

COMPACTNESS OF ISOSPECTRAL POTENTIALS

HAROLD DONNELLY

ABSTRACT. The Schrödinger operator $-\Delta + V$, of a compact Riemannian manifold M , has pure point spectrum. Suppose that V_0 is a smooth reference potential. Various criteria are given which guarantee the compactness of all V satisfying $\text{spec}(-\Delta + V) = \text{spec}(-\Delta + V_0)$. In particular, compactness is proved assuming an a priori bound on the $W_{s,2}(M)$ norm of V , where $s > n/2 - 2$ and $n = \dim M$. This improves earlier work of Brüning. An example involving singular potentials suggests that the condition $s > n/2 - 2$ is appropriate. Compactness is also proved for non-negative isospectral potentials in dimensions $n \leq 9$.

1. INTRODUCTION

Let M be a compact Riemannian manifold. The associated Laplace operator is given in local coordinates by

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x_j} \right).$$

Clearly, this is a second order elliptic differential operator. If V is a smooth potential function, there is an associated Schrödinger operator $-\Delta + V$. Because M is compact, these Schrödinger operators have pure point spectrum consisting of eigenvalues λ_i with associated eigenfunctions ϕ_i ,

$$(-\Delta + V)\phi_i = \lambda_i \phi_i.$$

Suppose that V_0 is a smooth reference potential. The set of smooth potentials satisfying $\text{spec}(-\Delta + V) = \text{spec}(-\Delta + V_0)$ is denoted by $Is(V_0)$, an abbreviation for isospectral. An outstanding open problem is whether or not $Is(V_0)$ is always compact. If the dimension n of M is at most three, then compactness of $Is(V_0)$ was proved by Brüning [4].

In higher dimensions, Brüning proved compactness given the additional hypothesis of a bound for the $W_{s,2}(M)$ norm of V , $s > 3(n/2) - 2$. Here $W_{s,2}(M)$ denotes the Sobolev space of functions with s weak derivatives lying in $L_2(M)$.

In Theorem 4.1, we show that $Is(V_0)$ is compact given a bound for the $W_{s,2}(M)$ norm of $V \in Is(V_0)$, where $s > (n/2) - 2$. Since one has such a bound for the $L_2(M)$ norm of V , this recovers the compactness of $Is(V_0)$, when $n \leq 3$. The improvement in the lower bound of s is achieved by developing the method of [4]. An example involving singular potentials suggests that the condition $s > (n/2) - 2$ is appropriate.

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A number of alternative compactness criteria are derived in Section 5. We show the compactness of non-negative isospectral potentials in dimensions $n \leq 9$. Also, if one assumes isospectrality for more than $(n/2) - 1$ different Schrödinger operators of the form $-\Delta + \gamma V$, $\gamma \neq 0$, the compactness is assured. Finally, in the case of the flat torus, the set of isospectral potentials with non-negative Fourier coefficients is shown to be compact.

All of our proofs are based upon the heat equation asymptotics and the Sobolev embedding theorems. We present two different approaches to the asymptotic expansion of the heat kernel. For the case of the flat torus, the work of Bañuelos and Sa Barreto [2] is developed to give a formula in terms of the Fourier expansion of the potential function $V \in C^\infty(M)$. For more general compact Riemannian manifolds, Duhamel's principle is employed to generate $\exp(-t(-\Delta + V))$ as a perturbation of $\exp(t\Delta)$. The Sobolev embedding theorem and generalized Hölder inequality are used to evaluate the successive terms which appear in the asymptotic expansion. A case-by-case inductive argument leads to Theorem 4.1.

2. HEAT ASYMPTOTICS FOR TORI

Suppose that $M = R^n/\mathcal{L}$ is a flat torus, where \mathcal{L} denotes a lattice. If $V \in C^\infty(M)$, then the Schrödinger operator $-\Delta + V$ has pure point spectrum. The purpose of this section is to derive an asymptotic expansion for the trace of the corresponding heat kernel $K(t, x, y)$:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_x + V(x)\right)K(t, x, y) &= 0, \\ K(0, x, y) &= \delta_{x,y}, \end{aligned}$$

where $\delta_{x,y}$ denotes the Dirac delta distribution. The terms in the asymptotic expansion will be specified as polynomials in the Fourier coefficients V_k of $V(x)$:

$$V(x) = \sum_{k \in \mathcal{L}^*} V_k e^{i\langle k, x \rangle},$$

where \mathcal{L}^* is the dual lattice of \mathcal{L} . Our formula is analogous to the result of Bañuelos and Sa Barreto [2] for scattering theory on R^n . The Fourier transform is replaced by the Fourier series. Although some details are different, the general outline of the method is the same for both R^n and R^n/\mathcal{L} .

The heat kernel $K_0(t, x, y)$ for the Laplacian $-\Delta$ satisfies

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_x\right)K_0(t, x, y) &= 0, \\ K_0(0, x, y) &= \delta_{x,y}. \end{aligned}$$

Taking the difference between the two heat equations gives

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_x\right)[K(t, x, y) - K_0(t, x, y)] &= -V(x)K(t, x, y), \\ (2.1) \quad K(t, x, y) - K_0(t, x, y) &= 0. \end{aligned}$$

We use (2.1) to set up an iteration scheme expressing $K(t, x, y)$ as a perturbation of $K_0(t, x, y)$.

