ON SOME CONSTANTS IN THE SUPERCUSPIDAL CHARACTERS OF $\text{GL}_l$, $l$ A PRIME $\neq p$

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Abstract. The article gives explicit values of some constants which appear in the character formula for the irreducible supercuspidal representation of $\text{GL}_l(F)$ for $F$ a local field of the residual characteristic $p \neq l$.

Introduction

Let $l$ be a prime, let $F$ be a non-Archimedean local field of residual characteristic $p \neq l$, let $A/F$ be a central simple algebra of reduced degree $l$ and let $E$ be a field such that $F \subseteq E \subseteq A$. Then either $A = D_l$, a division algebra of index $l$ over $F$ or $A = M_l(F)$, the algebra of all $l \times l$ matrices. Moreover, every compact mod center Cartan subgroup of $A^\times$ is of the form $E^\times$ for some such $E$, and every irreducible supercuspidal representation of $A^\times$ corresponds to a quasi-character of some such $E^\times$ ([17], [6]).

The character formula for the irreducible supercuspidal representations of $A^\times$ has been extensively studied by many mathematicians (see especially [9] for this topic). Debacker ([8],[9]) got the formula for $\text{GL}_l(F)$ under the assumption $p > l$. But it contains some undetermined constants (see Remark 3.14). The aim of this paper is to compute the constants explicitly and get complete formulas, valid on the regular elliptic set, for the supercuspidal characters of $\text{GL}_l(F)$ which correspond to characters of $E^\times$ for some such $E$, and every irreducible supercuspidal representation of $A^\times$ corresponds to a quasi-character of some such $E^\times$ ([17], [6]).

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bijections is unknown, but in the case \( n \) is a prime \( \neq p \) it is known that the two bijections coincide \((12)\). Therefore, it suffices to determine the character formula on either \( \text{GL}_l(F) \) or \( D_l^* \). Thus we only treat the \( \text{GL} \) case.

Let us summarize the contents of this paper, indicating its organization. Section 1 is devoted to the review of the construction of an irreducible supercuspidal representation \( \pi_\theta \) (resp. \( \pi_\theta' \)) of \( \text{GL}_l(F) \) (resp. \( D_l^* \)) from a generic quasi-character \( \theta \) of \( E^\times \) and the known results about the representation. We note that \( \pi_\theta \) is not always monomial, i.e., induced from a one-dimensional representation, but it can be written as a \( \mathbb{Q} \)-linear combination of monomial representations. In fact \( \pi_\theta \) is written as a \( \mathbb{Q} \)-linear combination of the forms \( \text{ind}_{H}^{\text{GL}_l(F)}(\rho_\theta) \), where \( H \) is a compact mod center subgroup of \( \text{GL}_l(F) \) and \( \rho_\theta \) is a quasi-character of \( H \).

In section 2, we compute the character of \( \pi_\theta \) up to some root numbers. Let \( G = \text{GL}_l(F) \), let \( B \) be the normalizer of an Iwahori subgroup of \( G \) containing \( H \) and let \( \eta_\theta = \text{ind}_{H}^{B}(\rho_\theta) \). Since we treat only regular elliptic conjugacy classes, we consider the character \( \chi_{\pi_\theta} \) on \( L^\times \), where \( L/F \) are extensions of fields of degree \( l \). Moreover the case \( L = E \) is essential. By the Frobenius formula and the result of Kutzko \((16)\), we have only to calculate the sum

\[
\chi_{\pi_\theta}(x) = \sum_{a \in H \setminus \overline{B}} \rho_\theta(axa^{-1})
\]

for \( x \in E \) in order to get the character formula of \( \pi_\theta \). Therefore it is essential to know when \( axa^{-1} \in H \), which is determined in Lemma 2.1. From this, we get the character formula of \( \eta_\theta \) except “at the depth” (Proposition 2.2). But this formula contains the undetermined Gauss sum part \( G(y,j) \), which is calculated later. From this we recover the formula of Debacker including the case \( l < p \). The proof of the formula is short and simple. Moreover since we use the property “intertwining implies conjugacy” of an \( E/F \)-minimal (very cuspidal in the terminology of Carayol \((5)\)) element as the key tool, the result may be extended to \( \text{GL}_n \), at least when \( n \) is prime to \( p \). The character of \( \pi_\theta \) “at the depth” can be calculated directly by taking the explicit matrix form of \( E^\times \) (Lemma 2.5). The character value at the depth has not been previously determined explicitly (see 5.6 in \((8)\)). We can reveal that the character value is represented by some Kloosterman sum.

Section 3 is devoted to the calculation of the Gauss sum part \( G(y,j) \); it appears in the character formula on \( E^\times \). For this purpose, the point is that we have only to treat the character of \( \pi_\theta \) on \( U_1^* = F^\times (1 + P_E) - F^\times (1 + P_E^2) \). For this calculation, we use the \( E^\times \)-module structure of various objects. We first assume \( E/F \) is a Galois extension since the \( E^\times \)-module structure can be described easily for this case. This part is analogous to section 1 of \((23)\), but everything becomes easier since we have only to treat \( U_1^* \). When \( E/F \) is non-Galois, we use the base change lift. Let \( \zeta \) be a primitive \( l \)-th root of unity and \( L = F(\zeta) \). Then \( L \) is an unramified extension of \( F \) and \( EL/L \) is Galois. Therefore we can use the tools of the Galois case for \( \text{GL}_l(L) \). Let \( \text{Gal}(L/F) = \langle \tau \rangle \). By the result of Bushnell-Henniart \((1)\), there is a base change lift \( \eta_L \) of \( \eta_\theta \) to \( H_1^L \) such that the twisted trace of \( \eta_L \) by \( \tau \) gives the trace of \( \eta_\theta \) (see Proposition 3.7 and Lemma 3.8). We remark that we need not assume that the characteristic of \( F \) is 0 since we do not use the Arthur-Clozel base change lift \((1)\). The method of calculating the twisted trace of \( \eta_L \) is similar to that of the Galois case. The complete character formula is stated as Theorem 3.13.
At the end of this Introduction, we compare our formula with the known results besides [8] and [9]. The same type of character formula for the division algebra case was given by Corwin, Moy and Sally [7] in the case \( l \neq p \). Their formulas agree with the result given in section 2. It contains some root numbers associated with a quadratic form as in [8]. They have shown that this root number is a root of unity when \( p \neq 2 \). In this paper, we have determined it completely including the case \( p = 2 \) in section 3. Moreover we find that the Kloosterman sum appears in the character formula. These are new results of this paper. In [25], the author gave the character formula of \( \pi^\theta \) for \( \text{GL}_3 \) by using the decomposition of \( \pi^\theta \) as an \( E^\times \)-module.

**Notation.** Let \( F \) be a non-archimedean local field. We denote by \( \mathcal{O}_F, P_F, \varpi_F, k_F \) and \( v_F \) the maximal order of \( F \), the maximal ideal of \( \mathcal{O}_F \), a prime element of \( P_F \), the residue field of \( F \) and the valuation of \( F \) normalized by \( v_F(\varpi_F) = 1 \). We set \( q \) to be the number of elements in \( k_F \). Henceforth we fix an additive character \( \psi \) of \( F \) whose conductor is \( P_F \), i.e., \( \psi \) is trivial on \( P_F \) and not trivial on \( \mathcal{O}_F \). For an extension \( E \) over \( F \), we denote by \( \text{tr}_E, n_E \) the trace and norm to \( F \), respectively. We set \( \psi_E = \psi \circ \text{tr}_E \). The trace of the matrix is denoted by \( \text{Tr} \). For an irreducible admissible representation \( \pi \) of \( \text{GL}_l(F) \), the conductor exponent of \( \pi \) is defined to be the integer \( f(\pi) \) such that the local constant \( \varepsilon(s, \pi, \psi) \) of Godement-Jacquet [11] is the form \( a q^{-s(f(\pi)-1)} \).

We call \( \pi \) minimal if

\[
f(\pi) = \min_{\eta} f(\pi \otimes (\eta \circ \text{Nr})),
\]

where \( \eta \) runs through the quasi-characters of \( F^\times \). Let \( G \) be a totally disconnected, locally compact group. We denote by \( \hat{G} \) the set of (equivalence classes of) irreducible admissible representations of \( G \). For a closed subgroup \( H \) of \( G \) and a representation \( \rho \) of \( H \), we denote by \( \text{Ind}_H^G \rho \) (resp. \( \text{ind}_H^G \rho \) ) the induced representation (resp. compactly induced representation) of \( \rho \) to \( G \). For a representation \( \pi \) of \( G \), we denote by \( \pi|_H \) the restriction of \( \pi \) to \( H \).

