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WEIGHTED ESTIMATES IN L^2 FOR LAPLACE'S EQUATION ON LIPSCHITZ DOMAINS

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Abstract. Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. For Laplace's equation $\Delta u = 0$ in Ω, we study the Dirichlet and Neumann problems with boundary data in the weighted space $L^2(\partial\Omega, \omega_\alpha d\sigma)$, where $\omega_\alpha(Q) = |Q-Q_0|^\alpha$, Q_0 is a fixed point on $\partial\Omega$, and $d\sigma$ denotes the surface measure on $\partial\Omega$. We prove that there exists $\varepsilon = \varepsilon(\Omega) \in (0, 2]$ such that the Dirichlet problem is uniquely solvable if $1 - d < \alpha < d - 3 + \varepsilon$, and the Neumann problem is uniquely solvable if $3 - d - \varepsilon < \alpha < d - 1$. If Ω is a C^1 domain, one may take $\varepsilon = 2$. The regularity for the Dirichlet problem with data in the weighted Sobolev space $L^2_1(\partial\Omega, \omega_\alpha d\sigma)$ is also considered. Finally we establish the weighted L^2 estimates with general A_p weights for the Dirichlet and regularity problems.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. We fix $Q_0 \in \partial\Omega$, and let

$$\omega_\alpha = \omega_\alpha(Q) = |Q - Q_0|^\alpha \quad \text{for} \quad \alpha > 1 - d, \tag{1.1}$$

and $d\sigma$ denote the surface measure on $\partial\Omega$. The main purpose of this paper is to study the solvability of the boundary value problems for Lapace's equation $\Delta u = 0$ in Ω with boundary data in $L^2(\partial\Omega, \omega_\alpha)$, the space of functions on $\partial\Omega$ which are square integrable with respect to the measure $\omega_\alpha d\sigma$. We obtain certain ranges of α for which the Dirichlet and Neumann problems as well as a regularity problem are uniquely solvable. In the cases of Dirichlet and regularity problems, these ranges are sharp. We also establish the weighted L^2 estimates with general A_p weights for the Dirichlet and regularity problems.

For a function F on Ω, the non-tangential maximal function of F is defined by

$$(F)^*(Q) = \sup\big\{|F(X)| : \ X \in \Omega, \, |X - Q| < 2\,\text{dist}(X, \partial\Omega)\big\} \tag{1.2}$$

for $Q \in \partial\Omega$. We will be interested in the solvability of the Dirichlet problem

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ u = f \in L^2(\partial\Omega, \omega_\alpha) \quad \text{on } \partial\Omega, \\ \|(u)^*\|_{L^2(\partial\Omega,\omega_\alpha)} < \infty, \end{cases} \tag{1.3}$$

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and the Neumann problem

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = g \in L^2(\partial\Omega, \omega_\alpha) \quad \text{on } \partial\Omega, \\ \|(\nabla u)^*\|_{L^2(\partial\Omega,\omega_\alpha)} < \infty, \end{cases} \tag{1.4}$$

where n denotes the outward unit norm to $\partial\Omega$. We remark that in (1.3) and (1.4), the boundary values are taken in the sense of non-tangential convergence, almost everywhere with respect to the surface measure on $\partial\Omega$.

The following are the main results of the paper.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain with connected boundary. Then there exists $\varepsilon = \varepsilon(\Omega) \in (0, 2]$ such that, given any $f \in L^2(\partial\Omega, \omega_\alpha)$ with $1 - d < \alpha < d - 3 + \varepsilon$, the Dirichlet problem* (1.3) *has a unique solution. Moreover, the solution u satisfies*

$$\int_{\partial\Omega} |(u)^*|^2 \,\omega_\alpha \, d\sigma + \int_\Omega |\nabla u(X)|^2 \,\delta(X)\, |X - Q_0|^\alpha dX \leq C \int_{\partial\Omega} |f|^2 \,\omega_\alpha \, d\sigma, \tag{1.6}$$

where $\delta(X) = dist(X, \partial\Omega)$, and C is a constant depending only on d, α and the Lipschitz character of Ω. If Ω is a C^1 domain, one may take $\varepsilon = 2$.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain with connected boundary. Then there exists $\varepsilon = \varepsilon(\Omega) \in (0, 2]$ such that, given any $g \in L^2(\partial\Omega, \omega_\alpha)$ with $3 - d - \varepsilon < \alpha < d - 1$ and $\int_{\partial\Omega} g \, d\sigma = 0$, the Neumann problem* (1.4) *has a unique (up to a constant) solution. Moreover, the solution u satisfies*

$$\int_{\partial\Omega} |(\nabla u)^*|^2 \,\omega_\alpha \, d\sigma + \int_\Omega |\nabla\nabla u(X)|^2 \,\delta(X)\, |X - Q_0|^\alpha \, dX \leq C \int_{\partial\Omega} |g|^2 \,\omega_\alpha \, d\sigma, \tag{1.8}$$

where C depends only on d, α and the Lipschitz character of Ω. If Ω is a C^1 domain, one may take $\varepsilon = 2$.

Let $L^2_1(\partial\Omega, \omega_\alpha) = \{f \in L^2(\partial\Omega, \omega_\alpha) : \; |\nabla_t f| \in L^2(\partial\Omega, \omega_\alpha)\}$, where $\nabla_t f$ denotes the tangential derivatives of f on $\partial\Omega$. We also study the regularity of solutions for the Dirichlet problem when the boundary data $f \in L^2_1(\partial\Omega, \omega_\alpha)$.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain with connected boundary. Then there exists $\varepsilon = \varepsilon(\Omega) \in (0, 2]$ such that, given any $f \in L^2_1(\partial\Omega, \omega_\alpha)$ with $3 - d - \varepsilon < \alpha < d - 1 + \varepsilon$, there exists a unique harmonic function u in Ω satisfying $u = f$ on $\partial\Omega$ and $(\nabla u)^* \in L^2(\partial\Omega, \omega_\alpha)$. Moreover, we have*

$$\int_{\partial\Omega} |(\nabla u)^*|^2 \,\omega_\alpha \, d\sigma + \int_\Omega |\nabla\nabla u(X)|^2 \,\delta(X)\, |X - Q_0|^\alpha \, dX \leq C \int_{\partial\Omega} |\nabla_t f|^2 \,\omega_\alpha \, d\sigma, \tag{1.10}$$

where C depends only on d, α and the Lipschitz character of Ω. If Ω is a C^1 domain, one may take $\varepsilon = 2$.

The main ingredients of the proofs of the theorems stated above are (1) the unweighted L^2 estimates, (2) certain localization techniques originated in [DK], and (3) the representation formulas in terms of the Green's and Neumann functions. We point out that the ranges of α in Theorems 1.5 and 1.9 are sharp. See Remarks 2.22 and 5.12. With estimates (1.8), (1.10) and their counterparts for the exterior domain $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$, we are also able to establish the invertibility of layer potentials on $\partial\Omega$. See Theorem 5.4. As a consequence, if $1 - d < \alpha < d - 3 + \varepsilon$, the unique solution of the Dirichlet problem (1.3) may be represented by a double layer potential with density in $L^2(\partial\Omega, \omega_\alpha)$. For the Neumann problem (1.4), the solution may be represented by a single layer potential with density in $L^2(\partial\Omega, \omega_\alpha)$ if $3 - d - \varepsilon < \alpha < d - 1$.

