

SMALL DEVIATIONS OF WEIGHTED FRACTIONAL PROCESSES AND AVERAGE NON-LINEAR APPROXIMATION

MIKHAIL A. LIFSHITS AND WERNER LINDE

ABSTRACT. We investigate the small deviation problem for weighted fractional Brownian motions in L_q -norm, $1 \leq q \leq \infty$. Let B^H be a fractional Brownian motion with Hurst index $0 < H < 1$. If $1/r := H + 1/q$, then our main result asserts

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\left\| \rho B^H \right\|_{L_q(0, \infty)} < \varepsilon \right) = -c(H, q) \cdot \|\rho\|_{L_r(0, \infty)}^{1/H},$$

provided the weight function ρ satisfies a condition slightly stronger than the r -integrability. Thus we extend earlier results for Brownian motion, i.e. $H = 1/2$, to the fractional case. Our basic tools are entropy estimates for fractional integration operators, a non-linear approximation technique for Gaussian processes as well as sharp entropy estimates for l_q -sums of linear operators defined on a Hilbert space.

1. INTRODUCTION

Let $B^H = (B^H(t))_{t \geq 0}$ be a fractional Brownian motion with Hurst index $H \in (0, 1)$, i.e. B^H is a centered Gaussian process with a.s. continuous paths and covariance

$$(1.1) \quad \mathbb{E} B^H(t) B^H(t') = \frac{1}{2} \left\{ |t|^{2H} + |t'|^{2H} - |t - t'|^{2H} \right\}, \quad t, t' \geq 0.$$

Given a measurable weight function $\rho : (0, \infty) \rightarrow [0, \infty)$ such that we have for some $q \in [1, \infty]$

$$\mathbb{P} \left(\left\| \rho B^H \right\|_{L_q(0, \infty)} < \infty \right) = 1,$$

we are mainly interested in the behaviour of

$$(1.2) \quad \log \mathbb{P} \left(\left\| \rho B^H \right\|_{L_q(0, \infty)} < \varepsilon \right)$$

as $\varepsilon \rightarrow 0$. For $\rho = \mathbf{1}_{[0,1]}$ this question was investigated in [24], [19], [11] and [17]. The final result is as follows: For $1 \leq q \leq \infty$ the limit

$$(1.3) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\left\| B^H \right\|_{L_q[0,1]} < \varepsilon \right) := -c(H, q)$$

exists with finite $c(H, q) > 0$.

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If $H = 1/2$, then B^H is a standard Wiener process W . In this case the behaviour of (1.2) was thoroughly investigated in [16]. To formulate the main result of this paper, we need the following notion: For $q \in [1, \infty]$ and $0 < H < 1$ define the number $r > 0$ by

$$(1.4) \quad \frac{1}{r} := H + \frac{1}{q}.$$

Using this $r > 0$ we construct a new norm for functions ρ on $(0, \infty)$ by

$$(1.5) \quad |\rho|_r := \left(\sum_{k=-\infty}^{\infty} 2^{krH} \|\rho\|_{L_q[2^k, 2^{k+1}]}^r \right)^{1/r}.$$

The main result in [16] asserts for $H = 1/2$, i.e. for Brownian motion,

$$(1.6) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P} \left(\|\rho W\|_{L_q(0, \infty)} < \varepsilon \right) = -c(1/2, q) \cdot \|\rho\|_{L_r(0, \infty)}^2$$

provided $|\rho|_r < \infty$. Observe that Hölder's inequality yields

$$\|\rho\|_{L_r(0, \infty)} := \|\rho\|_r \leq |\rho|_r,$$

yet it is not difficult to construct ρ 's with $\|\rho\|_r < \infty$ and $|\rho|_r = \infty$.

The main result of the present paper is the extension of (1.6) to all $H \in (0, 1)$, i.e. if $|\rho|_r < \infty$, then we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho B^H\|_{L_q(0, \infty)} < \varepsilon \right) = -c(H, q) \cdot \|\rho\|_{L_r(0, \infty)}^{1/H}$$

with $c(H, q)$ and r defined by (1.3) and (1.4), respectively.

When treating small deviation problems for ρB^H with $H \neq 1/2$, completely new methods are required. These methods are of analytical nature. It is well known that fractional Brownian motions are tightly related to fractional integration operators of Riemann–Liouville type (cf. [18]). Thus, by the results in [8] and [12] the small deviation problem for weighted fractional motions turns out to be equivalent to the study of the asymptotic behavior of the entropy numbers for certain weighted fractional integration operators.

It is convenient to solve this analytical problem in a more general context, namely to investigate the degree of compactness of l_q -sums of operators. More precisely, let \mathcal{H} be a Hilbert space and let T_j , $j = 1, 2, \dots$, be operators from \mathcal{H} into some Banach spaces E_j . If

$$E^q := l_q(E_j : j \in \mathbb{N}) = \left\{ x = (x_j)_{j \geq 1} : x_j \in E_j, \left\| (\|x_j\|) \right\|_{l_q} < \infty \right\},$$

then (under some obvious assumptions) the family $(T_j)_{j \geq 1}$ generates in a canonical way an operator T^q from \mathcal{H} into E^q . The main result in the functional analytic part of this paper is an estimate for the entropy numbers $e_n(T^q)$ by means of $e_n(T_j)$, $j \in \mathbb{N}$. Surprisingly, the proof of this requires completely different methods for the two cases $q < \infty$ and $q = \infty$, respectively. For $q < \infty$ so-called average Kolmogorov numbers play a central role. They describe how good a non-linear approximation of a Gaussian vector by finite rank vectors can be. These numbers were introduced and investigated in the theory of information-based complexity; their basic properties can be found in [6].

In the case $q = \infty$ the technique of average Kolmogorov numbers does not work. Instead we use strong results due to B. Carl about compactness properties of Hölder

operators. In our construction, the essential and (as far as we know) new idea is to investigate bundles of metric spaces and their metric entropy.

It is remarkable that we use analytic methods for proving probabilistic bounds while probabilistic tools work for proving analytic estimates. Let us note, however, that a part of our analytic arguments can be expressed directly in probabilistic language; see [7]. But we believe that our analytic methods are more general and could therefore also be applied for the investigation of other classes of fractional processes.

The paper is organized as follows. Section 2 provides a minimal necessary analytical background as e.g. the basic estimates of entropy numbers for l_q -sums of operators. Weighted fractional integration operators are treated in Section 3. The small deviation results are stated and proved in Section 4. Finally, the detailed exposition of the analytic approach is presented in Sections 5 and 6 for finite and infinite q , respectively. These last two sections are not necessary for those who are only interested in probabilistic applications. On the other hand, “pure analysts” may skip Section 4 and read Sections 5 and 6 directly.

2. ENTROPY NUMBERS OF l_q -SUMS OF OPERATORS

For any sequence of Banach spaces $(E_j)_{j \geq 1}$ and a number $q \in [1, \infty)$, we define the l_q -sum E^q by

$$E^q := l_q(E_j : j \in \mathbb{N}) = \left\{ x = (x_j)_{j \geq 1} : x_j \in E_j, \sum_{j=1}^{\infty} \|x_j\|^q < \infty \right\}$$

and endow it with the norm

$$\|x\|_q := \left(\sum_{j=1}^{\infty} \|x_j\|^q \right)^{1/q}, \quad x = (x_j)_{j \geq 1}.$$

If linear operators T_j from a Hilbert space \mathcal{H} into E_j satisfy

$$\sup_{\|h\| \leq 1} \sum_{j=1}^{\infty} \|T_j h\|^q < \infty,$$

then the operator $T^q : \mathcal{H} \rightarrow E^q$ defined by $T^q h := (T_j h)_{j \geq 1}$, $h \in \mathcal{H}$, is bounded. We call T^q the l_q -sum of the T_j 's. Assume now that each operator T_j has a certain degree of compactness. Then it is natural to investigate the degree of compactness of T^q . To be more precise, recall the notion of entropy numbers, a suitable measure of compactness. Let $T : \mathcal{H} \rightarrow E$ be a linear operator. Its n -th (dyadic) entropy number is then defined by

$$e_n(T) := \inf \left\{ \varepsilon > 0 : T(U_{\mathcal{H}}) \subseteq \bigcup_{j=1}^{2^{n-1}} \{T h_j + \varepsilon U_E\}, h_j \in U_{\mathcal{H}} \right\},$$

where $U_{\mathcal{H}}$ and U_E are the unit balls in \mathcal{H} and E , respectively. The previous question can now be formulated as follows. Suppose that

$$(2.1) \quad \sup_{n \geq 1} n^{1/\gamma} e_n(T_j) := \kappa_j < \infty$$

for a certain $\gamma > 0$ and all $j \in \mathbb{N}$. What can be said about $\sup_{n \geq 1} n^{1/\gamma} e_n(T^q)$?