1. **Construction of the representation**

Let \( l \neq p \) be an odd prime and let \( E \) be a ramified extension of \( F \) of degree \( l \). Then \( E \) can be embedded into \( M_l(F) \) and, up to conjugacy, the embedding is unique. Let \( G = \text{GL}_l(F) \). In this section, we review the construction of supercuspidal representations of \( G \) which are parameterized by the quasi-characters of \( E^\times \). Of course, this construction is well known ([5], [17]).

**Definition 1.1.** Let \( \theta \) be a quasi-character of \( E^\times \) and let \( f(\theta) \) be the exponent of the conductor of \( \theta \) i.e. the minimum integer such that \( \text{Ker} \theta \subset 1 + P^2_F \). Then \( \theta \) is called generic if \( f(\theta) \not\equiv 1 \mod l \). For a generic character \( \theta \) of \( E^\times \), \( \beta_\theta \in P^{-f(\theta)}_E - P^{2-f(\theta)}_E \) is defined by

\[
\theta(1 + x) = \psi_E(\beta_\theta x) \quad \text{for} \quad x \in P^{(f(\theta)+1)/2}_E.
\]

Then \( F(\beta_\theta) = E \). We denote by \( \hat{E}^\times_{\text{gen}} \) the set of generic quasi-characters of \( E^\times \).

We construct an irreducible supercuspidal representation of \( G = \text{GL}_l(F) \) from \( \theta \in \hat{E}^\times_{\text{gen}} \). For simplicity, we set \( \beta = \beta_\theta \). Since \( E/F \) is tamely ramified, there exists a prime element \( \varpi_E \) of \( \mathcal{O}_E \) satisfying \( \varpi^l_E \in F \). Put \( \varpi_F = \varpi^l_E \). We identify

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Let $B$ construct an irreducible representation of $M$, which is also an $O_E$-basis of $O_E$. From the lattice flag $\{P_k\}_{k \in \mathbb{Z}}$, we construct a maximal compact mod center subgroup. The construction of the representation is well known. For details, see \cite{17}.

**Definition 1.2.** For $i \in \mathbb{Z}$, set

$$A^i = \{ g \in M_1(F) | g(P_E^j) \subset P_E^{j+i} \text{ for all } j \in \mathbb{Z} \}.$$

Put $K = (A^0)^\times$, $B = E^\times K$ and $K^i = 1 + A^i$ for $i \geq 1$.

Then $K$ is an Iwahori subgroup of $G$ and $B$ is a normalizer of $K$. First we construct an irreducible representation of $B$ from a generic quasi-character of $E^\times$.

Let $\theta$ be a generic quasi-character of $E^\times$, i.e., $f(\theta) = n \neq 1 \mod l$. There exists an element $\beta \in P_E^{1-n}$ such that $\theta(1 + x) = \psi_E(\beta x)$ for $x \in P_E^m$, where $m = [(n + 1)/2]$. Define $\psi_\beta$ on $K^m$ by $\psi_\beta(1 + x) = \psi(\text{Tr}(\beta x))$ for $x \in A^m$. Then $\psi_\beta$ is a quasi-character of $K^m$. Put $H = E^\times K^m$ and define a quasi-character $\rho_\theta$ of $H$ by

$$\rho_\theta(h \cdot g) = \theta(h) \psi_\beta(g) \quad \text{for } h \in E^\times, \ g \in K^m.$$

Let $J$ be the normalizer of $\psi_\beta$ in $B$, i.e.,

$$J = \{ a \in B \mid \psi_\beta^a = \psi_\beta \},$$

where $\psi_\beta^a(x) = \psi_\beta(a^{-1}xa)$ for $x \in K^m$. Then $J = E^\times K^{m'}$, where $m' = [n/2]$. Put $\eta_\theta = \text{Ind}_H^B \rho_\theta$.

When $n$ is even, i.e., $n = 2m$, then $J = H = E^\times K^m$. By the Clifford theory, $\eta_\theta$ is an irreducible representation of $B$. We put

$$\kappa_\theta = \eta_\theta.$$

If $n + 1 = 2m - 1$ is odd, then $J = E^\times K^{m-1}$, so we have to determine an irreducible component of $\text{Ind}_H^B \rho_\theta$. For a subgroup $M \subset B$, we write $M^1 = M \cap F^\times K$. In particular, $H^1 = F^\times (1 + P_E)K^m$. It is well known and not difficult to show that

$$\text{Ind}_H^B \rho_\theta |_{J^1} = \text{Ind}_{H^1}^{J^1} \rho_\theta |_{J^1} = q \eta.$$

This $\eta$ can be extended to $J$ in $|E^\times |F^\times (1 + P_E)| = l$ ways. To determine the extension by $\theta$, we will express $\rho_\theta$ by a linear combination of $\text{Ind}_H^B \rho_\theta \otimes \chi (\chi \in (E^\times F^\times (1 + P_E)))$.

**Lemma 1.3.** Define the virtual representation $\kappa_\theta$ of $B$ by

$$\kappa_\theta = 1 - \left( \frac{q}{l} \right) q^{(l-1)/2} \sum_{\chi \in (E^\times F^\times (1 + P_E))} \eta_\theta \otimes \chi + \left( \frac{q}{l} \right) \eta_\theta,$$

where $\left( \frac{q}{l} \right)$ is the Legendre symbol. Then $\kappa_\theta$ is a real representation and an irreducible component of $\text{Ind}_H^B \rho_\theta$.

**Proof.** Let $\{\eta_1, \ldots, \eta_l\}$ be the set of the extensions of $\eta$ to $J$ and $(E^\times F^\times (1 + P_E)) = \{\chi_1, \ldots, \chi_l\}$. It follows from Lemma 3.5.35 in \cite{17} that

$$\text{Ind}_H^B \rho_\theta = \left( \frac{q^{(l-1)/2} - \left( \frac{q}{l} \right)}{l} \right) \sum_{i=1}^l \eta_i + \left( \frac{q}{l} \right) \eta_\theta.$$
Theorem 1.4. Let the notation be as above and denote by \( \kappa \) the equivalence classes of the supercuspidal representations of \( E/F \). Let \( \kappa_0 = \text{Ind}_H^B \kappa_0 \). Then the depth zero representation appears when \( \epsilon \) is trivial on \( \kappa_0 \). Therefore we get our proposition from (1.5). □

The following result is well known ([17], [21]).

**Remark.** 1. If \( \pi \in A_0(G) \) and \( f_{\min}(\pi) \equiv 0 \mod l \), then \( \pi \) can be constructed from a regular character \( \theta \) of \( L^\times \), where \( L \) is an unramified extension of \( F \) of degree \( l \). The characters of such representations on elliptic conjugacy classes were completely calculated in [23].

2. The representation \( \pi_\theta \) has depth \( f(\theta)/e(E/F) \) in terms of the Moy-Prasad filtration ([8]). Since \( e(E/F) = l > 1 \), the depth of the representation is positive. The depth zero representation appears when \( E/F \) is unramified.

By the following proposition, we have only to calculate \( \chi_\eta_0 \).

**Proposition 1.6.** The character \( \chi_{\kappa_0} \) of \( \kappa_0 \) on \( B \) is expressed by \( \chi_{\eta_0} \) as follows:

\[
\chi_{\kappa_0}(x) = \begin{cases} 
\left( \frac{q}{l} \right) \chi_{\eta_0}(x), & x \in B - F^\times K, \\
q^{-(l-1)/2} \chi_{\eta_0}(x), & x \in F^\times K.
\end{cases}
\]

Proof. Let \( \chi \in E^\times / F^\times (1 + P_E) \). Since \( \chi \) is trivial on \( F^\times (1 + P_E) \), \( \beta_\theta \otimes \chi = \beta_\theta \). Thus

\[
\rho_\theta \otimes \chi(h \cdot g) = \chi(h) \rho_\theta(h \cdot g) \quad \text{for} \quad h \in E^\times, \quad g \in K^m.
\]

Due to the isomorphisms \( B/F^\times K \simeq H/H^1 \simeq E^\times / F^\times (1 + P_E) \), we regard \( \chi \) as a character of \( H \) and \( B \). Then it follows from (1.8) that \( \rho_\theta \otimes \chi = \rho_\theta \otimes \chi \). By the well-known formula, we have

\[
\text{Ind}_H^B(\rho_\theta \otimes \chi) = (\text{Ind}_H^B \rho_\theta) \otimes \chi.
\]

Therefore we get

\[
\sum_{\chi \in (E^\times / F^\times (1 + P_E))} \eta_\theta \otimes \chi = \eta_\theta \otimes \left( \sum_{\chi \in B / F^\times K} \chi \right).
\]

For \( x \in B \),

\[
\sum_{\chi \in B / F^\times K} \chi(x) = \begin{cases} 
1, & x \in B - F^\times K, \\
0, & x \in F^\times K.
\end{cases}
\]

Therefore we get our proposition from [150]. □
To end this section, we quote the result of Kutzko [16] in the form that the character formula of $\pi_\theta$ on regular elliptic elements is essentially given by the one of $\kappa_\theta$.

