

EXTENSION-ORTHOGONAL COMPONENTS OF PREPROJECTIVE VARIETIES

CHRISTOF GEISS AND JAN SCHRÖER

ABSTRACT. Let Q be a Dynkin quiver, and let Λ be the corresponding preprojective algebra. Let $\mathcal{C} = \{C_i \mid i \in I\}$ be a set of pairwise different indecomposable irreducible components of varieties of Λ -modules such that generically there are no extensions between C_i and C_j for all i, j . We show that the number of elements in \mathcal{C} is at most the number of positive roots of Q . Furthermore, we give a module-theoretic interpretation of Leclerc's counterexample to a conjecture of Berenstein and Zelevinsky.

1. INTRODUCTION

Let k be an algebraically closed field. For a finitely generated k -algebra A let $\text{mod}_A(\mathbf{d})$ be the affine variety of (left) A -modules with dimension vector \mathbf{d} . Throughout, we only consider finite-dimensional modules.

For irreducible components $C_1 \subseteq \text{mod}_A(\mathbf{d}_1)$ and $C_2 \subseteq \text{mod}_A(\mathbf{d}_2)$, define

$$\text{ext}_A^1(C_1, C_2) = \min\{\dim \text{Ext}_A^1(M_1, M_2) \mid (M_1, M_2) \in C_1 \times C_2\}.$$

An irreducible component $C \subseteq \text{mod}_A(\mathbf{d})$ is *indecomposable* if it contains a dense subset of indecomposable A -modules. A general theory of irreducible components and their decomposition into indecomposable irreducible components was developed in [5]. Our aim is to apply this to the preprojective varieties.

If not mentioned otherwise, we always assume that Q is a Dynkin quiver of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ or \mathbb{E}_8 . By R^+ we denote the set of positive roots of Q , and by Λ we denote the preprojective algebra associated to Q ; see [20]. Let n be the number of vertices of Q , and let $\Lambda(\mathbf{d}) = \text{mod}_\Lambda(\mathbf{d})$, $\mathbf{d} \in \mathbb{N}^n$, be the variety of Λ -modules with dimension vector \mathbf{d} . The varieties $\Lambda(\mathbf{d})$ are called *preprojective varieties*. Since we consider only preprojective algebras of Dynkin type, the preprojective varieties coincide with the nilpotent varieties defined by Lusztig. We refer to [16, Section 12] for basic properties. Our main result is the following:

Theorem 1.1. *If $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ is a set of pairwise different indecomposable irreducible components such that $\text{ext}_\Lambda^1(C_i, C_j) = 0$ for all $i, j \in I$, then $|I| \leq |R^+|$.*

The upper bound given in the theorem seems to be optimal. For example, if Q is of type \mathbb{A}_n , then it is easy to construct sets $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ of pairwise

Received by the editors September 5, 2002 and, in revised form, October 7, 2003.

2000 *Mathematics Subject Classification.* Primary 14M99, 16D70, 16G20, 17B37.

The second author thanks the Nuffield Foundation (Grant Number NAL/00270/G) for financial support, and the IM UNAM, Mexico City, where most of this work was done.

different indecomposable irreducible components such that $\text{ext}_\Lambda^1(C_i, C_j) = 0$ for all $i, j \in I$ and $|I| = |R^+|$; see the end of Section 4.

As a consequence of the above theorem we get the following result on Λ -modules without self-extensions:

Corollary 1.2. *Let M be a Λ -module with $\text{Ext}_\Lambda^1(M, M) = 0$. Then the number of pairwise non-isomorphic indecomposable direct summands of M is at most $|R^+|$.*

Let U_v^- be the negative part of the quantized enveloping algebra of the Lie algebra corresponding to Q . We regard U_v^- as a $\mathbb{Q}(v)$ -algebra. Let \mathcal{B} be the canonical basis and \mathcal{B}^* the dual canonical basis of U_v^- ; see [2], [14], [16] or [18] for definitions. By [12, Section 5] and [15] the elements of \mathcal{B} (and thus of \mathcal{B}^*) correspond to the irreducible components of the preprojective varieties $\Lambda(\mathbf{d})$, $\mathbf{d} \in \mathbb{N}^n$. Let $b^*(C)$ be the dual canonical basis vector corresponding to an irreducible component C . We denote the structure constants of U_v^- with respect to the basis \mathcal{B}^* by $\lambda_{C,D}^E$, i.e.

$$b^*(C)b^*(D) = \sum_E \lambda_{C,D}^E b^*(E).$$

Following the terminology in [2], two dual canonical basis vectors $b^*(C)$ and $b^*(D)$ are called *multiplicative* if

$$b^*(C)b^*(D) = \lambda b^*(E)$$

for some irreducible component E and some $0 \neq \lambda \in \mathbb{Q}(v)$. One calls $b^*(C)$ and $b^*(D)$ *quasi-commutative* if

$$b^*(C)b^*(D) = \lambda b^*(D)b^*(C)$$

for some $0 \neq \lambda \in \mathbb{Q}(v)$. The following conjecture was stated in [2, Section 1]:

Conjecture 1.3 (Berenstein, Zelevinsky). *Two dual canonical basis vectors are multiplicative if and only if they are quasi-commutative.*

One direction of this conjecture was proved by Reineke [18, Corollary 4.5]. The other direction turned out to be wrong. Namely, Leclerc [13] constructed examples of quasi-commutative elements in \mathcal{B}^* which are not multiplicative. Using preprojective algebras, we give a module-theoretic interpretation of one of his examples.