The main objective of Section 5 is to provide the following answer.

Theorem 2.1. For $1 \leq j < \infty$ let T_j be operators from a separable Hilbert space \mathcal{H} into Banach spaces E_j . Suppose that for some $\gamma < 2$ we have (2.1). Then, if $1 \leq q < \infty$, this implies

$$(2.2) \quad \sup_{n \geq 1} n^{1/\gamma} e_n(T^q) \leq c \cdot \left(\sum_{j=1}^{\infty} \kappa_j^r \right)^{1/r}$$

with $1/r = 1/\gamma + 1/q - 1/2$ and $c > 0$ depending only on q and γ .

The notions of E^q and T^q make sense for $q = \infty$ as well. Yet, unfortunately, the proof of Theorem 2.1 cannot be extended to this case. Thus the following question remains open.

Problem. Let $\gamma < 2$ and $1/r := 1/\gamma - 1/2$. Suppose that operators $T_j : \mathcal{H} \rightarrow E_j$ satisfy (2.1). Does this imply for $T^\infty : \mathcal{H} \rightarrow E^\infty$ that

$$(2.3) \quad \sup_{n \geq 1} n^{1/\gamma} e_n(T^\infty) \leq c \cdot \left(\sum_{j=1}^{\infty} \kappa_j^r \right)^{1/r} ?$$

Our next aim is to formulate a result (proved in Section 6 below) asserting that (2.3) is valid under certain stronger assumptions than (2.1), namely, under certain additional properties of the E_j 's as well as of the type of continuity of the T_j 's. We start with specifying the image spaces E_j . Let (K_j, d_j) , $j \in \mathbb{N}$, be compact metric spaces. Then $C(K_j, d_j)$ denote the Banach spaces of continuous functions on K_j . This time compactness properties of the K_j 's are expressed in terms of (non-dyadic) entropy numbers. If (K, d) is a metric space, then its entropy numbers are defined by

$$\varepsilon_n(K, d) := \inf \left\{ \varepsilon > 0 : \exists A_1, \dots, A_n : K = \bigcup_{i=1}^n A_i, \text{diam}(A_i) \leq \varepsilon \right\}.$$

Next we have to specify the properties of the treated operators. Let T be an operator from \mathcal{H} with values in $C(K, d)$ for some compact metric space K . Given $\alpha \in (0, 1]$, it is said to be an α -Hölder operator (with α -Hölder norm $\|T\|_\alpha$) provided that

$$\|T\|_\alpha := \sup_{\|h\| \leq 1} \sup_{t \neq t'} \frac{|(Th)(t) - (Th)(t')|}{d(t, t')^\alpha} < \infty.$$

Now we are in a position to state the substitute for Theorem 2.1 in the case $q = \infty$.

Theorem 2.2. Let $T_j : \mathcal{H} \rightarrow C(K_j, d_j)$ be α -Hölder operators for some $\alpha \in (0, 1]$ and suppose that for each $j, n \in \mathbb{N}$ we have $\varepsilon_n(K_j, d_j) \leq \rho_j \cdot n^{-1/\sigma}$ for certain $\rho_j > 0$ and some $\sigma > 0$. If

$$T^\infty : \mathcal{H} \rightarrow l_\infty(C(K_j, d_j) : j \in \mathbb{N})$$

is defined by $T^\infty h := (T_j h)_{j \geq 1}$, then

$$e_n(T^\infty) \leq c \cdot \tilde{\rho} \cdot n^{-1/2 - \alpha/\sigma},$$

where $\tilde{\rho} := \left(\sum_{j=1}^{\infty} \|T_j\|_\alpha^{\sigma/\alpha} \rho_j^\sigma \right)^{\alpha/\sigma}$.

Remark 2.3. Let us compare Theorem 2.2 with Theorem 2.1. If the T_j 's from \mathcal{H} into $C(K_j, d_j)$ are 1-Hölder with $\|T_j\|_1 \leq 1$ (note that this can always be achieved by changing the metrics suitably) and if the compact spaces K_j satisfy for some $\gamma < 2$ the assumption

$$(2.4) \quad \varepsilon_n(K_j, d_j) \leq \kappa_j \cdot n^{-1/\gamma+1/2} ,$$

then we derive on one hand (see Theorem 6.2 below)

$$(2.5) \quad e_n(T_j) \leq c \cdot \kappa_j \cdot n^{-1/\gamma}$$

and on the other hand (by Theorem 2.2)

$$(2.6) \quad e_n(T^\infty) \leq c \cdot \left(\sum_{j=1}^\infty \kappa_j^r \right)^{1/r} \cdot n^{-1/\gamma} ,$$

where $1/r = 1/\gamma - 1/2$. For $q < \infty$, Theorem 2.1 allows us to derive (2.6) (with adequate r) directly from (2.5) while for $q = \infty$ we need (2.4) together with the Hölder condition for the T_j 's.

3. DEGREE OF COMPACTNESS OF FRACTIONAL INTEGRATION OPERATORS

For $H \in (0, 1)$ we shall investigate the following two integral operators:

$$(3.1) \quad (R_H f)(t) := \int_0^t (t-s)^{H-1/2} f(s) ds, \quad t \geq 0,$$

$$(3.2) \quad (B_H f)(t) := \int_{-\infty}^t \left[(t-s)^{H-1/2} - (-s)_+^{H-1/2} \right] f(s) ds, \quad t \geq 0.$$

As usual we set $(x)_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. It is well known that for any bounded interval $I \subseteq [0, \infty)$ and for any $p \in [1, \infty]$ both operators are bounded as mappings from $L_2(\mathbb{R})$ into $L_p(I)$. The operator R_H is usually called (up to some factor) a fractional integration operator of Riemann–Liouville type and B_H appears in a natural way in the theory of fractional Brownian motion.

For later purposes it is also important to investigate their difference $Q_H := B_H - R_H$ acting as follows:

$$(3.3) \quad (Q_H f)(t) := \int_{-\infty}^0 \left[(t-s)^{H-1/2} - (-s)^{H-1/2} \right] f(s) ds, \quad t \geq 0.$$

We need the following two known properties of these operators:

- (1) For $1 \leq p \leq \infty$ we have (cf. [12] or [2])

$$(3.4) \quad e_n(R_H : L_2(\mathbb{R}) \rightarrow L_p[0, 1]) \approx n^{-H-1/2} .$$

- (2) For each $H \in (0, 1)$ there exist constants $c_1, c_2 > 0$ such that (cf. [2])

$$(3.5) \quad e_n(Q_H : L_2(\mathbb{R}) \rightarrow L_p[0, 1]) \leq c_1 e^{-c_2 n^{1/3}} .$$

Consequently, in view of (3.4) and (3.5) we also have

$$(3.6) \quad e_n(B_H : L_2(\mathbb{R}) \rightarrow L_p[0, 1]) \approx n^{-H-1/2} .$$

Now let I be a bounded interval in $[0, \infty)$ and regard B_H as an operator from $L_2(\mathbb{R})$ into $L_p(I)$. Then, of course, the asymptotic behaviour of $e_n(B_H : L_2(\mathbb{R}) \rightarrow L_p(I))$ is of order $n^{-H-1/2}$ as well. For our later purposes it is important to know how the asymptotic depends on size and position of the interval I . The next theorem answers this question.

Theorem 3.1. *Let $I = [s_1, s_2]$ be a bounded interval in $[0, \infty)$. Regard B_H as an operator from $L_2(\mathbb{R})$ into $L_p(I)$ for some $p \in [1, \infty]$. Then there are operators S_H and F_H from $L_2(\mathbb{R})$ into $L_p(I)$ possessing the following properties:*

- (a) $B_H = S_H + F_H$,
- (b) $e_n(S_H) \leq c \cdot |I|^{H+1/p} \cdot n^{-H-1/2}$,
- (c) $\text{rank}(F_H) = 1$ and $\|F_H\| \leq C \cdot s_1^H \cdot |I|^{1/p}$.