We note that for $\omega_\alpha = |Q - Q_0|^\alpha$, it is well known that

$$\omega_\alpha \in A_q(\partial\Omega),\ q > 1 \quad \text{if and only if} \quad 1 - d < \alpha < (d-1)(q-1), \tag{1.11}$$

where $A_q(\partial\Omega)$ denotes the class of A_q weights on $\partial\Omega$. Thus it is natural to ask whether the weighted estimates in the theorems stated above would hold for general A_q weights. The question is also very interesting, given the close connection between the weighted L^2 estimates with A_q weights and the well understood (unweighted) L^p estimates. We remark that for Laplace's equation in a Lipschitz domain, the Dirichlet problem with data in $L^p(\partial\Omega)$ is uniquely solvable for $2 - \varepsilon < p \le \infty$ [D1, D2], while the Neumann problem with data in $L^p(\partial\Omega)$ and the regularity problem with data in $L^p_1(\partial\Omega)$ are uniquely solvable for $1 < p < 2 + \varepsilon$ [JK, V1, DK]. Furthermore, the solutions may be represented by layer potentials [V1, DK]. The ranges of p are known to be sharp.

In this paper we will show that there exists $\eta = \eta(\Omega) \in (0, 1]$ such that if $\omega \in A_{1+\eta}(\partial\Omega)$, the Dirichlet problem with data in $L^2(\partial\Omega, \omega)$ and the regularity problem with data in $L^2_1(\partial\Omega, \frac{1}{\omega})$ are uniquely solvable. See Theorems 7.2 and 7.6. We remark that by an extrapolation theorem of Rubio de Francia [R], the weighted estimates $\|(u)^*\|_{L^2(\partial\Omega,\omega)} \le C\, \|f\|_{L^2(\partial\Omega,\omega)}$ and

$$\|(\nabla u)^*\|_{L^2(\partial\Omega, \frac{1}{\omega})} \le C\, \|\nabla_t f\|_{L^2(\partial\Omega, \frac{1}{\omega})} \tag{1.12}$$

for $\omega \in A_{1+\eta}(\partial\Omega)$ imply the corresponding (unweighted) L^p estimates for $2/(1+\eta) < p < \infty$ and $1 < p < 2/(1-\eta)$, respectively. It follows from the sharpness of the ranges of p for the L^p estimates that our condition $\omega \in A_{1+\eta}(\partial\Omega)$ is also sharp. Note that by (1.11), $\omega_\alpha \in A_{1+\eta}(\partial\Omega)$ if and only if $1-d < \alpha < (d-1)\eta$. Hence our Theorems 1.5, 1.7 and 1.9 show that the weighted L^2 estimates hold for certain power weights which are not in the optimal $A_{1+\eta}$ class.

In view of (1.12), it would be interesting to see if a similar estimate holds for the Neumann problem. It would also be interesting to extend the results in this paper to higher order elliptic equations and second order elliptic systems. In a forthcoming paper [Sh2], we obtain some partial results on the weighted L^2 estimates with power weights for the second order elliptic systems with constant coefficients. We should remark that in the case of elliptic systems, the question of the sharp ranges of p for which one may solve the L^p boundary value problems remains open for $d \ge 4$. The partial results in [Sh2] are based on certain Morrey space estimates for the elliptic systems in [Sh1].

We mention that in order to solve the oblique derivative problem with L^p data for $p > 2$, Kenig and Pipher [KP1] established a L^2 weighted estimate with A_1 weights. We also point out that there exists an extensive literature on the solvablity of boundary value problems on piecewise smooth domains in weighted Sobolev spaces with power weights. We refer the reader to a recent monograph [KMR] by Kozlov, Maz'ya and Rossmann for references.

The paper is organized as follows. In Section 2, we prove Theorem 1.9 for the case $3-d-\alpha < \alpha < 1-d$. In Section 3, we establish the size and Hölder estimates for the Neumann function on Ω. These estimates are used in Section 4 to prove Theorem 1.7. The invertibility of layer potentials on $L^2(\partial\Omega, \omega_\alpha)$ is established in

Section 5, which also contains the proof of Theorem 1.5. In Section 6, we give the proof of Theorem 1.9 for the remaining case $d-1 \le \alpha < d-1+\varepsilon$. Finally we establish two weighted estimates with general A_p weights in Section 7.

Throughout this paper, we will use Ω to denote a bounded Lipschitz domain in $\mathbb{R}^d$, $d \ge 3$, with connected boundary. For $P \in \mathbb{R}^d$, $B(P,r)$ denotes the ball centered at P with radius $r > 0$.

2. Estimates in $L^2_1(\partial\Omega, \omega_\alpha)$ for the regularity problem

This section is devoted to the proof of Theorem 1.9 for the case $3-d-\varepsilon < \alpha < d-1$. The remaining case $d-1 \le \alpha < d-1+\varepsilon$ will be dealt with in Section 6. We remark that since ω_α is an A_∞ weight on $\partial\Omega$ for $\alpha > 1-d$, the second term on the left-hand side of (1.10) is dominated by a constant times the first term on the left. This follows easily from the estimate of the square function by the non-tangential maximal function, established in [D3] for Lipschitz domains. The same remark also applies to estimates (1.6) and (1.8).

For $f \in L^2(\partial\Omega)$, we let u_f denote the unique harmonic function u on Ω satisfying $(u)^* \in L^2(\partial\Omega)$ and $u = f$ on $\partial\Omega$.

Lemma 2.1. *There exists $\varepsilon \in (0,2]$ such that if f and $|\nabla_t f| \in L^2(\partial\Omega)$, then*

$$\int_{B(Q_0,r)\cap\partial\Omega} |\nabla u_f|^2 \, d\sigma \le C_\lambda \, r^\lambda \int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q-Q_0|+r\}^\lambda} \, d\sigma(Q) \tag{2.2}$$

for any $Q_0 \in \partial\Omega$ and $r > 0$, where $0 < \lambda < d-3+\varepsilon$.

Proof. Since estimate (2.2) becomes stronger as λ increases, it suffices to consider the case $\lambda > d-3$.

Fix $Q_0 \in \partial\Omega$. There exists $r_0 > 0$ depending only on the Lipschitz character of Ω such that, after a possible rotation of the coordinate system,

$$\Omega \cap B(Q_0,r_0) = \big\{(X',x_d) \in \mathbb{R}^d : \ x_d > \psi(X')\big\} \cap B(Q_0,r_0), \tag{2.3}$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz continuous. We may assume that $\psi(0) = 0$, $Q_0 = (0,0)$ and $0 < r < c_0 r_0$.

Let $\Delta_r = \{(X',\psi(X')) : \ |X'| \le r\}$. We choose $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on Δ_{8r}, $\varphi \equiv 0$ on $\partial\Omega \setminus \Delta_{9r}$, and $|\nabla\varphi| \le C/r$. Also for $E \subset \partial\Omega$, let

$$f_E = \frac{1}{|E|}\int_E f \, d\sigma. \tag{2.4}$$

We now write $f = g + h + f_{\Delta_{9r}}$, where $g = (f - f_{\Delta_{9r}})\varphi$ and $h = (f - f_{\Delta_{9r}})(1-\varphi)$. Then $u_f = u_g + u_h + f_{\Delta_{9r}}$ in Ω. By the L^2 regularity estimate [JK], we have

(2.5)
$$\begin{aligned}\int_{\Delta_r} |\nabla u_g|^2 \, d\sigma &\le C\int_{\partial\Omega} |\nabla_t g|^2 \, d\sigma \le C\int_{\Delta_{9r}} |\nabla_t f|^2 \, d\sigma + \frac{C}{r^2}\int_{\Delta_{9r}} |f - f_{\Delta_{9r}}|^2 \, d\sigma \\ &\le C\int_{\Delta_{9r}} |\nabla_t f|^2 \, d\sigma,\end{aligned}$$

where we have used Poincaré's inequality in the last inequality.

