

THE LIMIT SETS OF SCHOTTKY QUASICONFORMAL GROUPS ARE UNIFORMLY PERFECT

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ABSTRACT. In this paper we study Schottky quasiconformal groups. We show that the limit sets of Schottky quasiconformal groups are uniformly perfect, and that the limit set of a given discrete non-elementary quasiconformal group has positive Hausdorff dimension.

1. INTRODUCTION

The notion of *uniform perfectness* first appeared in [10]. A closed subset A of the complex plane \mathbb{C} is said to be uniformly perfect if there exists a constant $c > 0$ such that

$$A \cap \{z : cr \leq |z - a| \leq r\} \neq \emptyset,$$

for any $a \in A$ and $0 < r < \text{diam}(A)$, where $\text{diam}(A)$ is the Euclidean diameter of the set A . Pommerenke proved in [10] that a closed set $A \subset \mathbb{C}$ is uniformly perfect if, and only if, there is a constant $c > 0$ such that

$$\text{cap}(A \cap \{|z - a| \leq r\}) \geq cr,$$

for all $a \in A$ and $0 < r < \text{diam}(A)$, where cap , the *logarithmic capacity* of a compact set E , is defined by

$$\text{cap}(E) = \lim_{m \rightarrow \infty} \left(\max_{z_1, \dots, z_m \in E} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^m \prod_{\nu=1}^m |z_\mu - z_\nu| \right)^{\frac{1}{m(m-1)}}.$$

The idea of uniform perfectness was first introduced in [1]. In that paper, Beardon and Pommerenke showed that the boundary of a domain $\Omega \subset \mathbb{C}$ not containing a neighborhood of infinity is uniformly perfect if, and only if, there exists a positive number c such that

$$\lambda_\Omega(z) \geq \frac{c}{\text{dist}(z, \partial\Omega)},$$

where $\lambda_\Omega(z)$ is the density of the *Poincaré metric* of Ω , and $\text{dist}(z, \partial\Omega)$ is the Euclidean distance of z to $\partial\Omega$. In the same paper, they also showed that the limit set $L(G)$ of a classical plane *Schottky group* G is uniformly perfect.

Mañé and da Rocha [8] and Hinkkanen [6] proved independently that the *Julia sets* of rational functions are uniformly perfect.

In [7] Järvi and Vuorinen investigated the uniform perfectness of compact sets of $\overline{\mathbb{R}^n}$. They gave several equivalent conditions for uniform perfectness, and they

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showed that uniformly perfect sets have positive Hausdorff dimension and that the limit set of a non-elementary, finitely generated Kleinian group of $\overline{\mathbb{R}^n}$ is uniformly perfect.

In their 1987 paper [5], Gehring and Martin first introduced the notion of convergence groups in $\overline{\mathbb{R}^n}$. They studied the limit sets of discrete convergence groups and studied discrete quasiconformal groups, which are convergence groups. Tukia, in [11], extended the notion of convergence groups to compact Hausdorff spaces satisfying the first axiom of countability and investigated convergence groups in that setting.

Petra Bonfert-Taylor and Edward C. Taylor in [2] and [3] extended the Patterson-Sullivan theory to the quasiconformal group setting. They investigated the connection between the exponent of convergence of Poincaré series and the Hausdorff dimension of the limit sets of quasiconformal groups. They showed that the Hausdorff dimension of the conical limit set $L_c(G)$ has an upper bound; see Corollary 1.6. and Conjecture 1.7. in [3].

Our main results are the following theorems.

1.1. Theorem. *For an integer t with $t \geq 2$, let $G = \langle g_1, g_2, \dots, g_t \rangle$ be a Schottky K -quasiconformal group. Then the limit set $L(G)$ is a uniformly perfect set.*

1.2. Theorem. *Let G be a discrete non-elementary quasiconformal group. Then the Hausdorff dimension of the limit set $L(G)$ is positive.*

This paragraph is about the organization of this paper. Section 2 contains some notations and basic definitions. We talk about *moduli* of curve families, and basic properties of K -quasiconformal mappings in \mathbb{R}^n , then we recall some results about convergence groups and their limit sets, which are needed in this paper. In section 3, the notion of a *Schottky quasiconformal group* is introduced, and we give representations of the limit set $L(G)$ and the regular set $\Omega(G)$. In section 4, we show that the limit set $L(G)$ of a Schottky quasiconformal group is uniformly perfect. In section 5, we generalize our theorem in section 4, and we show that if G is a discrete non-elementary quasiconformal group, then the Hausdorff dimension of the limit set $L(G)$ is positive.

2. PRELIMINARIES

Let \mathbb{R}^n be the n -dimensional Euclidean space and let $\overline{\mathbb{R}^n}$ be the one-point compactification of \mathbb{R}^n . We write e_1, e_2, \dots, e_n for the standard basis of \mathbb{R}^n , where $e_1 = (1, 0, \dots, 0)$, etc. For $x \in \mathbb{R}^n$, we define

$$|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

We write $B^n(x, r)$ for the open ball centered at x with radius r , and write $S^{n-1}(x, r) = \partial B^n(x, r)$ for the boundary of $B^n(x, r)$.

The stereographic projection $\pi : \overline{\mathbb{R}^n} \rightarrow S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ is given by

$$\pi(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2}.$$

Clearly, for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we have

$$\pi(x) = \left(\frac{x_1}{|x|^2 + 1}, \dots, \frac{x_n}{|x|^2 + 1}, \frac{|x|^2}{|x|^2 + 1} \right),$$

and we define $\pi(\infty) = e_{n+1}$. The function $\pi : \overline{\mathbb{R}}^n \rightarrow S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ is a bijective map. We define the *chordal metric* on $\overline{\mathbb{R}}^n$ by $q(x, y) = |\pi(x) - \pi(y)|$ for $x, y \in \overline{\mathbb{R}}^n$. For subsets $X, Y \subset \overline{\mathbb{R}}^n$, the *diameter* of X is defined as $q(X) = \sup_{x, y \in X} q(x, y)$ and the *distance* between X and Y is given by $q(X, Y) = \inf_{x \in X, y \in Y} q(x, y)$.