Proof. Defining F_H by

$$(F_H f)(t) = \int_{-\infty}^{s_1} \left[(s_1 - s)^{H-1/2} - (-s)_+^{H-1/2} \right] f(s) ds, \quad t \in I,$$

the properties in (c) may be proved easily with

$$(3.7) \quad C^2 = \int_{-\infty}^1 \left[(1 - s)^{H-1/2} - (-s)_+^{H-1/2} \right]^2 ds.$$

If $S_H : L_2(\mathbb{R}) \rightarrow L_p(I)$ is defined by

$$(3.8) \quad (S_H f)(t) := \int_{-\infty}^t \left[(t - s)^{H-1/2} - (s_1 - s)_+^{H-1/2} \right] f(s) ds, \quad t \in I,$$

then decomposition (a) holds, and a simple scaling transformation gives

$$e_n(S_H) = |I|^{H+1/p} \cdot e_n(B_H : L_2 \rightarrow L_p[0, 1]).$$

Thus in view of (3.6) it follows that $e_n(S_H) \leq c \cdot |I|^{H+1/p} \cdot n^{-H-1/2}$. □

Let $\rho : (0, \infty) \rightarrow [0, \infty)$ be an L_q -weight function, i.e. it is measurable and satisfies $\rho \cdot \mathbf{1}_{[x,y]} \in L_q$ for any real numbers $0 < x < y$. Consequently, ρB_H is well defined as mapping from $L_2(\mathbb{R})$ into $L_q(0, \infty)$ and our aim is to find suitable estimates for its entropy numbers (in dependence of ρ).

Theorem 3.2. *For $0 < H < 1$, $1 \leq q \leq \infty$, let ρ be as above and regard ρB_H as mapping from $L_2(\mathbb{R})$ into $L_q(0, \infty)$. Then*

$$(3.9) \quad \sup_{n \geq 1} n^{H+1/2} e_n(\rho B_H) \leq c \cdot |\rho|_r$$

with r and $|\rho|_r$ defined by (1.4) and (1.5), respectively.

Proof. For the dyadic intervals $\Delta_k := [2^k, 2^{k+1}]$, $k \in \mathbb{Z}$, we define operators B_H^k , $k \in \mathbb{Z}$, by

$$(3.10) \quad (B_H^k f)(t) = \int_{-\infty}^t \left[(t - s)^{H-1/2} - (-s)_+^{H-1/2} \right] f(s) ds, \quad t \in \Delta_k.$$

If $f \in L_2(\mathbb{R})$, then we have

$$(3.11) \quad \|\rho B_H f\|_q = \left\| \sum_{k=-\infty}^{\infty} (\rho \cdot \mathbf{1}_{\Delta_k}) (B_H^k f) \right\|_q \leq \left(\sum_{k=-\infty}^{\infty} \|\rho\|_{L_q(\Delta_k)}^q \cdot \|B_H^k f\|_{\infty}^q \right)^{1/q}.$$

Setting

$$(3.12) \quad \rho_k := \|\rho\|_{L_q(\Delta_k)},$$

the operator

$$B_{H,\rho}^{\infty} : L_2(\mathbb{R}) \rightarrow l_q(C(\Delta_k) : k \in \mathbb{Z})$$

is defined by $B_{H,\rho}^\infty f := (\rho_k \cdot B_H^k f)_{k \in \mathbb{Z}}$. Estimate (3.11) may now be rewritten as

$$(3.13) \quad \|\rho B_H f\|_q \leq \|B_{H,\rho}^\infty f\|_{l_q(C(\Delta_k))}$$

for $f \in L_2(\mathbb{R})$. This is the place where the l_q -sums from Section 2 come into consideration. It is easy to see (cf. [16], Lemma 4.2) that (3.13) implies

$$(3.14) \quad e_n(\rho B_H) \leq e_n(B_{H,\rho}^\infty),$$

thus it remains to estimate the right-hand side of (3.14). This is done for $q < \infty$ and $q = \infty$ by different methods.

Let us start with $1 \leq q < \infty$. By Theorem 3.1 the operator $\rho_k B_H^k$ (with ρ_k defined by (3.12) and B_H^k by (3.10)) can be written as

$$(3.15) \quad \rho_k B_H^k = \rho_k S_H^k + \rho_k F_H^k,$$

where $S_H^k, F_H^k : L_2(\mathbb{R}) \rightarrow L_\infty(\Delta_k)$ possess the properties stated in Theorem 3.1 for $p = \infty$. In particular, assertion (b) of Theorem 3.1 yields

$$(3.16) \quad \sup_{n \geq 1} n^{1/2+H} e_n(\rho_k S_H^k) \leq c \cdot \rho_k \cdot 2^{kH}$$

while its assertion (c) leads to (use 1.3.36 in [5])

$$(3.17) \quad \begin{aligned} \sup_{n \geq 1} n^{1/2+H} e_n(\rho_k F_H^k) &\leq c \cdot \rho_k \cdot \sup_{n \geq 1} 2^{n-1} e_n(F_H^k) \\ &\leq c \cdot \rho_k \cdot \|F_H^k\| \leq c \cdot \rho_k \cdot 2^{kH}. \end{aligned}$$

We now apply Theorem 2.1 with $1/\gamma = H + 1/2$, i.e. it holds that

$$1/r = 1/\gamma + 1/q - 1/2 = H + 1/q,$$

hence (3.16) leads to

$$(3.18) \quad \sup_{n \geq 1} n^{1/2+H} e_n((\rho_k S_H^k)_{k \in \mathbb{Z}}) \leq c \cdot \left(\sum_{k=-\infty}^{\infty} \rho_k^r 2^{kHr} \right)^{1/r} = c \cdot |\rho|_r.$$

By similar reason (3.17) implies

$$(3.19) \quad \sup_{n \geq 1} n^{1/2+H} e_n((\rho_k F_H^k)_{k \in \mathbb{Z}}) \leq c \cdot |\rho|_r$$

and since (3.15) yields $B_{H,\rho}^\infty = (\rho_k S_H^k)_{k \in \mathbb{Z}} + (\rho_k F_H^k)_{k \in \mathbb{Z}}$ we derive (3.9) from (3.14), (3.18) and (3.19). This completes the proof in the case $q < \infty$.

Assume now $q = \infty$. Our aim is to apply Theorem 2.2. Therefore, we have to evaluate the Hölder continuity of B_H^k as well as the degree of compactness of Δ_k . By trivial calculations $\rho_k B_H^k$ is H -Hölder with H -Hölder norm

$$(3.20) \quad \|\rho_k B_H^k\|_H = C \cdot \rho_k, \quad k \in \mathbb{Z},$$

with C defined in (3.7). On the other hand, of course, it holds that

$$(3.21) \quad \varepsilon_n(\Delta_k, |\cdot|) \approx |\Delta_k| \cdot n^{-1}.$$

Consequently, an application of Theorem 2.2 (with $\sigma = 1$ and $\alpha = H$) leads by (3.20) and (3.21) to

$$\sup_{n \geq 1} n^{1/2+H} e_n(B_{H,\rho}^\infty) \leq c \cdot \left(\sum_{k=-\infty}^{\infty} \rho_k^{1/H} |\Delta_k| \right)^H = c \cdot |\rho|_r.$$

Note that $1/r = H$ in this case. Finally (3.14) completes the proof for $q = \infty$. \square

4. SMALL DEVIATIONS OF WEIGHTED FRACTIONAL PROCESSES

Let W be a Wiener process on $[0, \infty)$. Then, if $0 < H < 1$, we consider the following two fundamental fractional processes:

$$(4.1) \quad \begin{aligned} R^H(t) &:= \int_0^t (t-s)^{H-1/2} dW(s), \quad t \geq 0, \\ B^H(t) &:= \int_{-\infty}^t \left[(t-s)^{H-1/2} - (-s)_+^{H-1/2} \right] dW(s), \quad t \geq 0. \end{aligned}$$

The process $(R^H(t))_{t \geq 0}$ is usually called the Riemann–Liouville process of order H . It makes sense for all $H > 0$, yet since we are mainly interested in the relation between R^H and B^H we restrict ourselves to $H < 1$. The process $(B^H(t))_{t \geq 0}$ is up to some factor the classical fractional Brownian motion as introduced in (1.1) (with Hurst index H). Of course, for $H = 1/2$ both processes coincide with W .

The processes R^H and B^H are tightly related with the operators R_H and B_H introduced in (3.1) and (3.2), respectively. Indeed, if $(f_k)_{k \geq 1}$ is any ONB in $L_2(\mathbb{R})$ and if the ξ_k 's are $\mathcal{N}(0, 1)$ -distributed i.i.d. variables, then the random functions

$$\begin{aligned} t &\rightarrow \sum_{k=1}^{\infty} \xi_k (R_H f_k)(t), \quad t \geq 0, \quad \text{and} \\ t &\rightarrow \sum_{k=1}^{\infty} \xi_k (B_H f_k)(t), \quad t \geq 0, \end{aligned}$$

are Riemann–Liouville and fractional Brownian motion processes of order H , respectively.