Expanding in Fourier series, the heat kernels are represented by

$$K(t, x, y) = \sum_{n,m} \hat{K}(t, n, m) e^{i\langle n, x \rangle} e^{-i\langle m, y \rangle},$$

$$K_0(t, x, y) = \sum_{n,m} \hat{K}_0(t, n, m) e^{i\langle n, x \rangle} e^{-i\langle m, y \rangle},$$

where the sums runs over $(n, m) \in \mathcal{L}^* \times \mathcal{L}^*$. Since the complex exponentials form an orthonormal basis for the eigenfunctions of $-\Delta$, one has

$$\hat{K}_0(t, n, m) = e^{-t|n|^2} \delta_{n,m}.$$

The key step of the perturbation method is provided by

Proposition 2.2. *The Fourier coefficients of the heat kernels for $-\Delta$ and $-\Delta + V$ satisfy*

$$\hat{K}(t, n, m) = \hat{K}_0(t, n, m) - \sum_{\ell} V_{\ell} \int_0^t e^{(s-t)|n|^2} \hat{K}(s, n - \ell, m) ds.$$

Proof. Taking the product of the Fourier series and rearranging terms gives

$$\begin{aligned} V(x)K(t, x, y) &= \sum_{\ell} V_{\ell} e^{i\langle \ell, x \rangle} \sum_{n,m} \hat{K}(t, n, m) e^{i\langle n, x \rangle} e^{-i\langle m, y \rangle} \\ &= \sum_{\ell, p, m} V_{\ell} \hat{K}(t, p - \ell, m) e^{i\langle p, x \rangle} e^{-i\langle m, y \rangle}. \end{aligned}$$

Consequently, solving (2.1) eigenspace by eigenspace gives the ordinary differential equations

$$\left(\frac{d}{dt} + |n|^2\right)(\hat{K}(t, n, m) - \hat{K}_0(t, n, m)) = - \sum_{\ell} V_{\ell} \hat{K}(t, n - \ell, m)$$

or equivalently,

$$e^{-t|n|^2} \frac{d}{dt} (e^{t|n|^2} [\hat{K}(t, n, m) - \hat{K}_0(t, n, m)]) = - \sum_{\ell} V_{\ell} \hat{K}(t, n - \ell, m).$$

Proposition 2.2 follows by integration in t and use of the initial conditions $K(0, x, y) = K_0(0, x, y)$. □

Substituting the formula of Proposition 2.2 into itself yields

Proposition 2.3. $\hat{K}(t, n, m) = \hat{K}_0(t, n, m) - \sum_{\ell_1} V_{\ell_1} \int_0^t e^{(s_1-t)|n|^2}$

$$\begin{aligned} &\hat{K}_0(s_1, n - \ell_1, m) ds_1 + \sum_{\ell_1, \ell_2} V_{\ell_1} V_{\ell_2} \int_0^t \int_0^{s_1} e^{(s_1-t)|n|^2} \\ &e^{(s_2-s_1)|n-\ell_1|^2} \hat{K}(s_2, n - \ell_1 - \ell_2, m) ds_2 ds_1. \end{aligned}$$

Repeating the process again gives

Proposition 2.4. $\hat{K}(t, n, m) = \hat{K}_0(t, n, m) - \sum_{\ell_1} V_{\ell_1}$

$$\begin{aligned} & \int_0^t e^{(s_1-t)|n|^2} \hat{K}_0(s_1, n - \ell_1, m) ds_1 \\ & + \sum_{\ell_1, \ell_2} V_{\ell_1} V_{\ell_2} \int_0^t \int_0^{s_1} e^{(s_1-t)|n|^2} e^{(s_2-s_1)|n-\ell_1|^2} \\ & \hat{K}_0(s_2, n - \ell_1 - \ell_2, m) ds_2 ds_1 - \sum_{\ell_1, \ell_2, \ell_3} V_{\ell_1} V_{\ell_2} V_{\ell_3} \\ & \int_0^t \int_0^{s_1} \int_0^{s_2} e^{(s_1-t)|n|^2} e^{(s_2-s_1)|n-\ell_1|^2} \\ & e^{(s_3-s_2)|n-\ell_1-\ell_2|^2} \hat{K}(s_3, n - \ell_1 - \ell_2 - \ell_3, m) ds_3 ds_2 ds_1. \end{aligned}$$

One proves by induction that the typical term produced at the k th stage of the iteration is

$$\begin{aligned} E_k(t, n, m) &= (-1)^k \sum_{\ell_1, \ell_2, \dots, \ell_k} V_{\ell_1} V_{\ell_2} \dots V_{\ell_k} \int_0^t \int_0^{s_1} \int_0^{s_2} \\ & \dots \int_0^{s_{k-1}} e^{(s_1-t)|n|^2} e^{(s_2-s_1)|n-\ell_1|^2} e^{(s_2-s_2)|n-\ell_1-\ell_2|^2} \\ & \dots e^{(s_k-s_{k-1})|n-\ell_1-\ell_2-\dots-\ell_{k-1}|^2} \hat{K}_0(s_k, n - \ell_1 - \ell_2 - \dots - \ell_k, m) \\ & ds_k ds_{k-1} \dots ds_2 ds_1. \end{aligned}$$

The Fourier coefficients are rapidly decreasing because $V \in C^\infty(M)$. Consequently, E_k is trace class and

$$Tr(E_k) = \sum_n E_k(t, n, n).$$

Moreover, one has

Proposition 2.5. $Tr(E_1) = (4\pi t)^{-n/2} vol(M)(-t)V_0 + 0(e^{-c/t})$ and for $k \geq 2$,

$$\begin{aligned} Tr(E_k) &= (4\pi t)^{-n/2} vol(M)(-t)^k \sum_{\ell_1+\ell_2+\dots+\ell_k=0} V_{\ell_1} V_{\ell_2} \dots V_{\ell_k} \\ & \int_0^1 \int_0^{\lambda_1} \int_0^{\lambda_2} \dots \int_0^{\lambda_{k-1}} e^{-tA_k(\lambda, \ell)} d\lambda_k d\lambda_{k-1} \dots d\lambda_2 d\lambda_1 + 0(e^{-c/t}), \end{aligned}$$

where

$$\begin{aligned} A_k(\lambda, \ell) &= \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) |\ell_1 + \ell_2 + \dots + \ell_j|^2 \\ & - \left| \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) (\ell_1 + \ell_2 + \dots + \ell_j) \right|^2. \end{aligned}$$

Proof.

$$Tr(E_1) = - \sum_{\ell_1} V_{\ell_1} \sum_n \int_0^t e^{(s_1-t)|n|^2} \hat{K}_0(s_1, n - \ell_1, n) ds_1.$$

However, $\hat{K}_0(s_1, n - \ell_1, n) = 0$, unless $n = n - \ell_1$, i.e. $\ell_1 = 0$. So

$$\begin{aligned} Tr(E_1) &= -V_0 \sum_n \int_0^t e^{(s_1-t)|n|^2} e^{-s_1|n|^2} ds_1 \\ &= -V_0 t \sum_n e^{-t|n|^2}. \end{aligned}$$

The desired formula for $Tr(E_1)$ now follows from the standard heat equation asymptotics for the unperturbed operator $-\Delta$ on $M = R^n/\mathcal{L}$.