Let $B_q(x, r)$ denote the chordal ball

$$B_q(x, r) = \{y \in \overline{\mathbb{R}}^n : q(x, y) < r\}.$$

A *Möbius transformation* acting on $\overline{\mathbb{R}}^n$ is a finite composition of reflections in spheres and hyperplanes. We write $GM(\overline{\mathbb{R}}^n)$ for the group of all Möbius transformations. A *Möbius group* is a subgroup of $GM(\overline{\mathbb{R}}^n)$. Let D and D' be domains in $\overline{\mathbb{R}}^n$. A homeomorphism $f : D \rightarrow D'$ is a *conformal mapping* if $f \in C^1$, and

$$|f'(x)h| = \max_{|s|=1} |f'(x)s||h|$$

for any $x \in D$ and $h \in \mathbb{R}^n$. If D and D' are domains in $\overline{\mathbb{R}}^n$, then a homeomorphism $f : D \rightarrow D'$ is conformal if f is conformal in $D \setminus \{\infty, f^{-1}(\infty)\}$. Möbius transformations are conformal mappings. By Liouville's theorem, when $n \geq 3$, the only conformal mappings in $\overline{\mathbb{R}}^n$ are Möbius transformations.

Let Γ be a family of curves in $\overline{\mathbb{R}}^n$, and let $\mathcal{F}(\Gamma)$ be the set of *admissible functions*, i.e., non-negative Borel functions $\rho : \overline{\mathbb{R}}^n \rightarrow \mathbb{R} \cup \{\infty\}$ such that

$$\int_{\gamma} \rho ds \geq 1$$

for any locally rectifiable $\gamma \in \Gamma$. For $p \geq 1$ the *p-modulus* of Γ is defined as

$$M_p(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^p dm.$$

The *p-modulus* M_p is an outer measure in the space of all curve families in $\overline{\mathbb{R}}^n$. When $p = n$ we denote $M_p(\Gamma)$ by $M(\Gamma)$. The quantity $M(\Gamma)$ is conformally invariant, i.e., if $f : D \rightarrow D'$ is a conformal mapping and Γ is a curve family in D , then we have

$$M(f\Gamma) = M(\Gamma),$$

where $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$. From now on we denote $f\Gamma$ by Γ' .

2.1. Definition. Let $f : D \rightarrow D'$ be a homeomorphism. The inner and outer dilatations $K_I(f)$ and $K_O(f)$ of the mapping f are defined by

$$K_I(f) = \sup \frac{M(\Gamma')}{M(\Gamma)}, \quad K_O(f) = \sup \frac{M(\Gamma)}{M(\Gamma')},$$

where the suprema are taken over all curve families Γ in D such that $M(\Gamma)$ and $M(\Gamma')$ are not simultaneously zero or ∞ . The *maximal dilatation* of the mapping f is defined as $K(f) = \max\{K_I(f), K_O(f)\}$. The mapping f is said to be *K-quasiconformal* if $K(f) \leq K < \infty$, for some constant K . Equivalently, the mapping f is *K-quasiconformal* if, and only if,

$$\frac{M(\Gamma)}{K} \leq M(\Gamma') \leq KM(\Gamma)$$

for every curve family Γ in D .

Let f be K_1 -quasiconformal and let g be K_2 -quasiconformal. Then the inverse function f^{-1} is K_1 -quasiconformal, and $f \circ g$ is K_1K_2 -quasiconformal.

Let G be a group of self-homeomorphisms of a domain D . We say that the group G is a K -quasiconformal group if every element of G is a K -quasiconformal self-mapping of D . We say that G is a quasiconformal group if it is a K -quasiconformal group for some constant K . Möbius groups are 1-quasiconformal groups.

2.2. Definition (Gehring and Martin). A group G of self-homeomorphisms of $\overline{\mathbb{R}^n}$ is said to be a *convergence group* if each infinite subfamily of G has a sequence $\{g_i\}$ of distinct elements such that one of the following is true:

- (1) There exists a self-homeomorphism g of $\overline{\mathbb{R}^n}$ with the property that

$$\lim_{i \rightarrow \infty} g_i = g, \quad \lim_{i \rightarrow \infty} g_i^{-1} = g^{-1}$$

uniformly on $\overline{\mathbb{R}^n}$.

- (2) There exist x_0 and y_0 in $\overline{\mathbb{R}^n}$ (possibly $x_0 = y_0$) with the property that

$$\lim_{i \rightarrow \infty} g_i = y_0, \quad \lim_{i \rightarrow \infty} g_i^{-1} = x_0$$

uniformly on compact subsets of $\overline{\mathbb{R}^n} \setminus \{x_0\}$ and $\overline{\mathbb{R}^n} \setminus \{y_0\}$, respectively, or simply, c-uniformly in $\overline{\mathbb{R}^n} \setminus \{x_0\}$ and $\overline{\mathbb{R}^n} \setminus \{y_0\}$, respectively.

In the above definition, the points y_0 and x_0 are called the attracting and repelling points of $\{g_i\}$, respectively. Both points are called limit points of G . Also the group G is said to be *discrete* if (1) never happens, and the group G is said to be *precompact* if (1) always happens. By Väisälä [12], Gehring and Martin [5] quasiconformal groups are convergence groups, and so are Möbius groups.

