \int_{\mathbb{R}} |k_{iy}(x)|^2 |f(x) - 1|^2 \, dx.$$

Applying both parts of Lemma 1.8, we deduce that

$$\int_{\mathbb{R}} |k_{iy}(x)|^2 |f(x) - 1|^2 \, dx \leq C \left( \int_{\mathbb{R}} |k_{iy}(x)|^2 ||f(x)|^2 - 1| \, dx + \frac{|s|}{y^3} \right).$$

Then,

$$\int_{\mathbb{R}} |k_{iy}(x)|^2 ||f(x)|^2 - 1| \, dx = \int_{\mathbb{R}} |k_{iy}(x)|^2 |w(x+s) - w(x)| \, \frac{dx}{w(x)}. \tag{2.17}$$

For $|s| \leq 1$, we use that $w \in \mathrm{Lip}_\gamma(\mathbb{R})$

$$\begin{aligned}\int_{\mathbb{R}} |k_{iy}(x)|^2 |w(x+s) - w(x)| \, \frac{dx}{w(x)} &\leq \int_{-3}^{3} |k_{iy}(x)|^2 |w(x+s) - w(x)| \, \frac{dx}{w(x)} \\ &\leq C \, \frac{|s|^\gamma}{y^2}.\end{aligned}$$

For $|s| > 1$,

$$\begin{aligned}&\int_{\mathbb{R}} |k_{iy}(x)|^2 |w(x+s) - w(x)| \, \frac{dx}{w(x)} \\ &\quad = \int_{\{|x| \leq 2\} \cup \{|x+s| \leq 2\}} |k_{iy}(x)|^2 |w(x+s) - w(x)| \, \frac{dx}{w(x)} \leq \frac{C}{y^2}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} |k_{iy}(x)|^2 |w(x+s) - w(x)| \, \frac{dx}{w(x)} \leq C \, \frac{\min\{|s|^\gamma, 1\}}{y^2}.$$

Combining the estimates obtained, we get

$$||\Pi_0(\cdot+s)-\Pi_0||_\sigma^2 \le C\left\{\frac{\min\{|s|^\gamma,1\}}{y^2}+\frac{s^2}{y^3}+\frac{|s|}{y^3}\right\}.$$

We now turn back to (2.13). The bound above together with (2.14) imply that

$$\begin{aligned}\int_r^\infty A^2(s)\,ds &\le C\lim_{y\to+\infty}\frac{y^2}{r^3}\int_{\mathbb{R}}\left\{\frac{\sin rs/4}{s}\right\}^4\left\{\frac{\min\{|s|^\gamma,1\}}{y^2}+\frac{s^2}{y^3}+\frac{|s|}{y^3}\right\}ds\\ &= \frac{C}{r^3}\int_{\mathbb{R}}\left\{\frac{\sin rs/4}{s}\right\}^4\min\{|s|^\gamma,1\}\,ds.\end{aligned}$$

The latter integral is less than or equal to $C/r^\gamma$, and the theorem is proved. □

2.4. The proof of Theorem 2.2 is close in spirit to that of Theorem 2.1. Once again, we begin by choosing the parameters that define the measure of a K-system.

Let $0<\gamma_0<1$. We pick $0<\beta<1$ such that $\gamma_0+\dim_{\mathrm{H}} E_\beta=1$ (see Section 2.3). Consider a large parameter $r$. Define $w_r$ as

$$w_r(x)=\begin{cases}\min\{1,\kappa_r|x+1|^\gamma\}, & x\le -1,\\ 0, & x\in E_\beta,\\ \min\{1,\kappa_r|x-a|^\gamma,\kappa_r|x-b|^\gamma\}, & x\in J_{nk}=(a,b),\\ \min\{1,\kappa_r|x-1|^\gamma\}, & x\ge 1,\end{cases}$$

where $\kappa_r$ is a positive increasing function with the property $\kappa_r\to+\infty$ as $r\to+\infty$. Notice that $w_r$ also depends on $\beta$. The function $w_r$ is even and lies in $\mathrm{Lip}_\gamma(\mathbb{R})$. We have $w_r=0$ on $E_\beta$ and $w_r=1$ outside $[-2,2]$. A precise choice of $\{\kappa_r\}$ and $\gamma$ will be made later.