Marsh and Reineke [17] conjectured that the multiplicative behaviour of dual canonical basis vectors should be related to sets of irreducible components with Ext^1 vanishing generically between them. This was the principal motivation for our work.

The paper is organized as follows: In Section 2 we review the main results from [5]. In Section 3 we recall known results for the case that Λ is an algebra of finite or tame representation type. The proof of Theorem 1.1 and its corollary can be found in Section 4. Finally, Section 5 is devoted to the interpretation of Leclerc’s example.

2. VARIETIES OF MODULES—DEFINITIONS AND KNOWN RESULTS

In this section, we work with arbitrary finite quivers.

2.1. Let $Q = (Q_0, Q_1)$ be a finite quiver, where Q_0 denotes the set of vertices and Q_1 the set of arrows of Q . Assume that $|Q_0| = n$. For an arrow α let s_α be its starting vertex and e_α its end vertex. An element $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^n$ is called a *dimension vector* for Q . A *representation* of Q with dimension vector \mathbf{d} is a matrix tuple $M = (M_\alpha)_{\alpha \in Q_1}$ with $M_\alpha \in \text{M}_{d_{e_\alpha} \times d_{s_\alpha}}(k)$. A *path* of length $l \geq 1$ in Q is a

sequence $p = \alpha_1 \cdots \alpha_l$ of arrows in Q_1 such that $s\alpha_i = e\alpha_{i+1}$ for $1 \leq i \leq l-1$. Define $sp = s\alpha_l$ and $ep = e\alpha_1$. For a representation M and a path $p = \alpha_1 \cdots \alpha_l$ define $M_p = M_{\alpha_1} \cdots M_{\alpha_l}$, which is a matrix in $M_{d_{ep} \times d_{sp}}(k)$. A *relation* for Q is a k -linear combination $\sum_{i=1}^t \lambda_i p_i$ of paths p_i of length at least two such that $sp_i = sp_j$ and $ep_i = ep_j$ for all $1 \leq i, j \leq t$. A representation M satisfies such a relation if $\sum_{i=1}^t \lambda_i M_{p_i} = 0$. Given a set ρ of relations for Q , let $\text{rep}_{(Q,\rho)}(\mathbf{d})$ be the affine variety of representations of Q with dimension vector \mathbf{d} which satisfy all relations in ρ .

2.2. One can interpret this construction in a module-theoretic way. Namely, let kQ be the path algebra of Q , and let $A = kQ/(\rho)$, where (ρ) is the ideal generated by the elements in ρ . Then $\text{mod}_A(\mathbf{d}) = \text{rep}_{(Q,\rho)}(\mathbf{d})$ is the affine *variety of A -modules* with dimension vector \mathbf{d} . If $A = kQ/(\rho)$ is finite-dimensional, then A is a basic algebra. In this case, the vertices of Q correspond to the isomorphism classes of simple A -modules. The entry d_i , $i \in Q_0$, of \mathbf{d} is the multiplicity of the simple module corresponding to i in a composition series of any $M \in \text{mod}_A(\mathbf{d})$. The group $\text{GL}(\mathbf{d}) = \prod_{i \in Q_0} \text{GL}_{d_i}(k)$ acts on $\text{mod}_A(\mathbf{d})$ by conjugation, i.e.

$$g \cdot M = (g_{e\alpha} M_{\alpha} g_{s\alpha}^{-1})_{\alpha \in Q_1}.$$

The orbit of M under this action is denoted by $\mathcal{O}(M)$. There is a 1-1 correspondence between the set of orbits in $\text{mod}_A(\mathbf{d})$ and the set of isomorphism classes of A -modules with dimension vector \mathbf{d} . For further details on varieties of modules, in particular on the close relation between representations of bounded quivers and modules over finite-dimensional algebras, we refer to [3].