To estimate ∇u_h on Δ_r, we note that $u_h = 0$ on Δ_{8r}. First we use an argument originated in [DK] to reduce the surface integral to a solid integral. For $\tau > 0$, let

$$D_\tau = \big\{(X',x_d) \in \mathbb{R}^d : \ |X'| < \tau \text{ and } \psi(X') < x_d < \psi(X') + \tau\big\}. \tag{2.6}$$

Then $D_\tau \subset \Omega$ if $\tau \le c_0 r_0$. We apply a Rellich identity [JK] on the Lipschitz domain D_τ to obtain

$$\int_{\Delta_r} |\nabla u_h|^2 \, d\sigma \le C \int_{\Omega\cap\partial D_{tr}} |\nabla u_h|^2 \, d\sigma \tag{2.7}$$

for $t \in (1,2)$. By integrating both sides of (2.7) in $t \in (1,2)$, we have

$$\int_{\Delta_r} |\nabla u_h|^2 \, d\sigma \le \frac{C}{r} \int_{D_{2r}} |\nabla u_h|^2 \, dX \le \frac{C}{r^3} \int_{D_{3r}} |u_h|^2 \, dX, \tag{2.8}$$

where we have used the Caccioppoli inequality in the second inequality.

Next, let $G^X(Y) = G(X,Y)$ denote the Green's function for the Laplacian on Ω. Then

$$\begin{aligned} |u_h(X)| &= \left| \int_{\partial\Omega} \frac{\partial G^X}{\partial n} \, h(Q) \, d\sigma(Q) \right| \\ &\le \int_{\partial\Omega\setminus\Delta_{8r}} \left|\nabla G^X\right| |f - f_{\Delta_{9r}}| \, d\sigma. \end{aligned} \tag{2.9}$$

Since $G^X = 0$ on $\partial\Omega$, we may use the same argument as in the proof of (2.7)-(2.8) to show that

$$\int_{\Delta_{2^j r}\setminus\Delta_{2^{j-1}r}} |\nabla G^X|^2 \, d\sigma \le \frac{C}{(2^j r)^3} \int_{D_{2^{j+1}r}\setminus D_{2^{j-2}r}} |G(X,Y)|^2 \, dY, \tag{2.10}$$

where $4 \le j \le J$ and $2^J r \sim c_0 r_0$. It is well known that there exists $\eta = \eta(\Omega) \in (0,1)$ such that for $X, Y \in \Omega$,

$$|G(X,Y)| \le \frac{C \, \{\operatorname{dist}(X,\partial\Omega)\}^\eta}{|X-Y|^{d-2+\eta}}. \tag{2.11}$$

Thus, for $X \in D_{3r}$,

$$\int_{\Delta_{2^j r}\setminus\Delta_{2^{j-1}r}} |\nabla G^X|^2 \, d\sigma \le \frac{C \, r^{2\eta}}{(2^j r)^{d-1+2\eta}}. \tag{2.12}$$

To finish the proof, we note that, by Poincaré's inequality,

$$\begin{aligned} |f_{\Delta_{2^j r}} - f_{\Delta_{2^{j-1}r}}| &\le \frac{C}{|\Delta_{2^j r}|} \int_{\Delta_{2^j r}} |f - f_{\Delta_{2^j r}}| \, d\sigma \\ &\le C \, (2^j r)^{\frac{3-d}{2}} \left\{ \int_{\Delta_{2^j r}} |\nabla_t f|^2 \, d\sigma \right\}^{1/2} \\ &\le C \, (2^j r)^{\frac{\lambda+3-d}{2}} \left\{ \int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q|+r\}^\lambda} \, d\sigma(Q) \right\}^{1/2} \end{aligned}$$

for $\lambda > 0$. It follows by summation that, if $\lambda > d-3$,

$$|f_{\Delta_{2^j r}} - f_{\Delta_{9r}}| \le C \, (2^j r)^{\frac{\lambda+3-d}{2}} \left\{ \int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q|+r\}^\lambda} \, d\sigma(Q) \right\}^{1/2}. \tag{2.13}$$

Again, by Poincaré's inequality

$$\tag{2.14}\begin{aligned}\int_{\Delta_{2^j r}} |f - f_{\Delta_{9r}}|^2\, d\sigma &\le 2\int_{\Delta_{2^j r}} |f - f_{\Delta_{2^j r}}|^2 d\sigma + C(2^j r)^{d-1}|f_{\Delta_{2^j r}} - f_{\Delta_{9r}}|^2\\ &\le C\,(2^j r)^{\lambda+2}\int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q|+r\}^\lambda}\, d\sigma(Q).\end{aligned}$$

This, together with (2.12) and Hölder inequality, gives

$$\tag{2.15}\int_{\Delta_{2^j r}\setminus\Delta_{2^{j-1} r}} |\nabla G^X|\,|f - f_{\Delta_{9r}}|\, d\sigma \le \frac{C\, r^\eta}{(2^j r)^{\frac{d-3-\lambda}{2}+\eta}} \left\{\int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q|+r\}^\lambda}\, d\sigma(Q)\right\}^{1/2},$$

where $X \in D_{3r}$ and $4 \le j \le J$. Similarly, it is not hard to see that for $X \in D_3$,

$$\tag{2.16}\int_{\partial\Omega\setminus\Delta_{c_0 r_0}} |\nabla G^X|\,|f - f_{\Delta_{9r}}|\, d\sigma \le C\, r^\eta \left\{\int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q|+r\}^\lambda}\, d\sigma(Q)\right\}^{1/2}.$$

In view of (2.9), (2.15) and (2.16), we have proved that if $X \in D_{3r}$ and $d-3 < \lambda < d-3+2\eta$,

$$\tag{2.17}|u_h(X)| \le C\, r^{\frac{\lambda-d+3}{2}} \left\{\int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q|+r\}^\lambda}\, d\sigma(Q)\right\}^{1/2}.$$

By (2.8), this yields

$$\tag{2.18}\int_{\Delta_r} |\nabla u_h|^2\, d\sigma \le C\, r^\lambda \int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q|+r\}^\lambda}\, d\sigma(Q)$$

for $d-3 < \lambda < d-3+2\eta$. Finally it follows from (2.5) and (2.18) that

$$\tag{2.19}\begin{aligned}\int_{\Delta_r} |\nabla u_f|^2\, d\sigma &\le 2\int_{\Delta_r} |\nabla u_g|^2\, d\sigma + 2\int_{\Delta_r} |\nabla u_h|^2\, d\sigma\\ &\le C\, r^\lambda \int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q|+r\}^\lambda}\, d\sigma(Q)\end{aligned}$$

for $d-3 < \lambda < d-3+2\eta$. The proof of estimate (2.2) with $\varepsilon = 2\eta$ is now complete. □

We now give the

Proof of Theorem 1.9 *for the case* $3-d-\varepsilon < \alpha < d-1$. Fix $Q_0 \in \partial\Omega$. Let $\omega_\alpha = \omega_\alpha(Q) = |Q-Q_0|^\alpha$. If $0 \le \alpha < d-1$, then $\frac{1}{\omega_\alpha} \in A_p(\partial\Omega)$ for any $p > 1$. The desired result is a consequence of Theorem 7.6 in Section 7 for the general A_p weights.