Let G be a group of self-homeomorphisms of a domain D in $\overline{\mathbb{R}^n}$. The group G is said to be *discontinuous* at $x \in D$ if there is a neighborhood U of x in D such that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G$. We say that the group G is discontinuous if G is discontinuous at some point $x \in D$. The group G is said to be *properly discontinuous* in an open set $O \subset D$ if for each compact set $F \subset O$, we have $g(F) \cap F = \emptyset$ for all but finitely many $g \in G$. We say that the group G is *discrete* if there is no sequence of distinct elements $\{g_n\} \subset G$ such that $g_n \rightarrow g$ uniformly on compact subsets of D , or c-uniformly in D , where g is a self-homeomorphism of D . Let G be a convergence group of self-homeomorphisms of the domain D . The regular set of G , denoted by $\Omega(G)$, is given by $\Omega(G) = \{x \in D : G \text{ is discontinuous at } x\}$, and the limit set $L(G)$ is defined by $L(G) = \overline{D} \setminus \Omega(G)$. Let X be a subset of $\overline{\mathbb{R}^n}$. We say that the set X is G -invariant if $f(X) = X$ for every $f \in G$. If the number of elements of $L(G)$ is less than or equal to two, we say that G is an *elementary convergence group*. Otherwise G is called a *non-elementary convergence group*.

2.3. Theorem (Gehring and Martin). *Let G be a convergence group in $\overline{\mathbb{R}^n}$. Let $\Omega(G)$ and $L(G)$ be the regular set and the limit set of G , respectively. The following statements hold:*

- (1) *The set $\Omega(G)$ is open, and the set $L(G)$ is closed.*
- (2) *The intersection $\Omega(G) \cap L(G)$ is the empty set.*
- (3) *The union $\Omega(G) \cup L(G)$ is $\overline{\mathbb{R}^n}$.*
- (4) *Both $\Omega(G)$ and $L(G)$ are G -invariant.*

Let G be a convergence group. By Gehring and Martin [5], and Tukia [11], an element $g \in G$ is *elliptic* if the group generated by g is precompact. For a non-elliptic element $g \in G$ we say that g is *parabolic* if the number of the fixed points of g is one, and g is *loxodromic* if the number of the fixed points of g is two. A non-elliptic element $g \in G$ can fix at most two points.

2.4. Theorem (Tukia). *If G is a non-elementary convergence group, then the limit set $L(G)$ is an infinite perfect set, and $L(G)$ is in the accumulation set of any orbit $Gx = \{f(x) : f \in G\}$, for $x \in D$, and thus if $x \in L(G)$, we have*

$$\overline{Gx} = L(G).$$

See also [5].

Let E, F, D be subsets of $\overline{\mathbb{R}^n}$. We use $\Delta(E, F; D)$ to denote the family of all closed paths joining E and F in D ; see [12] and [13]. Let $D \subset \overline{\mathbb{R}^n}$ be a domain and let E be a compact subset of D . Then we call the pair (D, E) a *condenser* and the capacity of the condenser is $\text{cap}(D, E) = M(\Delta(E, \partial D; D))$. A ring domain D is a domain such that the complement of D has exactly two components E and F . In this case, we write $D = R(E, F)$. The modulus of D is given by

$$\text{mod}(D) = \left(\frac{M(\Delta(E, F; \overline{\mathbb{R}^n}))}{\omega_{n-1}} \right)^{\frac{1}{1-n}},$$

where $\omega_{n-1} > 0$ is the $(n-1)$ -dimensional measure of S^{n-1} .

The proof of the following lemma can be found in [13].

2.5. Lemma. *For two numbers r and s with $0 < r < s < 1$ and $x \in \overline{\mathbb{R}^n}$, the domain $D = B_q^n(x, s) \setminus \overline{B}_q^n(x, r)$ is called a chordal ring. Then the modulus of the chordal ring D is given by*

$$\text{mod}(D) = \log \left(\frac{s\sqrt{1-r^2}}{r\sqrt{1-s^2}} \right).$$

A ring domain D is called a *Teichmüller ring* if the complementary components of D are $[-e_1, 0]$ and $[se_1, \infty]$, where $s > 0$. We use $\tau_n(s)$ to denote the modulus of the family of all curves connecting the two complementary components of D , and we also briefly write $\tau(s)$ instead of $\tau_n(s)$. The function $\tau(s)$ is decreasing and

$$\lim_{s \rightarrow 0^+} \tau(s) = \infty, \quad \lim_{s \rightarrow \infty} \tau(s) = 0.$$

Let X be a subset of $\overline{\mathbb{R}^n}$ and let $D = R(E, F)$ be a ring domain. Then we say that D *separates* X if $X \cap D = \emptyset$, $X \cap E \neq \emptyset$ and $X \cap F \neq \emptyset$. Let X and Y be two subsets of $\overline{\mathbb{R}^n}$. We say that the ring domain $D = R(E, F)$ separates the sets X and Y if we have $X \subset E$, $Y \subset F$ or $Y \subset E$, $X \subset F$.

2.6. Definition. Let $X \subset \overline{\mathbb{R}^n}$ be a closed subset containing at least two points. Then we say that X is a *uniformly perfect set* if there exists a constant C such that for any ring domain $D \subset \overline{\mathbb{R}^n}$ separating X , the modulus of D is bounded above by the constant C .

The following two lemmas are due to Järvi and Vuorinen; see [7].