Repeating, in essence, computations from (2.11), we deduce that

$$\int_{-2}^2 w_r^{-1}(x)\,dx\le C,$$

provided $0<\gamma<\gamma_0$ (see (2.10)). Taking $j_r=\dfrac{\log(C_0\kappa_r)}{\gamma(\log 2-\log(1-\beta))}$, we have

$$\begin{aligned}\int_{-2}^2\log\frac{1}{w_r(x)}\,dx &\le C\sum_{k\le j_r}2^k\int_0^{\kappa_r^{-1/\gamma}}\log\frac{1}{\kappa_r x^\gamma}\,dx\\ &\quad + C\sum_{k\ge j_r}2^k\int_0^{\beta(1-\beta)^k2^{-k}}\log\frac{1}{\kappa_r x^\gamma}\,dx\le C\frac{\log\kappa_r}{\kappa_r^{\gamma_0/\gamma}}.\end{aligned}$$

*Proof of Theorem 2.2.* As before, we put $d\sigma=d\sigma_{ac}+d\sigma_s$, where $\sigma_s$ is an even singular continuous measure supported on $E_\beta$,

$$\int_{-2}^2 d\sigma(x)=\frac{4}{2\pi},$$

and $\sigma_{ac}$ is the absolutely continuous measure described in the theorem. The measure defines a K-system with a real-valued coefficient $A\in W^{1,2}(\mathbb{R}_+)$ (see Section 1.2). For any $r>0$, consider outer functions $v_r\in H^\infty(\mathbb{C}_+)$ with the property $w_r=|v_r|^2$ and normalized as in Remark 1.7. We also take an outer function

$\Pi \in H^\infty(\mathbb{C}_+)$ so that $\sigma'_{ac} = 1/(2\pi|\Pi|^2)$. Moreover, we let $v_{r0} = \chi_{\mathbb{R}\setminus E_\beta} k_{iy} v_r \Pi$ and $\Pi_0 = \chi_{\mathbb{R}\setminus E_\beta} k_{iy} \Pi$.

Lemmas 2.4 and 2.5 give us the following inequalities:

$$\begin{aligned} \int_r^\infty A^2(s)\,ds &\le C \lim_{y\to+\infty} y^2 \inf_{F\in\boldsymbol{\mathcal{F}}_r} ||\Pi_0 - F||_\sigma^2 \\ &\le C \lim_{y\to+\infty} y^2 \left\{ ||\Pi_0 - v_{r0}||_\sigma^2 + \inf_{F\in\boldsymbol{\mathcal{F}}_r} ||v_{r0} - F||_\sigma^2 \right\}. \end{aligned} \tag{2.18}$$

The first summand in this expression can be easily estimated. Namely, we have

$$\begin{aligned} 2\pi||\Pi_0 - v_{r0}||_\sigma^2 &= ||k_{iy}(v_r - 1)||_2^2 \\ &= ||k_{iy} v_r||^2 - \frac{1}{4\pi y} + \frac{2}{4\pi y}(1 - \mathrm{Re}\ v_r(iy)) \le \frac{C}{y}\,(1 - v_r(iy)), \end{aligned}$$

because $||v_r||_\infty = 1$ and $v_r(iy) \in \mathbb{R}$ (see Remark 1.7). Furthermore, using the fact that $\log w_r$ is supported on $[-2, 2]$ and $1 - e^{-x} \le x$ for $x \in \mathbb{R}$, we proceed as follows:

$$\begin{aligned} 1 - v_r(iy) &= 1 - \exp\left( \frac{1}{2\pi} \int_{\mathbb{R}} \frac{y}{x^2 + y^2} \log w_r(x)\,dx \right) \\ &\le \frac{1}{2\pi} \int_{-2}^{2} \frac{y}{x^2 + y^2} \log w_r^{-1}(x)\,dx. \end{aligned}$$

So, we come to

$$\begin{aligned} \lim_{y\to+\infty} y^2 ||\Pi_0 - v_{r0}||_\sigma^2 &\le C \lim_{y\to+\infty} y^2 \int_{-2}^{2} \frac{\log w_r^{-1}(x)}{x^2 + y^2}\,dx \\ &\le C \int_{-2}^{2} \log w_r^{-1}(x)\,dx \le C\,\frac{\log \kappa_r}{\kappa_r^{\gamma_0/\gamma}}. \end{aligned}$$

We turn to the second term in (2.18). By Lemma 1.9, we get

$$\inf_{F\in\boldsymbol{\mathcal{F}}_r} ||v_{r0} - F||_\sigma^2 \le \frac{C}{r^3} \int_{\mathbb{R}} \left\{ \frac{\sin rs/4}{s} \right\}^4 ||v_{r0}(\cdot + s) - v_{r0}||_\sigma^2\,ds. \tag{2.19}$$