2.3. Given irreducible components $C_i \subseteq \text{mod}_A(\mathbf{d}_i)$, $1 \leq i \leq t$, we consider all A -modules with dimension vector $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_t$, which are isomorphic to $M_1 \oplus \cdots \oplus M_t$, where $M_i \in C_i$ for all i . By

$$C_1 \oplus \cdots \oplus C_t$$

we denote the corresponding subset of $\text{mod}_A(\mathbf{d})$. This is the image of the map

$$\text{GL}(\mathbf{d}) \times C_1 \times \cdots \times C_t \longrightarrow \text{mod}_A(\mathbf{d}),$$

$$(g, M_1, \dots, M_t) \mapsto g \cdot \left(\bigoplus_{i=1}^t M_i \right).$$

We call $C_1 \oplus \cdots \oplus C_t$ the *direct sum* of the components C_i . It follows that the closure $\overline{C_1 \oplus \cdots \oplus C_t}$ is irreducible. For an irreducible component C , define $C^n = \bigoplus_{i=1}^n C$. We call C *indecomposable* if C contains a dense subset of indecomposable A -modules. The following result from [5] is an analogue of the Krull-Remak-Schmidt Theorem.

Theorem 2.1. *If $C \subseteq \text{mod}_A(\mathbf{d})$ is an irreducible component, then*

$$C = \overline{C_1 \oplus \cdots \oplus C_t}$$

for some indecomposable irreducible components $C_i \subseteq \text{mod}_A(\mathbf{d}_i)$, $1 \leq i \leq t$. The components C_1, \dots, C_t are uniquely determined by this, up to reordering. The above direct sum is called the canonical decomposition of C .

However, the closure of a direct sum of irreducible components is not in general an irreducible component. The next result is also proved in [5].

Theorem 2.2. *If $C_i \subseteq \overline{\text{mod}_A(\mathbf{d}_i)}$, $1 \leq i \leq t$, are irreducible components, and $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_t$, then $\overline{C_1 \oplus \cdots \oplus C_t}$ is an irreducible component of $\text{mod}_A(\mathbf{d})$ if and only if $\text{ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$.*

Instead of taking direct sums of the modules in two irreducible components, one can take extensions. Let $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$ be dimension vectors, let $G = \text{GL}(\mathbf{d}_1) \times \text{GL}(\mathbf{d}_2)$, and let S be a G -stable subset of $\text{mod}_A(\mathbf{d}_1) \times \text{mod}_A(\mathbf{d}_2)$. We denote by $\mathcal{E}(S)$ the $\text{GL}(\mathbf{d})$ -stable subset of $\text{mod}_A(\mathbf{d})$ corresponding to all modules M which belong to a short exact sequence

$$0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

with $(M_1, M_2) \in S$; see [5] for more details.

For an irreducible component $C \subseteq \text{mod}_A(\mathbf{d})$ let

$$\mu_g(C) = \dim C - \max\{\dim \mathcal{O}(M) \mid M \in C\}$$

be the *generic number of parameters* of C . Thus $\mu_g(C) = 0$ if and only if C contains a dense orbit $\mathcal{O}(M)$. For example, if P is a projective A -module, then $\text{Ext}_A^1(P, P) = 0$. This implies that the closure of the orbit $\mathcal{O}(P)$ is an irreducible component, and we get $\mu_g(\overline{\mathcal{O}(P)}) = 0$. Also, if $C = \overline{C_1 \oplus \cdots \oplus C_t}$, then

$$\mu_g(C) = \sum_{i=1}^t \mu_g(C_i).$$

3. THE FINITE AND TAME CASES

Let A be a finite-dimensional k -algebra. Then A is called *representation-finite* if there are only finitely many isomorphism classes of indecomposable A -modules. The algebra A is *tame* if A is not representation-finite, and if for all dimension vectors \mathbf{d} the indecomposable A -modules in $\text{mod}_A(\mathbf{d})$ can be parametrized by a finite number of affine lines. Otherwise A is called *wild*. For precise definitions we refer to [4, Section 5.3]. The preprojective algebras of Dynkin type are selfinjective. We refer to [9] for the theory of representation-finite selfinjective algebras. The general theory of arbitrary representation-finite algebras is explained in [8]. Introductions to Auslander-Reiten theory and the representation theory of finite-dimensional algebras can be found in [1] and [19].

Proposition 3.1. *Let Q be a Dynkin quiver, and let Λ be the associated preprojective algebra. Then the following hold:*

- Λ is representation-finite if and only if Q is of type \mathbb{A}_i , $i \leq 4$;
- Λ is tame if and only if Q is of type \mathbb{A}_5 or \mathbb{D}_4 .

The above proposition is well known to the experts. Let us sketch a proof: There always exists a simply connected Galois covering $F : \tilde{\Lambda} \rightarrow \Lambda$ of Λ . In the cases \mathbb{A}_i , $i \leq 4$, one can construct all indecomposable $\tilde{\Lambda}$ -modules via the knitting procedure of preprojective components. One gets that $\tilde{\Lambda}$ is locally representation-finite; thus Λ is representation-finite in these cases. For \mathbb{A}_5 and \mathbb{D}_4 the algebra $\tilde{\Lambda}$ is the repetitive algebra of a tubular algebra. It follows that the push-down functor $\text{mod}(\tilde{\Lambda}) \rightarrow \text{mod}(\Lambda)$ is dense, and that Λ is tame in these two cases. We refer to [6, Section 6] and [11] for more details. In all other cases, one can show that the algebra $\tilde{\Lambda}$ contains wild full convex subalgebras; thus also Λ is wild. For the basics of covering theory we refer to [7] and [10].