For $k \geq 2$, we observe that the diagonality of \hat{K}_0 implies that

$$\hat{K}_0(s_k, n - \ell_1 - \ell_2 - \dots - \ell_k, n) = 0,$$

unless $\ell_1 + \ell_2 + \dots + \ell_k = 0$. Consequently, we have

$$\begin{aligned} Tr(E_k) &= (-1)^k \sum_{\ell_1+\ell_2+\dots+\ell_k=0} V_{\ell_1} V_{\ell_2} \dots V_{\ell_k} \sum_n \\ &\int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} e^{(s_1-t)|n|^2} e^{(s_2-s_1)|n-\ell_1|^2} \\ &e^{(s_3-s_2)|n-\ell_1-\ell_2|^2} \dots e^{(s_k-s_{k-1})|n-\ell_1-\ell_2-\dots-\ell_{k-1}|^2} \\ &e^{-s_k|n|^2} ds_k ds_{k-1} \dots ds_2 ds_1. \end{aligned}$$

The change of variables $s_k = t\lambda_k$ gives

$$\begin{aligned} Tr(E_k) &= (-t)^k \sum_{\ell_1+\ell_2+\dots+\ell_k=0} V_{\ell_1} V_{\ell_2} \dots V_{\ell_k} \\ &\sum_n \int_0^1 \int_0^{\lambda_1} \int_0^{\lambda_2} \dots \int_0^{\lambda_{k-1}} e^{(\lambda_1-1)t|n|^2} e^{(\lambda_2-\lambda_1)t|n-\ell_1|^2} \\ &e^{(\lambda_3-\lambda_2)t|n-\ell_1-\ell_2|^2} \dots e^{(\lambda_k-\lambda_{k-1})t|n-\ell_1-\ell_2-\dots-\ell_{k-1}|^2} \\ &e^{-t\lambda_k|n|^2} d\lambda_k d\lambda_{k-1} \dots d\lambda_2 d\lambda_1. \end{aligned}$$

By completing the square to rewrite the exponent, we obtain

$$\begin{aligned} Tr(E_k) &= (-t)^k \sum_{\ell_1+\ell_2+\dots+\ell_k=0} V_{\ell_1} V_{\ell_2} \dots V_{\ell_k} \sum_n \\ &\int_0^1 \int_0^{\lambda_1} \int_0^{\lambda_2} \dots \int_0^{\lambda_{k-1}} e^{-t|n+\sum_{j=1}^{k-1}(\lambda_{j+1}-\lambda_j)(\ell_1+\ell_2+\dots+\ell_j)|^2} \\ &e^{t \left[\left| \sum_{j=1}^{k-1} (\lambda_{j+1}-\lambda_j)(\ell_1+\ell_2+\dots+\ell_j) \right|^2 + \sum_{j=1}^{k-1} (\lambda_{j+1}-\lambda_j) \left| \ell_1+\ell_2+\dots+\ell_j \right|^2 \right]} \\ &d\lambda_k d\lambda_{k-1} \dots d\lambda_2 d\lambda_1. \end{aligned}$$

The standard heat equation asymptotics with coefficients in a flat bundle give

$$\begin{aligned} &\sum_n e^{-t|n+\sum_{j=1}^{k-1}(\lambda_{j+1}-\lambda_j)(\ell_1+\ell_2+\dots+\ell_j)|^2} \\ &= (4\pi t)^{-n/2} \text{vol}(M) + O(e^{-c/t}). \end{aligned}$$

Substitution completes the proof of Proposition 2.5. □

If $A_k(\lambda, \ell)$ are defined as in Proposition 2.5, then set for $k \geq 2$ and $p \geq 0$,

$$I_{k,p} = \frac{1}{p!} \int_0^1 \int_0^{\lambda_1} \cdots \int_0^{\lambda_{k-1}} \sum_{\ell_1 + \ell_2 + \dots + \ell_k = 0} (A_k(\lambda, \ell))^p V_{\ell_1} V_{\ell_2} \cdots V_{\ell_k} d\lambda_k d\lambda_{k-1} \cdots d\lambda_1.$$

The asymptotic expansion for the trace of the heat kernel is given by

Theorem 2.6. *As $t \downarrow 0$,*

$$Tr(K) \sim (4\pi t)^{-n/2} vol(M) \sum_{m=0}^{\infty} a_m(V) t^m,$$

where $a_0 = 1$, $a_1 = -V_0$, and for $m \geq 2$,

$$a_m(V) = (-1)^m \sum_{\substack{k+p=m \\ k \geq 2, p \geq 0}} I_{k,p}.$$

Proof. Since $Tr(E_k) = c_k t^k + 0(e^{-c/t})$, it follows from our perturbation expansion of the heat kernel K that

$$Tr(K) \sim Tr(K_0) + \sum_{k=1}^{\infty} Tr(E_k).$$

One now deduces Theorem 2.6 from Proposition 2.5 and the expansion of $\exp(-tA_k(\lambda, \ell))$ in Taylor series. □

In Section 5, we will apply Theorem 2.6 to obtain compactness of certain isospectral potentials. This will be based upon proper utilization of the terms $I_{2,p}$. A brief calculation verifies that

$$I_{2,p} = c_p \int_M |\nabla^p V|^2.$$

The coefficient c_p is positive.

