UPPER BOUNDS FOR THE NUMBER OF SOLUTIONS
OF A DIOPHANTINE EQUATION

M. Z. GARAЕV

Abstract. We give upper bound estimates for the number of solutions of a
certain diophantine equation. Our results can be applied to obtain new lower
bound estimates for the $L_1$-norm of certain exponential sums.

1. Introduction

Throughout the text the following notations will be used:

$A \ll B$ means $|A| \leq cB$ with some absolute constant $c$.

$A \ll_{a,b,...} B$ means $|A| \leq cB$ with some constant $c$

which may depend only on $a, b, ...$

$N$ is an integer parameter, $N \geq 3$. We also assume that $f(n)$ is an integer for
integers $n, 1 \leq n \leq N$.

the Littlewood conjecture which states that

$$\int_0^1 \left| \sum_{n=1}^N \exp(2\pi i \alpha f(n)) \right| d\alpha \gg \log N. \tag{1}$$

The relation [11, p. 67]

$$\int_0^1 \left| \sum_{n=1}^N \exp(2\pi i \alpha n) \right| d\alpha = \frac{4}{\pi^2} \log N + O(1)$$

shows that the order $\log N$ in (1) is sharp. However, for a wide class of sequences
$f(n)$ estimate (1) can be improved.

Let $f(1) < f(2) < \ldots < f(N)$ and $J = J(N)$ be the number of solutions of the
equation

$$f(x) + f(y) = f(u) + f(v), \quad 1 \leq x, y, u, v \leq N.$$

Theorem (A. A. Karatsuba [6]). For any coefficients $\gamma_n, |\gamma_n| = 1$, the inequality

$$I = I(N) = I(N, f) := \int_0^1 \left| \sum_{n=1}^N \gamma_n \exp(2\pi i \alpha f(n)) \right| d\alpha \geq \left( \frac{N^3}{J} \right)^{1/2}$$

is valid.
If \( f(n) \) is a polynomial with integer coefficients, \( \deg f > 1 \), then \( J \ll N^{2+\varepsilon} \) for any \( \varepsilon > 0 \) and \( N > N_0(\varepsilon, f) \). Therefore, Karatsuba’s theorem gives the estimate \( I(N) \gg N^{\frac{1}{2}} - \varepsilon \) for \( N > N_0(\varepsilon, f) \). Note that on the other hand we have

\[
I(N) \leq \int_0^1 \left| \sum_{n=1}^N \gamma_n \exp(2\pi i f(n)) \right|^2 \, d\alpha = N^{\frac{1}{2}}.
\]

Another example is when \( f(n) \) is a very fast increasing sequence, say \( f(n) = a^n \) with integer \( a > 1 \). Then the theorem implies

\[
N^{\frac{1}{2}} \ll I(N) \ll N^{\frac{1}{2}}.
\]

It should be pointed out that in this case A. A. Karatsuba carried out another approach and obtained an asymptotic formula for \( I(N, f) \), and later M. A. Korolev for any positive number \( p \) obtained an asymptotic formula for the \( L_p \)-norm of the sum (none of these results is published).

S. V. Konyagin [8], using the work of G. Elekes, M. B. Nathanson and I. Z. Ruzsa [2], proved that if

\[
0 < f(2) - f(1) < f(3) - f(2) < \ldots < f(N) - f(N - 1),
\]

then \( J \ll N^{\frac{3}{2}} \), and thus \( I \gg N^{\frac{3}{2}} \). In particular, for Bochkarev’s sequence [1]

\[
f(n) = [2^\log n]^\beta
\]

the estimate \( I \gg N^{\frac{3}{2}} \) holds for any fixed \( \beta > 1 \) and \( N \geq N_0(\beta) \). The same holds for the more general sequence [6]

\[
f(n) = [e^{A \log n}]^\beta,
\]

where \( A > 0, \beta > 1 \) are fixed numbers.

For a new proof of Konyagin’s result see our work [3].

In the present paper we give a new upper bound for \( J(N) \). In particular, we slightly improve (see Corollary 2) one of the main results of [5], give another proof of Konyagin’s estimate, and slightly improve it for Bochkarev’s sequence.