We now assume $3-d-\varepsilon < \alpha < 0$, where $\varepsilon > 0$ is the same as in Lemma 2.1. Let $f \in L^2(\partial\Omega, \omega_\alpha)$ such that $|\nabla_t f| \in L^2(\partial\Omega, \omega_\alpha)$. Since $L^2(\partial\Omega, \omega_\alpha) \subset L^2(\partial\Omega)$ for $\alpha < 0$, the uniqueness follows from an integration by parts. We only need to show that

$$\tag{2.20}\int_{\partial\Omega} |(\nabla u_f)^*|^2\, \omega_\alpha\, d\sigma \le C\int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma.$$

Also, since $\omega_\alpha \in A_q(\partial\Omega)$ for any $q > 1$, by Theorem 7.2, it suffices to show that

$$\tag{2.21}\int_{\partial\Omega} |\nabla u_f|^2\, \omega_\alpha\, d\sigma \le C\int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma.$$

To this end, we choose $\lambda > 0$ so that $-\alpha < \lambda < d-3+\varepsilon$. It follows from (2.2) that

$$\begin{aligned}
&\int_{\partial\Omega} |\nabla u_f(Q)|^2\, |Q-Q_0|^\alpha\, d\sigma(Q)\\
&\quad\le |\alpha| \int_0^\infty r^{\alpha-1} \left\{ \int_{B(Q_0,r)\cap\partial\Omega} |\nabla u_f(Q)|^2\, d\sigma(Q) \right\} dr\\
&\quad\le C \int_0^\infty r^{\alpha+\lambda-1} \left\{ \int_{\partial\Omega} \frac{|\nabla_t f(Q)|^2}{\{|Q-Q_0|+r\}^\lambda}\, d\sigma(Q) \right\} dr\\
&\quad= C \int_{\partial\Omega} |\nabla_t f(Q)|^2 \left\{ \int_0^\infty \frac{r^{\alpha+\lambda-1}}{\{|Q-Q_0|+r\}^\lambda}\, dr \right\} d\sigma(Q)\\
&\quad\le C \int_{\partial\Omega} |\nabla_t f(Q)|^2 |Q-Q_0|^\alpha\, d\sigma(Q).
\end{aligned}$$

Estimate (2.21) is proved.

Finally we point out that, if Ω is a C^1 domain, estimate (2.11) on the Green's function holds for any $\eta \in (0,1)$. Thus in this case, one may take $\varepsilon = 2$ in Lemma 2.1. It follows that Theorem 1.9 holds for $1-d < \alpha < d-1$ on C^1 domains. □

Remark 2.22. The condition $3-d-\varepsilon < \alpha < d-1+\varepsilon$ in Theorem 1.9 is sharp. Indeed, let $\mathcal{O} \subset \mathbb{S}^{d-1}$ be the complement of a small spherical cap centered at the north pole. Consider the functions of form $u = u(r,\theta) = r^\lambda v(\theta)$ on the cone $\Gamma = \mathbb{R}_+ \times \mathcal{O}$ in spherical coordinates. Note that $\Delta u = 0$ in Γ if $\Delta_{\mathbb{S}^{d-1}} v = -\lambda(\lambda+d-2)v$ on $\mathcal{O}$, where $\Delta_{\mathbb{S}^{d-1}}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. Let $\lambda(\lambda+d-2)$ be the first eigenvalue of $-\Delta_{\mathcal{S}^{d-1}}$ on $\mathcal{O}$ with the Dirichlet boundary condition and v_λ the corresponding eigenfunction. It is known that $\lambda \to 0$ as $\sigma(\mathcal{S}^{d-1} \setminus \mathcal{O}) \to 0$ (see e.g. [KMR]). Let Ω be a bounded domain such that $\Omega \subset \Gamma$, $\Omega \cap B(0,1) = \Gamma \cap B(0,1)$, and $\partial\Omega \setminus \{0\}$ is smooth. Let $u_\lambda(r,\theta) = r^\lambda v_\lambda(\theta)$. Note that $(\nabla u_\lambda)^* \sim |Q|^{\lambda-1}$ near $Q_0 = 0$. It follows that $(\nabla u_\lambda)^* \in L^2(\partial\Omega, \omega_\alpha)$ if and only if $\alpha > 3-d-2\lambda$. This implies that for $\alpha \le 3-d-2\lambda$ and $f = u_\lambda|_{\partial\Omega} \in L^2_1(\partial\Omega)$, the regularity problem has no solution with property $(\nabla u)^* \in L^2(\partial\Omega, \omega_\alpha)$. For otherwise, u would have to agree with u_λ, the unique solution for the case $\alpha = 0$.

Note that, $\widetilde{u}_\lambda(r,\theta) = r^{2-d-\lambda} v_\lambda(\theta)$ is also harmonic in Ω, and $(\nabla \widetilde{u}_\lambda)^* \in L^2(\partial\Omega, \omega_\alpha)$ if and only if $\alpha > d-1+2\lambda$. Let $h = \widetilde{u}_\lambda|_{\partial\Omega} \in L^2_1(\partial\Omega)$ and F be the harmonic function on Ω such that $F = h$ on $\partial\Omega$ and $(\nabla F)^* \in L^2(\partial\Omega)$. Then $F - \widetilde{u}_\lambda \equiv 0$ on $\partial\Omega$ and $\big(\nabla(F-\widetilde{u}_\lambda)\big)^* \in L^2(\partial\Omega, \omega_\alpha)$ if $\alpha > d-1+2\lambda$. Since $\big(\nabla(F-\widetilde{u}_\lambda)\big)^* \notin L^2(\partial\Omega)$, we have $F - \widetilde{u}_\lambda \not\equiv 0$ in Ω. This means that if $\alpha > d-1+2\lambda$, the uniqueness fails for the regularity problem on Ω.

Remark 2.23. Theorem 1.9 has a counterpart in the exterior domain $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$. Indeed, one can show that there exists $\varepsilon = \varepsilon(\Omega) \in (0,2]$ such that if $f \in L^2_1(\partial\Omega, \omega_\alpha)$ and $3-d-\varepsilon < \alpha < d-1$, then there exists a unique harmonic function u on Ω_- satisfying $(\nabla u)^*_e \in L^2(\partial\Omega, \omega_\alpha)$, $u = f$ on $\partial\Omega$, and $|u(X)| = O(|X|^{2-d})$ as $|X| \to \infty$. Here $(\nabla u)^*_e$ denotes the non-tangential maximal function of ∇u with respect to Ω_-. Moreover, we have

$$\|(\nabla u)^*_e\|_{L^2(\partial\Omega,\omega_\alpha)} \le C \left\{ \|\nabla_t f\|_{L^2(\partial\Omega,\omega_\alpha)} + \|f\|_{L^2(\partial\Omega,\omega_\alpha)} \right\}. \tag{2.24}$$

Also, if Ω is a C^1 domain, one may take $\varepsilon = 2$. The proof, which we omit, is similar to that of Theorem 1.9. We will need estimate (2.24) in Section 5.

3. Estimates of the Neumann function

To prove Theorem 1.7, we will need to represent the solutions of the L^2 Neumann problem in terms of the so-called Neumann function $N(X,Y)$. The following estimates are crucial to us:

$$|N(X,Y)| \le \frac{C}{|X-Y|^{d-2}} \quad \text{for any} \quad X,Y \in \overline{\Omega}, \tag{3.1}$$

$$|N(X,Y_1)-N(X,Y_2)| \le \frac{C\,|Y_1-Y_2|^\delta}{|X-Y_1|^{d-2+\delta}} \tag{3.2}$$

if $X,Y_1,Y_2 \in \overline{\Omega}$ and $|Y_1-Y_2| \le \frac{1}{2}|X-Y_1|$, where $\delta=\delta(\Omega)\in(0,1)$. We remark that estimates (3.1)-(3.2) were established by Kenig and Pipher [KP2] for the divergence form elliptic operators on star-like Lipschitz domains. However we have not been able to find a reference in the case of the Laplacian on general Lipschitz domains, although the estimates seem to be well known to experts. In this section we give the proof of estimates (3.1)-(3.2) for the sake of completeness.