Obviously,

$$||v_{r0}(\cdot + s) - v_{r0}||_\sigma^2 = ||v_{r0}(\cdot + s) - v_{r0}||_{\sigma_s}^2 + ||v_{r0}(\cdot + s) - v_{r0}||_{\sigma_{ac}}^2.$$

The bound for the first norm is easy:

$$\begin{aligned} \int_{\mathbb{R}} |v_{r0}(x + s) - v_{r0}(x)|^2\,d\sigma_s(x) &= \int_{\mathbb{R}} |v_{r0}(x + s)|^2\,d\sigma_s(x) \\ &\le C\,\min\{\kappa_r |s|^\gamma, 1\} \int_{-1}^{1} |k_{iy}(x + s)|^2\,d\sigma_s(x) \\ &\le C\,\frac{\min\{\kappa_r |s|^\gamma, 1\}}{y^2}. \end{aligned}$$

As for the second norm, we have

$$\begin{aligned}\int_{\mathbb{R}} |v_{r0}(x+s)-v_{r0}(x)|^2\,d\sigma_{ac}(x) &\le C\int_{\mathbb{R}} |(k_{iy}v_r\Pi)(x+s)-(k_{iy}v_r\Pi)(x)|^2\,dx\\ &\le C\Big\{\int_{\mathbb{R}} |k_{iy}(x+s)-k_{iy}(x)|^2|(v_r\Pi)(x+s))|^2\,dx\\ &\quad+\int_{\mathbb{R}} |k_{iy}(x)|^2|(v_r\Pi)(x+s)-(v_r\Pi)(x)|^2\,dx\Big\}.\end{aligned}$$

We notice that $|(v_r\Pi)(\cdot+s)|^2\le 1$, and, by Lemma 1.6,

$$\int_{\mathbb{R}} |k_{iy}(x+s)-k_{iy}(x)|^2|(v_r\Pi)(x+s))|^2\,dx \le C\,\frac{s^2}{y^3}.$$

Defining $f_r=(v_r\Pi)(\cdot+s)/(v_r\Pi)$, we get

$$\int_{\mathbb{R}} |k_{iy}(x)|^2|(v_r\Pi)(x+s)-(v_r\Pi)(x)|^2\,dx = \int_{\mathbb{R}} |k_{iy}(x)|^2|f_r(x)-1|^2|(v_r\Pi)(x)|^2\,dx.$$

The second claim of Lemma 1.8 along with $|v_r\Pi|^2\le 1$ shows that

$$\begin{aligned}&\int_{\mathbb{R}} |k_{iy}(x)|^2|f_r(x)-1|^2|(v_r\Pi)(x)|^2\,dx\\ &\qquad\le \int_{\mathbb{R}} |k_{iy}(x)|^2||f_r(x)|^2-1|\,dx+\frac{1}{4\pi y}|\operatorname{Re}(1-f_r(iy))|.\end{aligned}$$

Applying Lemma 1.8 once again and arguing as in (2.17), we get

$$\int_{\mathbb{R}} |k_{iy}(x)|^2||f_r(x)|^2-1|\,dx \le C\,\frac{\min\{\kappa_r|s|^\gamma,1\}}{y^2},$$

and

$$|\operatorname{Re}(1-f_r(iy))|\le C\frac{|s|}{y^2}.$$

Combining these bounds together, we obtain

$$\int_{\mathbb{R}} |v_{r0}(x+s)-v_{r0}(x)|^2\,d\sigma(x) \quad\le\quad C\left\{\frac{\min\{\kappa_r|s|^\gamma,1\}}{y^2}+\frac{s^2}{y^3}+\frac{|s|}{y^3}\right\}.$$

Looking at (2.18), (2.19), we infer that

$$\begin{aligned}&\int_r^\infty A^2(s)\,ds\\ &\qquad\le C\left[\frac{\log\kappa_r}{\kappa_r^{\gamma_0/\gamma}}+\lim_{y\to+\infty}\frac{y^2}{r^3}\int_{\mathbb{R}}\left\{\frac{\sin rs/4}{s}\right\}^4\left\{\frac{\min\{\kappa_r|s|^\gamma,1\}}{y^2}+\frac{s^2}{y^3}+\frac{|s|}{y^3}\right\}ds\right]\\ &\qquad\le C\left\{\frac{\kappa_r}{r^\gamma}+\frac{\log\kappa_r}{\kappa_r^{\gamma_0/\gamma}}\right\}.\end{aligned}$$