2. THE RESULTS

For a given integer \( l, 1 \leq l < N \), let \( J_l = J_l(N) \) denote the number of solutions of the equation

\[
f(x) + f(y) = f(x + l) + f(z), \quad 1 \leq x < x + l \leq z < y \leq N.
\]

**Theorem 1.** For any real number \( \varepsilon, 0 < \varepsilon < 1 \), we have

\[
J \ll N^{2+\varepsilon} + N^\varepsilon \sum_{1 \leq l \leq N^{1-\varepsilon}} J_l.
\]

Note that if \( f(n) \) satisfies condition (2), then \( J_l < ln \). Indeed, it would follow from (3) that \( z < y < z + l \). For \( z \) we have at most \( N \) possibilities, and once \( z \) is fixed we have less than \( l \) possibilities for \( y \). Besides, for fixed \( y, z \), we have at most one solution of (3) in variable \( x \). Therefore, \( J_l < ln \). Taking \( \varepsilon = \frac{1}{2} \) and applying the Karatsuba theorem we obtain Konyagin’s estimate:

**Corollary 1.** For the sequence (2) we have \( I \gg N^{\frac{3}{2}} \).
Corollary 2. Let \( f(x) = [F(x)] \), where the real valued function \( F(x) \) is three times continuously differentiable on the segment \([1,N]\), \( F'(x) > 0 \), \( F''(x) > 0 \), \( F'''(x) < 0 \).

Then
\[
J \ll (F'(1)^{-1} + 1) N^{5/2} + N^2 F''(N)^{-1}.
\]

In particular, if \( F'(1) \geq 1 \), then
\[
I \gg \min \left( N^{5/4}, N^{5/2} F''(N)^{5/4} \right).
\]

In the case \( F(x) = Ax^\alpha \), \( A > 0 \), \( 1 < \alpha \leq 3/2 \), Corollary 2 gives
\[
J \ll_{\alpha, A} N^{4-\alpha}.
\]

Note that for \( 1 < \alpha < \frac{3}{2} \) this estimate was established and applied to the Waring-Goldbach problem by I. I. Piatetski-Shapiro [10].

In order to prove Corollary 2, we use a result from [5] which states that if \( F'(1) \geq 1 \), then
\[
J \leq \frac{6N}{F''(N)} + 3N.
\]

Taking \( \varepsilon = \frac{1}{4} \) and applying Theorem 1 we obtain Corollary 2 in this case. If \( F'(1) < 1 \), then we reduce it to the first case as it was done in [5].

Conjecture. Let \( f(x) = [Ax^\alpha] \), \( A > 0 \), \( 1 < \alpha < 2 \). Then
\[
J \ll_{\alpha, A} N^{4-\alpha}.
\]

The validity of the conjecture would also have an important application to the Waring-Goldbach problem with a small non-integer exponent. For more details we refer the reader to [4].

Theorem 2. Let \( A > 0 \), \( \beta > 1 \), and \( f(n) = [e^{A(n)^\beta}] \). Then
\[
J \ll_{A, \beta} N^{\frac{3(1-\beta)}{2}} \log \log N.
\]

Corollary 3. Under the assumption of Theorem 2 the estimate
\[
I \gg_{A, \beta} N^{\frac{1}{4}} (\log N)^{\frac{3(\beta-1)}{4}} (\log \log N)^{-\frac{1}{4}}
\]
holds for all \( N > N_0(A, \beta) \).

It is interesting to investigate \( J \) for more general rapidly increasing sequences, in particular for \( f(n) = [e^{An^\beta}] \), where \( A > 0 \), \( 0 < \beta < 1 \). In this connection we would like to stress an unpublished work of S. V. Konyagin, where for \( \beta > \frac{1}{3} \) he obtains an asymptotic formula \( J \sim 2N^2 \).