Let $\Gamma(X,Y)=c_d/|X-Y|^{d-2}$ denote the fundamental solution for Laplace's equation in $\mathbb{R}^d$, where $c_d=[(2-d)|\partial B(0,1)|]^{-1}$. Fix $X\in\Omega$, let v^X denote the unique harmonic function in Ω such that $(\nabla v^X)^* \in L^2(\partial\Omega)$,

$$\frac{\partial v^X}{\partial n} = \frac{\partial \Gamma^X}{\partial n} - \frac{1}{|\partial\Omega|} \quad \text{on} \quad \partial\Omega \tag{3.3}$$

where $\Gamma^X(\cdot)=\Gamma(X,\cdot)$, and

$$\int_\Omega v^X(Y)\,dY = 0. \tag{3.4}$$

For $X,Y\in\Omega$, the Neumann function is defined by

$$N(X,Y) = v^X(Y) - \Gamma(X,Y). \tag{3.5}$$

Using Green's formulas, it is not hard to show that $v^X(Y)=v^Y(X)$ for $X\neq Y$. Hence

$$N(X,Y) = N(Y,X) \quad \text{for} \quad X,Y\in\Omega \text{ and } X\neq Y. \tag{3.6}$$

Lemma 3.7. *There exist $r_0>0$, $\delta\in(0,1)$, and $c_0>0$ depending only on d and the Lipschitz character of Ω, such that if $u\in H^1(B(P,r)\cap\Omega)$, $\Delta u=0$ in $B(P,r)\cap\Omega$ and $\frac{\partial u}{\partial n}=g\in L^\infty(B(P,r)\cap\partial\Omega)$ on $B(P,r)\cap\partial\Omega$ for some $P\in\partial\Omega$ and $0<r<r_0$, then*

$$|u(X_1)-u(X_2)| \le C\left(\frac{|X_1-X_2|}{r}\right)^\delta \left[\left(\frac{1}{r^d}\int_{B(P,r)\cap\Omega}|u|^2\,dX\right)^{1/2} + r\,\|g\|_\infty\right] \tag{3.8}$$

for any $X_1,X_2\in B(P,c_0r)\cap\Omega$.

Proof. There exists $r_1>0$ depending only on d and the Lipschitz character of Ω such that for any $P\in\Omega$, one may find a Lipschitz function $\psi:\mathbb{R}^{d-1}\to\mathbb{R}$ so that

$$\Omega\cap B(P,r_1) = \left\{(X',x_d)\in\mathbb{R}^d:\ x_d>\psi(X')\right\}\cap B(P,r_1) \tag{3.9}$$

after a possible rotation of the coordinate system. We may assume that $\psi(0)=0$ and $P=(0,0)$.

Let D_τ be defined by (2.6) and

$$\widetilde{D}_\tau = \left\{(X',x_d)\in\mathbb{R}^d:\ |X'|<\tau \text{ and } \psi(X')-\tau<x_d<\psi(X')+\tau\right\}. \tag{3.10}$$

Note that $D_r \subset \Omega$ if $0 < r < c_1 r_1 = r_0$. For $X = (X', x_d) \in \mathbb{R}^d$, let $X^* = (X', 2\psi(X') - x_d)$. We extend u to $\widetilde{D}_r$ by letting $\widetilde{u} = u$ on D_r, and $\widetilde{u}(X) = u(X^*)$ for $X \in \widetilde{D}_r \setminus D_r$. Then $\widetilde{u} \in H^1(\widetilde{D}_r)$ and is a weak solution of $L\widetilde{u} = -2g\,d\mu$ in $\widetilde{D}_r$, where L is the divergence form elliptic operator with bounded measurable coefficients on $\mathbb{R}^d$ constructed in [DK, Theorem 2.15], and $d\mu$ is the surface measure on $\Delta_r = \{(X', \psi(X')) \in \mathbb{R}^d : \ |X'| < r\}$. Let $G^L(X,Y)$ denote the Green's function for L on $B(0, 2r_1)$ and

$$w(X) = 2\int_{\Delta_r} G^L(X,Q)\, g(Q)\, d\sigma(Q). \tag{3.11}$$

Then $L(\widetilde{u} - w) = 0$ in $\widetilde{D}_r$. It follows from the theory of De Giorgi-Nash-Moser that there exists $\delta > 0$ depending on $\|\nabla\psi\|_\infty$ such that

$$|(\widetilde{u} - w)(X_1) - (\widetilde{u} - w)(X_2)| \le C\left(\frac{|X_1 - X_2|}{r}\right)^\delta \left\{\frac{1}{r^d}\int_{\widetilde{D}_r} |\widetilde{u} - w|^2\, dX\right\}^{1/2} \tag{3.12}$$

for any $X_1, X_2 \in \widetilde{D}_{r/2}$. Using the well-known estimates on the Green's function G^L, it is not hard to see that

$$\begin{aligned} &|w(X)| \le C\, r\|g\|_\infty \quad \text{for any } X \in \widetilde{D}_r,\\ &|w(X_1) - w(X_2)| \le C\, r\|g\|_\infty \left(\frac{|X_1 - X_2|}{r}\right)^\delta \quad \text{for any } X_1, X_2 \in \widetilde{D}_r. \end{aligned} \tag{3.13}$$

Estimate (3.8) follows easily from (3.12)-(3.13). □

Remark 3.14. Estimate (3.8) implies that

$$\sup_{X \in B(P, c_0 r)\cap\Omega} |u(X)| \le C\left[\left(\frac{1}{r^d}\int_{B(P,r)\cap\Omega} |u(X)|^2\, dX\right)^{1/2} + r\,\|g\|_\infty\right]. \tag{3.15}$$

Remark 3.16. If Ω is a C^1 domain, one may take δ in Lemma 3.7 to be any number in $(0,1)$. This, however, does not follow directly from our proof, since the coefficients of the operator L are still not continuous. To deal with the C^1 case, we use the fact that the Neumann problem in L^p is solvable for any $1 < p < \infty$ [FJR]. This implies that if $u \in H^1(\Omega)$ and $\Delta u = 0$ in Ω, then

$$\|\nabla u\|_{L^q(\Omega)} \le C\|\frac{\partial u}{\partial n}\|_{L^p(\partial\Omega)}$$

where $q = dp/(d-1)$. By a localization argument, one may show that if $u \in H^1(B(P,r)\cap\Omega)$, $\Delta u = 0$, in $B(P,r)\cap\Omega$ and $\frac{\partial u}{\partial n} = g \in L^\infty(B(P,r)\cap\partial\Omega)$ on $B(P,r)\cap\partial\Omega$ for some $P \in \partial\Omega$ and $r > 0$ small, we have

$$\left(\frac{1}{r^d}\int_{B(P,cr)\cap\Omega} |\nabla u|^q\, dX\right)^{1/q} \le C_p\left\{\left(\frac{1}{r^d}\int_{B(P,r)\cap\Omega} |\nabla u|^p\, dX\right)^{1/p} + \|g\|_\infty\right\} \tag{3.17}$$

for any $1 < p < \infty$ and $q = dp/(d-1)$. It follows by an iteration argument that

$$\left(\frac{1}{r^d}\int_{B(P,cr)\cap\Omega} |\nabla u|^p\, dX\right)^{1/p} \le C_p\left\{\left(\frac{1}{r^d}\int_{B(P,r)\cap\Omega} |\nabla u|^2\, dX\right)^{1/2} + \|g\|_\infty\right\} \tag{3.18}$$

for any $p > d$. By Sobolev imbedding and the Caccioppoli inequality, this gives estimate (3.8) with $\delta = 1 - (d/p)$. We omit the details.