To optimize the estimate, pick $\kappa_r=r^{\gamma^2/(\gamma+\gamma_0)}$ with $\gamma=\gamma_0-\varepsilon$ and $\varepsilon\to+0$. This choice of parameters proves the statement of the theorem. □

## 3. Schrödinger and Dirac operators

3.1. In this section, we apply results obtained for Krein systems to Dirac and one-dimensional Schrödinger operators.

Consider a K-system (1.4) with a coefficient $A$. We define the functions $\phi(x,\lambda) = \mathrm{Re}\, e^{-i\lambda x}P(2x,\lambda)$ and $\psi(x,\lambda) = \mathrm{Im}\, e^{-i\lambda x}P(2x,\lambda)$. An easy computation shows that the functions $\phi, \psi$ are solutions of the following Dirac system:

$$\left\{ \begin{array}{rcl} \phi' & = & -\lambda\psi - a_1\phi + a_2\psi, \\ \psi' & = & \lambda\phi + a_2\phi + a_1\psi, \end{array} \right. \tag{3.1}$$

with boundary conditions $\phi(0) = 1, \psi(0) = 0$. Here, $a_1(x) = 2\,\mathrm{Re}\, A(2x)$ and $a_2(x) = 2\,\mathrm{Im}\, A(2x)$. This allows us to say (see [8], [18], [26]) that $\rho_{Dir} = 2\sigma$, where $\rho_{Dir}$ is the spectral measure of the Dirac system (3.1).

Using this simple relation between the measures of Krein and Dirac systems, we immediately obtain the following corollaries of Theorem 2.1 and Theorem 2.2.

**Theorem 3.1.** *Let $0 < \gamma_0 < 1$. For any $\gamma$, $0 < \gamma < \gamma_0$, there exists a Dirac system* (3.1) *with the properties:*

i) *The coefficient $a_2(x) = 0$ for all $x > 0$.*
ii) $\dim_{\mathrm{H}} \mathrm{supp}\ \rho_{Dir,s} = 1 - \gamma_0$.
iii) *The coefficient $a_1$ satisfies the inequality*

$$\int_x^\infty a_1^2(s)\, ds \le \frac{C}{(1+x)^\gamma}.$$

**Theorem 3.2.** *Let $0 < \gamma_0 < 1$. For any $\gamma$, $0 < \gamma < \gamma_0$, there exists a Dirac system* (3.1) *with the properties:*

i) *The coefficient $a_2(x) = 0$ for all $x > 0$.*
ii) $\dim_{\mathrm{H}} \mathrm{supp}\ \rho_{Dir,s} = 1 - \gamma_0$, *the density $\rho'_{Dir,\,ac} \in C^\infty(\mathbb{R})$ is even and*

$$\rho'_{Dir,ac}(x) = 1/\pi \left\{ \begin{array}{ll} 1/2, & |x| \le 1, \\ 1, & |x| > 2. \end{array} \right.$$

iii) *The coefficient $a_1$ satisfies the estimate*

$$\int_x^\infty a_1^2(s) ds \le \frac{C}{(1+x)^{\gamma/2}}.$$

It is likely that Theorem 0.2 has a direct analog for Dirac operator (3.1).

3.2. When the coefficient $a_1$ is absolutely continuous and $a_2 = 0$, we deduce from (3.1) that

$$\begin{array}{rcl} \psi'' - q\psi + \lambda^2\psi & = & 0, \\ \phi'' - q_1\phi + \lambda^2\phi & = & 0, \end{array}$$

where $q = a_1^2 + a_1'$, $q_1 = a_1^2 - a_1'$. The corresponding boundary conditions are

$$\begin{array}{ll} \psi(0) = 0, & \psi'(0) = \lambda, \\ \phi(0) = 1, & \phi'(0) + a_1(0)\phi(0) = 0. \end{array}$$

Therefore, the spectral measure $\rho_{Sch}$ of the Schrödinger operator

$$L_q y = -y'' + qy \tag{3.2}$$

with *Dirichlet boundary condition* $y(0) = 0$ is related to $\sigma$ by the formula

$$\rho_{Sch}(\lambda) = 4 \int_0^{\lambda^{1/2}} \xi^2 d\sigma(\xi), \tag{3.3}$$

where $\lambda > 0$.