2.7. Lemma. *Let X be a closed subset of $\overline{\mathbb{R}^n}$ containing at least two points. Then X is uniformly perfect if, and only if, there is a constant C such that the modulus of any chordal ring domain separating X is bounded above by the constant C .*

2.8. Lemma. Let $D = R(E, F)$ be a ring domain in $\overline{\mathbb{R}^n}$. Then the following inequality is true:

$$\text{cap}(D) \geq 2^{1-n} \tau \left(\frac{2q(E, F)}{\min\{q(E), q(F)\}} \right),$$

where τ is the Teichmüller ring capacity.

3. SCHOTTKY QUASICONFORMAL GROUPS

For $t \geq 2$ let $B_1, B_2, \dots, B_{2t-1}$ and B_{2t} be t pairs of open balls with disjoint closures in $\overline{\mathbb{R}^n}$. Set $C_1 = \partial B_1, C_2 = \partial B_2, \dots, C_{2t-1} = \partial B_{2t-1}$ and $C_{2t} = \partial B_{2t}$. The classical Schottky group is a Möbius group Γ such that Γ is finitely generated by t Möbius transformations g_1, g_2, \dots, g_t of $\overline{\mathbb{R}^n}$ with the property that

$$g_i(B_{2i-1}) = \overline{\mathbb{R}^n} \setminus \overline{B_{2i}}$$

for all i with $1 \leq i \leq t$.

3.1. Theorem. Let Γ be a classical Schottky group defined as above. Then Γ is a free group and every element of Γ is loxodromic. In addition, the group Γ is a discrete Möbius group.

See [4] and [9].

Discrete Möbius groups are also called *Kleinian groups*. In the rest of this section, we assume that $B_1, B_2, \dots, B_{2t-1}$ and B_{2t} are open *quasiballs* with disjoint closures, i.e., for each i with $1 \leq i \leq 2t$, the set B_i is the image of a ball in $\overline{\mathbb{R}^n}$ under some quasiconformal mapping of $\overline{\mathbb{R}^n}$. For $1 \leq i \leq 2t$, we set $C_i = \partial B_i$.

3.2. Definition. A group $G = \langle g_1, g_2, \dots, g_t \rangle$ is called a *Schottky quasiconformal group* acting on $\overline{\mathbb{R}^n}$ if G is K -quasiconformal group on $\overline{\mathbb{R}^n}$ for some $K \geq 1$, and

$$g_i(B_{2i-1}) = \overline{\mathbb{R}^n} \setminus \overline{B_{2i}}$$

for all i between 1 and t .

We give an example of a Schottky quasiconformal group. Let Γ be a classical Schottky group. Let g be a K -quasiconformal mapping on $\overline{\mathbb{R}^n}$. Then the conjugate $g \circ \Gamma \circ g^{-1}$ of Γ under g is a Schottky K^2 -quasiconformal group.

Notation. Set $A_1 = \bigcup_{i=1}^{2t} \overline{B}_i$, $D_1 = \overline{\mathbb{R}^n} \setminus A_1$. We use Σ to denote D_1 throughout the remainder of this paper. Clearly the set Σ is open.

3.3. Lemma. The Schottky quasiconformal group $G = \langle g_1, g_2, \dots, g_t \rangle$ is a discontinuous group, and hence it is a discrete group.

Notation. Consider B_1 ; there are $2t - 1$ smaller quasiballs inside B_1 , which are images of $B_1, B_3, B_4, \dots, B_{2t}$ under the K -quasiconformal mapping g_1^{-1} . Similarly, for each of the remaining B_i , there are $2t - 1$ smaller quasiballs inside the B_i which are the images of certain $2t - 1$ original quasiballs under g_j or g_j^{-1} for some j between 1 and t . Clearly, there are a total of $2t(2t - 1)$ such smaller quasiballs, called the second generation quasiballs of the group G . For i with $1 \leq i \leq 2t(2t - 1)$, we denote these quasiballs by B_i^2 , and set

$$A_2 = \bigcup_{i=1}^{2t(2t-1)} \overline{B}_i^2.$$

The complement of A_2 is denoted by

$$D_2 = \overline{\mathbb{R}^n} \setminus A_2.$$

The set A_2 is closed and D_2 is open.

Similarly, we can define A_p as the closure of the union of all p -th generation open quasiballs, and D_p is the complement of A_p . The set A_p is closed and D_p is open. Also we have

$$A_1 \supset A_2 \supset \dots \supset A_p \supset \dots$$

and

$$\Sigma \subset D_2 \subset \dots \subset D_p \subset \dots.$$

3.4. Definition. Let $G = \langle g_1, g_2, \dots, g_t \rangle$ be a Schottky quasiconformal group. We know that G is a free group. For any non-identity $g \in G$, g can be written uniquely as

$$g = \prod_{i=1}^q g_{t_i}^{s_i},$$

where $g_{t_i} \in \{g_1, g_2, \dots, g_t\}$ and $g_{t_i} \neq g_{t_{i+1}}$ for $1 \leq i \leq q - 1$, each s_i is an integer with $s_i \neq 0$ for $1 \leq i \leq q$, for some positive integer q . We define the index of g by

$$\text{ind}(g) = \sum_{i=1}^q |s_i|,$$

for $g \neq I$. We define $\text{ind}(I) = 0$.

3.5. Lemma. Let G be a Schottky quasiconformal group. For $m \geq 1$, the sets A_m and D_m are defined as before. Then we have the following:

$$(1) \quad D_m = \left(\bigcup_{\text{ind}(g) \leq m-1} g(\overline{\Sigma}) \right) \cup \left(\bigcup_{\text{ind}(g)=m} g(\Sigma) \right),$$

and the complement of D_m is given by

$$(2) \quad A_m = \bigcup \{g(\overline{B}_i) : \text{ind}(g) = m, 1 \leq i \leq 2t, g(\overline{B}_i) \subset A_1\},$$

where the set $A_1 = \bigcup_{i=1}^{2t} \overline{B}_i$.