We are now in a position to give the proof of estimates (3.1)-(3.2).

Theorem 3.19. *Let $N(X,Y)$ be the Neumann function defined in* (3.5)*. Then estimates* (3.1)-(3.2) *hold for some $C = C(\Omega) > 0$ and $\delta = \delta(\Omega) \in (0,1)$. Moreover, if Ω is a C^1 domain, estimate* (3.2) *holds for any $\delta \in (0,1)$.*

Proof. We will give the proof of estimate (3.1). Estimate (3.2) follows easily from (3.1) by Lemma 3.7 and Remark 3.16.

To see (3.1), we fix $X_0, Y_0 \in \Omega$ and let $r = |X_0 - Y_0|/4$. Suppose that $f \in C_0^\infty(\Omega \cap B(Y_0, r))$ and $\int_\Omega f\, dX = 0$. Then there exists a unique $w \in H^1(\Omega)$ such that $\Delta w = f$ in Ω, $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$, and $\int_\Omega w\, dX = 0$. By the definition of $N(X,Y)$ and Green's formula,

$$w(X) = -\int_\Omega N(X,Y) f(Y)\, dY. \tag{3.20}$$

Since $\int_\Omega w\, dX = 0$, we may use the Sobolev inequality and energy estimate to obtain

$$\begin{aligned} \|w\|^2_{L^{\frac{2d}{d-2}}(\Omega)} &\le C\|\nabla w\|^2_{L^2(\Omega)} = C\left|\int_\Omega w\, f\, dX\right| \\ &\le C\|w\|_{L^{\frac{2d}{d-2}}(\Omega)} \|f\|_{L^{\frac{2d}{d+2}}(\Omega)}. \end{aligned} \tag{3.21}$$

It follows that

$$\|w\|_{L^{\frac{2d}{d-2}}(\Omega)} \le C\, \|f\|_{L^{\frac{2d}{d+2}}(\Omega)}. \tag{3.22}$$

Now, since $\Delta w = 0$ in $\Omega \cap B(X_0, r)$ and $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$, by Remark 3.14 as well as the interior estimates for harmonic functions, we have

$$\begin{aligned} |w(X_0)| &\le C\left\{\frac{1}{r^d}\int_{B(X_0,r)\cap\Omega} |w(Y)|^2\, dY\right\}^{1/2} \\ &\le \frac{C}{r^{\frac{d-2}{2}}}\, \|w\|_{L^{\frac{2d}{d-2}}(\Omega)} \le \frac{C}{r^{\frac{d-2}{2}}}\, \|f\|_{L^{\frac{2d}{d+2}}(\Omega)}. \end{aligned} \tag{3.23}$$

In view of (3.20), this implies that

$$\left\{\int_{\Omega\cap B(Y_0,r)} |N(X_0,Y)|^{\frac{2d}{d-2}}\, dY\right\}^{\frac{d-2}{2d}} \le \frac{C}{r^{\frac{d-2}{2}}} \tag{3.24}$$

by duality.

Finally we apply (3.15) and the interior estimates to the harmonic function $u(X) = N(X_0, X)$ in $\Omega \cap B(Y_0, r)$. With Hölder inequality and (3.24), we may conclude that

$$\begin{aligned} |N(X_0,Y_0)| &\le C\left\{\left(\frac{1}{r^d}\int_{\Omega\cap B(Y_0,r)} |N(X_0,Y)|^{\frac{2d}{d-2}}\, dY\right)^{\frac{d-2}{2d}} + \frac{r}{|\partial\Omega|}\right\} \\ &\le \frac{C}{r^{d-2}}. \end{aligned} \tag{3.25}$$

The proof is complete. □

Remark 3.26. The Neumann function for the exterior domain $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$ may be constructed in a similar manner. Indeed, for $X \in \Omega_-$, we let v^X be the unique harmonic function in Ω_- such that $(\nabla v^X)^* \in L^2(\partial\Omega)$, $\frac{\partial v^X}{\partial n} = \frac{\partial \Gamma^X}{\partial n}$ on $\partial\Omega$, and $|v^X(Y)| = O(|Y|^{2-d})$ as $|Y| \to \infty$. We define $N(X,Y) = v^X(Y) - \Gamma(X,Y)$ as before. The estimates (3.1)-(3.2) may be proved by the same argument as in the bounded case.

4. The proof of Theorem 1.7

In this section we give the proof of Theorem 1.7.

We begin by noting that if u is a harmonic function in Ω such that $(\nabla u)^* \in L^2(\partial\Omega)$ and $\frac{\partial u}{\partial n} = f \in L^2(\partial\Omega)$ on $\partial\Omega$, then

$$u(X) = \int_{\partial\Omega} N(X,Q) f(Q)\, d\sigma(Q) + \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u\, d\sigma. \tag{4.1}$$

This follows easily from the definition of the Neumann function and Green's formula.

Lemma 4.2. *Let u be a harmonic function in Ω such that $(\nabla u)^* \in L^2(\partial\Omega)$ and $\frac{\partial u}{\partial n} = f$ on $\partial\Omega$. Then for any $Q_0 \in \partial\Omega$ and $r > 0$,*

$$\int_{\substack{Q\in\partial\Omega \\ |Q-Q_0|>r}} |(\nabla u)^*(Q)|^2\, d\sigma(Q) \le C_\lambda \int_{\partial\Omega} \left\{ \frac{|Q-Q_0|}{|Q-Q_0|+r} \right\}^\lambda |f(Q)|^2\, d\sigma(Q), \tag{4.3}$$

where $0 \le \lambda < d-1$.

Proof. As in the proof of Lemma 2.1, we may assume that $Q_0 = (0,0)$ and

$$\Omega \cap B(Q_0, r_0) = \left\{ (X', x_d) \in \mathbb{R}^d :\ x_d > \psi(X') \right\} \cap B(Q_0, r_0), \tag{4.4}$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz continuous. Clearly it suffices to consider the case $0 < r < r_0$.

Recall that $\Delta_r = \{(X', x_d) :\ |X'| < r\}$. Let $f = g + h$ where $g = (f - f_{\Delta_r})\chi_{\Delta_r}$. Let

$$v(X) = \int_{\partial\Omega} N(X,Q)\, g(Q)\, d\sigma(Q) \quad \text{and} \quad w(X) = \int_{\partial\Omega} N(X,Q)\, h(Q)\, d\sigma(Q). \tag{4.5}$$

Then $u = v + w + k$ where k is a constant. By the L^2 estimate [JK],

$$\begin{aligned} \int_{\partial\Omega\setminus\Delta_{8r}} |(\nabla w)^*|^2\, d\sigma &\le C \int_{\partial\Omega} |h|^2\, d\sigma \le C\, r^{d-1} |f_{\Delta_r}|^2 + C \int_{\partial\Omega\setminus\Delta_r} |f|^2\, d\sigma \\ &\le \frac{C}{r^\lambda} \int_{\Delta_r} |f(Q)|^2\, |Q|^\lambda\, d\sigma(Q) + C \int_{\partial\Omega\setminus\Delta_r} |f|^2\, d\sigma, \end{aligned} \tag{4.6}$$

where $0 \le \lambda < d-1$.