It is known (cf. [11] and [17]) that for any $q \in [1, \infty]$ there exist the finite positive small ball limits

$$(4.2) \quad c(H, q) := -\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|R^H\|_{L_q[0,1]} \leq \varepsilon \right)$$

$$(4.3) \quad = -\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|B^H\|_{L_q[0,1]} \leq \varepsilon \right).$$

Suppose now we are given a (measurable) weight function $\rho : (0, \infty) \rightarrow [0, \infty)$ such that for some $q \in [1, \infty]$ we have $\mathbb{P} \left(\|\rho B^H\|_{L_q(0, \infty)} < \infty \right) = 1$. In this case one may ask whether or not $\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho B^H\|_{L_q(0, \infty)} \leq \varepsilon \right)$ exists as well and if so, how it depends on ρ . Similar questions may be posed for R^H .

A first step toward an answer to the above questions is given by the following estimates.

Theorem 4.1. *For $q \in [1, \infty]$ let $r > 0$ be defined by (1.4). Then*

$$(4.4) \quad \begin{aligned} &\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho R^H(t)\|_q \leq \varepsilon \right) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho B^H(t)\|_q \leq \varepsilon \right) \geq -c \cdot |\rho|_r^{1/H} \end{aligned}$$

for some $c > 0$ depending only on q and H .

Proof. The first inequality is a direct consequence of Anderson's inequality (see [1] or [14], Chapter 11). The second one immediately follows from Theorem 3.2 and the results in [8] and [12] linking small deviations and entropy numbers. \square

Metric entropy tools are too crude for getting sharp small deviation estimates as stated in (4.2) or (4.3). For this purpose, more precise probabilistic methods are necessary. Nevertheless, Theorem 4.1 will play an important role in eliminating some disturbing remainder term in Theorem 4.6 below.

Before proceeding further, we state two lemmas for later use. The first one is a partial solution of the so-called correlation conjecture. It was first proved in the necessary generality in [9] as an extension of [23]; see also Theorem 2.14 in [13]. We formulate it here in the form appropriate for our own use.

Lemma 4.2. *For every $\eta \in (0, 1)$ there exists a constant $K = K(\eta) > 0$ such that for every $u > 0$ and every pair of centered jointly Gaussian random vectors X_1, X_2 with values in a Banach space E one has*

$$\mathbb{P}(\|X_1 + X_2\| \leq u) \geq \mathbb{P}(\|X_1\| \leq (1 - \eta)u) \mathbb{P}(\|X_2\| \leq u/K) .$$

The next result is a direct consequence of properties of Q_H defined in (3.3).

Lemma 4.3. *Let $I \subseteq [0, \infty)$ be a finite interval and let $(I_j)_{j=1}^m$ be a finite partition of I into intervals $I_j = [s_j, s_{j+1}]$ with $s_1 < \dots < s_{m+1}$. Define the stochastic process $Z^H = (Z^H(t))_{t \in I}$ by*

$$(4.5) \quad Z^H(t) := \int_{-\infty}^{s_j} \left[(t - s)^{H-1/2} - (-s)_+^{H-1/2} \right] dW(s) , \quad t \in I_j .$$

If $\rho = \sum_{j=1}^m \rho_j \mathbf{1}_{I_j}$ for some $\rho_j \geq 0$, then for each $\gamma > 0$ and $q \in [1, \infty]$ there is a constant $c > 0$ (depending on q, γ, ρ and I) such that

$$\log \mathbb{P} \left(\|\rho Z^H\|_{L_q(I)} \leq \varepsilon \right) \geq -c \cdot \varepsilon^{-\gamma} , \quad \varepsilon > 0 .$$

Proof. Define an operator $T_H^j : L_2(\mathbb{R}) \rightarrow L_q(I_j)$ by

$$\begin{aligned} (T_H^j f)(t) &:= \int_{-\infty}^{s_j} \left[(t - s)^{H-1/2} - (-s)_+^{H-1/2} \right] f(s) ds \\ &= \int_{-\infty}^{s_j} \left[(t - s)^{H-1/2} - (s_j - s)^{H-1/2} \right] f(s) ds \\ &+ \int_{-\infty}^{s_j} \left[(s_j - s)^{H-1/2} - (-s)_+^{H-1/2} \right] f(s) ds, \end{aligned}$$

where the first operator is equivalent to $|I_j|^{H+1/q} Q_H$ while the second one has rank one. Thus by (3.5) and estimate 1.3.36 in [5] it follows that

$$(4.6) \quad e_n(\rho_j T_H^j) \leq c_1 \cdot e^{-c_2 n^{1/3}}$$

for certain $c_1, c_2 > 0$ depending on ρ_j, q, H and I_j . Of course, by the additivity of entropy numbers estimate (4.6) remains valid (with different constants) for $e_n(\sum_{j=1}^m \rho_j T_H^j)$. In particular, this implies

$$(4.7) \quad e_n \left(\sum_{j=1}^m \rho_j T_H^j \right) \leq c \cdot n^{-1/\theta}$$

for any $\theta < 2$. Finally, by Theorem 5.1 in [12] we derive from (4.7)

$$\log \mathbb{P} \left(\|\rho Z^H\|_{L_q(I)} \leq \varepsilon \right) \geq -c' \cdot \varepsilon^{-2\theta/(2-\theta)},$$

completing the proof. □

Our next aim is to precise the lower bound in (4.4) for ρ 's possessing bounded support.

Theorem 4.4. *Let $I \subseteq [0, \infty)$ be a bounded interval and suppose $\rho \in L_q(I)$ for some $q \in [1, \infty)$. Then*

$$(4.8) \quad \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho B^H\|_{L_q(I)} \leq \varepsilon \right) \geq -c(H, q) \cdot \|\rho\|_{L_r(I)}^{1/H},$$

where r and $c(H, q)$ are as in (1.4) and (1.5), respectively.

Proof. In a first step we prove (4.8) for interval step functions ρ , i.e. we assume

$$(4.9) \quad \rho = \sum_{j=1}^m \rho_j \mathbf{1}_{I_j}$$

for certain $\rho_j \geq 0$ and intervals $I_j = [s_j, s_{j+1}]$ with $s_1 < \dots < s_{m+1}$ and $I = [s_1, s_{m+1}]$. Consequently, for such a function ρ we may represent ρB^H as

$$(4.10) \quad \rho B^H = \rho Z^H + \sum_{j=1}^m \rho_j R_j^H$$

with Z^H defined in (4.5) and

$$(4.11) \quad R_j^H(t) := \int_{s_j}^t (t-s)^{H-1/2} dW(s), \quad t \in I_j.$$

Note that R_1^H, \dots, R_m^H are independent processes satisfying

$$(4.12) \quad \|R_j^H\|_{L_q(I_j)} \stackrel{d}{=} |I_j|^{H+1/q} \|R^H\|_{L_q[0,1]}$$

with R^H given by (4.1). Hence, from (4.2) and (4.12) we derive

$$(4.13) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho_j R_j^H\|_{L_q(I_j)} \leq \varepsilon) = -c(H, q) \cdot \rho_j^{1/H} \cdot |I_j|^{1+1/Hq}$$

for $1 \leq j \leq m$.

Next, we use the following summation rule for the small ball asymptotic (see e.g. [10] or Corollary 3.1 in [13]). It asserts that if for each member of a finite family $(\zeta_j)_{j=1}^m$ of non-negative independent random variables one has

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \log \mathbb{P} (\zeta_j \leq \varepsilon) = -b_j,$$

then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \log \mathbb{P} \left(\sum_{j=1}^m \zeta_j \leq \varepsilon \right) = - \left(\sum_{j=1}^m b_j^{1+\alpha} \right)^{1+\alpha}.$$

We apply this rule with $\alpha = 1/Hq$ and $\zeta_j = \|\rho_j R_j^H\|_{L_q(I_j)}^q$ and obtain by (4.13)

$$(4.14) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\left\| \sum_{j=1}^m \rho_j R_j^H \right\|_{L_q(I)} \leq \varepsilon \right) \\ &= -c(H, q) \cdot \left(\sum_{j=1}^m \rho_j^{\frac{1}{H+1/q}} |I_j| \right)^{1+1/Hq} = -c(H, q) \cdot \|\rho\|_{L_r(I)}^{1/H}. \end{aligned}$$

From Lemma 4.2 and (4.10) one derives for each $\eta \in (0, 1)$ the following inequality:

$$\begin{aligned} & \mathbb{P} \left(\|\rho B^H\|_{L_q(I)} \leq \varepsilon \right) \\ & \geq \mathbb{P} \left(\|\rho Z^H\|_{L_q(I)} \leq \varepsilon/K(\eta) \right) \cdot \mathbb{P} \left(\left\| \sum_{j=1}^m \rho_j R_j^H \right\|_{L_q(I)} \leq (1-\eta)\varepsilon \right). \end{aligned}$$

In view of Lemma 4.3 this leads to

$$(4.15) \quad \begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho B^H\|_{L_q(I)} \leq \varepsilon \right) \\ & \geq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\left\| \sum_{j=1}^m \rho_j R_j^H \right\|_{L_q(I)} \leq (1-\eta)\varepsilon \right). \end{aligned}$$

Combining (4.15) with (4.14) and taking the limit $\eta \rightarrow 0$ in the right-hand side completes the proof for interval step functions ρ .