In addition, if two mappings f and g are in the group G , and $f \neq g$, then

$$f(\Sigma) \cap g(\Sigma) = \emptyset.$$

If two mappings f and g are in the group G with $f \neq g$, and $\text{ind}(f) = \text{ind}(g)$, then

$$f(\overline{B}_i) \cap g(\overline{B}_i) = \emptyset,$$

when $f(\overline{B}_i)$ and $g(\overline{B}_i)$ are both inside the set A_1 .

Proof. It is easy to check this by induction. We omit the proof. □

3.6. Lemma. Let $G = \langle g_1, g_2, \dots, g_t \rangle$ be the Schottky quasiconformal group defined as above, and for any integer $p \geq 1$, let A_p be the closure of the union of all the p -th generation open quasiballs. We define

$$d_p = \max_{E \subset A_p} q(E),$$

where the maximum is taken over all components E of A_p . Then the limit of the sequence $\{d_p\}$ exists, and in fact, the limit is zero.

Proof. Clearly $\{d_p\}$ is a decreasing sequence and so it has a limit d . We suppose that $d > 0$. By the Cantor diagonal process, we can choose a sequence of sets $\{\Delta_p\}$ with Δ_p a component of A_p such that the following hold:

$$\Delta_i \supset \Delta_{i+1}$$

for i with $1 \leq i < \infty$, and

$$\lim_{p \rightarrow \infty} q(\Delta_p) = d.$$

For each set Δ_p with $1 \leq p < \infty$, there exists a quasiconformal mapping $f_p \in G$ such that

$$\Delta_p = f_p(\overline{\mathbb{R}^n} \setminus B_{j_p})$$

for some j_p with $1 \leq j_p \leq 2t$. Since there are only a finite number of B_{j_p} and there are infinitely many $f_p \in G$, there exists at least one B_{j_p} , denoted by B'_0 , such that

$$(3) \quad \Delta_p = f_p(\overline{\mathbb{R}^n} \setminus B'_0)$$

for infinitely many $f_p \in G$. Without loss of generality, we assume that (3) is true for all p with $1 \leq p < \infty$.

On the other hand, the group G is a discrete convergence group. For the sequence $\{f_p\}$ of distinct elements of G , there exists a subsequence of $\{f_p\}$ which is still denoted by $\{f_p\}$ without danger of confusion, and there exist two points $x'_0, y'_0 \in \overline{\mathbb{R}^n}$ such that

$$\lim_{p \rightarrow \infty} f_p = y'_0, \quad \lim_{p \rightarrow \infty} f_p^{-1} = x'_0$$

uniformly on compact subsets of $\overline{\mathbb{R}^n} \setminus \{x'_0\}$ and $\overline{\mathbb{R}^n} \setminus \{y'_0\}$, respectively. But this contradicts (3), and we complete our proof for this lemma. \square

3.7. Theorem. *Let $G = \langle g_1, g_2, \dots, g_t \rangle$ be a Schottky quasiconformal group. For each i with $i \geq 1$, let A_i be the closure of the union of all the i -th generation open quasiballs and let D_i be the complement of A_i with respect to $\overline{\mathbb{R}^n}$. Then the limit set $L(G)$ is given by*

$$(4) \quad L(G) = \bigcap_{i=1}^{\infty} A_i,$$

and the regular set $\Omega(G)$ is given by

$$(5) \quad \Omega(G) = \bigcup_{i=1}^{\infty} D_i.$$

The proof is easy and we omit it.

4. UNIFORM PERFECTNESS OF $L(G)$

In this section, we show that the limit set of a Schottky quasiconformal group is uniformly perfect, and by a result of Järvi and Vuorinen [7] the limit set then has positive Hausdorff dimension.

In the following, we denote by $U(x, r, R)$ a chordal ring centered at x with radii r and R .

Proof of Theorem 1.1. We suppose that the limit set $L(G)$ is not uniformly perfect. Then there exists a sequence of distinct chordal rings $U_m(x_m, r_m, R_m)$ separating $L(G)$ with $0 < r_m < R_m < 1$, such that

$$(6) \quad \text{mod}(U_m(x_m, r_m, R_m)) \rightarrow \infty.$$

For simplicity, we write $U_m = U_m(x_m, r_m, R_m)$ and $V_m = B_q(x_m, r_m)$, where $B_q(x_m, r_m)$ is a chordal ball centered at x_m with radius r_m . Without loss of generality, we may assume that $x_m \rightarrow x_0$ for some $x_0 \in \overline{\mathbb{R}^n}$. Since $L(G)$ is perfect, we have

$$(7) \quad \lim_{m \rightarrow \infty} r_m = \lim_{m \rightarrow \infty} R_m = 0.$$

By Theorem 3.7, $L(G) = \bigcap_{m=1}^{\infty} A_m$, where A_m is the closure of the union of all the m -th generation quasiballs. Let m_0 be a fixed positive integer. Then there is one component of A_{m_0} which contains x_0 , and we denote by Δ'_{m_0} the interior of that component. Since $R_m \rightarrow 0$ there exists a positive integer M_1 such that when $m \geq M_1$, we have $U_m \subset \Delta'_{m_0}$. Pick a fixed integer m_1 with $m_1 \geq M_1$; then we have

$$U_{m_1} \subset \Delta'_{m_0}.$$

Note that the chordal ring domain U_{m_1} separates $L(G)$. Thus there exists a positive integer $M_2 > m_0$ such that whenever $m \geq M_2$, the set U_{m_1} separates the set A_m .