If Λ is representation-finite, and if $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ is a maximal set of pairwise different indecomposable irreducible components such that $\text{ext}_\Lambda^1(C_i, C_j) = 0$ for all i, j , then $|I| = |R^+|$. This was proved by Marsh and Reineke, compare also [2].

For a tame algebra A one has $\mu_g(C) \leq 1$ for any indecomposable irreducible component $C \subseteq \text{mod}_A(\mathbf{d})$. For Λ tame a complete classification of the indecomposable irreducible components, and a necessary and sufficient condition for $\text{ext}_\Lambda^1(C, D) = 0$ for any two irreducible components C and D , were obtained in [11].

If Λ is of wild representation type, one should expect irreducible components C with $\text{ext}_\Lambda^1(C, C) \neq 0$. Thus, maybe one should study sets $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ of irreducible components with the weaker condition $\text{ext}_\Lambda^1(C_i, C_j) = 0$ for all $i \neq j$. However, we do not know how to generalize Theorem 1.1 to this case.

For the two tame cases, the following was proved in [11]:

Theorem 3.2. *Assume that Q is of type A_5 or D_4 . Then the following hold:*

- (1) *For any irreducible component $C \subseteq \Lambda(\mathbf{d})$ we have $\text{ext}_\Lambda^1(C, C) = 0$.*
- (2) *If $C \subseteq \Lambda(\mathbf{d})$ is an indecomposable irreducible component, then we have $\mu_g(C) = 0$ or $\mu_g(C) = 1$. For suitable \mathbf{d} there exists an indecomposable irreducible component $C \subseteq \Lambda(\mathbf{d})$ with $\mu_g(C) = 1$.*
- (3) *Let $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ be a maximal set of pairwise different indecomposable irreducible components such that $\text{ext}_\Lambda^1(C_i, C_j) = 0$ for all i, j . Then there is at most one C_i with $\mu_g(C_i) = 1$. In this case, we have $|I| = |R^+| - 1$, and we get $|I| = |R^+|$, otherwise.*

This leads us to the following conjecture for arbitrary Dynkin quivers of type A_n, D_n, E_6, E_7 or E_8 :

Conjecture 3.3. *If $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ is a maximal set of pairwise different indecomposable irreducible components such that $\text{ext}_\Lambda^1(C_i, C_j) = 0$ for all i, j , then*

$$|I| = |R^+| - \sum_{i \in I} \mu_g(C_i).$$

4. PROOF OF THEOREM 1.1

As before, let Q be a Dynkin quiver, and let

$$R^+ = \{\mathbf{a}_i \mid 1 \leq i \leq N\}$$

be the set of positive roots of Q . From now on $N = |R^+|$ will always be the number of positive roots of Q .

By Gabriel's Theorem there is a 1-1 correspondence between R^+ and the set of isomorphism classes of indecomposable kQ -modules. This correspondence associates to a root \mathbf{a}_i the isomorphism class $[M(\mathbf{a}_i)]$ of an indecomposable kQ -module $M(\mathbf{a}_i)$ with dimension vector \mathbf{a}_i . By the Krull-Remak-Schmidt Theorem each kQ -module is isomorphic to a unique (up to reordering) direct sum of the indecomposable modules $M(\mathbf{a}_i)$. For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ set

$$M_\alpha = \bigoplus_{i=1}^N M(\mathbf{a}_i)^{\alpha_i} \text{ and } C_\alpha = \overline{\pi_{\mathbf{d}}^{-1}(\mathcal{O}(M_\alpha))},$$

where \mathbf{d} is the dimension vector of M_α and

$$\pi_{\mathbf{d}} : \Lambda(\mathbf{d}) \longrightarrow \text{mod}_{kQ}(\mathbf{d})$$

is the canonical projection map. Let $\text{Irr}(\Lambda)$ be the set of irreducible components of the varieties $\Lambda(\mathbf{d})$, $\mathbf{d} \in \mathbb{N}^n$. The three maps

$$\begin{aligned} \mathbb{N}^N &\longrightarrow \{\mathcal{O}(M) \mid M \in \text{mod}_{kQ}(\mathbf{d}), \mathbf{d} \in \mathbb{N}^n\} \longrightarrow \text{Irr}(\Lambda) \longrightarrow \mathcal{B}^*, \\ \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) &\mapsto \mathcal{O}(M_{\boldsymbol{\alpha}}) \mapsto C_{\boldsymbol{\alpha}} \mapsto b^*(C_{\boldsymbol{\alpha}}), \end{aligned}$$

are all bijective.

Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^N$. By Theorem 2.2 the closure $\overline{C_{\boldsymbol{\alpha}} \oplus C_{\boldsymbol{\beta}}}$ is an irreducible component if and only if $\text{ext}_{\Lambda}^1(C_{\boldsymbol{\alpha}}, C_{\boldsymbol{\beta}}) = \text{ext}_{\Lambda}^1(C_{\boldsymbol{\beta}}, C_{\boldsymbol{\alpha}}) = 0$. In this case, we have $\overline{C_{\boldsymbol{\alpha}} \oplus C_{\boldsymbol{\beta}}} = C_{\boldsymbol{\alpha} + \boldsymbol{\beta}}$.

Let $\boldsymbol{\delta}_j = (\delta_{1j}, \dots, \delta_{Nj})$, $1 \leq j \leq N + 1$, be non-zero pairwise different elements in \mathbb{N}^N such that $C_{\boldsymbol{\delta}_j}$ is an indecomposable irreducible component for all j . To get a contradiction, we assume that $\text{ext}_{\Lambda}^1(C_{\boldsymbol{\delta}_i}, C_{\boldsymbol{\delta}_j}) = 0$ for all $1 \leq i, j \leq N + 1$. For $\mathbf{m} = (m_1, \dots, m_{N+1}) \in \mathbb{N}^{N+1}$, define

$$C(\mathbf{m}) = C_{\Delta \mathbf{m}}, \text{ where } \Delta = (\delta_{ij}) \in \mathbb{N}^{N \times (N+1)}.$$

Thus $\boldsymbol{\delta}_j$ is the j th column of the matrix Δ . By our assumption we get

$$C(\mathbf{m}) = \overline{C_{\boldsymbol{\delta}_1}^{m_1} \oplus \dots \oplus C_{\boldsymbol{\delta}_{N+1}}^{m_{N+1}}}.$$

Since the $C_{\boldsymbol{\delta}_j}$ are indecomposable, the above is the canonical decomposition of the irreducible component $C(\mathbf{m})$.

Now there exist some elements $\mathbf{m} = (m_1, \dots, m_{N+1}) \neq \mathbf{1} = (1, \dots, 1_{N+1})$ in \mathbb{N}^{N+1} such that

$$\Delta \mathbf{1} = \Delta \mathbf{m} \in \mathbb{N}^N.$$

This implies $C(\mathbf{m}) = C(\mathbf{1})$. Thus, we get a contradiction to the unicity of the canonical decomposition of irreducible components; see Theorem 2.1.

In fact, let $0 \neq \mathbf{z} \in \mathbb{Z}^{N+1}$ with $\Delta \mathbf{z} = 0$. Clearly, such a \mathbf{z} always exists. Since all entries of Δ are non-negative, \mathbf{z} has at least one negative entry. Let

$$l = -\min\{z_i \mid 1 \leq i \leq N + 1\};$$

then trivially $\mathbf{1} = (l, l, \dots, l)$ and $\mathbf{m} = \mathbf{1} + \mathbf{z}$ are as required. This finishes the proof of Theorem 1.1.

Corollary 1.2 follows immediately from the fact that an orbit $\mathcal{O}(N) \subseteq \text{mod}_A(\mathbf{d})$ of an A -module N is open provided $\text{Ext}_A^1(N, N) = 0$. Clearly, $\mathcal{O}(N)$ is open if and only if the closure $\overline{\mathcal{O}(N)}$ is an irreducible component. Then we use Theorems 1.1 and 2.2.

Finally, assume that Q is of type \mathbb{A}_n , and let S_1, \dots, S_n denote the (isomorphism classes of) simple Λ -modules. Set

$$H = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq n, 1 \leq j \leq n + 1 - i\},$$

and notice that $|H| = |R^+|$. For $(i, j) \in H$ there exists a unique (up to isomorphism) Λ -module $L_{(i,j)}$ with $\text{top}(L_{(i,j)}) \cong S_i$ and $\text{soc}(L_{(i,j)}) \cong S_j$ that admits S_1 as a composition factor. It is easy to check that $\text{Ext}_{\Lambda}^1(L_{(i,j)}, L_{(p,q)}) = 0$ for all $(i, j), (p, q) \in H$. Thus

$$\mathcal{C} = \{\overline{\mathcal{O}(L_{(i,j)})} \mid (i, j) \in H\}$$

is a set of pairwise different indecomposable irreducible components with $\text{ext}_{\Lambda}^1(C, D) = 0$ for all $C, D \in \mathcal{C}$, and we have $|\mathcal{C}| = |R^+|$.