Now let $\rho \in L_q(I)$ be arbitrary. For $\delta > 0$ we find (cf. Lemma 4.4 in [16]) an interval step function ψ such that

$$(4.16) \quad \|\bar{\rho}\|_{L_q(I)} < \delta \quad \text{and} \quad \|\psi\|_{L_r(I)} = \|\rho\|_{L_r(I)},$$

where $\bar{\rho} := [\rho - \psi]_+$. We have $\rho \leq \psi + \bar{\rho}$, hence another application of Lemma 4.2 yields

$$(4.17) \quad \begin{aligned} & \mathbb{P} \left(\|\rho B^H\|_{L_q(I)} \leq \varepsilon \right) \geq \mathbb{P} \left(\|\psi B^H + \bar{\rho} B^H\|_{L_q(I)} \leq \varepsilon \right) \\ & \geq \mathbb{P} \left(\|\psi B^H\|_{L_q(I)} \leq (1-\eta)\varepsilon \right) \cdot \mathbb{P} \left(\|\bar{\rho} B^H\|_{L_q(I)} \leq \varepsilon/K(\eta) \right). \end{aligned}$$

Since by (4.16)

$$\|\bar{\rho} B^H\|_{L_q(I)} \leq \|\bar{\rho}\|_{L_q(I)} \cdot \|B^H\|_{L_\infty(I)} \leq \delta \cdot \|B^H\|_{L_\infty(I)}$$

from (4.3) we derive

$$(4.18) \quad \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\bar{\rho} B^H\|_{L_q(I)} \leq \varepsilon/K(\eta) \right) \geq -c(H, \infty) \cdot (K(\eta) \cdot \delta)^{1/H}.$$

On the other hand (recall that ψ is an interval step function), by the first step and by (4.16) it follows that

$$(4.19) \quad \begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\psi B^H\|_{L_q(I)} \leq (1-\eta)\varepsilon \right) \\ & \geq -c(H, q) \cdot \|\psi\|_{L_r(I)}^{1/H} \cdot (1-\eta)^{1/H} = -c(H, q) \cdot \|\rho\|_{L_r(I)}^{1/H} \cdot (1-\eta)^{1/H}. \end{aligned}$$

Combining (4.17), (4.18) and (4.19) finally leads to

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho B^H\|_{L_q(I)} \leq \varepsilon \right) \\ & \geq -c(H, q) \cdot \|\rho\|_{L_r(I)}^{1/H} \cdot (1-\eta)^{1/H} - c(H, \infty) \cdot (K(\eta) \cdot \delta)^{1/H}, \end{aligned}$$

hence, if we first take the limit $\delta \rightarrow 0$ and then $\eta \rightarrow 0$, we obtain (4.8). □

Theorem 4.5. *For any weight function ρ on $[0, \infty)$ and any $q \in [1, \infty)$ one has*

$$(4.20) \quad \limsup_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho R^H\|_{L_q(0, \infty)} \leq \varepsilon \right) \leq -c(H, q) \cdot \|\rho\|_{L_r(0, \infty)}^{1/H}$$

with $r > 0$ defined by (1.4).

Proof. Again we start with proving (4.20) for interval step functions. Thus suppose ρ may be written as $\rho = \sum_{j=1}^m \rho_j \mathbf{1}_{I_j}$ with ρ_j 's, I and I_j 's as in (4.9). We now decompose

$$(4.21) \quad \rho R^H = \rho \tilde{Z}^H + \sum_{j=1}^m \rho_j R_j^H$$

with R_j^H 's defined by (4.11) and with

$$\tilde{Z}^H(t) := \int_0^{s_j} (t - s)^{H-1/2} dW(s), \quad t \in I_j.$$

By Anderson's inequality

$$\mathbb{P} \left(\|\rho Z^H\|_{L_q(I)} \leq \varepsilon \right) \leq \mathbb{P} \left(\|\rho \tilde{Z}^H\|_{L_q(I)} \leq \varepsilon \right)$$

for Z^H defined by (4.5), hence Lemma 4.3 lets us conclude that

$$(4.22) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho \tilde{Z}^H\|_{L_q(I)} \leq \varepsilon \right) = 0.$$

On the other hand, by (4.14) we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\left\| \sum_{j=1}^m \rho_j R_j^H \right\|_{L_q(I)} \leq \varepsilon \right) = -c(H, q) \cdot \|\rho\|_{L_r(I)}^{1/H}.$$

Next we apply Lemma 4.2 to (4.21) and obtain

$$\begin{aligned} -c(H, q) \cdot \|\rho\|_{L_r(I)}^{1/H} &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\left\| \sum_{j=1}^m \rho_j R_j^H \right\|_{L_q(I)} \leq \varepsilon \right) \\ &\geq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho R^H\|_{L_q(I)} \leq (1 - \eta) \varepsilon \right) \\ &\quad + \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho \tilde{Z}^H\|_{L_q(I)} \leq \varepsilon / K(\eta) \right) \end{aligned}$$

which by (4.22) implies

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} \left(\|\rho R^H\|_{L_q(I)} \leq \varepsilon \right) \leq -c(H, q) \cdot (1 - \eta)^{1/H} \cdot \|\rho\|_{L_r(I)}^{1/H}.$$

Thus (4.20) follows by taking $\eta \rightarrow 0$.

Let us now treat the general case. First note that it suffices to prove (4.20) for bounded ρ 's supported on a bounded closed interval $I \subset (0, \infty)$. Since I is strictly separated from the critical points 0 and ∞ , for a given $\delta > 0$ we find a (non-negative) interval step function ψ on I satisfying

$$(4.23) \quad \|\psi - \rho\|_{L_r(I)} < \delta$$

as well as

$$(4.24) \quad \left| \|\rho\|_{L_r(I)}^{1/H} - \|\psi\|_{L_r(I)}^{1/H} \right| < \delta.$$

Because of $\psi \leq \rho + [\psi - \rho]_+$ from Lemma 4.2 and the first step we derive

$$\begin{aligned}
 -c(H, q) \cdot \|\psi\|_{L_r(I)}^{1/H} &\geq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\psi R^H\|_{L_q(I)} \leq \varepsilon) \\
 &\geq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho R^H\|_{L_q(I)} \leq (1 - \eta) \varepsilon) \\
 (4.25) \qquad &+ \liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|[\psi - \rho]_+ R^H\|_{L_q(I)} \leq \varepsilon/K(\eta)) .
 \end{aligned}$$

The last term in (4.25) can be estimated by Theorem 4.4, hence by (4.23) and (4.24) it follows that

$$\begin{aligned}
 &\limsup_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho R^H\|_{L_q(I)} \leq \varepsilon) \\
 &\leq (1 - \eta)^{1/H} \left\{ -c(H, q) \cdot \|\psi\|_{L_r(I)}^{1/H} + c(H, q) \cdot K(\eta)^{1/H} \cdot \|\psi - \rho\|_r^{1/H} \right\} \\
 &\leq (1 - \eta)^{1/H} \left\{ -c(H, q) \cdot (\|\rho\|_{L_r(I)}^{1/H} - \delta) + c(H, q) \cdot K(\eta)^{1/H} \cdot \delta^{1/H} \right\} .
 \end{aligned}$$

Finally, first taking $\delta \rightarrow 0$ and then $\eta \rightarrow 0$ proves (4.20) for arbitrary ρ 's. □

Combining the previous results we obtain the following.