Pick an integer m_2 with $m_2 \geq M_2$; then U_{m_1} separates A_{m_2} . In particular, then $U_{m_1} \cap A_{m_2} = \emptyset$. Clearly, we have $m_2 > m_0$.

Now consider the set $\Delta'_{m_0} \cap L(G)$. If $\Delta'_{m_0} \cap L(G) \subset V_{m_1}$, where V_{m_1} is the chordal ball centered at x_{m_1} with radius r_{m_1} , then there exists a K -quasiconformal mapping $f \in G$ such that

$$f^{-1}(\Delta'_{m_0} \setminus A_{m_0+1}) = \Sigma,$$

and therefore the ring domain $f^{-1}(U_{m_1})$ separates some component B_j of the interior of A_1 and $(A_1 \setminus \overline{B_j}) \cap L(G)$. We write $f^{-1}(U_{m_1}) = R(E, F)$, where $E \supset B_j$ and $F \supset (A_1 \setminus \overline{B_j}) \cap L(G)$. Then we have

$$(8) \quad \min\{q(E), q(F)\} \geq \min_{1 \leq i \leq 2t} \{q(B_i \cap L(G))\} > 0.$$

Note also that

$$(9) \quad q(E, F) \leq \max_{1 \leq j \leq 2t} \{q(B_j, (A_1 \setminus \overline{B_j}) \cap L(G))\} < \infty.$$

Thus we have

$$(10) \quad \text{cap}(U_{m_1}) \geq \frac{2^{1-n}}{K} \tau \left(\frac{2 \max_{1 \leq j \leq 2t} \{q(B_j, (A_1 \setminus \overline{B_j}) \cap L(G))\}}{\min_{1 \leq i \leq 2t} \{q(B_i \cap L(G))\}} \right) > 0,$$

and this is a contradiction.

From now on we suppose that the chordal ring domain U_{m_1} separates $\Delta'_{m_0} \cap L(G)$.

First for i with $m_0 \leq i \leq m_2 - 1$ let us consider the set A_i . There are finitely many components of the set $\Delta'_{m_0} \cap A_i$. Now we need to use some ad hoc definitions. For each component Δ of the interior of $\Delta'_{m_0} \cap A_i$, we call $\Delta \cap L(G)$ an *effective set* of $\Delta'_{m_0} \cap A_i$.

We define, for i with $m_0 \leq i \leq m_2 - 1$, that

$$\Theta_i = \{\Delta \cap L(G) : \Delta \text{ is a component of the interior of } \Delta'_{m_0} \cap A_i\},$$

i.e., Θ_i is the set of all the effective sets of $\Delta'_{m_0} \cap A_i$. Clearly the number of the elements of the set Θ_{m_0+1} is given by

$$\#\Theta_{m_0+1} = 2t - 1.$$

Here and later we use the symbol $\#$ to denote the number of elements in a set.

In general, for i between $m_0 + 1$ and $m_2 - 1$, the number of the elements of Θ_i is given by

$$(11) \quad \#\Theta_i = (2t - 1)^{i-m_0},$$

and clearly, $\#\Theta_{m_0} = 1$.

For a given effective set $E_1 \in \Theta_i$, an effective set $F_1 \in \Theta_i$ is said to be a *close relative set* of E_1 if F_1 and E_1 are inside the same element of the set Θ_{i-1} as point sets.

In the following, we proceed in two steps.

(1) Let us pick an effective set E_1 from Θ_{m_2-1} , and consider all the close relative sets of E_1 . We denote by E'_1 the set of all the close relative sets of E_1 , together with the set E_1 .

If there exist two effective sets $\Delta_1 \cap L(G)$ and $\Delta_2 \cap L(G)$ in the set E'_1 such that the chordal ring domain U_{m_1} separates the two sets $\Delta_1 \cap L(G)$ and $\Delta_2 \cap L(G)$, where Δ_1 and Δ_2 are two components of the interior of A_{m_2-1} which are inside Δ' , one component of the interior of A_{m_2-2} , then there exists a K -quasiconformal mapping $f \in G$ with

$$(12) \quad f^{-1}(\Delta' \setminus A_{m_2-1}) = \Sigma.$$

Note that the ring domain $f^{-1}(U_{m_1})$ separates two sets $B_i \cap L(G)$ and $B_j \cap L(G)$, where B_i and B_j are two distinct components of the set $\overline{\mathbb{R}^n} \setminus \overline{\Sigma}$. We write $f^{-1}(U_{m_1}) = R(E, F)$. Therefore we have

$$\min\{q(E), q(F)\} \geq \min_{1 \leq i \leq 2t} \{q(B_i \cap L(G))\} > 0.$$

In addition, the following is true:

$$q(E, F) \leq \max_{1 \leq i, j \leq 2t} \{q(B_i \cap L(G), B_j \cap L(G))\} < \infty$$

and therefore,

$$(13) \quad \text{cap}(U_{m_1}) \geq \frac{2^{1-n}}{K} \tau \left(\frac{2 \max_{1 \leq i, j \leq 2t} \{q(B_i \cap L(G), B_j \cap L(G))\}}{\min_{1 \leq i \leq 2t} \{q(B_i \cap L(G))\}} \right) > 0,$$

but this is a contradiction since m_1 is arbitrarily chosen.

If there are no two effective sets in E'_1 which are separated by the chordal ring domain U_{m_1} , we consider all the remaining effective sets in Θ_{m_2-1} , and repeat the above process.

Now let us go through the second step.

(2) We assume that there are no two effective sets in Θ_{m_2-1} , which are close relative sets to each other, and which are separated by the chordal ring domain U_{m_1} . We consider all the effective sets in Θ_{m_2-2} and repeat step (1). If there are no two effective sets in Θ_{m_2-2} , which are close relative sets to each other, and which are separated by the chordal ring domain U_{m_1} , we consider all the effective sets in Θ_{m_2-3} , and so on. Since the chordal ring domain U_{m_1} separates $\Delta'_{m_0} \cap L(G)$, after a finite number of repeating steps we will come to the situation in the first step.