To estimate $(\nabla v)^*$ on $\partial\Omega \setminus \Delta_{8r}$, we note that $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega \setminus \Delta_r$. Also for $X \in \Omega$ and $\operatorname{dist}(X, \Delta_r) \ge c\, r$, in view of estimate (3.1) and (4.5), we have

$$|v(X)| \le \frac{C}{|X|^{d-2}} \int_{\Delta_r} |f(Q)|\, d\sigma(Q). \tag{4.7}$$

Let $E_j = \Delta_{2^j r} \setminus \Delta_{2^{j-1} r}$, where $4 \le j \le J$ and $2^J r \sim r_0$. For $Q \in E_j$, let

$$M_1(F)(Q) = \sup \left\{ |F(X)| :\ X \in \gamma(Q) \text{ and } |X - Q| \le \theta 2^j r \right\}, \tag{4.8}$$

$$M_2(F)(Q) = \sup \left\{ |F(X)| :\ X \in \gamma(Q) \text{ and } |X - Q| \ge \theta 2^j r \right\}, \tag{4.9}$$

where $\gamma(Q) = \{X \in \Omega : |X - Q| < 2\,\mathrm{dist}(X, \partial\Omega)\}$ and θ is chosen so that for $Q \in E_j$, $M_1(F)(Q)$ is less than the non-tangential maximal function of F with respect to the domain $D_{2^{j+1}r} \setminus D_{2^{j-2}r}$ defined in (2.6). Clearly $(\nabla v)^* \le M_1(\nabla v) + M_2(\nabla v)$.

Note that if $X \in \gamma(Q)$ and $|X - Q| \ge \theta 2^j r$, we may use the interior estimate and (4.7) to obtain

$$
\begin{aligned}
|\nabla v(X)| &\le \frac{C}{(2^j r)^{d+1}} \int_{B(X, c2^j r)} |v(Y)|\, dY \\
&\le \frac{C}{(2^j r)^{d-1}} \int_{\Delta_r} |f|\, d\sigma \\
&\le \frac{C\, r^{\frac{d-\lambda-1}{2}}}{(2^j r)^{d-1}} \left\{ \int_{\Delta_r} |f(Q)|^2 |Q|^\lambda\, d\sigma(Q) \right\}^{1/2},
\end{aligned} \tag{4.10}
$$

where $0 \le \lambda < d - 1$. It follows that

$$
\int_{E_j} |M_2(\nabla v)|^2\, d\sigma \le \frac{C}{(2^j)^{d-1} r^\lambda} \int_{\Delta_r} |f(Q)|^2\, |Q|^\lambda\, d\sigma(Q). \tag{4.11}
$$

For $M_1(\nabla v)$ on E_j, we use the L^2 estimate on the Lipschitz domain $D_\tau \setminus D_{\tau/4}$, where $\tau \in (2^j r, 2^{j+1} r)$ to obtain

$$
\int_{E_j} |M_1(\nabla v)|^2\, d\sigma \le C \int_{\Omega \cap \partial(D_\tau \setminus D_{\tau/4})} |\nabla v|^2\, d\sigma. \tag{4.12}
$$

Integrating both sides of (4.12) in $\tau \in (2^j r, 2^{j+1} r)$ then yields

$$
\begin{aligned}
\int_{E_j} |M_1(\nabla v)|^2\, d\sigma &\le \frac{C}{2^j r} \int_{D_{2^{j+1}r} \setminus D_{2^{j-2}r}} |\nabla v|^2\, dX \\
&\le \frac{C}{(2^j r)^3} \int_{D_{2^{j+2}r} \setminus D_{2^{j-3}r}} |v|^2\, dX,
\end{aligned} \tag{4.13}
$$

where the second inequality follows from the Cacciopoli inequality. This, together with (4.7), gives

$$
\int_{E_j} |M_1(\nabla v)|^2\, d\sigma \le \frac{C}{(2^j)^{d-1} r^\lambda} \int_{\Delta_r} |f(Q)|^2\, |Q|^\lambda\, d\sigma(Q). \tag{4.14}
$$

It follows from estimates (4.11) and (4.14) by summation that

$$
\int_{\Delta_{c_0 r_0} \setminus \Delta_{8r}} |(\nabla v)^*|^2\, d\sigma \le \frac{C}{r^\lambda} \int_{\Delta_r} |f(Q)|^2\, |Q|^\lambda\, d\sigma(Q). \tag{4.15}
$$

The same argument can also be used to show that

$$
\begin{aligned}
\int_{\partial\Omega \setminus \Delta_{c_0 r_0}} |(\nabla v)^*|^2\, d\sigma &\le C \int_{\Omega \cap B(0, c_1 r_0)} |v|^2\, dX \\
&\le \frac{C}{r^\lambda} \int_{\Delta_r} |f(Q)|^2\, |Q|^\lambda\, d\sigma(Q).
\end{aligned} \tag{4.16}
$$

Thus we have

$$(4.17)\qquad \begin{aligned} \int_{\partial\Omega\setminus\Delta_{8r}} |(\nabla u)^*|^2\, d\sigma &\le 2\int_{\partial\Omega\setminus\Delta_{8r}} |(\nabla v)^*|^2\, d\sigma + 2\int_{\partial\Omega\setminus\Delta_{8r}} |(\nabla w)^*|^2\, d\sigma \\ &\le \frac{C}{r^\lambda}\int_{\Delta_r} |f(Q)|^2\, |Q|^\lambda\, d\sigma(Q) + C\int_{\partial\Omega\setminus\Delta_r} |f|^2\, d\sigma \\ &\le C\int_{\partial\Omega} \left\{\frac{|Q|}{|Q|+r}\right\}^\lambda |f(Q)|^2\, d\sigma(Q). \end{aligned}$$

From this, estimate (4.3) follows easily. The proof is complete. □

Lemma 4.18. *Let u be a harmonic function in Ω such that $(\nabla u)^* \in L^2(\partial\Omega)$ and $\frac{\partial u}{\partial n} = f$ on $\partial\Omega$. Then there exists $\varepsilon = \varepsilon(\Omega) \in (0, 2]$ such that for any $Q_0 \in \partial\Omega$ and $r > 0$,*

$$(4.19)\qquad \int_{\substack{Q\in\partial\Omega \\ |Q-Q_0|<r}} |\nabla u|^2\, d\sigma \le C_\lambda\, r^\lambda \int_{\partial\Omega} \frac{|f(Q)|^2}{\{|Q-Q_0|+r\}^\lambda}\, d\sigma(Q),$$

where $0 < \lambda < d - 3 + \varepsilon$. If Ω is a C^1 domain, one may take $\varepsilon = 2$.

Proof. We may assume that $Q_0 = (0, 0)$ and

$$(4.20)\qquad \Omega \cap B(Q_0, r_0) = \left\{(X', x_d) \in \mathbb{R}^d :\ x_d > \psi(X')\right\} \cap B(Q_0, r_0),$$

where $r_0 = r_0(\Omega) > 0$ and $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz continuous. Also we only need to consider the case $0 < r < c_0 r_0$.