**Theorem 1.7.** Let $x$ be an regular elliptic element of $G$.

1. If $F(x)/F$ is ramified and $x \not\in F^\times(1 + P_{F(x)}^n)$,
   \[\chi_{\pi_\theta}(x) = \chi_{\kappa_\theta}(x).\]

2. If $F(x)/F$ is unramified and $x \not\in F^\times(1 + P_{F(x)}^{[n/l]+1})$,
   \[\chi_{\pi_\theta}(x) = 0.\]

**Proof.** These are obtained by applying Proposition 5.5 in [16] to our case. □

2. **Calculation of the Character**

Now we begin to calculate the characters of the representations constructed in the previous section. In this section, we shall get a character formula up to some root numbers. These root numbers are calculated explicitly in the next section.

Henceforth we fix a generic character $\theta$ and put $\rho = \rho_\theta$, $\eta = \eta_\theta$ and so on. Since $E/F$ is a totally tamely ramified extension, there exists a prime element $\varpi_E$ of $O_E$ such that $\varpi_E \in P_F - P_F^2$. Put $\varpi_E = \varpi_F$. As in the previous section, we identify $M_1(F)$ with $\text{End}_F(E)$ by the $F$-basis $\{\varpi_E, \varpi_E^2, \ldots, \varpi_E, 1\}$, which is an $O_F$-basis of $O_E$. Thus we get the explicit matrix forms of various objects:

\[\varpi_E = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
\varpi_F & 0 & \cdots & \cdots & 0
\end{pmatrix},\]

\[A^0 = \begin{cases}
(a_{1i} \cdots a_{1j}) & a_{ij} \in O_F & \text{if } i \leq j \\
(a_{1i} \cdots a_{1j}) & a_{ij} \in P_F & \text{if } i > j
\end{cases},\]

\[A^1 = \begin{cases}
(a_{1i} \cdots a_{1j}) & a_{ij} \in O_F & \text{if } i < j \\
(a_{1i} \cdots a_{1j}) & a_{ij} \in P_F & \text{if } i \geq j
\end{cases}.\]

If $q \equiv 1 \mod l$, $F$ has a primitive $l$-th root of unity $\zeta$ and $E/F$ is a cyclic extension. Let $\sigma$ be a generator of $\text{Gal}(E/F)$ determined by $\sigma \varpi_E = \varpi_E \zeta$. We denote the diagonal matrix $\text{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta)$ by $\xi$. Then $\xi$ satisfies $\xi^l = 1$ and $\xi x^q = x^q$ for $x \in E$.

Define a natural ring morphism $R$ from $A^0$ to $k_F^l$ by the identification of $A^0/A^1$ with $k_F^l$. We note that $R(O_E^0) = \{(a_0, \ldots, a_{l-1}) \mid a \in k_F\}$ and if $R(a) = (a_0, a_1, \ldots, a_{l-1})$, $R(\varpi_E \varpi_E^{-1}) = (\alpha_1, \alpha_2, \ldots, \alpha_0)$. For convenience, we extend the subscript to $\mathbb{Z}$ by putting $\alpha_i = \alpha_i \mod l$. The next lemma is the key tool for the character calculation.
Lemma 2.1. Let \( x \in P_{E}^{i} - (F + P_{E}^{i+1}) \), let \( g \in B \) and let \( j \) be a positive integer. If \( g x g^{-1} \in E^x K^j \), then

\[
g \in \begin{cases} E^x (1 + A^j) & \text{if } q \not\equiv 1 \mod l, \\ \bigcup_{k=0}^{l-1} E^x (1 + A^j) \xi^k & \text{if } q \equiv 1 \mod l. \end{cases}
\]

Proof. We may assume \( g \in A_0 \) by replacing \( g \) by \( \varpi_E^{-k} g \) if \( g \in A_k \). Let \( x = \varpi_E x_0 \) for \( x_0 \in O_E^x \) and \( R(g) = (\alpha_0, \alpha_1, \ldots, \alpha_{l-1}) \). Then

\[
R(g x g^{-1} x^{-1}) = (\alpha_0 \alpha_1^{-1}, \alpha_1 \alpha_2^{-1}, \ldots, \alpha_{l-1} \alpha_{l-1}^{-1}),
\]

where \( \alpha_s = \alpha_{s \mod l} \) for \( s \in \mathbb{Z} \). Since \( k_E = k_F \), \( g x g^{-1} x^{-1} \in E^x (1 + A^j) \) implies \( R(g x g^{-1} x^{-1}) \in \Delta = \{ (\alpha, \ldots, \alpha) \mid \alpha \in k_F \} \). By virtue of \( i \not\equiv 0 \mod l \), we have

\[
\alpha_k = (\zeta^j)^k \alpha_0, \quad 0 \leq k \leq l - 1,
\]

for some \( l \)-th root of unity \( \zeta^j \). If \( q \equiv 1 \mod l \), \( \xi \varpi_E \xi^{-1} = \xi \varpi_E \). Thus we get

\[
g \in \begin{cases} E^x (1 + A^j) & \text{if } q \not\equiv 1 \mod l, \\ \bigcup_{k=0}^{l-1} E^x (1 + A^j) \xi^k & \text{otherwise}. \end{cases}
\]

Thus we may assume \( g - 1 \in A^k - (P_{E}^{k+1} + A^{k+1}) \) for some \( k \geq 1 \). Put \( g - 1 = \varpi_E g_0 \) and \( R(g_0) = (\gamma_0, \gamma_1, \ldots, \gamma_{l-1}) \). Since

\[
g x g^{-1} x^{-1} \equiv 1 + (g - 1) - x (g - 1) x^{-1} \mod A^{k+1}
\]

\[
= 1 + \varpi_E (g_0 - x g_0 x^{-1}) \mod A^{k+1},
\]

\( R(g_0 - x g_0 x^{-1}) = (\gamma_0 - \gamma_i, \gamma_i - \gamma_{i+1}, \ldots, \gamma_l - \gamma_{l-1}) \). Therefore \( g x g^{-1} x^{-1} \in E^x K^{k+1} \) contradicts \( g - 1 \in A^k - (P_{E}^{k+1} + A^{k+1}) \). It implies that if \( g x g^{-1} x^{-1} \in E^x K^j \),

\[
g \in \begin{cases} E^x (1 + A^j) & \text{if } q \not\equiv 1 \mod l, \\ \bigcup_{k=0}^{l-1} E^x (1 + A^j) \xi^k & \text{if } q \equiv 1 \mod l. \end{cases}
\]

\( \Box \)

Put \( U_{-1} = E^x, U_0 = F^x O_E^x, U_i = F^x (1 + P_{E}^i) \) for \( i \geq 1 \) and \( U_i^{*} = U_i - U_{i+1} \) for \( j \geq 1 \). The previous lemma gives the character of \( \eta_0 \) on \( E^x \times U_{n-1} \). We remark that Aut\(_{F} E = \{ 1 \} \) if \( q \not\equiv 1 \mod l \).