Theorem 4.6. *Let ρ be a weight function on $(0, \infty)$ such that for some $q \in [0, \infty)$ we have $|\rho|_r < \infty$. Recall that $1/r = H + 1/q$. Then*

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho R^H\|_{L_q(0,\infty)} \leq \varepsilon) &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho B^H\|_{L_q(0,\infty)} \leq \varepsilon) \\
 (4.26) \qquad \qquad \qquad &= -c(H, q) \cdot \|\rho\|_{L_r(0,\infty)}^{1/H} .
 \end{aligned}$$

Proof. Since $|\rho|_r < \infty$, for a given $\delta > 0$ we may split ρ into the sum of two non-negative functions $\rho^{(1)}$ and $\rho^{(2)}$ such that $\rho^{(1)}$ is supported on a closed bounded interval in $(0, \infty)$ and that $|\rho^{(2)}|_r < \delta$. Hence Theorem 4.4 applies to $\rho^{(1)}$ and gives

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho^{(1)} B^H\|_{L_q(0,\infty)} \leq \varepsilon) \geq -c(H, q) \cdot \|\rho\|_{L_r(0,\infty)}^{1/H}$$

while by Theorem 4.1 we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho^{(2)} B^H\|_{L_q(0,\infty)} \leq \varepsilon) \geq -c \cdot |\rho^{(2)}|_r^{1/H} \geq -c \cdot \delta^{1/H} .$$

Applying Lemma 4.2, for each $\eta \in (0, 1)$ it follows that

$$\begin{aligned}
 &\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho B^H\|_{L_q(0,\infty)} \leq \varepsilon) \\
 &\geq -c(H, q) \cdot \|\rho\|_{L_r(0,\infty)}^{1/H} \cdot (1 - \eta)^{-1/H} - c \cdot K(\eta)^{1/H} \cdot \delta^{1/H} .
 \end{aligned}$$

Taking $\delta \rightarrow 0$ and then $\eta \rightarrow 0$ finally implies that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho B^H\|_{L_q(0,\infty)} \leq \varepsilon) \geq -c(H, q) \cdot \|\rho\|_{L_r(0,\infty)}^{1/H} .$$

By Anderson's inequality, the same is true for R^H .

On the other hand, Anderson's inequality and Theorem 4.5 let us conclude that

$$\begin{aligned}
 &\limsup_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho B^H\|_{L_q(0,\infty)} \leq \varepsilon) \\
 &\leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \log \mathbb{P} (\|\rho R^H\|_{L_q(0,\infty)} \leq \varepsilon) \leq -c(H, q) \cdot \|\rho\|_{L_r(0,\infty)}^{1/H}
 \end{aligned}$$

completing the proof. □

Some comments to the case $q = \infty$ which is not covered by Theorem 4.4, Theorem 4.5 and by Theorem 4.6 above. The results are also valid in this case with ρ replaced by its regularization ρ^* defined by

$$\rho^*(s) := \lim_{\delta \rightarrow 0} \operatorname{ess\,sup}_{\{x: |x-s| < \delta\}} \rho(s).$$

The proofs practically do not change. We only have to use Lemma 4.5 in [16] instead of Lemma 4.4 to approximate the weight function by interval step functions. For more details we refer to [16].

5. AVERAGE KOLMOGOROV NUMBERS AND THEIR APPLICATION

The proof of Theorem 2.1 relies upon properties of so-called average Kolmogorov numbers of operators. For a separable Hilbert space \mathcal{H} we denote by $\mathcal{G}(\mathcal{H}, E)$ the set of operators T from \mathcal{H} into E such that the series

$$(5.1) \quad X_T := \sum_{k=1}^{\infty} \xi_k T f_k$$

converges a.s. in E for some (each) ONB $(f_k)_{k \geq 1}$ in \mathcal{H} . Here $(\xi_k)_{k \geq 1}$ is an i.i.d. sequence of $\mathcal{N}(0, 1)$ -distributed random variables. Recall that (with suitable $T \in \mathcal{G}(\mathcal{H}, E)$) any centered Gaussian E -valued vector may be represented in the form (5.1).

For $T \in \mathcal{G}(\mathcal{H}, E)$ we may now define its n -th average Kolmogorov number by

$$g_n(T) := g_n(X_T) = \inf \left\{ \left(\mathbb{E} \|Q_N X_T\|^2 \right)^{1/2} : \dim N < n \right\}$$

with X_T defined by (5.1). In particular, $l(T) := g_1(T) = \left(\mathbb{E} \|X_T\|^2 \right)^{1/2}$ is the well-known l -norm of T and $[\mathcal{G}(\mathcal{H}, E), l]$ is a Banach space (cf. [22] for more details).

The following properties of $g_n(T)$ are either straightforward or may be found in [6]. For the definition of the “ordinary” Kolmogorov numbers $d_n(T)$, we refer to [21].

(a) For $q \in [1, \infty)$ set

$$g_n^{(q)}(T) := \inf \left\{ \left(\mathbb{E} \inf_{y \in N} \|X_T - y\|^q \right)^{1/q} : \dim(N) < n \right\}.$$

Then there exist two constants $c_1, c_2 > 0$ depending only on q such that

$$c_1 g_n(T) \leq g_n^{(q)}(T) \leq c_2 g_n(T).$$

(b) By a deep result in [20] there exists a universal $c > 0$ such that for all $T \in \mathcal{G}(\mathcal{H}, E)$

$$n^{1/2} \cdot d_{2n-1}(T) \leq c \cdot g_n(T).$$

Consequently, by Carl’s inequality

$$\sup_{n \geq 1} n^{1/\beta+1/2} e_n(T) \leq c(\beta) \cdot \sup_{n \geq 1} n^{1/\beta+1/2} d_n(T)$$

(see e.g. [22] or [4] for a proof) it follows that

$$(5.2) \quad \sup_{n \geq 1} n^{1/\beta+1/2} e_n(T) \leq c \cdot \sup_{n \geq 1} n^{1/\beta} g_n(T)$$

for any $\beta > 0$ and with $c > 0$ depending only on β .

(c) The following is a slightly improved version of a result due to G. Pisier (see [22], page 141). If $T^* : E^* \rightarrow \mathcal{H}$ denotes the dual of $T \in \mathcal{G}(\mathcal{H}, E)$, then for some $c_1, c_2 > 0$ we have

$$g_n(T) \leq c_1 \cdot \sum_{k \geq c_2 n} k^{-1/2} e_k(T^*) .$$

By combining this bound with the estimates from [25], we obtain

$$(5.3) \quad \sup_{n \geq 1} n^{1/\beta} g_n(T) \leq c \cdot \sup_{n \geq 1} n^{1/\beta+1/2} e_n(T)$$

for any $\beta > 0$.

The main advantage of the numbers g_n (in contrast to the entropy numbers) is that they are well related to the l_q -summation of operators, as we shall see in a moment. Before doing so, let us first state conditions ensuring $T^q \in \mathcal{G}(\mathcal{H}, E^q)$ provided each operator T_j belongs to $\mathcal{G}(\mathcal{H}, E_j)$.

Theorem 5.1. *If the operators $T_j \in \mathcal{G}(\mathcal{H}, E_j)$ satisfy*

$$(5.4) \quad \sum_{j=1}^{\infty} l(T_j)^q < \infty ,$$

then their l_q -sum T^q belongs to $\mathcal{G}(\mathcal{H}, E^q)$ and

$$(5.5) \quad l(T^q) \leq c_q \cdot \left[\sum_{j=1}^{\infty} l(T_j)^q \right]^{1/q} .$$

Proof. For $m \geq 0$ let $P_m : E^q \rightarrow E^q$ be defined by

$$P_m(x) := (x_1, \dots, x_m, 0, 0, \dots), \quad x = (x_j)_{j \geq 1} .$$

Of course, we have $P_m \circ T^q \in \mathcal{G}(\mathcal{H}, E^q)$ and, if $n > m$, then by the equivalence of all moments of Gaussian vectors it follows that

$$(5.6) \quad l(P_n \circ T^q - P_m \circ T^q) \leq c_q \left[\sum_{j=m+1}^n l(T_j)^q \right]^{1/q} .$$

Hence, by (5.4) and the completeness of $[\mathcal{G}(\mathcal{H}, E^q), l]$ we get $T^q \in \mathcal{G}(\mathcal{H}, E^q)$ as asserted.

From (5.6) we derive

$$l(P_n \circ T^q) \leq c_q \left[\sum_{j=1}^n l(T_j)^q \right]^{1/q}$$

which, of course, implies (5.5) by taking the limit $n \rightarrow \infty$. □

The next result is the key for the proof of Theorem 2.1.