Let the measure $\sigma$ and the real-valued coefficient $A$ be as in Theorems 2.1 or 2.2. We already mentioned that $A \in W^{1,2}(\mathbb{R}_+)$. The standard arguments from [6, Sect. 2] also prove that $A \in W^{m,2}(\mathbb{R}_+)$ for any integer $m$.

**Lemma 3.3.** *The measures $\sigma$ of Theorems 2.1 and 2.2 yield the real-valued coefficients $A \in W^{m,2}(\mathbb{R}_+)$ for any integer $m$.*

*Sketch of the proof.* First, we consider a K-system (1.4) with a real-valued coefficient $A$ lying in the Schwartz class. In the corresponding Dirac system (3.1), $a_2(x) = 0$, $a_1(x) = 2A(2x)$, and $\rho_{Dir} = 2\sigma$. For (3.1), we have relations [27], [9] analogous to the well-known Faddeev-Zakharov trace formulas [10]. Writing them for the half-line in terms of $\sigma$ and $A$ [27, Sect. 8], we obtain

$$\int_{\mathbb{R}} \lambda^{2j} \log 2\pi\sigma'_{ac}(\lambda)\, d\lambda = (-1)^{j+1} 2\pi \int_0^\infty d_{2j+1}(s) ds, \tag{3.4}$$

where $j = 0, 1, \ldots$. For $l = 1, 2, \ldots$, the functions $d_l$ are given by the following recursive relations:

$$d_1(s) = A^2(s), \quad d_{l+1} = -A \frac{d}{dx}\left(A^{-1} d_l\right) - \sum_{m+n=l} d_m d_n.$$

Integrating by parts, we have

$$\begin{aligned} \int_{\mathbb{R}} \lambda^{2j} \log 2\pi\sigma'_{ac}(\lambda)\, d\lambda &= -2\pi \int_0^\infty \left[A^{(j)}(s)\right]^2 ds \\ &+ \int_0^\infty P_j(A(s), A'(s), \ldots, A^{(j-1)}(s))\, ds \\ &+ Q_j(A(0), A'(0), \ldots, A^{(2j)}(0)), \end{aligned} \tag{3.5}$$

where $P_j, Q_j$ are certain polynomials. An analysis similar to [20, Theorem 4] shows that

$$\int_0^\infty |P_j(A, A', \ldots, A^{(j-1)})|\, ds \le \varepsilon ||A^{(j)}||_2^2 + C(\varepsilon, \|A\|_{W^{j-1,2}(\mathbb{R}_+)}) \tag{3.6}$$

with arbitrary $\varepsilon > 0$.

The second part of the proof consists in approximating the coefficient $A$ from Theorems 2.1 and 2.2 by properly chosen auxiliary coefficients $A_n$. Let $\varphi$ be a nonnegative function from $C^\infty(\mathbb{R})$ with support contained in $(-1, 1)$ and $\int_{\mathbb{R}} \varphi(s)\, ds = 1$. We take $\varphi_n(s) = n\varphi(ns)$ for an integer $n$. Smearing $\sigma$ with $\varphi_n$, we get the sequence of absolutely continuous measures $\{\sigma_n\}$ with densities

$$\sigma'_n(t) = \int_{\mathbb{R}} \varphi_n(t - s) d\sigma(s).$$

It is plain that $\sigma_n$ has the following properties:

i) the measures $d\sigma_n$ converge weakly to $d\sigma$ on $[-3, 3]$,
ii) $\sigma'_n \in C^\infty(\mathbb{R})$,
iii) $\sigma'_n(t) = 1/(2\pi)$ outside the interval $[-3, 3]$.

Each $\sigma_n$ generates a K-system (1.4) with coefficient $A_n$ via the solution of the inverse spectral problem described in Section 1.2. By the construction,

i) the coefficients $A_n$ belong to the Schwartz class,

ii) for every fixed $j$, $A_n^{(j)}$ converges to $A^{(j)}$ uniformly on any compact.