Proposition 2.2. Let \( x \in U_i^{*} \) for \( -1 \leq i < n - 1 \). If \( i > 0 \), \( x \) is written in the form \( x = c (1 + y) \) for \( c \in F \) and \( y = \varpi_E y_0 \in \varpi_E O_E^x \). For \( u \in k_F^x \) and \( j \in (\mathbb{Z} / l \mathbb{Z}) \) such that \( j \not\equiv 0 / 2 \), we define the Gaussian sum \( G(u, j) \) by

\[
G(u, j) = \sum_{(\alpha_0, \ldots, \alpha_{l-1}) \in k_F^x / \Delta} \psi \left( \sum_{k=0}^{l-1} u (\alpha_{k+1} - \alpha_k) \xi^{k+j} \right),
\]

where \( \Delta = \{ (\alpha, \ldots, \alpha) \mid \alpha \in k_F \} \). Then \( \chi_{\eta_0} \) on \( U_i^{*} \) is given as follows:

\[
\chi_{\eta_0}(x) = \begin{cases} \sum_{\sigma \in \text{Aut}_F E} \theta(\sigma x), & i = -1, \\ q_i / 2 (l-1) \sum_{\sigma \in \text{Aut}_F E} \theta(\sigma x), & i > 0 \text{ and } n - i \text{ even}, \\ q_{i+1} / 2 (l-1) \sum_{\sigma \in \text{Aut}_F E} \theta(\sigma x) \\ G(\beta \varpi_E^{-1} y_0 (\sigma \varpi_E / \varpi_E)^i, c), & i > 0 \text{ and } n - i \text{ odd}, \end{cases}
\]

where \( c = i^{-1} (n + i - 1) / 2 \) satisfies \( c \not\equiv 1 / 2 \mod l \).
Proof. Put $a_x = axa^{-1}$ for $a, x \in G$. First we treat the case $x \in U_{\ast}^E = E^\times - F^\times O^E_F$.

Since
\[ \chi_{\eta^p}(x) = \sum_{a \in H \setminus B} \rho_0(a_x), \]
the proof follows immediately from Lemma 2.1.

Now we treat the case $x = c(1 + y)$ for $c \in F$ and $y \in P_L^F - (F + P_L^{i+1})$. We may assume $c = 1$ since $F^\times$ is the center of $B$. For $1 + k \in K^{1-(n+i)/2}$ and $a \in B$, we have
\[ \chi_{\eta^p}(1 + y) = \sum_{a \in H \setminus B} \rho_0(\eta^p(1 + y)) \]
\[ = C \sum_{1 + k \in K^{1-(n+i)/2}} \sum_{a \in H \setminus B} \rho_0(a^{1+k}(1 + y)), \]
where $C = \sum_{1 + k \in K^{1-(n+i)/2}} \rho_0(a^{1+k}(1 + y))$. In the above expression,
\[ \rho_0(a^{1+k}(1 + y)) = \rho_0(1 + a^{1+k}(ky - yk)) \]
\[ = \rho_0(1 + a^{1+k}) \rho_0(1 + a^{(1 + y)^{-1}}(ky - yk)) \]
\[ = \rho_0(1 + a^{1+k}) \psi(Tr(y^{a-1} \beta(1 + y)^{-1} (ky - yk))) \]
\[ = \rho_0(1 + a^{1+k}) \psi(Tr(y^{a-1} \beta - a^{-1} \beta y)(1 + y)^{-1} k) \]
since $yk^2 \in A^a$ and $a(1 + y)^{-1} (ky - yk) a^{-1} \in A^m$. If $y^{a-1} \beta - a^{-1} \beta y \not\in A^{1-(n+i)/2}$, the map $k \mapsto \psi(Tr(y^{a-1} \beta - a^{-1} \beta y)(1 + y)^{-1} k))$ is a non-trivial character of $A^{1-(n+i)/2} / A^{1-(n+i)/2}$; thus
\[ \sum_{k \in A^{1-(n+i)/2}} \psi(Tr(y^{a-1} \beta - a^{-1} \beta y)(1 + y)^{-1} k) = 0. \]

By Lemma 3.3 in [4], $y^{a-1} \beta - a^{-1} \beta y \in A^{1-(n+i)/2}$ is equivalent to $a^{-1} \beta \in E^\times K^{1-(n+i)/2}$. Thus it follows from Lemma 2.1 that
\[ \chi_{\eta^p}(1 + y) = \sum_{\sigma \in \Aut_F E} \sum_{1 + a \in H \setminus E^\times K^{1-(n+i)/2}} \rho_0(1 + (1 + a)^\sigma y(1 + a)^{-1}). \]

By virtue of $(1 + y)^{-1}(1 + (1+a)y) \in K^m$ and $(1 + y)^{-1}(1 + (1+a)y) \equiv 1 + (1+y)^{-1}((ay - ya) + (ya - ay)a) \mod K^n$,
\[ \rho_0(1 + (1+a)y) = \theta(1 + y) \psi_\beta((1 + y)^{-1} (ay - ya)) \psi_\beta((1 + y)^{-1} (ya - ay)a). \]

Since $\psi_\beta((1 + y)^{-1} (ay - ya)) = \psi(Tr(y \beta(1 + y)^{-1} - \beta(1 + y)^{-1} y a) = 1$, $\psi_\beta((1 + y)^{-1} (ya - ay)a) = \psi_\beta((ya - ay)a)$ and $|E^\times K^j / E^\times K^n| = q^{1-j(m-j)}$, we obtain
\[ \chi_{\eta^p}(1 + y) = \begin{cases} q^{m-(n-i)/2} \sum_{\sigma \in \Aut_F E} \theta(1 + \sigma y), & n - i \text{ even}, \\ q^{m-(n+i+1)/2} \sum_{\sigma \in \Aut_F E} \theta(1 + \sigma y) S(n - i, \sigma), & n - i \text{ odd}, \end{cases} \]
Proof. By the definition of $k_p$ of unity when $p = 1$, we have
\[ S(n - i, \sigma) = \sum_{a \in A(n - i + 1)/2 \cap A(n - i - 1)/2} \psi_p((\sigma y a - a^\sigma y)a). \]

Now we may assume $n - i$ odd and $\sigma = 1$. Put $y = w_{E}^{a} y_0$, $a = \frac{n - i - 1}{2} a_0$ and $S = S(n - i, 1)$. Since $(y a - ay)a$ equals
\[ w_{E}^{n - 1}(y_0 w_{E}^{-(n - i - 1)/2} a_0 \frac{n - i - 1}{2} - w_{E}^{-(n - i - 1)/2} a_0 \frac{n + i + 1}{2} y_0 a_0, \]
we have by way of the map $R : A_0/A_1 \to k_F^1$ that
\[ S = \sum_{(\alpha_j) \in k_F^1/\Delta} \psi_{\beta}(\sum_{j=0}^{l-1} \beta w_{E}^{n - 1} y_0(\alpha_j - (n - i - 1)/2 - \alpha_j - (n + i + 1)/2)\alpha_j). \]
(The subscript is extended to $\mathbb{Z}$ by $\alpha_j = \alpha_j \mod l$.) First replacing the subscript $j$ by $j + (n + i - 1)/2$ and then replacing $\alpha_ij$ by $\alpha_j$, we get $S = G(\beta w_{E}^{n - 1} y_0, c)$. By virtue
\[ \text{of } n \not\equiv 1 \mod l \text{ and } i \not\equiv 0 \mod l, \ c = i^{-1}(n + 1 - i)/2 \text{ satisfies } c \not\equiv 1/2 \mod l. \]

Remark 2.3. It is proved that the Gauss sum $q^{-(l-1)/2} G(u, j)$ is an eighth root of unity when $p \neq 2$ in [2] and [3]. This is the Gauss sum attached to quadratic form over $k_F$ when $p \neq 2$. But it is not easy to calculate the determinant of the quadratic form. Moreover we need to treat the case $p = 2$ separately.

Next we calculate the character on $K^{n - 1} - K^n$. We state the character formula including the case $x \notin E$. The calculation of the character on $K^{n - 1} - K^n$ ("at the depth") seems difficult in [3]. Here we find it is written explicitly by the Kloosterman sum.

Definition 2.4. For $a \in k_F^\times$, we define the Kloosterman sum $\text{Kl}(a)$ by
\[ (2.5) \quad \text{Kl}(a) = \sum_{(y_0, \ldots, y_{l-1}) \in k_F^1/\Delta} \psi(y_0 + \cdots + y_{l-1}). \]

Theorem 2.5. Let $x = 1 + w_{E}^{n - 1} x_0$ for $x_0 = \text{diag}(k_0, \ldots, k_{l-1}) \ (k_i \in O_E^\times)$. Then
\[ \chi_{\eta_0}(x) = q^{(l-1)(m-1)} \text{Kl} \left( \beta_{0} w_{E}^{n - 1} \prod_{j=0}^{l-1} k_j \right). \]

(Since $\beta_{0} w_{E}^{n - 1} \in O_E^\times$ and $k_{F}^{x} = k_{E}^{x}$, we regard $\beta_{0} w_{E}^{n - 1} \mod P_E$ as an element of $k_{E}^{x}$.)