3. HEAT ASYMPTOTICS FOR COMPACT MANIFOLDS

Let M be a compact Riemannian manifold and let Δ be its Laplacian acting on functions. The initial value problem for the heat equation is

$$\begin{aligned} (\Delta - \frac{\partial}{\partial t})f(x, t) &= 0, \\ f(x, 0) &= g(x). \end{aligned}$$

There is a fundamental solution $K_0(t, x, y)$ so that

$$f(x, t) = \int_M K_0(t, x, y)g(y)dy.$$

Suppose that n denotes the dimension of M . One has an asymptotic expansion valid in a neighborhood of the diagonal in $M \times M$, given by [2]:

$$K_0(t, x, y) = (4\pi t)^{-n/2} \exp\left(\frac{-r^2(x, y)}{4t}\right)(\theta_0(x, y) + \theta_1(x, y)t + \theta_2(x, y)t^2 + \dots).$$

Here $r(x, y)$ denotes the geodesic distance between x and y . The terms $\theta_i(x, y)$ are determined by iterative solution of the transport equations. Upon restriction to the diagonal, the coefficients $\theta_i(x, x)$ are $O(n)$ invariant polynomials in the components

of the curvature tensor of M and its covariant derivatives, and thus given by pairwise contraction of indices.

Suppose that V is a smooth potential function defined on M . The corresponding heat equation problem is

$$(\Delta - V - \frac{\partial}{\partial t})f(x, t) = 0 \quad f(x, 0) = g(x).$$

Again, there is a fundamental solution $K(t, x, y)$ so that

$$f(x, t) = \int_M K(t, x, y)g(y)dy.$$

Using Duhamel's principle, one generates a perturbative expansion for $K(t, x, y)$ in terms of $K_0(t, x, y)$. At the first stage, one has

$$K(t, x, y) - K_0(t, x, y) = - \int_0^t \int_M K_0(s, x, z)V(z)K(t - s, z, y)dzds.$$

Substitution for $K(t - s, z, y)$ in the integral leads to

$$\begin{aligned} K(t, x, y) - K_0(t, x, y) &= - \int_0^t \int_M K_0(s_1, x, z_1)V(z_1)K_0(t - s_1, z_1, y), \\ dz_1ds_1 &+ \int_0^t \int_0^{t-s_1} \int_{M \times M} K_0(s_1, x, z_1)V(z_1)K_0(s_2, z_1, z_2), \\ &V(z_2)K(t - s_1 - s_2, z_2, y)dz_2dz_1ds_2ds_1. \end{aligned}$$

One proves by induction that the typical term at the k th stage of the induction argument is

$$\begin{aligned} E_k(t, x, y) &= (-1)^k \int_0^t \int_0^{t-s_1} \int_0^{t-s_1-s_2} \dots \int_0^{t-s_1-s_2-\dots-s_{k-1}} \\ &\int_{M^k} K_0(s_1, x, z_1)V(z_1)K_0(s_2, z_1, z_2)V(z_2)K_0(s_3, z_2, z_3) \\ &V(z_3) \dots K_0(s_k, z_{k-1}, z_k)V(z_k)K(t - s_1 - s_2 - \dots - s_k, z_k, y) \\ &dz_kdz_{k-1} \dots dz_1ds_kds_{k-1} \dots ds_1. \end{aligned}$$

Here $M^k = M \times M \times \dots \times M$ denotes the k -fold Cartesian product.

By working in normal coordinates centered at x , using the known asymptotic expansion for $K_0(t, x, y)$, and expanding V in Taylor series, one deduces

Proposition 3.1. *There is an asymptotic expansion as $t \downarrow 0$:*

$$K(t, x, x) \sim (4\pi t)^{-n/2}(1 + a_1(x)t + a_2(x)t^2 + \dots).$$

Each term $a_i(x)$ is given by an $O(n)$ invariant polynomial in the components of the curvature tensor R of M , the potential function V and their covariant derivatives of higher order.

If λ_i are the eigenvalue of $-\Delta + V$, then

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} = \int_M K(t, x, x)dx.$$

Consequently, as $t \downarrow 0$,

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} \sim (4\pi t)^{-n/2}(\text{vol } M + \int_M a_1(x)t + \int_M a_2(x)t^2 + \dots).$$

We are primarily interested in the dependence of the $\int a_i(x)dx$ upon the potential function V and its covariant derivatives. Some simplifications of the integrals are possible using partial integration. One readily verifies that

$$\begin{aligned} \int_M a_1(x)dx &= \int_M [V(x) + b_1(x)]dx, \\ \int_M a_2(x)dx &= \frac{1}{2} \int_M [V^2(x) + b_{21}(x)V(x) + b_{22}(x)]dx, \end{aligned}$$

where b_1, b_{21} , and b_{22} depend upon the metric but not upon the potential.

For the general term, there is the following description:

Proposition 3.2. *If $j \geq 3$, then*

$$\int_M a_j = c_j \int_M |\nabla^{j-2}V|^2 + \sum_{k=0}^j \sum_{\substack{\alpha \in \mathbb{Z}_+^k \\ |\alpha| \leq \ell(k)}} \int_M P_{\alpha_1}^k(V)P_{\alpha_2}^k(V) \dots P_{\alpha_k}^k(V),$$

where each $P_{\alpha_i}^k$ is a differential operator with coefficients depending only upon the metric of M .

For all k , we have $\text{ord } P_{\alpha_i}^k \leq j - 3$ and $\sum_{i=1}^k \text{ord } P_{\alpha_i}^k \leq 2(j - 3)$. Moreover, if $k \geq 3$, then

$$\text{ord } P_{\alpha_i}^k \leq j - k \quad \sum_{i=1}^k \text{ord } P_{\alpha_i}^k \leq 2(j - k).$$

Remark 1. The last sentence refines the corresponding description of the terms given by Brüning [4].

Remark 2. If $k = 0$, the terms depend upon the metric, but not upon V . If $k = 1$, then by partial integration, any derivatives applied to V may be moved over to R and its covariant derivatives. So, it may be assumed that $\text{ord } P_{\alpha_1}^1 = 0$.