3. Proof of Theorem 1

Denote
\[
S(\alpha) := \sum_{1 \leq n \leq N} e^{2\pi i \alpha f(n)}.
\]

For a given integer \( s, 1 \leq s \leq [N^{\varepsilon}] \), put
\[
I_s = \{ n \in \mathbb{Z} : (s-1)N^{1-\varepsilon} < n \leq sN^{1-\varepsilon} \},
\]
and for \( s = \lfloor N^\varepsilon \rfloor + 1 \) put
\[
I_s = \{ n \in \mathbb{Z} : (s - 1)N^{1-\varepsilon} < n \leq N \}.
\]

Then
\[
|S(\alpha)|^4 = \left| \sum_{1 \leq n \leq N} e^{4\pi i\alpha f(n)} + 2 \sum_{1 \leq n < m \leq N} e^{2\pi i\alpha (f(n)+f(m))} \right|^2 \\
\ll N^2 + \left| \sum_{1 \leq n < m \leq N} e^{2\pi i\alpha (f(n)+f(m))} \right|^2 \\
\ll N^2 + \left| \sum_{s \leq 1+N^\varepsilon} \sum_{n \in I_s} \sum_{n < m \leq N} e^{2\pi i\alpha (f(n)+f(m))} \right|^2,
\]
whence
\[
|S(\alpha)|^4 \ll N^2 + N^\varepsilon \sum_{s \leq 1+N^\varepsilon} \sum_{n \in I_s} \sum_{n < m \leq N} e^{2\pi i\alpha (f(n)+f(m))}.
\]

Therefore,
\[
J = \int_0^1 |S(\alpha)|^4 \, d\alpha \ll N^2 + N^\varepsilon \sum_{s \leq 1+N^\varepsilon} \sum_{n,n_1,m,m_1} J_s \ll N^2 + N^\varepsilon J',
\]
where the prime means that the inside summation is taken over the integers \( n, n_1, m, m_1 \) with conditions
\[
f(n) + f(m) = f(n_1) + f(m_1), \ n \in I_s, n_1 \in I_s, n < m \leq N, n_1 < m_1 \leq N,
\]
and \( J' \) is the number of solutions of the equation
\[
f(n) + f(m) = f(n_1) + f(m_1), \ n-n_1 \leq N^{1-\varepsilon}, n < m \leq N, n_1 < m_1 \leq N.
\]

Theorem 1 now follows from
\[
J' \ll N^2 + \sum_{l \leq N^{1-\varepsilon}} J_l.
\]

4. PROOF OF THEOREM 2

Obviously \( J \leq 8J_1 \), where \( J_1 \) is the number of solutions of the equation
\[
f(x) + f(y) = f(u) + f(v), \ 1 \leq x, y, u, v \leq N, \ x \geq y, u \geq v, \ x \geq u.
\]

From
\[
e^{A(\log x)^\beta} \leq e^{A(\log u)^\beta + 2}
\]

it follows that
\[
0 \leq x - u \leq cN(\log N)^{1-\beta}
\]

with some \( c = c(A, \beta) > 0 \). Hence \( J \leq 8J_2 \) where \( J_2 \) is the number of solutions of the equation
\[
f(x) - f(u) = f(v) - f(y), \ 1 \leq x, y, u, v \leq N, \ 0 \leq x - u \leq N_1.
\]

Here \( N_1 = cN(\log N)^{1-\beta} \).

Let \( T_1(n) \) denote the number of solutions of the equation
\[
f(x) - f(u) = n, \ 1 \leq x, u \leq N, \ 0 \leq x - u \leq N_1,
\]
and let $T_2(n)$ denote the number of solutions of the equation

$$f(y) - f(v) = n, \; 1 \leq y, v \leq N.$$ 

Then

$$J^2 \leq 64J_2^2 = 64 \left( \sum_n T_1(n)T_2(n) \right)^2 \leq 64 \sum_n T_1^2(n) \sum_n T_2^2(n) = 64J_3,$$

where $J_3 := \sum_n T_1^2(n)$ is equal to the number of solutions of the equation

$$f(x) - f(u) = f(v) - f(y)$$

subject to

$$1 \leq x, y, u, v \leq N, \; 0 \leq x - u, \; v - y \leq N_1.$$ 

Therefore $J \leq 64N^2 + 64J_4,$ where $J_4$ denotes the number of solutions of the same equation subject to

$$1 \leq x, y, u, v \leq N, \; 1 \leq x - u, \; v - y \leq N_1.$$ 

From the inequality $|a + b|^2 \leq 2|a|^2 + 2|b|^2$ and the relation

$$J_4 = \int_0^1 \left| \sum_{1 \leq x - u \leq N_1} e^{2\pi i \alpha(f(x) - f(u))} \right|^2 \, d\alpha,$$