Proof. By the definition of $\eta_0$, we have
\[ \chi_{\eta_0}(1 + w_{E}^{n} \text{diag}(k_0, \ldots, k_{l-1})) = q^{(l-1)(m-1)} \sum_{a \in E^x K^1 \setminus B} \psi(\text{Tr} \beta a w_{E}^{n} \text{diag}(k_0, \ldots, k_{l-1}))a^{-1}). \]

It follows from [22] and [23] that the set $\{\text{diag}(1, y_1, \ldots, y_{l-1}) \mid y_i \in k_F^1\}$ makes a complete system of representatives of $E^x K^1 \setminus B$. For convenience, put $y_0 = 1$. Since
\[ w_{E} \text{diag}(y_0, y_1, \ldots, y_{l-1})w_{E}^{n - 1} = \text{diag}(y_0, \ldots, y_{l-1}, 1), \]
we have
\[
\text{Tr} \, \beta \, \text{diag}(y_0, y_1, \ldots, y_{l-1}) \, \text{diag}(k_0, \ldots, k_{l-1}) \, \text{diag}(y_0, y_1, \ldots, y_{l-1})^{-1} \\
\equiv \beta \, \text{diag}(y_0, y_1, \ldots, y_{l-1}) - 1
\]
By replacing \( y_i \) by \( k_i y_{i-n+1}/y_i \), we get our lemma. \( \square \)

On \( K^n \), the character of \( \pi = \pi_\theta \) becomes a constant function on regular elliptic conjugacy classes.

**Lemma 2.6.** Let \( x \) be an regular elliptic element in \( K^n \). Then
\[
\chi_\pi(x) = q^{(n-2)(l-1)/2}(q^l - 1)/q - 1.
\]

**Proof.** We use the Deligne-Kazhdan correspondence ([10], [22]). There exists an irreducible representation \( \pi' \) of \( \mathbb{D} \times \mathbb{L} \) such that \( \chi_\pi = \chi_{\pi'} \) on the regular elliptic conjugacy classes.

Since the correspondence preserves the conductoral exponents, \( \pi' \) is trivial on \( 1 + P_E \). Thus \( \chi_\pi \) is also constant on \( K^n \). This constant is expressed by the local character expansion and equals \( q^{(n-2)(l-1)/2}(q^l - 1)/q - 1 \) (see for example [5], 7.4). \( \square \)

The character formula on regular elliptic conjugacy classes outside \( E^\times \) can be easily obtained.

**Lemma 2.7.** Let \( x \) be an regular elliptic element of \( B \). If \( x \) satisfies the condition that \( F(x) \not\equiv E \) and \( x \) is not conjugate to an element of \( F^\times K^n \), then \( \chi_\pi(x) = 0 \).

**Proof.** See Lemma 3.3 in [16]. \( \square \)

3. Calculation of Gauss sums

In this section, we determine the Gauss sum part \( G(y,n - i) \) explicitly. Since \( G(y,n - i) \) depends only on \( n - i \) mod \( l \) and \( y \) mod \( P_E \), we have only to treat the character of \( \eta_\theta \) on \( U_1^* \) by making \( n \) big enough. We have only to treat the case that \( n \) is even by replacing \( n \) by \( n + l \) if necessary.

**Lemma 3.1.** Assume \( n = 2m \). Then for \( x \in U_1^* \),
\[
\chi_{\eta_\theta}(x) = \sum_{\sigma \in \text{Aut}_F} \sum_{a \in H \backslash E^\times K^{m-1}} \rho_\theta(a^\sigma xa^{-1}).
\]

**Proof.** It follows from Lemma [2.1] that \( axa^{-1} \in H \) implies \( a \in E^\times K^{m-1} \). Hence our lemma. \( \square \)

For the calculation of the sum in the above lemma, we use the \( E^\times \)-module structure of various objects. When \( E/F \) is a Galois extension, it is easy to treat. Thus we first assume \( E/F \) is Galois, i.e., \( q \equiv 1 \mod l \). We recall that \( \xi \) is the diagonal matrix \( \text{diag}(1, \xi^{l-1}, \xi^{l-2}, \ldots, \xi) \), where \( \xi \) is an \( l \)-th root of unity in \( F \) and
\(\xi\) satisfies \(\xi^l = 1\) and \(\xi x \xi^{-1} = \sigma x\) for \(x \in E\), where \(\sigma\) is the generator of \(\text{Gal}(E/F)\) determined by \(\sigma E = E \xi\). By the explicit matrix form of \(E\) and \(A_i\), we obtain

\[
\begin{align*}
M_l(F) &= E \oplus \cdots \oplus E \xi \oplus \cdots \oplus E \xi^{l-1}, \\
A^0 &= O_E \oplus \cdots \oplus O_E \xi \oplus \cdots \oplus O_E \xi^{l-1}, \\
A^1 &= P_E \oplus \cdots \oplus P_E \xi \oplus \cdots \oplus P_E \xi^{l-1}, \\
A^{l-1} &= P_E^{l-1} \oplus \cdots \oplus P_E^{l-1} \xi \oplus \cdots \oplus P_E^{l-1} \xi^{l-1}.
\end{align*}
\]

(3.2)

Lemma 3.2. A complete system of representatives of \(H \backslash E^x K^{m-1}\) is given by

\[
\{1 + \omega_{E}^{-1} \alpha_1 \xi + \cdots + \omega_{E}^{-1} \alpha_{l-1} \xi^{l-1} \mid \alpha_i \in k_F\}.
\]

Proof. This is obvious from (3.2).

For \(a = 1 + \alpha_1 \xi + \cdots + \alpha_{l-1} \xi^{l-1} \in A^{m-1}, \rho(\alpha x a^{-1})\) for \(x \in U_1^e\) can be expressed explicitly in terms of \(\alpha_1, \ldots, \alpha_{l-1}\). First, we determine the coefficients of \(a^{-1}\) with respect to the \(E\)-basis \(\{1, \xi, \ldots, \xi^{l-1}\}\).

Lemma 3.3. For \(a = \sum_{j=0}^{l-1} \alpha_j \xi^j \ (\alpha_j \in E)\), define \(\Lambda(a) \in M_l(E)\) by

\[
\Lambda(a) = (\sigma \alpha_{i-j} \mod l)_{0 \leq i, j \leq l-1}
\]

and let \(\Lambda_k(a)\) be the \((1, k+1)\)-cofactor of \(\Lambda(a)\). Then

\[
a^{-1} = \sum_{j=0}^{l-1} \Lambda_j(a) \xi^j.
\]

Proof. Our lemma follows from Cramer’s formula.

Lemma 3.4. Assume \(n = 2m\) and \(3(m-1) \geq 2m\). Let \(c \in F^x, y \in P_{E}^{m-1}\) and \(a = 1 + \sum_{j=1}^{l-1} \alpha_j \xi^j \in K^{m-1}\). Then

\[
\rho_\theta(\alpha c(1+y)a^{-1}) = \theta(c(1+y))\psi_E \left( \sum_{j=1}^{l-1} (\beta \alpha_j \sigma^j \alpha_{l-j} - \sigma^{-j} \beta \alpha_{l-j} \sigma^{-j} \alpha_j) y \right).
\]