Remark 3. If $k \geq 3$, the terms independent of R , and its covariant derivatives, satisfy $\sum_{i=1}^k \text{ord } P_{\alpha_i}^k = 2(j - k)$. This agrees with the results of Section 2 concerning the flat metric. For the metric dependent terms, one has $\sum_{i=1}^k \text{ord } P_{\alpha_i}^k < 2(j - k)$.

4. COMPACTNESS AND BOUNDS IN SOBOLEV SPACES

Suppose that V_0 is a given smooth potential function defined on M . The set of all isospectral potentials, denoted $Is(V_0)$, consists of smooth potentials V satisfying $\text{spec}(-\Delta + V) = \text{spec}(-\Delta + V_0)$. If s is a non-negative integer, then $W_{s,2}(M)$ is the Sobolev space of functions with s weak derivatives lying in $L_2(M)$. Moreover, $\|V\|_{s,2}$ denotes the norm of V as an element of $W_{s,2}(M)$. Let $Is(V_0, s, c_0)$ be the members of $Is(V_0)$ which satisfy $\|V\|_{s,2} \leq c_0$, for a positive constant c_0 .

The main purpose of this section is to prove

Theorem 4.1. *If $s > \frac{n}{2} - 2$ and $c_0 > 0$, then $Is(V_0, s, c_0)$ is compact in $C^\infty(M)$.*

Remark. If $s > 3(\frac{n}{2}) - 2$, then Brüning [4] proved that $Is(V_0, s, c_0)$ is compact in $C^\infty(M)$. The improved bound $s > \frac{n}{2} - 2$ is achieved by developing his method.

The following corollary was established already in [4]:

Corollary 4.2. *If $n \leq 3$, then $Is(V_0)$ is compact in $C^\infty(M)$.*

Proof. In Section 3, we observed that the second heat equation coefficient satisfies

$$\int_M a_2(x)dx = \frac{1}{2} \int_M [V^2(x) + b_{21}(x) + b_{22}(x)V(x)]dx,$$

where b_{21} and b_{22} depend only upon the underlying metric of M . Consequently,

$$\int_M V^2(x)dx \leq c_1 \left(\int_M a_2(x)dx + c_3 + c_4 \left(\int_M V^2(x)dx \right)^{1/2} \right).$$

Since $\int_M a_2$ is determined by $\text{spec}(-\Delta + V)$, it follows that $\|V\|_{0,2}$ is bounded for $V \in Is(V_0)$. □

The remainder of this section is devoted to the proof of Theorem 4.1. One proceeds by induction on j , using the following consequences of Proposition 3.2:

$$\|V\|_{j-2,2}^2 \leq c \left(1 + \sum_{\substack{3 \leq k \leq j \\ \alpha \in \mathbb{Z}_+^k}} \int_M |P_{\alpha_1}^k(V)P_{\alpha_2}^k(V) \dots P_{\alpha_k}^k(V)| \right)$$

for $j \geq 3$. The bound for $\|V\|_2$ is given above in the proof of Corollary 4.2. In general, a bound on $\|V\|_{j-3,2}$ is assumed. We first attempt to bound the right-hand side by a function of $\|V\|_{j-3,2}$. If this does not succeed, we bound the right-hand side by a multiple of $\|V\|_{j-2,2}^\beta$, $\beta < 2$. When we show that V is bounded in all the Sobolev spaces, then the result follows from the Rellich compactness criterion [1]. By taking products with circles, it suffices to treat the case of dimension $n \geq 3$.

In addition to the norm of V in $W_{s,2}(M)$, we will need to employ the norm of V in $L_p(M)$. It is denoted by $\|V\|_p$, $1 \leq p \leq \infty$. Those versions of the Sobolev embedding theorems, which will be used, are compiled in Section 6. The appendix also includes an interpolation result which follows from Hölder’s inequality.

We consider a typical term T in the above summation. It is convenient to drop the double subscript and the superscript, so that P_i denotes $P_{\alpha_i}^k$. Let us write

$$T = P_1(V)P_2(V) \dots P_\ell(V)P_{\ell+1}(V) \dots P_k(V),$$

where $n > 2(j - \text{ord}P_i - 3)$ for $i \leq \ell$, and $n \leq 2(j - \text{ord}P_i - 3)$ for $i \geq \ell + 1$. We apply the Sobolev embedding theorem as summarized in Proposition 6.1. When $i \leq \ell$, $\|P_i V\|_{r_i} \leq c\|V\|_{j-3,2}$, with $r_i = 2n/[n - 2(j - \text{ord}P_i - 3)]$ because $P_i V \in W_{j-3-\text{ord}P_i,2}$. For $i \geq \ell + 1$, and odd dimension n , $\|P_i V\|_\infty \leq c\|V\|_{j-3,2}$. If n is even and $i > \ell + 1$, there is the weaker estimate $\|P_i V\|_{r_i} \leq c\|V\|_{j-3,2}$, for all $r_i < \infty$.

Theorem 4.1 is proved by establishing appropriate bounds on our typical term T . A case-by-case discussion serves to clarify the various ideas involved. We begin with

Lemma 4.3. *If n is odd, $j > \frac{n}{2} + 3$, and $\|V\|_{j-3,2} \leq c_1$, then $\int_M T \leq c_2$, where c_2 depends upon c_1 .*

Proof. Note that $\int |T| \leq c \int |P_1(V)||P_2(V)| \dots |P_\ell(V)|$, because $\|P_i(V)\|_\infty$ is bounded for $i \geq \ell + 1$. According to Proposition 3.2, we have $\text{ord}(P_i) \leq j - 3$, for all i . $\ell = 1$ or 2 , the desired estimate then follows from the Cauchy–Schwarz inequality.

Assume $\ell \geq 3$, $i \leq \ell$, and $r_i = 2n/[n - 2(j - \text{ord}P_i - 3)]$. If $\frac{1}{r_1} + \frac{1}{r_2} + \dots + \frac{1}{r_\ell} \leq 1$, the generalized Hölder’s inequality gives

$$\int_M |T| \leq c \|P_1(V)\|_{r_1} \|P_2(V)\|_{r_2} \dots \|P_\ell(V)\|_{r_\ell},$$

and the lemma follows.