We also have

$$\int_{\mathbb{R}} \lambda^{2j} |\log 2\pi\sigma_n'(\lambda)|\, d\lambda \le C_j \tag{3.7}$$

uniformly in $n$ (the details are in [6, Sect. 2]). Let us show that the sequence $A_n$ is uniformly bounded in $W^{m,2}(\mathbb{R}_+)$ for any $m$. We will do this by induction on $m$. For $m = 0$, the statement follows from [6]. Recalling (3.5) and inequalities (3.6), (3.7), we get

$$\begin{aligned} &2\pi \int_0^\infty \left[A_n^{(m)}(s)\right]^2 ds \\ &\qquad \le C(\|A_n\|_{W^{m-1,2}(\mathbb{R}_+)} + |Q_m(A_n(0), A_n'(0), \dots, A_n^{(2m)}(0))| + C_m) \le C_m'. \end{aligned}$$

The constant $C_m'$ is independent of $n$. Since $A_n^{(m)}$ converges to $A^{(m)}$ uniformly on any compact set, we deduce that $A \in W^{m,2}(\mathbb{R}_+)$. The proof is finished. □

**Theorem 3.4.** *For any $0 < \gamma_0 < 1$ and $0 < \gamma < \gamma_0$, there is a potential $q$ with the properties:*

i) *The spectral measure of $L_q$ (see* (3.2)*) has a singular continuous component $\rho_{Sch,s}$ such that* $\dim_{\mathrm{H}} \operatorname{supp} \rho_{Sch,\,s} = 1 - \gamma_0$.

ii) *The following estimate holds:*

$$\int_x^\infty q^2(s)\, ds \le \frac{C}{(1+x)^\gamma}.$$

*Proof.* Let $\sigma$ and $A$ be as in Theorem 2.1. Consider $a_1(x) = 2A(2x)$ and the Schrödinger operator with Dirichlet boundary conditions and potential $q = a_1' + a_1^2$. By Lemma 3.3, $a_1$ is bounded. Since $q = a_1' + a_1^2$, it suffices to show that

$$\int_x^\infty {a_1'}^2(s) ds \le \frac{C}{(1+x)^\gamma}. \tag{3.8}$$

We use an inequality from [11],

$$\|a_1'\|_2 \le C_m \|a_1\|_2^{1-1/m} \|a_1^{(m)}\|_2^{1/m},$$

and Lemma 3.3 to obtain (3.8). The required properties of $\rho_{Sch}$ now follow from Theorem 2.1 and (3.3). □

Similarly, the following corollary of Theorem 3.2 can be proved.

**Theorem 3.5.** *For any $0 < \gamma_0 < 1$ and $0 < \gamma < \gamma_0$, there exists a potential $q$ such that*

i) *The spectral measure $\rho_{Sch}$ of the Schrödinger operator* (3.2) *has the properties* $\dim_{\mathrm{H}} \operatorname{supp} \rho_{Sch,\,s} = 1 - \gamma_0$ *and* $\rho_{Sch,\,ac}' \in C^\infty(\mathbb{R}_+), \rho_{Sch,\,ac}' > 0$.

ii) *The following estimate holds:*

$$\int_x^\infty q^2(s) ds \le \frac{C}{(1+x)^{\gamma/2}}.$$

After this paper was submitted for publication, Damanik-Killip-Simon [4] obtained a criterion for $q \in L^2(\mathbb{R}_+)$ in terms of the spectral measure $\rho_{Sch}$.

## 4. Orthogonal polynomials

The following theorems are counterparts of Theorems 2.1 and 2.2 for orthogonal polynomials on $\mathbb{T}$. Their proofs follow word for word the proofs of the results for Krein systems. The only difference is that we need to use inequality (1.3) instead of Lemmas 2.4 and 2.5. That is why the arguments below are omitted.

**Theorem 4.1.** *For given $0 < \gamma_0 < 1$ and $0 < \gamma < \gamma_0$, there exists a measure $\sigma$ with the properties:*

i) $\dim_{\mathrm{H}} \operatorname{supp} \sigma_s = 1 - \gamma_0$.
ii) *The sequence $\{a_n\}$ of Verblunsky coefficients is such that*

$$\sum_{k=n}^{\infty} |a_k|^2 \le \frac{C}{n^{\gamma}}.$$

**Theorem 4.2.** *Let $0 < \gamma_0 < 1$. Then, for any $0 < \gamma < \gamma_0$, there exists a singular continuous measure $\sigma_s$ so that*

i) $\dim_{\mathrm{H}} \operatorname{supp} \sigma_s = 1 - \gamma_0$.
ii) *The sequence $\{a_n\}$ associated to $d\sigma = dm/2 + d\sigma_s$ satisfies the condition*

$$\sum_{k=n}^{\infty} |a_k|^2 \le \frac{C}{n^{\gamma/2}}.$$

An extensive discussion of these results can be found in [29, Sect. 2.11].

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