Proof. It is obvious that we may assume \(c = 1\). Since

\[
g^{-1} a g a^{-1} = 1 + (g^{-1} (a-1) g - (a-1)) a^{-1} = 1 + \left( \sum_{j=1}^{l-1} (\sigma^j g g^{-1} - 1) \alpha_j \xi^j \right) a^{-1},
\]

\[
\sum_{j=1}^{l-1} (\sigma^j g g^{-1} - 1) \alpha_j \xi^j \in A^m\) and \(\text{Tr}(\beta x \xi^j) = 0\) for all \(x \in E\), we have

\[
\rho_\theta(g^{-1} a g a^{-1}) = \psi_\beta \left( \sum_{j=1}^{l-1} (\sigma^j g g^{-1} - 1) \alpha_j \xi^j a^{-1} \right) = \psi_\beta \left( \sum_{j=1}^{l-1} (\sigma^j g g^{-1} - 1) \alpha_j \sigma^j (f_{i-j}(a)) \right),
\]
where \( f_j(a) \in E \) is defined by \( a^{-1} = \sum_{j=0}^{l-1} f_j(a) \xi^j \). Put \( g = 1 + y \). In the last equation, \( \beta \in P_E^{1-n}, f_{l-j} \in P_E^{m-1} \) and \( \sigma'g^{-1} - 1 = \sigma'y - y \mod P_E^{2m-2} \). Thus we get

\[
\rho_\theta(g^{-1}ag^{-1}) = \psi_E \left( \sum_{j=1}^{l-1} \sigma'^j \beta f_{l-j}(a) \sigma^{-j} \alpha_j - \beta \sigma'(f_{l-j}(a)) \alpha_j y \right)
\]

by virtue of \( \text{tr}_E u\sigma'v = \text{tr}_E \sigma^{-j}uv \) for any \( u, v \in E \). It follows from Lemma 3.3 that

\[
f_{l-j}(a) = \frac{\Lambda_{l-j}(a)}{\det \Lambda(a)} \equiv \alpha_{l-j} \mod P_E^{2m-2}.
\]

By the assumption \( 3m - 3 \geq 2m \), we obtain the desired formula. \( \square \)

**Proposition 3.5.** Assume \( q \equiv 1 \mod l \).

1. If \( n = 2m \) and \( m \geq 3 \),

\[
\chi_{\eta_0}(x) = q^{(l-1)/2} \sum_{j=0}^{l-1} \theta(\sigma^j x) \quad \text{for} \quad x \in U_1^*.
\]

2. For any integer \( j \not\equiv 1/2 \mod l \) and \( y \in \mathcal{O}_F^\times \), \( G(y, j) = q^{(l-1)/2} \).

**Proof.** By Lemmas 3.1, 3.2 and 3.4 we have for \( c \in F^\times \) and \( y \in 1 + P_E \)

\[
\chi_{\eta_0}(c(1+y)) = \sum_{i=0}^{l-1} \theta(c(1+\sigma^i y)) \sum_{(\alpha_1, \ldots, \alpha_{l-1}) \in (P_E^{m-1}/P_E^m)^{l-1}} f(\alpha_1, \ldots, \alpha_{l-1}; \sigma^i y),
\]

where

\[
f(\alpha_1, \ldots, \alpha_{l-1}; y) = \psi_E \left( \sum_{j=1}^{l-1} (3\alpha_j \sigma^j \alpha_{l-j} - \sigma^{-j} \beta \alpha_{l-j} \sigma^{-j} \alpha_j) y \right).
\]

Put \( S_j = \{(\alpha_1, \ldots, \alpha_{l-1}) \in (P_E^{m-1}/P_E^m)^{l-1} | \alpha_k = 0 \text{ for } k < j, \alpha_j \neq 0 \} \) and \( I_j(y) = \sum_{(\alpha_1, \ldots, \alpha_{l-1}) \in S_j} f(\alpha_1, \ldots, \alpha_{l-1}; y) \). Then

\[
\chi_{\eta_0}(c(1+y)) = \sum_{i=0}^{l-1} \theta(c(1+\sigma^i y)) \sum_{j=1}^{l-1} I_j(\sigma^i y).
\]

If \( \alpha_1 = \cdots = \alpha_{(l-1)/2} = 0 \), \( f(\alpha_1, \ldots, \alpha_{l-1}; y) = 0 \). Thus we have

\[
\sum_{j=1}^{l-1} I_j(y) = q^{(l-1)/2}.
\]

For \( 1 \leq j \leq (l-1)/2 \), \( I_j(y) \) is proportional to

\[
\sum_{\alpha_{l-j} \in P_E^l/P_E^{l+1}} \psi_E((3\alpha_j \sigma^j \alpha_{l-j} - \sigma^{-j} \beta \alpha_{l-j} \sigma^{-j} \alpha_j) y).
\]

Since \( \alpha_j \neq 0 \), the map \( \alpha_{l-j} \mapsto \beta \alpha_j \sigma^j \alpha_{l-j} - \sigma^{-j} \beta \alpha_{l-j} \sigma^{-j} \alpha_j \) is a bijection from \( P_E^{m-1}/P_E^m \) to \( k_F \). Therefore \( I_j(y) = 0 \). Consequently we get the first part of our lemma. \( G(y, n-1) = q^{(l-1)/2} \) follows from Proposition 2.2 and the first part. Since \( G(y, j) \) depends only on \( y \) mod \( l \) an \( y \in k_F^\times \), \( G(y, j) = q^{(l-1)/2} \) for any \( j \). \( \square \)
Next we assume $q - 1 \not\equiv 0 \mod l$ and $n = 2m$. In this situation, it is rather difficult to describe $E^\times$-module structure of various objects since $F$ has no $l$-th primitive root of unity and $E/F$ is not Galois. In order to apply the result of the Galois case, we use the base change lift of simple characters by Bushnell-Henniart \[4\]. Let $\zeta$ be a primitive $l$-th root of unity and $L = F(\zeta)$. Then $L/F$ is an unramified extension of degree $d$, where $d$ is the smallest integer satisfying $q^d \equiv 1 \mod l$. The generator $\tau$ of $\Gal(L/F)$ is determined by $\tau^k = \zeta^k$, where $k = r(l-1)/d$ and $r$ is a generator of $(\mathbb{Z}/l\mathbb{Z})^\times$. We add the subscript $L$ to the base changed objects.

Then $\mathbb{M}_L(L) = \mathbb{M}_L(F) \otimes_F L$ and $L_E = E \otimes_F L \simeq EL$. $E_L$ is a ramified Galois extension over $L$ of degree $l$, an unramified extension over $E$ of degree $d$ with $\Gal(E_L/E) = \Gal(L/F) = \langle \tau \rangle$ and a non-Abelian Galois extension over $F$ of degree $ld$. (We embed $E$ into $E_L$ by the map $x \mapsto x \otimes 1$.)

As in the previous section, we identify $\mathbb{M}_L(L)$ with $\End_L E_L$ and $G_L = GL_l(L)$ with $\Aut_L E_L$ by the $L$-basis $\{\varpi_E^{-1}, \ldots, \varpi_E, 1\}$ of $E_L$, which is also an $\mathcal{O}_L$-basis of $\mathcal{O}_{E_L}$. By the lattice flag $\{P^j_{E_L}\}_{j \in \mathbb{Z}}$, we define

\[A_i^L = \{f \in \mathbb{M}_L(L) | f(P^j_{E_L}) \subset P^{j+i}_{E_L} \ \text{for all} \ \ j \in \mathbb{Z}\} \]

Put $K_L = (A_0^L)^\times, B_L = E_L^\times K_L, K^i_L = 1 + A_i^L$ for $i \geq 1$ and $H_L = L^\times (1 + P_{E_L})K_L^m$.

**Definition 3.6.** Let $\theta$ be a generic character of $E^\times$ with $f(\theta) = n$ and $\theta(1 + x) = \psi(\Tr_E(\beta x))$ for $x \in P^m_{E_L}$. We define a base change lift $\theta_L$ of $\theta$ to $L^\times$ by $\theta_L = \theta \otimes_{n_{E_L/E}}$. Then $\theta_L(1 + x) = \psi_L(\Tr_{E_L}(\beta x))$ for $x \in P^m_{E_L}$. (Recall $m = [(n+1)/2]$.) Let $H^1 = F^\times (1 + P_E)K^m \subset H$. The base change lift $\rho_L$ of $\rho|_{H^1}$ to $H^1_L = L^\times (1 + P_{E_L})K^m_L$ is defined by

\[\rho_L(h \cdot g) = \theta_L(h)\psi_L(\beta(g - 1)) \quad \text{for} \quad h \in L^\times (1 + P_{E_L}), \ g \in K^m_L.\]

Now we apply the result of Bushnell-Henniart \[4\] to our case and get the character relation between $\rho_L$ and $\rho$. Put $U_{E_L,i} = L^\times (1 + P_{E_L}^i)$ for $i > 0$ and $U_{E_L,i}^0 = U_{E_L,i} - U_{E_L,i+1}$. By (12.19) Corollary in \[4\] and the fact $\langle \tau \rangle$-fixed space $\langle \tau \rangle L^\times K^m_L$ is equal to $F^\times K^i$, we get the following result.