Theorem 5.2. *Let $\beta > 0$ be given and suppose we have*

$$(5.7) \quad \sup_{n \geq 1} n^{1/\beta} g_n(T_j) := \kappa_j < \infty .$$

Then

$$\sup_{n \geq 1} n^{1/\beta} g_n(T^q) \leq c_q \cdot \left(\sum_{j=1}^{\infty} \kappa_j^r \right)^{1/r}$$

with $1/r = 1/\beta + 1/q$ and c_q depending only on q .

Proof. Of course, we may assume $\sum \kappa_j^r < \infty$, hence $\sum \kappa_j^q < \infty$ as well. Consequently,

$$\sum_{j=1}^{\infty} l(T_j)^q = \sum_{j=1}^{\infty} g_1(T_j)^q \leq \sum_{j=1}^{\infty} \kappa_j^q < \infty$$

which implies $T^q \in \mathcal{G}(\mathcal{H}, E^q)$ by Theorem 5.1.

Let $n_j \geq 1$ be any sequence of integers with $m := \sum_{j=1}^{\infty} (n_j - 1) < \infty$. For this sequence, we choose subspaces $N_j \subseteq E_j$ satisfying $\dim(N_j) \leq n_j - 1$ and

$$\mathbb{E} \|Q_{N_j} X_{T_j}\|^q \leq 2^q g_{n_j}^{(q)}(T_j)^q.$$

Next we define $N \subseteq E^q$ by

$$N := \{x = (x_j)_{j \geq 1} : x_j \in N_j\}$$

and observe that $\dim(N) \leq m$. Hence, by property (a) and (5.7),

$$\begin{aligned} g_{m+1}^{(q)}(T^q)^q &\leq \mathbb{E} \|Q_N X_{T^q}\|^q = \sum_{j=1}^{\infty} \mathbb{E} \|Q_{N_j} X_{T_j}\|^q \leq 2^q \sum_{j=1}^{\infty} g_{n_j}^{(q)}(T_j)^q \\ (5.8) \quad &\leq c^q \sum_{j=1}^{\infty} g_{n_j}(T_j)^q \leq c^q \sum_{j=1}^{\infty} \kappa_j^q \cdot n_j^{-q/\beta}. \end{aligned}$$

For a fixed natural number n we now choose integers n_j satisfying

$$(5.9) \quad n_j - 1 < n \cdot \frac{\kappa_j^r}{\sum_{i=1}^{\infty} \kappa_i^r} \leq n_j$$

for all $j \geq 1$. Of course, only a finite number of the n_j 's can be different from 1, thus $m := \sum_{j=1}^{\infty} (n_j - 1) < \infty$ and by (5.9) it follows that $m < n$. From this, (5.8) and (5.9) we derive

$$\begin{aligned} g_n^{(q)}(T^q) &\leq g_{m+1}^{(q)}(T^q) \leq c \cdot n^{-1/\beta} \left(\sum_{j=1}^{\infty} \kappa_j^{q - qr/\beta} \right)^{1/q} \cdot \left(\sum_{i=1}^{\infty} \kappa_i^r \right)^{1/\beta} \\ &= c \cdot n^{-1/\beta} \left(\sum_{j=1}^{\infty} \kappa_j^r \right)^{1/r} \end{aligned}$$

by the choice of r . Combining this with property (a) of the g_n 's completes the proof. \square

Proof of Theorem 2.1. Let $\beta > 0$ be defined by $1/\beta := 1/\gamma - 1/2$. In view of (5.3) assumption (2.1) leads to $\sup_{n \geq 1} n^{1/\beta} g_n(T_j) \leq c_1(\beta) \cdot \kappa_j$ for all $j = 1, 2, \dots$. Hence, Theorem 5.2 yields

$$(5.10) \quad \sup_{n \geq 1} n^{1/\beta} g_n(T^q) \leq c_2(q, \beta) \cdot \left(\sum_{j=1}^{\infty} \kappa_j^r \right)^{1/r},$$

where $1/r = 1/\beta + 1/q = 1/\gamma + 1/q - 1/2$. Finally, from (5.2) and (5.10) we derive

$$\sup_{n \geq 1} n^{1/\gamma} e_n(T^q) = \sup_{n \geq 1} n^{1/\beta+1/2} e_n(T^q) \leq c_3(q, \beta) \cdot \left(\sum_{j=1}^{\infty} \kappa_j^r \right)^{1/r}$$

as asserted.

Problem. Suppose that the operators T_j are defined on some L_p -space with $1 < p < \infty$ and satisfy (2.1). What is the largest possible $r > 0$ (depending on γ, q and p) for which (2.2) holds? Which restrictions on the parameter γ are necessary?

6. BUNDLES OF METRIC SPACES

For $j \in \mathbb{N}$ let (K_j, d_j) be compact metric spaces and suppose that in each space K_j a special point t_j^* is marked. Then we define the bundle space (K_∞, d_∞) as a (disjoint) union of the K_j 's where all t_j^* 's are identified with one point $t^* \in K_\infty$. The metric d_∞ on K_∞ is then given by

$$d_\infty(t_1, t_2) := \begin{cases} d_j(t_1, t_2) & \text{if } t_1, t_2 \in K_j, \\ d_{j_1}(t_1, t_{j_1}^*) + d_{j_2}(t_2, t_{j_2}^*) & \text{if } t_1 \in K_{j_1}, t_2 \in K_{j_2}, j_1 \neq j_2. \end{cases}$$

Observe that (K_∞, d_∞) is not necessarily compact. It is so (as can easily be seen) iff we have $\text{diam}(K_j) \rightarrow 0$ as $j \rightarrow \infty$. Assuming this, it is natural to ask how certain compactness properties of the K_j 's carry over to those of the bundle space K_∞ . The next theorem answers this question.

Theorem 6.1. *Suppose the family of metric spaces (K_j, d_j) generates the bundle space (K_∞, d_∞) and assume that for some $\sigma > 0$ and for any $j \geq 1$ there exist $\rho_j > 0$ such that*

$$(6.1) \quad \varepsilon_n(K_j, d_j) < \rho_j \cdot n^{-1/\sigma}, \quad n \in \mathbb{N}.$$

Then

$$(6.2) \quad \varepsilon_n(K_\infty, d_\infty) \leq 2^{2/\sigma+1} \tilde{\rho} \cdot n^{-1/\sigma}, \quad n \in \mathbb{N},$$

with $\tilde{\rho} = \left(\sum_{j=1}^{\infty} \rho_j^\sigma \right)^{1/\sigma}$.

We omit the elementary and relatively simple proof of Theorem 6.1.

Before proceeding further we need certain compactness properties of Hölder operators with values in a space of continuous functions. More precisely, the following may be found in [5], Sect. 5.10.

Theorem 6.2. *Let \mathcal{H} be a Hilbert space and let $\rho_0, \sigma > 0$ and $\alpha \in (0, 1]$. Then there exists a constant $c = c(\sigma, \alpha) > 0$ such that the following is valid: If (K, d) is a compact space with*

$$\varepsilon_n(K, d) \leq \rho_0 \cdot n^{-1/\sigma}, \quad n \in \mathbb{N},$$

then for each α -Hölder operator $T : \mathcal{H} \rightarrow C(K, d)$ we have

$$e_n(T) \leq c \cdot \rho_0^\alpha \cdot \|T\|_\alpha \cdot n^{-1/2-\alpha/\sigma}, \quad n \in \mathbb{N}.$$

If $C_0(K_j, d_j)$ denotes the set of $f \in C(K_j, d_j)$ with $f(t_j^*) = 0$, the l_∞ -sum of the $C_0(K_j, d_j)$'s may obviously be viewed as $\tilde{C}_0(K_\infty, d_\infty)$, the space of functions defined on K_∞ , continuous on every K_j , and vanishing at t^* . Note that such functions need not be continuous on the entire space K_∞ . Suppose now we are given a family $T_j, j \in \mathbb{N}$, of operators, mapping a Hilbert space \mathcal{H} into $C(K_j, d_j)$. Our aim is to paste them together to a single operator T^∞ . To do so, we have to assume that each operator T_j maps even into $C_0(K_j, d_j)$, i.e. if $t_j^* \in K_j$ are the marked points, then we suppose that for $j \in \mathbb{N}$ we have

$$(6.3) \quad (T_j h)(t_j^*) = 0, \quad h \in \mathcal{H}.$$

Under this assumption $T^\infty : \mathcal{H} \rightarrow \tilde{C}_0(K_\infty, d_\infty)$ with

$$(T^\infty h)(t) := (T_j h)(t) \quad \text{for } t \in K_j$$

is well defined.