It remains to verify that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \dots + \frac{1}{r_\ell} \leq 1$. This reads

$$\sum_{i=1}^{\ell} \frac{n - 2(j - \text{ord}P_i - 3)}{2n} \leq 1$$

or equivalently $(n - 2j + 6)\ell + 2 \sum_{i=1}^{\ell} \text{ord}P_i \leq 2n$.

Since $k \geq \ell \geq 3$, Proposition 3.2 guarantees that $\sum_{i=1}^{\ell} \text{ord}P_i \leq 2(j - k)$. So, it suffices to establish the inequality $(n - 2j + 6)\ell + 4(j - k) \leq 2n$, that is, $(\ell - 2)n + 6\ell - 4k \leq (2\ell - 4)j$.

Because $\ell \geq 3$, division by $2(\ell - 2)$ reduces the required estimate to $\frac{n}{2} + \frac{3\ell - 2k}{\ell - 2} \leq j$ or equivalently $\frac{n}{2} + 3 - (\frac{2k - 6}{\ell - 2}) \leq j$. Clearly, $(\frac{2k - 6}{\ell - 2}) \geq 0$ when $3 \leq \ell \leq k$. So, Lemma 4.3 holds whenever $j \geq \frac{n}{2} + 3$. □

The case of even-dimensional manifolds is more delicate. If $\ell \neq 2$, a similar argument applies. If $\ell = 2$, more care is required. Details are given in the next two lemmas.

Lemma 4.4. *If n is even and $\|V\|_{j-3,2} \leq c_1$, $j > \frac{n}{2} + 3$, and $\ell \neq 2$, then $\int_M |T| \leq c_2$, where c_2 depends upon c_1 .*

Proof. By the generalized Hölder’s inequality

$$\int_M |T| \leq c \|P_1(V)\|_{r_1} \|P_2(V)\|_{r_2} \dots \|P_k(V)\|_{r_k},$$

provided $\frac{1}{r_1} + \frac{1}{r_2} + \dots + \frac{1}{r_k} \leq 1$. If $i \geq \ell + 1$, then we may take r_i to be arbitrarily large. So, it suffices to have $\frac{1}{r_1} + \frac{1}{r_2} + \dots + \frac{1}{r_\ell} < 1$. Each $r_i \geq 2$, so the case $\ell = 1$ follows.

Assume $\ell \geq 3$. The remainder of the argument is the same as in Lemma 4.3, with strict inequality replacing greater than or equal. This leads to the condition $j > \frac{n}{2} + 3$. □

Lemma 4.5. *If $n \geq 3$, $\ell = 2$, $j > \frac{n}{2} + 1$, and $\|V\|_{j-3,2} \leq c_1$, then $\int_M T \leq c_2(\|V\|_{j-2,2} + 1)$, where c_2 depends upon c_1 .*

Proof. If $\frac{1}{r_1} + \frac{1}{r_2} < 1$, then the method of Lemma 4.4 applies to yield $\int_M |T| \leq c_2$, where c_2 depends upon c_1 . If $k = \ell = 2$, and $\frac{1}{r_1} + \frac{1}{r_2} \leq 1$, then the result follows from Hölder’s inequality.

Assume $\ell = 2$, $k > \ell$, and $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then $r_1 = r_2 = 2$ and $\text{ord}P_1 = \text{ord}P_2 = j - 3$. From the generalized Hölder inequality, we get $\int_M |T| \leq c \|P_1 V\|_{r_1 + \epsilon}$. If ϵ is sufficiently small, then $r_1 + \epsilon < \frac{2n}{n-2}$. By the Sobolev embedding theorem $\|P_1 V\|_{r_1 + \epsilon} \leq c \|P_1 V\|_{1,2} \leq \tilde{c} \|V\|_{j-2,2}$. □

By taking products with circles, the proof of Theorem 4.1 may be reduced to the case $n \geq 3$. The inductive argument is completed above for $j > \frac{n}{2} + 3$. However, to start the induction a more sophisticated method is required.

The final step in the proof of Theorem 4.1 is provided by

Lemma 4.6. *If $n \geq 3$, $j > \frac{n}{2} + 1$, and $\|V\|_{j-3,2} \leq c_1$, then*

$$\int_M |T| \leq c_2(1 + \|V\|_{j-2,2}^\beta),$$

where $\beta < 2$ and c_2 depends upon c_1 .

Proof. If $\ell = 1$, then the result follows from the generalized Hölder’s inequality, since $r_1 \geq 2$. If $\ell = 2$, we may apply Lemma 4.5, to get the result with $\beta = 1$. So, for the rest of the proof, we assume $\ell \geq 3$.

Suppose $n > 2(j - \text{ord}P_i - 2)$, for two values of $i \leq \ell$, say $i = 1, 2$. Let $s_i = 2n/[n - 2(j - \text{ord}P_i - 2)] > r_i = 2n/[n - 2(j - \text{ord}P_i - 3)]$. The interpolation estimate of Proposition 6.2 gives $\|P_i V\|_{r_i+\epsilon} \leq \|P_i V\|_{r_i}^{\alpha_i} \|P_i V\|_{s_i}^{1-\alpha_i}$, $0 < \alpha_i < 1$. So, by the Sobolev embedding $\|P_i V\|_{r_i+\epsilon} \leq c \|V\|_{j-3,2}^{\alpha_i} \|V\|_{j-2,2}^{1-\alpha_i}$, for $i = 1, 2$.

The generalized Hölder’s inequality yields

$$\begin{aligned} \int_M |T| &\leq c \|P_1 V\|_{r_1+\epsilon} \|P_2 V\|_{r_2+\epsilon} \|P_3 V\|_{r_3} \cdots \|P_\ell V\|_{r_\ell} \\ &\leq c \|V\|_{j-2,2}^\beta, \text{ with } \beta < 2, \end{aligned}$$

provided $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{r_3} + \frac{1}{r_4} + \dots + \frac{1}{r_\ell} < 1$.