it follows that

$$J \leq 64N^2 + 64J_4 \leq 64N^2 + 128J_5 + 128J_6,$$

where

$$J_5 = \int_0^1 \left| \sum_{1 \leq x - u \leq N_2} e^{2\pi i \alpha(f(x) - f(u))} \right|^2 \, d\alpha,$$

$$J_6 = \int_0^1 \left| \sum_{N_2 < x - u \leq N_1} e^{2\pi i \alpha(f(x) - f(u))} \right|^2 \, d\alpha,$$

and

$$N_2 = N_1 (\log N)^{-10^\beta} = cN(\log N)^{1-11^\beta}.$$ 

To prove Theorem 2, we obtain upper bounds for $J_5$ and $J_6$.

**Estimate of $J_5$.** From (5) it follows that $J_5$ is equal to the number of solutions of the corresponding diophantine equation. Therefore, the wider the range of summation over $x$ and $u$, the larger the value of the integral on the right-hand side of (5). Hence,

$$J_5 \leq \int_0^1 \left| \sum_{l \leq N \leq N_2^{-1}} \sum_{1 \leq x - u \leq N_2} e^{2\pi i \alpha(f(x) - f(u))} \right|^2 \, d\alpha.$$
whence, by Cauchy inequality,

\[ J_5 \ll N N_2^{-1} \sum_{l \leq N N_2^{-1}} \int_0^1 \left| \sum_{(l-1)N_2 < x \leq l N_2} e^{2\pi i \alpha (f(x) - f(u))} \right|^2 d\alpha. \]

For a fixed \( l \) the integral on the right-hand side is not greater than the number of solutions of the equation

\[ f(x_1 + (l-1)N_2) - f(u_1 + (l-1)N_2) = f(v_1 + (l-1)N_2) - f(y_1 + (l-1)N_2) \]

with

\[ 1 \leq x_1, y_1, u_1, v_1 \leq 2N_2. \]

This number, according to Konyagin’s estimate, is \( \ll N_2^{5/2} \). Therefore,

\[ J_5 \ll N_2^{5/2} N_2^{-2} N_2^{5/2}, \]

whence, by (7),

\[ J_5 \ll N^{5/2} \log N \frac{(1-\beta)}{2} \log \log N. \]

**Estimate of \( J_6 \).** Note that \( J_6 \), defined by (6), is equal to the number of solutions of the equation

\[ f(x) - f(u) = f(v) - f(y) \]

subject to the condition

\[ 1 \leq x, y, u, v \leq N, N_2 < x - u \leq N_1, N_2 < v - y \leq N_1. \]

Let us prove that \( |x - v| \ll N_1 \log \log N \). Without loss of generality we may suppose \( x \geq v \).

If \( u \leq v \), then \( x - v \leq x - u \) and we are done in this case. Otherwise, from (9), we have

\[ e^{A \log x} - e^{A \log u} < 2 e^{A \log v}, \]

whence

\[ e^{A \log x} - A \log v - e^{A \log u} - A \log v \leq 2. \]

Therefore,

\[ e^{A \log u} - A \log v \left( e^{A \log x} - A \log u - 1 \right) \leq 2. \]

From \( e^r \geq 1 + r \) it follows that

\[ e^{A \log u} - A \log v \left( A \log x - A \log u \right) \leq 2. \]

On the other hand, for some \( t \in (u, x) \) we have

\[ A \log x - A \log u = A \beta (x - u) t^{-1} (\log t)^{-1}, \]

which, in view of (10), is \( \gg N_2 N^{-1} (\log N)^{-1} \). Therefore, by (7),

\[ A \log x - A \log u \gg (\log N)^{-10 \beta}. \]

Hence, from (11),

\[ e^{A \log u} - A \log v \ll (\log N)^{10 \beta}, \]
whence
\[(\log u)^\beta - (\log v)^\beta \ll_{A,\beta} \log \log N.\]

Then, for some real \(t \in (v, u)\), we have
\[(u - v)t^{-1}(\log t)^{\beta - 1} \ll_{A,\beta} \log \log N.\]

Therefore,
\[u - v \ll_{A,\beta} N(\log N)^{1-\beta} \log \log N \ll_{A,\beta} N_1 \log \log N.\]

Together with \(0 < x - u \leq N_1\) we conclude that \(x - v \ll_{A,\beta} N_1 \log \log N\).