5. INTERPRETATION OF LECLERC'S EXAMPLE

In the following, we use the notation introduced at the beginning of Section 4. Reineke proved in [18, Lemma 4.6] that the multiplicativity of $b^*(C_\alpha)$ and $b^*(C_\beta)$ implies that

$$b^*(C_\alpha)b^*(C_\beta) = v^m b^*(C_{\alpha+\beta})$$

for some $m \in \mathbb{Z}$. He also showed that $\lambda_{C,D}^E \neq 0$ if and only if $\lambda_{D,C}^E \neq 0$. This follows from [18, Proposition 4.4]. Thus one direction of Conjecture 1.3 holds. Namely, if two dual canonical basis vectors are multiplicative, then they are quasi-commutative. The following related problem should be of interest:

Problem 5.1. Describe the elements $\alpha, \beta \in \mathbb{N}^N$ such that

$$C_{\alpha+\beta} = \overline{C_\alpha \oplus C_\beta}.$$

As mentioned in the introduction, Leclerc recently constructed in [13] counterexamples for the other direction of the Berenstein-Zelevinsky Conjecture. We give a module-theoretic interpretation of one of his examples:

Let Q be the quiver of type \mathbb{A}_5 with arrows $a_i : i+1 \rightarrow i$, $1 \leq i \leq 4$. Thus Λ is given by the quiver

$$1 \begin{array}{c} \xrightarrow{\bar{a}_1} \\ \xleftarrow{a_1} \end{array} 2 \begin{array}{c} \xrightarrow{\bar{a}_2} \\ \xleftarrow{a_2} \end{array} 3 \begin{array}{c} \xrightarrow{\bar{a}_3} \\ \xleftarrow{a_3} \end{array} 4 \begin{array}{c} \xrightarrow{\bar{a}_4} \\ \xleftarrow{a_4} \end{array} 5$$

and the set of relations

$$\{a_1\bar{a}_1, \bar{a}_1a_1 - a_2\bar{a}_2, \bar{a}_2a_2 - a_3\bar{a}_3, \bar{a}_3a_3 - a_4\bar{a}_4, \bar{a}_4a_4\}.$$

Now R^+ contains exactly 15 elements, namely for each $1 \leq i \leq j \leq 5$ there is a positive root $[i, j] = (d_l)_{1 \leq l \leq 5}$ with $d_l = 1$ for $i \leq l \leq j$, and $d_l = 0$, otherwise. We identify $\mathbb{N}R^+$ with \mathbb{N}^{15} by fixing a linear ordering on R^+ , namely

$$\begin{aligned} & [1, 1] < [1, 2] < [1, 3] < [1, 4] < [1, 5] < [2, 2] < [2, 3] < [2, 4] < [2, 5] \\ & < [3, 3] < [3, 4] < [3, 5] < [4, 4] < [4, 5] < [5, 5]. \end{aligned}$$

Define

$$\begin{aligned} \alpha &= [1, 2] + [2, 4] + [3, 3] + [4, 5], \\ \beta &= [1, 2] + [1, 4] + [2, 3] + [2, 5] + [3, 4] + [4, 5]. \end{aligned}$$

Thus, regarded as elements in \mathbb{N}^{15} ,

$$\begin{aligned} \alpha &= (0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0), \\ \beta &= (0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0). \end{aligned}$$

In [13] Leclerc showed that

$$b^*(C_\alpha)^2 = v^{-2} (b^*(C_{\alpha+\alpha}) + b^*(C_\beta)).$$

This is obviously a counterexample to the Berenstein-Zelevinsky Conjecture. Now define

$$\begin{aligned} \beta_1 &= [1, 2] + [2, 3] + [3, 4] + [4, 5], \\ \beta_2 &= [1, 4] + [2, 5]. \end{aligned}$$

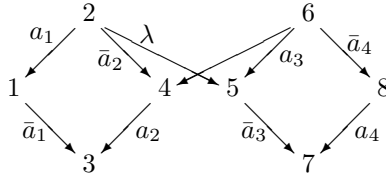
Thus we have $\beta = \beta_1 + \beta_2$.

Proposition 5.2. *Let $\alpha, \beta, \beta_1, \beta_2$ be as above. Then the following hold:*

- (1) *The irreducible components C_α, C_{β_1} and C_{β_2} are indecomposable with $\mu_g(C_\alpha) = 1$ and $\mu_g(C_{\beta_1}) = \mu_g(C_{\beta_2}) = 0$.*
- (2) *We have $C_{\alpha+\alpha} = \overline{C_\alpha \oplus C_\alpha}$ and $C_\beta = \overline{C_{\beta_1} \oplus C_{\beta_2}}$. Thus*

$$b^*(C_\alpha)^2 = v^{-2} (b^*(\overline{C_\alpha \oplus C_\alpha}) + b^*(\overline{C_{\beta_1} \oplus C_{\beta_2}})).$$

Proof. For $\lambda \in k \setminus \{0, 1\}$ let M_λ be the 8-dimensional Λ -module where the arrows of Λ operate on a basis $\{1, \dots, 8\}$ as in the following picture:



Thus, for example $a_1 \cdot 2 = 1, \bar{a}_2 \cdot 2 = 4 + \lambda 5, a_3 \cdot 6 = 4 + 5, \bar{a}_1 \cdot 1 = 3$, etc. Note that M_λ lies in C_α .

The modules M_λ are indecomposable and $\dim \text{End}_\Lambda(M_\lambda) = 3$. It is known that the preprojective varieties $\Lambda(\mathbf{d})$ are equidimensional of dimension

$$\sum_{\alpha \in Q_1} d_{s\alpha} d_{e\alpha};$$

see for example [16, Section 12].