To satisfy this last inequality, it suffices to have

$$\sum_{i=1}^2 \frac{n - 2(j - \text{ord}P_i - 2)}{2n} + \sum_{i=3}^{\ell} \frac{n - 2(j - \text{ord}P_i - 3)}{2n} < 1$$

or equivalently

$$\sum_{i=1}^{\ell} \frac{n - 2(j - \text{ord}P_i - 3)}{2n} - \frac{2}{n} < 1,$$

which may be written as

$$\sum_{i=1}^{\ell} [n - 2(j - \text{ord}P_i - 3)] - 4 < 2n.$$

So, we need to establish the inequality $(n - 2j + 6)\ell + 2 \sum_{i=1}^{\ell} \text{ord}P_i < 2n + 4$. By Proposition 3.2, and the fact that $\ell \leq k$, we know $\sum_{i=1}^{\ell} \text{ord}P_i \leq 2(j - k)$. Thus, it suffices to verify $(n - 2j + 6)\ell + 4(j - k) < 2n + 4$.

This last inequality may be rewritten as $(\ell - 2)n + 6\ell - 4k - 4 < (2\ell - 4)j$. Since $\ell \geq 3$, division by $2(\ell - 2)$ gives

$$\frac{n}{2} + \frac{3\ell - 2k - 2}{\ell - 2} < j$$

or equivalently

$$\frac{n}{2} + 3 - \left(\frac{2k - 4}{\ell - 2}\right) < j.$$

Because $k \geq \ell \geq 3$, $\frac{2k-4}{\ell-2} \geq \frac{2k-4}{k-2} = 2$. So $j > \frac{n}{2} + 1$ suffices.

Since $\ell \geq 3$, it only remains to treat the case where $n \leq 2(j - \text{ord}P_i - 2)$, for two values of $i \leq \ell$, say $i = 1, 2$. Then, if $2 < s < \infty$, Proposition 6.1 gives $\|P_i V\|_s \leq c \|P_i V\|_{j - \text{ord}P_i - 2, 2} \leq \tilde{c} \|V\|_{j-2, 2}$. For $2 < t < s$, we may interpolate using Proposition 6.2 to get $\|P_i V\|_t \leq \|P_i V\|_s^\alpha \|P_i V\|_2^{1-\alpha}$, $0 < \alpha < 1$. Applying the Sobolev embedding theorem and the fact that $\text{ord}(P_i) \leq j - 3$, yields $\|P_i V\|_t \leq c \|V\|_{j-2, 2}^\alpha \|V\|_{j-3, 2}^{1-\alpha}$.

Because one may take t to be arbitrarily large, the result follows from the generalized Hölder's inequality provided that $\frac{1}{r_3} + \frac{1}{r_4} + \dots + \frac{1}{r_\ell} < 1$. If $\ell = 3$, the result follows since $r_3 \geq 2$. So assume $\ell \geq 4$ in the remainder of the proof. Substituting the definitions of the r_i , it suffices to have $\sum_{i=3}^\ell n - 2(j - \text{ord}P_i - 3) < 2n$.

This last inequality may be rewritten as $(\ell - 2)(n - 2j + 6) + 2 \sum_{i=3}^\ell \text{ord}P_i < 2n$. Recall that $k \geq \ell$ and apply Proposition 3.2 to give $\sum_{i=3}^\ell \text{ord}P_i \leq 2(j - k)$. So, it is enough to require $(\ell - 2)(n - 2j + 6) + 4(j - k) < 2n$ or equivalently $(\ell - 4)n + 6(\ell - 2) - 4k < (2\ell - 8)j$.

If $\ell = 4$, then $k \geq 4$ and $6(\ell - 2) - 4k = 12 - 4k < 0$. So assume $\ell \geq 5$. Dividing by $2(\ell - 4)$, it suffices to have $n/2 + [3(\ell - 2) - 2k]/(\ell - 4) < j$, that is, $n/2 + 3 - (2k - 6)/(\ell - 4) < j$. For k fixed, the left-hand side is maximized when $\ell = k$. So, we only need to require $n/2 + 3 - (2k - 6)/(k - 4) < j$, i.e. $n/2 + 1 - 2/(k - 4) < j$. Since $k \geq 5$, it suffices to have $j > n/2 + 1$.

The proof of Lemma 4.6 is complete. This also finishes the proof of Theorem 4.1. □

Example 4.7. Consider the singular potential $V = r^{-2}$, where $r < 1$ is the Euclidean distance from the origin in R^n . For this potential, each term in the heat polynomial a_j is of order r^{-2j} . Moreover, if $s < n/2 - 2$, then $V \in W_{s, 2}(B_1)$. Here B_1 denotes the unit ball centered at the origin in R^n . By smoothing V , one obtains potentials in C^∞ whose limit in $W_{s, 2}(B_1)$ is V . This suggests that the condition $s \geq \frac{n}{2} - 2$ is needed for $|\nabla^{j-2} V|^2$ to dominate the other terms in a_j . For odd dimensions n , the condition agrees with our hypothesis $s > \frac{n}{2} - 2$ in Theorem 4.1.

If n is even, we need to discuss the borderline case $s = \frac{n}{2} - 2$. Consider $V_\epsilon = \epsilon^\epsilon r^{-2-\epsilon}$, defined when $r > \epsilon$ in R^n . This gives a family of potentials in $W_{s, 2}(B_1 - B_\epsilon)$ with norm bounded independent of ϵ . We extend these to all of B_1 with bounded Sobolev norms. In the limit as $\epsilon \downarrow 0$, each term in the heat polynomial is of comparable order. So, the hypothesis $s > \frac{n}{2} - 2$, in Theorem 4.1, is also appropriate for even values of n .

5. ALTERNATIVE COMPACTNESS CRITERIA

If $n \geq 4$, then it is not known whether or not $Is(V_0)$ is compact. Theorem 4.1 guarantees compactness if one requires an a priori bound for the $W_{s, 2}(M)$ norm of the potentials under consideration, $s > \frac{n}{2} - 2$. In this section, we describe some other conditions which guarantee compactness.