Thus, for some \(c_1 = c_1(A, \beta) > 0\), we have
\[(12)\ |x - v| \leq c_1 N_1 \log \log N.\]

We can split the range of variation of \(x\) into intervals of length at most
\[c_1 N_1 \log \log N.\]

The number of such intervals is
\[\ll_{A,\beta} N(\log N)^{-1} = (\log N)^{\beta - 1}(\log \log N)^{-1}.\]

It then follows that there exists \(l, 1 \leq l \ll_{A,\beta} (\log N)^{\beta - 1}(\log \log N)^{-1}\), such that
\[(13)\ J_7 \ll_{A,\beta} (\log N)^{\beta - 1}(\log \log N)^{-1} J_7,\]

where \(J_7\) denotes the number of solutions of (9) subject to conditions (10), (12) and
\[(l - 1)c_1 N_1 \log \log N < x \leq l c_1 N_1 \log \log N.\]

Hence, in view of (12),
\[(k - 1)c_1 N_1 \log \log N < x \leq (k + 1)c_1 N_1 \log \log N\]
and
\[(k - 1)c_1 N_1 \log \log N < v \leq (k + 1)c_1 N_1 \log \log N,\]

where \(k = \max(1, l - 1)\). Then, taking (10) into account, we have
\[J_7 \ll_{A,\beta} \int_0^1 \left| \sum_{n < \log \log N} \sum_{(k-1)L + (n-1)N_1 \leq x \leq (k-1)L + nN_1 \atop 1 \leq x - u \leq N_1} e^{2\pi i \alpha (f(x) - f(u))} \right|^2 d\alpha,\]

where \(L = c_1 N_1 \log \log N\).

We again use the fact that the wider the range of summation over \(x\) and \(u\) the larger the value of the integral. This gives
\[J_7 \ll_{A,\beta} \log \log N \sum_{n < \log \log N} \left| \sum_{(k-1)L + (n-1)N_1 \leq x \leq (k-1)L + nN_1 \atop 1 \leq x - u \leq N_1} e^{2\pi i \alpha (f(x) - f(u))} \right|^2 d\alpha.\]

By Cauchy inequality
\[J_7 \ll_{A,\beta} \log \log N \sum_{n < \log \log N} \int_0^1 \left| \sum_{(k-1)L + (n-1)N_1 \leq x \leq (k-1)L + nN_1 \atop 1 \leq x - u \leq N_1} e^{2\pi i \alpha (f(x) - f(u))} \right|^2 d\alpha.\]
It then follows that for some fixed \( n = n_0 \ll \log \log N \) we have

\[
J_7 \ll (\log \log N)^2 \left| \sum_{L \leq x \leq L+n_0 N_1} e^{2\pi i \alpha (f(x)-f(u))} \right|^2 da.
\]

The latter integral does not exceed the number of solutions of the equation

\[
f_1(x_1) - f_1(u_1) = f_1(v_1) - f_1(y_1)
\]

with \( 1 \leq x_1, y_1, u_1, v_1 \leq 2N_1 \) and \( f_1(z) = f((k-1)L + (n_0 - 1)N_1 + z) \). From Konyagin’s estimate we conclude that this integral is \( \ll_{A,\beta} N_{1}^{5/2} \). Hence

\[
J_7 \ll (\log \log N)^2 N_{1}^{5/2}.
\]

Therefore, by (13) and (7), we obtain

\[
J_6 \ll (\log N)^{\beta-1}(\log \log N)^{-1} J_7 \ll_{A,\beta} N_{1}^{2} (\log N)^{\frac{3(1-\beta)}{2}} \log \log N.
\]

This estimate, by virtue of (8) and (4), proves Theorem 2.

Acknowledgement

The author would like to thank the referee for valuable comments and suggestions.

References


Instituto de Matemáticas UNAM, Campus Morelia, AP. Postal 61-3 (Xangari) CP 58089, Morelia, Michoacán, México
E-mail address: garaev@matmor.unam.mx

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use