Thus each irreducible component of $\Lambda(1, 2, 2, 2, 1)$ has dimension $2+4+4+2 = 12$. The group $\text{GL}(1, 2, 2, 2, 1)$ acts as described in Section 2 on $\Lambda(1, 2, 2, 2, 1)$ and has dimension 14. Thus we get $\dim \mathcal{O}(M_\lambda) = 14 - 3 = 11$. One checks easily that M_λ and M_μ are isomorphic if and only if $\lambda = \mu$. This implies

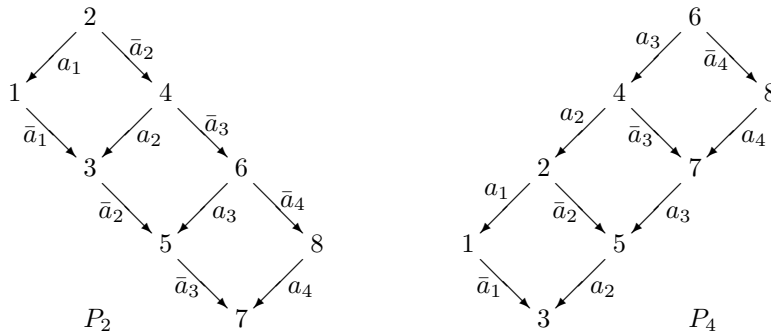
$$\dim \overline{\{\mathcal{O}(M_\lambda) \mid \lambda \in k \setminus \{0, 1\}\}} = 11 + 1 = 12.$$

We get

$$C_\alpha = \overline{\{\mathcal{O}(M_\lambda) \mid \lambda \in k \setminus \{0, 1\}\}}.$$

Thus C_α is an indecomposable irreducible component with $\mu_g(C_\alpha) = 1$.

Next, let P_2 and P_4 be the indecomposable projective Λ -modules corresponding to the vertices 2 and 4, respectively. These modules are both 8-dimensional, and with the same convention as above we may describe them as follows:



We have $\text{Ext}_\Lambda^1(P_i, P_j) = 0$ for all $i, j \in \{2, 4\}$. This follows directly from the projectivity of both modules. From this and the above pictures we get

$$\begin{aligned} C_{\beta_1} &= \overline{\mathcal{O}(P_2)}, \\ C_{\beta_2} &= \overline{\mathcal{O}(P_4)}, \\ C_\beta &= \overline{C_{\beta_1} \oplus C_{\beta_2}}. \end{aligned}$$

In particular, C_{β_1} and C_{β_2} are indecomposable irreducible components with $\mu_g(C_{\beta_1}) = \mu_g(C_{\beta_2}) = 0$. This finishes the proof. \square

For irreducible components $C \subseteq \Lambda(\mathbf{d})$ and $D \subseteq \Lambda(\mathbf{e})$, define

$$\mathcal{V}(C, D) = \bigcap_{U \subseteq C, V \subseteq D} \left\{ E \subseteq \Lambda(\mathbf{d} + \mathbf{e}) \text{ irred. comp.} \mid E \subseteq \overline{\mathcal{E}(U \times V)} \right\},$$

where U (resp. V) runs through all non-empty $\text{GL}(\mathbf{d})$ -stable (resp. $\text{GL}(\mathbf{e})$ -stable) open subsets of C (resp. D); see Section 2 for the definition of $\mathcal{E}(U \times V)$.

Using the previous proposition and some well-known results on the representation theory of the algebra Λ (see [11] and [19]), one can show that

$$\mathcal{V}(C_\alpha, C_\alpha) = \{C_{\alpha+\alpha}, C_\beta\}.$$

Note that $\text{ext}_\Lambda^1(C_\alpha, C_\alpha) = 0$, since $\text{Ext}_\Lambda^1(M_\lambda, M_\mu) = 0$ for all $\lambda \neq \mu$. But one can show that $\dim \text{Ext}_\Lambda^1(M_\lambda, M_\lambda) = 2$; see [11, Section 6]. For any M_λ there is a short exact sequence

$$0 \longrightarrow M_\lambda \longrightarrow M_\lambda(2) \longrightarrow M_\lambda \longrightarrow 0,$$

where $M_\lambda(2)$ is the module of quasi-length two in the same Auslander-Reiten component as M_λ (it is known that M_λ lies in a homogeneous tube). In addition to this ‘natural’ self-extension, there exists a short exact sequence

$$0 \longrightarrow M_\lambda \longrightarrow P_2 \oplus P_4 \longrightarrow M_\lambda \longrightarrow 0.$$

Motivated by our above analysis, one might conjecture the following:

- Conjecture 5.3.** (1) If $E \in \mathcal{V}(C, D) \cup \mathcal{V}(D, C)$, then $\lambda_{C,D}^E \neq 0$.
 (2) If irreducible components C and D contain non-empty stable open subsets $U \subseteq C$ and $V \subseteq D$ such that $\text{Ext}_\Lambda^1(M, N) = 0$ for all $M \in U, N \in V$, then $b^*(C)$ and $b^*(D)$ are multiplicative.