**Proposition 3.7.** Let $x \in U_{E_L,1}$. Between the set

\[\{g \in H^1 \setminus (E^\times K^m)^1 | gn_{E_L/E}(x)g^{-1} \in H^1\}\]

and the set

\[\{h \in H^1_L \setminus (E^\times L^\times K^m)^1 | hx^\tau h^{-1} \in H^1_L\},\]

there is a bijection $\psi$ with the property

\[\rho_L(\psi(g)x^\tau(\psi(g))^{-1}) = \rho(gn_{E_L/E}(x)g^{-1}).\]

Combining this with Lemma 3.1 we have:

**Lemma 3.8.** For $x \in U_{E_L,1}$,

\[\chi_{n_0}(n_{E_L/E}(x)) = \sum_{a \in H^1_L \setminus (E^\times L^\times K^m)^1} \rho_L(ax^\tau a^{-1}).\]

Since $n_{E_L/E}(L^\times (1 + P_{E_L}^1)) = F^\times (1 + P_E^1)$, it suffices to calculate the right-hand side of (3.3) for $x \in U_{E_L,1}^\ast$. 

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As in the Galois case, set \( \xi = \text{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{l-1}) \in M_l(L) \). Then \( \xi \) satisfies \( \xi^l = 1, \tau \xi = \xi^k \) and

\[
\xi x^{-1} = \sigma x \quad \text{for any} \quad x \in E_L,
\]

where \( \sigma \) is the generator of Gal\((E_L/L)\) determined by \( \sigma \varpi_E = \varpi_E \zeta \). Moreover we have \( \tau \sigma \tau^{-1} = \sigma^k \) and

\[
\begin{align*}
M_l(L) &= E_L \oplus E_L \xi \oplus \cdots \oplus E_L \xi^{l-1} \\
A^0_L &= O_E \oplus O_E \xi \oplus \cdots \oplus O_E \xi^{l-1} \\
A^1_L &= P_E \oplus P_E \xi \oplus \cdots \oplus P_E \xi^{l-1} \\
A^{-1}_L &= P_E^{-1} \oplus P_E^{-1} \xi \oplus \cdots \oplus P_E^{-1} \xi^{l-1}.
\end{align*}
\]

(3.4)

We note that any element of \( K^1_L \) can be written in the form \( (1 + \alpha_1 \xi + \alpha_2 \xi^2 + \cdots + \alpha_{l-1} \xi^{l-1}) \) for \( \alpha_i \in P_E \).

**Lemma 3.9.** Let \( i < m \) and \( a = 1 + \alpha_1 \xi + \alpha_2 \xi^2 + \cdots + \alpha_{l-1} \xi^{l-1} \) for \( \alpha_j \in O_E \) and \( x \in U_{E,i}^\times \). Then \( ax \tau^{-1} \in H_L \) is equivalent to \( \alpha_j \in P_{E,i}^m \) and \( \tau \alpha_j = \tau \alpha_h \) for \( j = 0, 1, \ldots, d - 1 \) and \( h = 1, r, \ldots, r^{(l-1)/d} - 1 \). (The subscript of \( \alpha_j \) is extended to \( \mathbb{Z} \) by \( \alpha_j = \alpha_j \mod l \).)

**Proof.** It follows from Lemma 3.2 that if \( ax \tau^{-1} \in H_L \), then there exist \( \gamma_0 \in O_E^\times \) and \( \gamma_j \in P_{E,i}^m \) for \( 1 \leq j \leq l - 1 \) such that

\[
(1 + \alpha_1 \xi + \cdots + \alpha_{l-1} \xi^{l-1})x = \gamma_0 (1 + \gamma_1 \xi + \gamma_2 \xi^2 + \cdots + \gamma_{l-1} \xi^{l-1})
\]

(1 + \tau \alpha_1 \xi + \tau \alpha_2 \xi^2 + \cdots + \tau \alpha_{l-1} \xi^{l-1}).

This implies

\[
x = \gamma_0 (1 + \gamma_1 \xi + \gamma_2 \xi^2 + \cdots + \gamma_{l-1} \xi^{l-1})
\]

\[
\alpha_k \tau \xi = \gamma_0 (\gamma_k + \tau \alpha_1 + \gamma_{l-k} \xi^{-1} \tau \alpha_2 + \cdots + \gamma_{2k} \xi^{-2k} \tau \alpha_{l-1})
\]

\[
\alpha_{l-k} \tau^{-1} = \gamma_0 (\gamma_{l-k} + \gamma_{l-k} \xi^{-1} \tau \alpha_1 + \cdots + \gamma_{l-k} \xi^{-2k} \tau \alpha_{l-1}).
\]

Thus we have

\[
\alpha_{hk} \tau \xi = x \tau \alpha_h \mod P_{E,i}^m \quad (h \in (\mathbb{Z}/l\mathbb{Z})^\times).
\]

By eliminating \( \alpha_{hk}, \alpha_{hk^2}, \ldots, \alpha_{hk^{l-1}} \), we get

\[
\alpha_h = n_{E,i}\tau \xi^{-1} \alpha_h \mod P_{E,i}^m.
\]

Since \( n_{E,i}\tau \xi^{-1} \in 1 + P_{E,i}^m + P_{E,i}^{m-1} \), and \( \alpha_k \in P_{E,i}^m \) by \( \tau \alpha_k = \tau \alpha_h \mod P_{E,i}^m \) for \( j = 0, 1, \ldots, d - 1 \) and \( h = 1, r, \ldots, r^{(l-1)/d} - 1 \), we obtain \( \alpha_k \in P_{E,i}^m \) and \( \alpha_{hk^i} = \tau \alpha_h \mod P_{E,i}^m \) for \( j = 0, 1, \ldots, d - 1 \) and \( h = 1, r, \ldots, r^{(l-1)/d} - 1 \).

**Lemma 3.10.** Assume \( n = 2m \) and \( m \geq 3 \). Let \( x \in 1 + P_{E,L} + P_{E,L}^2 \) and \( a = 1 + \sum_{i=1}^{(l-1)/d} \sum_{j=1}^d \tau \alpha_{i+1} \xi^{r \cdot k^j} \) for \( \alpha_i \in P_{E,i}^m \). Then

\[
\rho_L (ax \tau^{-1} x^{-1}) = \psi_E \left( \sum_{i=1}^{(l-1)/d} \text{tr}_{E,i}(u_i - \sigma^{-i} u_i) \text{tr}_{E,i}(x - 1) \right),
\]

where \( u_i = \gamma \alpha_i \tau, \sigma^{-i} \alpha_i \tau \).
Proof. It follows from Lemma 3.3 that \( \tau a a^{-1} \in H_L \). Since \( \rho_L(\tau a a^{-1}) = 1 \), we have
\[
\rho_L(ax^\tau a^{-1}g^{-1}) = \rho_L(axa^{-1}g^{-1}).
\]
By the same way as Lemma 3.3, we have
\[
\rho_L(ax^\tau a^{-1}x^{-1}) = \psi_{E_L} \left( \sum_{i=1}^{(l-1)/d} \sum_{j=1}^{d} (v_{i,j} - \sigma^{-r_i} v_{i,j}) (x-1) \right),
\]
where \( v_{i,j} = \beta^{\tau_i} \alpha_{r_i} \tau^i \alpha_{-r_i} \). Since \( \sigma^{-r_i} \tau^i = \tau^i \sigma^{-r_i} \) and \( \tau \beta = \beta \), we have
\[
\sum_{j=1}^{d} (v_{i,j} - \sigma^{-r_i} v_{i,j}) = \sum_{j=1}^{d} (\beta^{\tau_i} \alpha_{r_i} \tau^i \alpha_{-r_i} - \tau^i \sigma^{-r_i} \beta^{\tau_i} \sigma^{-r_i} \alpha_{r_i} \tau^i \alpha_{-r_i})
\]
\[
= \mathrm{tr}_{E_L/E}(\beta \alpha_{r_i} \tau^i \alpha_{-r_i} - \tau^i \beta \sigma^{-r_i} \alpha_{r_i} \alpha_{-r_i}).
\]
This implies (3.5). \( \square \)

It is time to get the character value of \( \chi_\eta \) on \( U_1^* \).

**Proposition 3.11.** Assume \( q \not\equiv 1 \mod l \).

1. Let \( x \in 1 + P_{E_L} - P_{E_L}^2 \) and \( n = 2m > 6 \). Then
\[
\chi_\eta(n_{E_L/E}(x)) = \left( \frac{q}{l} \right) q^{(l-1)/2} \theta(n_{E_L/E}(x)).
\]
2. For any integer \( j \not\equiv 1/2 \mod l \) and \( y \in \mathcal{O}_F^\times \), \( G(y,j) = \left( \frac{q}{l} \right) q^{(l-1)/2} \).