Given a real constant c , let $Is(V_0, c)$ denote the set of all smooth potentials satisfying $\text{spec}(-\Delta + V) = \text{spec}(-\Delta + V_0)$ and $V \geq c$. Our first result is

Proposition 5.1. *If $n \leq 9$, then $Is(V_0, c)$ is compact in the topology of $C^\infty(M)$, for any real constant c .*

Proof. Adding a constant to the potential shifts the spectrum by the same constant. So we may reduce to the case of non-negative potentials. If $n \leq 3$, then Corollary 4.2 gives a stronger result.

If $4 \leq n \leq 9$, we apply formulas for the heat coefficients a_3, a_4 , which were derived by Colin de Verdière [5], and for a_5 , derived by Bañuelos and Sa Barreto [2]. Their results are valid in locally Euclidean spaces. Using the third remark following Proposition 3.2, we see that in each case the metric dependent terms are of lower order. If n is 4 or 5, then one uses that

$$\int_M a_3(x)dx = -\frac{1}{6} \left(\int_M [V^3(x) + \frac{1}{2}|\nabla V(x)|^2]dx \right)$$

to get a bound in $W_{1,2}(M)$, when $V \geq 0$. The result then follows from Theorem 4.1. If n is 6 or 7, we need a bound in $W_{2,2}(M)$, which is obtained from

$$\int_M a_4(x)dx = \frac{1}{24} \left(\int_M [V^4(x) + 2V(x)|\nabla V(x)|^2 + \frac{1}{5}|\nabla^2 V(x)|^2]dx \right),$$

provided $V \geq 0$. Finally, if n is 8 or 9, we get a $W_{3,2}(M)$ bound using $V \geq 0$ and

$$\int_M a_5(x)dx = \frac{-1}{120} \left(\int_M [V^5(x) + \frac{3}{42}|\nabla^3 V(x)|^2 + 5V^2(x)|\nabla V(x)|^2 + \frac{15}{27}V(x)|\Delta V(x)|^2 + \frac{4}{9}V(x)|\nabla^2 V(x)|^2]dx \right).$$

This completes the proof of Proposition 5.1. □

Remark. The computations of Bañuelos and Sa Barreto used the continuous version of Theorem 2.6, involving the Fourier transform rather than the Fourier series. The same method applies for T^n as for R^n .

Another compactness criterion may be formulated by requiring isospectrality for a finite number of Schrödinger operators of the form $-\Delta + \gamma V_0$, $\gamma \neq 0$. Let $Is_\gamma(V_0)$ denote the set of all smooth potentials which satisfy $spec(-\Delta + \gamma V) = spec(-\Delta + \gamma V_0)$. Our next result is

Proposition 5.2. *If Γ is a non-empty set containing more than $\frac{n}{2} - 1$ non-zero real numbers, then $\bigcap_{\gamma \in \Gamma} Is_\gamma(V_0)$ is compact.*

Proof. The term bounding the $W_{s,2}(M)$ norm occurs in a_{s+2} . As a polynomial in γ , a_{s+2} is of order $s + 2$. We write $a_{s+2} = c_0 + c_1\gamma + \gamma^2 P_s(\gamma)$. The coefficient c_0 depends only upon the metric and c_1 is bounded because $\|V\|_2$ is bounded. If a_{s+2} is bounded, then so is $P_s(\gamma) = \gamma^{-2}(a_{s+2} - c_0 - c_1\gamma)$. Now P_s is of degree s . If $s + 1$ values are bounded, for a polynomial of degree s , then the coefficients of the polynomial are bounded by Lagrange interpolation. In particular, the constant term of P_s is bounded, which then bounds the $W_{s,2}(M)$ norm of V . Theorem 4.1 requires that $s + 1 > \frac{n}{2} - 1$. □

We now specialize to the case where $M = R^n/\mathcal{L}$ is a flat torus. For a given reference potential V_0 , we let $Is^+(V_0)$ denote the set of all smooth potentials V having non-negative Fourier coefficients and satisfying

$$spec(-\Delta + V) = spec(-\Delta + V_0).$$

One has

Proposition 5.3. *If V_0 is a smooth potential function defined on a flat torus, then $I_{s^+}(V_0)$ is compact.*

Proof. We apply Theorem 2.6. Note that the convexity of $f(x) = x^2$ implies that $A_k(\lambda, \ell) \geq 0$, for all values of k, λ , and ℓ . Since the Fourier coefficients of V are non-negative, $I_{k,p} \geq 0$ for all k and p . The isospectrality hypothesis fixes the terms $a_m(V)$, for all m . From these last two observations, we deduce that $I_{2,p}$ is bounded above. It was observed, in section two, that $I_{2,p}$ is a positive multiple of the integral $\int_M |\nabla^p V|^2$. This bounds all the Sobolev norms, and the result follows. \square

6. APPENDIX

This appendix summarizes certain standard results in a form convenient for reference. Suppose that M is a compact Riemannian manifold of dimension n . For positive integers s , we let $W_{s,2}(M)$ denote the Sobolev space of measurable functions with s weak derivatives lying in $L_2(M)$. We need the following special cases of the Sobolev embedding theorems [1]:

Proposition 6.1. *a. For $2m < n$, $W_{m,2}(M) \subset L_q(M)$, $2 \leq q \leq \frac{2n}{n-2m}$.
 b. For $2m = n$, $W_{m,2}(M) \subset L_q(M)$, $2 \leq q < \infty$.
 c. For $2m > n$, $W_{m,2}(M) \subset L_\infty(M)$.*

The norm of u in $L_p(M)$ is denoted by $\|u\|_p$. For interpolation, the following consequence of Hölder's inequality is quite useful:

Proposition 6.2. *If $0 < \alpha < 1$ and $1 \leq r < p < s$, then*

$$\|u\|_p \leq \|u\|_r^\alpha \|u\|_s^{1-\alpha},$$

provided that $\frac{1}{p} = \frac{\alpha}{r} + \frac{1-\alpha}{s}$.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907