ACKNOWLEDGEMENTS

We thank Robert Marsh and Markus Reineke for helpful and interesting discussions.

REFERENCES

- [1] *M. Auslander, I. Reiten, S. Smalø*, Representation theory of Artin algebras. Corrected reprint of the 1995 original. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge (1997), xiv+425pp. MR 98e:16011
- [2] *A. Berenstein, A. Zelevinsky*, String bases for quantum groups of type A_r . I.M. Gelfand Seminar, 51–89, Adv. Soviet Math. **16**, Part 1, Amer. Math. Soc., Providence, RI (1993). MR 94g:17019
- [3] *K. Bongartz*, A geometric version of the Morita equivalence. J. Algebra **139** (1991), no. 1, 159–171. MR 92f:16008
- [4] *W. Crawley-Boevey*, On tame algebras and bocses. Proc. London Math. Soc. (3) **56** (1988), no. 3, 451–483. MR 89c:16028

- [5] *W. Crawley-Boevey, J. Schröer*, Irreducible components of varieties of modules. *J. Reine Angew. Math.* **553** (2002), 201–220. MR 2004a:16020
- [6] *V. Dlab, C.M. Ringel*, The module theoretical approach to quasi-hereditary algebras. In: *Representations of algebras and related topics* (Kyoto, 1990), 200–224, Cambridge Univ. Press, Cambridge (1992). MR 94f:16026
- [7] *P. Dowbor, A. Skowroński*, Galois coverings of representation-infinite algebras. *Comment. Math. Helv.* **62** (1987), 311–337. MR 88m:16020
- [8] *P. Gabriel*, Auslander-Reiten sequences and representation-finite algebras. In: *Representation theory, I* (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), pp. 1–71, *Lecture Notes in Mathematics* **831**, Springer-Verlag, Berlin (1980). MR 82i:16030
- [9] *P. Gabriel*, Algèbres auto-injectives de représentation finie (d’après Christine Riedtmann). (French) [Self-injective algebras of finite representation (according to Christine Riedtmann)] *Bourbaki Seminar*, Vol. 1979/80, pp. 20–39, *Lecture Notes in Mathematics* **842**, Springer-Verlag, Berlin-New York (1981). MR 84j:16018
- [10] *P. Gabriel*, The universal cover of a representation-finite algebra. In: *Representations of algebras* (Puebla, 1980), 68–105, *Lecture Notes in Mathematics* **903**, Springer-Verlag, Berlin (1981). MR 83f:16036
- [11] *C. Geiß, J. Schröer*, Varieties of modules over tubular algebras. *Colloq. Math.* **95** (2003), no. 2, 163–183. MR 2004d:16026
- [12] *M. Kashiwara, Y. Saito*, Geometric construction of crystal bases. *Duke Math. J.* **89** (1997), 9–36. MR 99e:17025
- [13] *B. Leclerc*, Imaginary vectors in the dual canonical basis of $U_q(n)$. *Transform. Groups* **8** (2003), no. 1, 95–104. MR 2004d:17020
- [14] *B. Leclerc, M. Nazarov, J.-Y. Thibon*, Induced representations of affine Hecke algebras and canonical bases of quantum groups. In: *Studies in Memory of Issai Schur*, *Progress in Mathematics* **210**, 115–153, Birkhauser (2003). MR 2004d:17007
- [15] *G. Lusztig*, Canonical bases arising from quantized enveloping algebras. II. Common trends in mathematics and quantum field theories (Kyoto, 1990). *Progr. Theoret. Phys. Suppl. No.* **102** (1990), 175–201 (1991). MR 93g:17019
- [16] *G. Lusztig*, Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.* **4** (1991), no. 2, 365–421. MR 91m:17018
- [17] *R. Marsh, M. Reineke*, Private communication. Bielefeld, February 2002.
- [18] *M. Reineke*, Multiplicative properties of dual canonical bases of quantum groups. *J. Algebra* **211** (1999), 134–149. MR 99k:17034
- [19] *C.M. Ringel*, Tame algebras and integral quadratic forms. *Lecture Notes in Mathematics* **1099**, Springer-Verlag, Berlin (1984), xiii+376pp. MR 87f:16027
- [20] *C.M. Ringel*, The preprojective algebra of a quiver. *Algebras and modules II* (Geiranger, 1996), 467–480, *CMS Conf. Proc.* **24**, Amer. Math. Soc., Providence, RI (1998). MR 99i:16031

INSTITUTO DE MATEMÁTICAS, UNAM, CIUDAD UNIVERSITARIA, 04510 MEXICO D.F., MEXICO
E-mail address: christof@math.unam.mx

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, UNITED KINGDOM
E-mail address: jschroer@maths.leeds.ac.uk