There are two effective sets of some set Θ_j for $m_0 \leq j \leq m_2 - 1$ which are close relative sets to each other such that these two effective sets are separated by the chordal ring domain U_{m_1} . The remaining proof is exactly the same as for step (1). This shows that there are always contradictions with the assumption that the limit set $L(G)$ is not uniformly perfect. This completes the proof of Theorem 1.1. \square

5. SCHOTTKY-TYPE QUASICONFORMAL GROUPS

In this section, we deal with Schottky-type quasiconformal groups. We show that the limit set of a given Schottky-type quasiconformal group is uniformly perfect, and this generalizes our main result in the last section. Then we show that if a given discrete quasiconformal group contains two loxodromic elements with disjoint fixed point sets, then the limit set of the quasiconformal group has positive Hausdorff dimension. Further we prove that discrete non-elementary quasiconformal groups have limit sets with positive Hausdorff dimension.

For $t \geq 2$ let $D_1, D_2, \dots, D_{2t-1}, D_{2t}$ be $2t$ continua with non-empty interiors and disjoint closures. We suppose that there are t quasiconformal mappings g_1, g_2, \dots, g_t such that

$$g_i(\mathring{D}_{2i-1}) = \overline{\mathbb{R}^n} \setminus \overline{D}_{2i},$$

where \mathring{D}_{2i-1} is the interior of the set D_{2i-1} .

Let $G = \langle g_1, g_2, \dots, g_t \rangle$ be the group generated by the quasiconformal mappings g_1, g_2, \dots, g_t . We say that the group G is a Schottky-type quasiconformal group if G is a quasiconformal group.

5.1. Lemma. *Let $G = \langle g_1, g_2, \dots, g_t \rangle$ be a Schottky-type quasiconformal group. Then the group G is a discontinuous group, and hence it is a discrete group.*

5.2. Lemma. *Let $G = \langle g_1, g_2, \dots, g_t \rangle$ be a Schottky-type quasiconformal group. Then the limit set $L(G)$ of the group G is a perfect set.*

Proof. The group G is a quasiconformal group, and thus it is a convergence group. By Lemma 5.1 the group G is a discrete convergence group. Also note that the limit set $L(G)$ contains more than three points. Therefore the limit set $L(G)$ is a perfect set. \square

Let A_1 be the closure of the union of all the sets $D_1, D_2, \dots, D_{2t-1}$ and D_{2t} . First let us consider the mappings g_1 and g_1^{-1} . The images of $D_2, D_3, \dots, D_{2t-1}$ and D_{2t} under the mapping g_1 are all inside the set D_2 . The images of all the sets $D_1, D_3, \dots, D_{2t-1}$ and D_{2t} under g_1^{-1} are all inside the set D_1 . Similarly, for any i with $2 \leq i \leq t$, we consider the mappings g_i and g_i^{-1} . The images of $D_1, D_2, \dots, D_{2t-1}, D_{2t}$ under the mapping g_i are all inside the set D_{2i} , and the images of $D_1, D_2, \dots, D_{2i-1}, D_{2i+1}, \dots, D_{2t-1}, D_{2t}$ under the mapping g_i^{-1} are all inside the set D_{2i-1} . We call all these image sets the second level sets of the group G . We denote by A_2 the closure of the union of all the second level sets of G . In general, we can define A_p to be the closure of the union of all the p -th level sets of the group G .

For $1 \leq p < \infty$, set

$$D_p = \overline{\mathbb{R}^n} \setminus A_p,$$

the complement of A_p with respect to $\overline{\mathbb{R}^n}$. Clearly A_p is closed and D_p is open. In addition,

$$\begin{aligned} A_1 \supset A_2 \supset \dots \supset A_p \supset \dots, \\ D_1 \subset D_2 \subset \dots \subset D_p \subset \dots. \end{aligned}$$

5.3. Lemma. For $1 \leq p < \infty$, let r_p be the maximum of the diameters of all the components of A_p . Then the limit of the sequence $\{r_p\}$ exists, and

$$\lim_{p \rightarrow \infty} r_p = 0.$$

5.4. Lemma. Let $G = \langle g_1, g_2, \dots, g_t \rangle$ be a Schottky-type quasiconformal group of $\overline{\mathbb{R}^n}$. Then the limit set $L(G)$ of the group G is given by

$$L(G) = \bigcap_{p=1}^{\infty} A_p.$$

5.5. Theorem. Let $G = \langle g_1, g_2, \dots, g_t \rangle$ be a Schottky-type quasiconformal group of $\overline{\mathbb{R}^n}$. Then the limit set $L(G)$ of the group G is uniformly perfect.

The proofs for the above two lemmas and the last theorem are exactly the same as for the Schottky quasiconformal group case. We omit the proofs for the lemmas and the theorem.

Next we consider the general discrete quasiconformal groups and prove the following theorem.

5.6. Theorem. Let G be a discrete quasiconformal group. If there are two loxodromic elements g_1 and g_2 in the group G such that

$$\text{fix}(g_1) \cap \text{fix}(g_2) = \emptyset,$$

where $\text{fix}(g_1)$ and $\text{fix}(g_2)$ are fixed point sets of g_1 and g_2 respectively, then the Hausdorff dimension $\dim_H L(G)$ of the limit set $L(G)$ is positive.