**Proof.** By Proposition 3.7, Lemmas 3.3, 3.9, 3.10 and 3.11, we have
\[
\chi_\eta(n_{E_L/E}(x)) = \theta_L(x) \sum_{(\alpha_{r_i})} \psi_E \left( \sum_{i=1}^{(l-1)/d} \mathrm{tr}_{E_L/E}(u_i - \sigma^{-r_i} u_i) \mathrm{tr}_{E_L/E}(x-1) \right),
\]
where \( u_i = \beta \alpha_{r_i} \tau^i \alpha_{-r_i} \) and \( (\alpha_{r_i})_{1 \leq i \leq (l-1)/d} \in (P_{E_L}^{m-1} / P_{E_L}^m)^{(l-1)/d} \). First we assume \((l - 1)/d\) is odd. Then \( \left( \frac{q}{l} \right) = -1 \), \( d \) is even and \( \tau^d \alpha_{r_i} = \alpha_{-r_i} \). Let \( E_i \) be the \((\sigma^r \tau^d/2)\)-fixed field. Then \( E_L/E_i \) is a quadratic unramified extension, \( \alpha_{r_i} \tau^d \alpha_{-r_i} = n_{E_i/E}(\alpha_{r_i}) \), \( n_{E_L/E}_i \) induces a surjection from \( \mathcal{O}_{E_L}^{m-2} \mathcal{O}_{E_i}/1 + P_{E_L} \) to \( \mathcal{O}_{E_i}^{m-2} \mathcal{O}_{E_i}/1 + P_{E_i} \) and each fiber of the induced map has \( q^{d/2 + 1} \) elements. Moreover the map \( x \mapsto \mathrm{tr}_{E_L/E_i}(x - \sigma^{-r_i} x) \) induces a surjective \( k_F \)-linear map from \( P_{E_i}^{m-2} / P_{E_i}^{m-1} \) to \( P_{E_i}^{-1} \mathcal{O}_{E_i} \). Thus we have
\[
\sum_{\alpha_{r_i} \in n_{E_i/E}^{-1} \mathcal{O}_{E_i} / P_{E_i}} \psi_E \left( \sum_{i=1}^{(l-1)/d} \mathrm{tr}_{E_L/E}(u_i - \sigma^{-r_i} u_i) \mathrm{tr}_{E_L/E}(x-1) \right) = 1 - (q^{d/2} + 1).
\]
Putting this into (3.6), we get
\[
\chi_\eta(n_{E_L/E}(x)) = \theta_L(x)(1 - (q^{d/2} + 1))^{(l-1)/d}
\]
\[
= -q^{(l-1)/2} \theta_L(x),
\]

and it follows from Proposition 2.2 that \( G(y, j) = -q^{(l-1)/2} \) for all \( y \in k_F \) and \( j \) odd. Now we assume \((l-1)/d \) is even. Then \((\frac{4}{l}) = 1 \) and it follows from the same argument as in the proof of Proposition 3.3 that
\[
\chi_{\eta}(n_{E_L}/E(x)) = \theta_L(x)|k_{E_L}|^{(l-1)/2d}
= q^{(l-1)/2}\theta_L(x).
\]

By Proposition 2.2 and the fact \( G(y, j) \) depends only on \( j \) mod \( l \), we have \( G(y, j) = q^{(l-1)/2} \) for all \( y \in k_F \) and \( j \).

Summing up the above results, we get the following formula for \( G(y, j) \).

**Proposition 3.12.** For \( y \in k_F^* \) and \( j \in (\mathbb{Z}/l\mathbb{Z}) \) such that \( j \neq 1/2 \),
\[
G(y, j) = \left(\frac{q}{l}\right) q^{(l-1)/2}.
\]

From Theorems 1.7, 2.7, Lemmas 2.6, 2.7 and Propositions 1.6, 2.2 and 3.12, we get the complete character table of \( \pi_\theta \).

**Theorem 3.13.** Let \( E \) be a ramified extension of \( F \) with degree \( l \neq p \), \( \theta \) a generic quasi-character of \( E^* \) with \( f(\theta) = n \) and \( \pi = \pi_\theta \) the irreducible supercuspidal representation of \( \text{GL}_l(F) \) defined in section 1. Put \( U_0 = F^* \mathcal{O}_E^* \), \( U_j = F^* (1 + P_E^j) \) and \( U_j^* = U_j - U_{j+1} \) for \( j \geq 1 \). Let \( x \) be an regular elliptic element of \( \text{GL}_n(F) \) and let \( \text{Aut}_F E \) be the group of automorphism of \( E \) over \( F \).

1. If \( F(x)/F \) is unramified, then

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \chi_\pi(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \notin F^*(1 + P_n_F(x)) )</td>
<td>0</td>
</tr>
<tr>
<td>( c(1 + y)(c \in F^*, y \in P_n_F(x)) )</td>
<td>( q^{(n-2)(l-1)/2} \frac{q^j - 1}{q-1} \theta(c) ).</td>
</tr>
</tbody>
</table>

2. If \( F(x)/F \) is ramified and \( F(x) \neq E \), then

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \chi_\pi(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \notin F^*(1 + P_n_F(x)) )</td>
<td>0</td>
</tr>
<tr>
<td>( c(1 + \alpha_{E_1}^{-1} \text{diag}(k_0, \ldots, k_{l-1}) + z) ) ( (c \in F^<em>, k_i \in k_F^</em>, z \in P_n_F(x)) )</td>
<td>( q^{(n-2)(l-1)/2} \theta(c) \text{Kl}(\beta \alpha_{E_1}^{-1}</td>
</tr>
<tr>
<td>( c(1 + y) ) ( (c \in F^*, y \in P_n_F(x)) )</td>
<td>( q^{(n-2)(l-1)/2} \frac{q^j - 1}{q-1} \theta(c). )</td>
</tr>
</tbody>
</table>

3. When \( x \in E \), then

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \chi_{\pi_\theta}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \in U_j^*(0 \leq j \leq n - 1) )</td>
<td>( \left(\frac{q}{l}\right) q^{n-j} \frac{q^{j(l-1)/2}}{\theta(\sigma_F)} )</td>
</tr>
<tr>
<td>( c(1 + \alpha_{E_1}^{-1} x_0) ) ( (c \in F^<em>, x_0 \in \mathcal{O}_E^</em>) )</td>
<td>( q^{(n-2)(l-1)/2} \frac{q^j - 1}{q-1} \theta(c) \text{Kl}(\beta \alpha_{E_1}^{-1} x_0) ).</td>
</tr>
<tr>
<td>( c(1 + y) ) ( (c \in F^<em>, y \in \mathcal{O}_E^</em>) )</td>
<td>( q^{(n-2)(l-1)/2} \frac{q^j - 1}{q-1} \theta(c) ).</td>
</tr>
</tbody>
</table>

(See 2.5 for the definition of the Kloosterman sum \( \text{Kl}(a) \).)

**Remark 3.14.** 1. Combining with the formula for the unramified case 2.3 (see Remark 3.9), we get the complete character table for all supercuspidal representations of \( \text{GL}_l(F) \) on regular elliptic conjugacy classes when \( p \neq l \).
2. For the case $p > l$, Debacker got the following character table for the supercuspidal representations of $GL_l(F)$ ([9], Lemma 15):

$$
\Theta_\pi(\gamma) = \begin{cases} 
C_0(t)\lambda(\sigma) \sum_{w \in W} \phi(w t) & \text{if } n(\gamma) = 0 \text{ and } \gamma \sim t \text{ with } t \in G', \\
C_0(t)\lambda(\sigma) \sum_{w \in W} \phi(w t)\gamma(X, \gamma) & \text{if } 0 < n(\gamma) < r \text{ and } \gamma \sim t, \text{ where } \\
\sum_{k \in G_x/M_0G_{x,0}} \phi(k t) & \text{if } n(\gamma) = r \text{ and } \gamma \sim t, \text{ where } \\
0 & \text{otherwise.}
\end{cases}
$$

(For the notations, see Appendix B.6.6 in [9].) In this paper, we determined $\gamma(X, \gamma)$ and

$$\sum_{k \in G_x/M_0G_{x,0}} \phi(k t)$$

explicitly.

REFERENCES


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