Now we are in a position to verify the following basic ingredient in the proof of Theorem 2.2.

Theorem 6.3. *Let $T_j : \mathcal{H} \rightarrow C(K_j, d_j)$ be α -Hölder operators for some $\alpha \in (0, 1]$ satisfying (6.3) and suppose that for each $j, n \in \mathbb{N}$ we have*

$$(6.4) \quad \varepsilon_n(K_j, d_j) < \rho_j \cdot n^{-1/\sigma}$$

for some $\rho_j > 0$ and $\sigma > 0$. Then

$$(6.5) \quad e_n(T^\infty) \leq c \cdot \tilde{\rho} \cdot n^{-1/2-\alpha/\sigma},$$

where

$$(6.6) \quad \tilde{\rho} := \left(\sum_{j=1}^\infty \|T_j\|_\alpha^{\sigma/\alpha} \rho_j^\sigma \right)^{\alpha/\sigma}.$$

Proof. First suppose that the T_j 's are 1-Hölder with $\|T_j\|_1 \leq 1$. We claim that then T^∞ is 1-Hölder with $\|T^\infty\|_1 \leq 1$. To verify this, take $h \in \mathcal{H}$ with $\|h\| \leq 1$. For t and t' in the same K_j it follows that

$$(6.7) \quad |(T^\infty h)(t) - (T^\infty h)(t')| = |(T_j h)(t) - (T_j h)(t')| \leq d_j(t, t') = d_\infty(t, t').$$

On the other hand, if $t_1 \in K_{j_1}$ and $t_2 \in K_{j_2}$ with $j_1 \neq j_2$, using (6.3) and (6.7) we get

$$\begin{aligned} |(T^\infty h)(t_1) - (T^\infty h)(t_2)| &\leq |(T_{j_1} h)(t_1)| + |(T_{j_2} h)(t_2)| \\ &= |(T_{j_1} h)(t_1) - (T_{j_1} h)(t_{j_1}^*)| + |(T_{j_2} h)(t_2) - (T_{j_2} h)(t_{j_2}^*)| \\ &\leq d_{j_1}(t_1, t_{j_1}^*) + d_{j_2}(t_2, t_{j_2}^*) = d_\infty(t_1, t_2) \end{aligned}$$

which proves our claim.

By Theorem 6.1 we derive from (6.4) that

$$(6.8) \quad \varepsilon_n(K_\infty, d_\infty) \leq c \cdot \left(\sum_{j=1}^\infty \rho_j^\sigma \right)^{1/\sigma} \cdot n^{-1/\sigma}.$$

Since T^∞ is 1-Hölder, it maps even into $C(K_\infty, d_\infty)$, thus Theorem 6.2 applies and leads by (6.8) to

$$(6.9) \quad e_n(T_\infty) \leq c \cdot \left(\sum_{j=1}^\infty \rho_j^\sigma \right)^{1/\sigma} \cdot n^{-1/2-1/\sigma}.$$

The general case now follows easily. Indeed, for α -Hölder operators T_j we change the metric on K_j to $\hat{d}_j(t, t') := d_j(t, t')^\alpha \cdot \|T_j\|_\alpha$. Then $T_j : \mathcal{H} \rightarrow C(K_j, \hat{d}_j)$ is 1-Hölder with $\|T_j\|_1 = 1$ and, furthermore,

$$\varepsilon_n(K_j, \hat{d}_j) = \|T_j\|_\alpha \cdot \varepsilon_n(K_j, d_j)^\alpha \leq \|T_j\|_\alpha \cdot \rho_j^\alpha \cdot n^{-\alpha/\sigma}.$$

If we now apply the result of the first step with $\tilde{\sigma} = \sigma/\alpha$ and $\tilde{\rho}_j = \|T_j\|_\alpha \cdot \rho_j^\alpha$, then we get (6.5) and (6.6) from (6.9), as asserted. \square

Proof of Theorem 2.2. Let us mark arbitrary points $t_j^* \in K_j$. With these points we introduce 1-dimensional operators $V_j : \mathcal{H} \rightarrow C(K_j, d_j)$ by $(V_j h)(t) := (T_j h)(t_j^*)$, $t \in K_j$, $h \in \mathcal{H}$. If $\tilde{T}_j := T_j - V_j$, then the family $(\tilde{T}_j)_{j \geq 1}$ satisfies (6.3) and since $\|\tilde{T}_j\|_\alpha = \|T_j\|_\alpha$, by Theorem 6.3 we conclude that

$$(6.10) \quad e_n(\tilde{T}^\infty) \leq c \cdot \tilde{\rho} \cdot n^{-1/2-\alpha/\sigma}$$

with $\tilde{\rho}$ from (6.6).

Define as before $V^\infty : \mathcal{H} \rightarrow l_\infty(C(K_j, d_j) : j \in \mathbb{N})$ by $V^\infty h := (V_j h)_{j \geq 1}$, $h \in \mathcal{H}$. We have $T^\infty = \tilde{T}^\infty + V^\infty$, thus it remains to estimate $e_n(V^\infty)$. First note that we may regard V^∞ as an operator from \mathcal{H} into $l_\infty = l_\infty(\mathbb{N})$ with $V^\infty h = ((T_j h)(t_j^*))_{j \geq 1}$, $h \in \mathcal{H}$. Set

$$(6.11) \quad \delta_j := \rho_j^\alpha \cdot \|T_j\|_\alpha$$

and define $S : \mathcal{H} \rightarrow l_\infty$ by

$$(6.12) \quad Sh := (\delta_j^{-1} \cdot (T_j h)(t_j^*))_{j \geq 1}.$$

By Theorem 6.2 assumption (6.4) leads to

$$e_n(T_j) \leq c \cdot \rho_j^\alpha \cdot \|T_j\|_\alpha \cdot n^{-\frac{1}{2} - \frac{\alpha}{\sigma}},$$

which implies, in particular,

$$\|T_j\| = e_1(T_j) \leq c \cdot \rho_j^\alpha \cdot \|T_j\|_\alpha = c \cdot \delta_j.$$

Hence, by

$$\|S\| = \sup_{\|h\| \leq 1} \sup_j \frac{(T_j h)(t_j^*)}{\delta_j} \leq \frac{\|T_j\|}{\delta_j} \leq c$$

the operator S defined by (6.12) is bounded from \mathcal{H} into l_∞ . Let the diagonal operator $D : l_\infty \rightarrow l_\infty$ be generated by the δ_j 's from (6.11). Then, by the construction, we have $V^\infty = D \circ S$.

We will now apply (a special case of) a deep result due to B. Carl (cf. [3]) which, when combined with an estimate of Tomczak-Jaegermann (cf. [25]), may be formulated in the following dual version.

Theorem 6.4. *Let T be an operator from a Hilbert space \mathcal{H} into l_q for some $q \in [1, \infty]$. If $T = D \circ S$ for some bounded operator $S : \mathcal{H} \rightarrow l_\infty$ and for a diagonal operator $D : l_\infty \rightarrow l_q$ generated by a sequence $(\delta_j)_{j \geq 1}$, then for $0 < \gamma < 2$ we have*

$$\sup_{n \geq 1} n^{1/\gamma} e_n(T) \leq c \cdot \|S\| \cdot \|(\delta_j)_{j \geq 1}\|_{r, \infty} \leq c \cdot \|S\| \cdot \left(\sum_{j=1}^\infty |\delta_j|^r \right)^{1/r},$$

where $1/r = 1/\gamma + 1/q - 1/2$.

Let us now apply Theorem 6.4 with δ_j from (6.11), $T = V^\infty$ and parameters $q = \infty$, $1/\gamma = 1/2 + \alpha/\sigma$, hence $1/r = 1/\gamma - 1/2 = \alpha/\sigma$. Doing so, we get

$$\sup_{n \geq 1} n^{1/2 + \alpha/\sigma} e_n(V^\infty) \leq c \cdot \left(\sum_{j=1}^{\infty} \delta_j^r \right)^{1/r} = c \cdot \tilde{\rho}$$

and combining this with (6.10) completes the proof.

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ST. PETERSBURG STATE UNIVERSITY, POSTBOX 104, 197372 ST. PETERSBURG, RUSSIA
E-mail address: `lifts@mail.rcom.ru`

INSTITUT FÜR STOCHASTIK, FRIEDRICH-SCHILLER-UNIVERSITÄT JENA, ERNST-ABBE-PLATZ 2,
07743 JENA, GERMANY
E-mail address: `lindew@minet.uni-jena.de`