Proof. Without loss of generality, we suppose that x_1 and y_1 are the repelling point and the attracting point of the loxodromic element g_1 , respectively. Similarly, we use x_2 and y_2 to denote the repelling and attracting points of the mapping g_2 , respectively. The four points x_1, y_1, x_2, y_2 are all distinct by assumption.

Now let U_1 and U_2 be neighborhoods of x_1 and x_2 respectively, and let V_1 and V_2 be neighborhoods of y_1 and y_2 respectively, and we can pick U_1, U_2, V_1, V_2 so that they are balls and have disjoint closures.

Note that

$$\lim_{i \rightarrow \infty} g_1^i = y_1, \quad \lim_{i \rightarrow \infty} g_1^{-i} = x_1$$

uniformly on compact subsets of $\overline{\mathbb{R}^n} \setminus \{x_1\}$ and $\overline{\mathbb{R}^n} \setminus \{y_1\}$ respectively, and

$$\lim_{i \rightarrow \infty} g_2^i = y_2, \quad \lim_{i \rightarrow \infty} g_2^{-i} = x_2$$

uniformly on compact subsets of $\overline{\mathbb{R}^n} \setminus \{x_2\}$ and $\overline{\mathbb{R}^n} \setminus \{y_2\}$ respectively. Thus we can choose an integer $k_1 \geq 1$ such that

$$g_1^{k_1}(\overline{\mathbb{R}^n} \setminus U_1) \subset V_1,$$

and similarly, we can pick an integer $k_2 \geq 1$ such that

$$g_2^{k_2}(\overline{\mathbb{R}^n} \setminus U_2) \subset V_2.$$

Let $G_1 = \langle g_1^{k_1}, g_2^{k_2} \rangle$ be the subgroup of G generated by the elements $g_1^{k_1}$ and $g_2^{k_2}$. It is clear that the group G_1 is a Schottky-type quasiconformal group, and hence the limit set $L(G_1)$ is a uniformly perfect set. The Hausdorff dimension of the limit set $L(G_1)$ is positive. On the other hand,

$$L(G) \supset L(G_1)$$

and therefore we have

$$\dim_H L(G) \geq \dim_H L(G_1) > 0.$$

This completes our proof. \square

Proof of Theorem 1.2. By the Corollary 6.15 in [5], the group G contains infinitely many loxodromic elements, no two of which have a common fixed point. We pick two loxodromic elements g_1 and g_2 of the group G such that

$$\text{fix}(g_1) \cap \text{fix}(g_2) = \emptyset.$$

By Theorem 5.6 we obtain

$$\dim_H L(G) > 0,$$

and this finishes our proof. \square

5.7. Corollary. *Let G be a discrete convergence group. If G contains two quasiconformal mappings which generate a non-elementary quasiconformal group, then the Hausdorff dimension of the limit set $L(G)$ is positive.*

5.8. Corollary. *Let G be a discrete non-elementary quasiconformal group. Then any closed non-empty G -invariant subset of $\overline{\mathbb{R}^n}$ has positive Hausdorff dimension.*

Proof. Let E be a closed non-empty G -invariant subset of $\overline{\mathbb{R}^n}$ and let $L(G)$ be the limit set of the group G . Then we have

$$E \supset L(G),$$

since $L(G)$ is the smallest closed non-empty G -invariant subset of $\overline{\mathbb{R}^n}$. Hence we have

$$\dim_H E \geq \dim_H L(G) > 0,$$

and this completes our proof. \square

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REFERENCES

- [1] A. F. Beardon and Ch. Pommerenke, *The Poincaré metric of plane domains*, J. London Math. Soc. (2) **18** (1978), 475-483. MR0518232 (80a:30020)
- [2] Petra Bonfert-Taylor and Edward C. Taylor, *Hausdorff dimension and limit sets of quasiconformal groups*, Mich. Math. J. **49** (2001), 243-257. MR1852301 (2002g:30018)
- [3] Petra Bonfert-Taylor and Edward C. Taylor, *The exponent of convergence and a theorem of Astala*, Indiana Univ. Math. J. **51** (2002), 607-623. MR1911047 (2003c:30014)
- [4] Lester R. Ford, *Automorphic functions*, Chelsea Publishing Company, New York, 1972.
- [5] F. W. Gehring and G. J. Martin, *Discrete quasiconformal groups I*, Proc. London Math. Soc. (3) **55** (1987), 331-358. MR0896224 (88m:30057)
- [6] Aimo Hinkkanen, *Julia Sets of Rational Functions are Uniformly Perfect*, Math. Proc. Camb. Phil. Soc. **113** (1993), 543-559. MR1207519 (94b:58084)

- [7] P. Järvi and M. Vuorinen, *Uniformly Perfect Sets and Quasiregular Mappings*, J. London Math. Soc. (2) **54** (1996), 515-529. MR1413895 (98d:30031)
- [8] R. Mañé and L. F. da Rocha, *Julia sets are uniformly perfect*, Proc. Amer. Math. Soc. **116** (1992), 251-257. MR1106180 (92k:58229)
- [9] B. Maskit, *Kleinian groups*, Springer-Verlag, 1987. MR0959135 (90a:30132)
- [10] Ch. Pommerenke, *Uniformly Perfect Sets and the Poincaré Metric*, Arch. Math. **32** (1979), 192-199. MR0534933 (80j:30073)
- [11] Pekka Tukia, *Convergence Groups and Gromov's Metric Hyperbolic Spaces*, New Zealand J. Math. **23** (1994), 157-187. MR1313451 (96c:30042)
- [12] Jussi Väisälä, *Lectures on n -Dimensional Quasiconformal Mappings*, Springer-Verlag, 1971. MR0454009 (56:12260)
- [13] Matti Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Springer-Verlag, 1988. MR0950174 (89k:30021)

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