Let $f = g + h$, where $g = (f - f_{\Delta_{8r}})\chi_{\Delta_{8r}}$. Let

$$(4.21)\qquad u_g(X) = \int_{\partial\Omega} N(X, Q)\, g(Q)\, d\sigma(Q) \quad \text{and} \quad u_h(X) = \int_{\partial\Omega} N(X, Q)\, h(Q)\, d\sigma(Q).$$

Then $u = u_g + u_h + k$ for some constant k. By the L^2 estimate,

$$(4.22)\qquad \int_{\Delta_r} |\nabla u_g|^2\, d\sigma \le C\int_{\partial\Omega} |g|^2\, d\sigma \le C\int_{\Delta_{8r}} |f|^2\, d\sigma(Q).$$

To estimate ∇u_h on Δ_r, we first use a Rellich identity on the domain D_τ for $\tau \in (r, 2r)$ to obtain

$$(4.23)\qquad \int_{\Delta_r} |\nabla u_h|^2\, d\sigma \le C\int_{\Delta_{2r}} |h|^2\, d\sigma + C\int_{\Omega\cap\partial D_\tau} |\nabla u_h|^2\, d\sigma.$$

Integrating both sides of (4.23) then yields

$$(4.24)\qquad \int_{\Delta_r} |\nabla u_h|^2\, d\sigma \le C\int_{\Delta_{2r}} |f|^2\, d\sigma + \frac{C}{r}\int_{D_{2r}} |\nabla u_h|^2\, d\sigma.$$

By the proof of the Cacciopoli inequality,

$$(4.25)\qquad \int_{D_{2r}} |\nabla u_h|^2\, d\sigma \le \frac{C}{r^2}\int_{D_{3r}} |u_h - \beta|^2\, dX + \frac{C}{r}\int_{\Delta_{3r}} \left|\frac{\partial u_h}{\partial n}\right| |u_h - \beta|\, d\sigma,$$

where β is an arbitrary constant. It follows from (4.24)-(4.25) that

$$(4.26)\qquad \int_{\Delta_r} |\nabla u_h|^2\, d\sigma \le C\int_{\Delta_{3r}} |f|^2\, d\sigma + \frac{C}{r^3}\int_{D_{3r}} |u_h - \beta|^2\, dX + \frac{C}{r^2}\int_{\Delta_{3r}} |u_h - \beta|^2\, d\sigma.$$

Now note that

$$u_h(X) = \int_{\partial\Omega\setminus\Delta_{8r}} N(X,Q) f(Q)\, d\sigma(Q) + f_{\Delta_{8r}} \int_{\Delta_{8r}} N(X,Q)\, d\sigma(Q)$$
$$= \int_{\partial\Omega\setminus\Delta_{8r}} \{N(X,Q) - N(0,Q)\}\, f(Q)\, d\sigma(Q) + \beta + f_{\Delta_{8r}} \int_{\Delta_{8r}} N(X,Q)\, d\sigma(Q),$$

where we have let

$$\beta = \int_{\partial\Omega\setminus\Delta_{8r}} N(0,Q)\, f(Q)\, d\sigma(Q). \tag{4.27}$$

It follows from estimates (3.1)-(3.2) that for $X \in D_{3r}$,

$$\begin{aligned} |u_h(X) - \beta| &\le C\, r^\delta \int_{\partial\Omega\setminus\Delta_{8r}} \frac{|f(Q)|}{|Q|^{d-2+\delta}}\, d\sigma(Q) + C\, r |f_{\Delta_{8r}}| \\ &\le C\, r |f_{\Delta_{8r}}| + C\, r^{\frac{3+\lambda-d}{2}} \left\{ \int_{\partial\Omega\setminus\Delta_{8r}} \frac{|f(Q)|^2}{|Q|^\lambda}\, d\sigma(Q) \right\}^{1/2}, \end{aligned} \tag{4.28}$$

where $0 < \lambda < d - 3 + 2\delta$. This, together with (4.26), gives

$$\int_{\Delta_r} |\nabla u_h|^2\, d\sigma \le C \int_{\Delta_{8r}} |f|^2\, d\sigma + C\, r^\lambda \int_{\partial\Omega\setminus\Delta_{8r}} \frac{|f(Q)|^2}{|Q|^\lambda}\, d\sigma(Q). \tag{4.29}$$

Thus

$$\begin{aligned} \int_{\Delta_r} |\nabla u|^2\, d\sigma &\le 2 \int_{\Delta_r} |\nabla u_g|^2 d\sigma + 2 \int_{\Delta_r} |\nabla u_h|^2\, d\sigma \\ &\le C \int_{\Delta_{8r}} |f|^2\, d\sigma + C\, r^\lambda \int_{\partial\Omega\setminus\Delta_{8r}} \frac{|f(Q)|^2}{|Q|^\lambda}\, d\sigma(Q) \\ &\le C\, r^\lambda \int_{\partial\Omega} \frac{|f(Q)|^2}{\{|Q| + r\}^\lambda}\, d\sigma(Q), \end{aligned} \tag{4.30}$$

where $0 < \lambda < d - 3 + 2\delta$. Estimate (4.19) with $\varepsilon = 2\delta$ now follows.

Finally we call that if Ω is a C^1 domain, we may take δ to be any number in $(0,1)$ (see Remark 3.16). This implies that (4.19) holds for any $0 < \lambda < d - 1$. □

We are now ready to give to

Proof of Theorem 1.7. First we point out that the case $3 - d - \varepsilon < \alpha < 0$ follows from Lemma 4.18 by the same argument as in the proof of Theorem 1.9, given in Section 2.

The case $0 < \alpha < d - 1$ will follow from Lemma 4.2, as well as the solvability of the L^p Neumann problem for $1 < p < 2$ [DK]. Indeed, since $L^2(\partial\Omega, \omega_\alpha d\sigma) \subset L^p(\partial\Omega)$ for some $p = p(\alpha) \in (1,2)$ by Hölder's inequality, the uniqueness (up to a constant) follows directly from the uniqueness in L^p. To prove the existence, we fix $g \in L^2(\partial\Omega, \omega_\alpha d\sigma)$ with $\int_{\partial\Omega} g\, d\sigma = 0$, where $\omega_\alpha = \omega_\alpha(Q) = |Q - Q_0|^\alpha$. Let u be a harmonic function in Ω such that $(\nabla u)^* \in L^p(\partial\Omega)$ and $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$. We need to show that

$$\|(\nabla u)^*\|_{L^2(\partial\Omega,\omega_\alpha d\sigma)} \le C\, \|g\|_{L^2(\partial\Omega,\omega_\alpha d\sigma)}. \tag{4.31}$$

To this end, we let

$$g_j(Q) = \begin{cases} g(Q) & \text{for } Q \in \partial\Omega \setminus B(Q_0, \frac{1}{j}), \\ g_{B(Q_0,\frac{1}{j})\cap\partial\Omega} & \text{for } Q \in B(Q_0, \frac{1}{j}) \cap \partial\Omega. \end{cases} \tag{4.32}$$

It is easy to see that $g_j \in L^2(\partial\Omega)$, $\int_{\partial\Omega} g_j\, d\sigma = 0$, and $g_j \to g$ in $L^2(\partial\Omega, \omega_\alpha d\sigma)$ as $j \to \infty$.

Let u_j be a harmonic function in Ω such that $(\nabla u_j)^* \in L^2(\partial\Omega)$ and $\frac{\partial u_j}{\partial n} = g_j$ on $\partial\Omega$. Choose $\lambda \in (\alpha, d-1)$. We multiply both sides of (4.3) by $r^{\alpha-1}$ and integrate the resulting inequality in $r \in (0,\infty)$. This gives

$$\begin{aligned} &\int_{\partial\Omega} |(\nabla u_j)^*|^2 |Q-Q_0|^\alpha \, d\sigma(Q) \\ &\quad \le C \int_0^\infty r^{\alpha-1} \left\{ \int_{B(Q_0,r)\cap\partial\Omega} |(\nabla u_j)^*|^2 \, d\sigma(Q) \right\} dr \\ &\quad \le C \int_0^\infty \left\{ \int_{\partial\Omega} \left(\frac{|Q-Q_0|}{|Q-Q_0|+r} \right)^\lambda |g_j(Q)|^2 \, d\sigma(Q) \right\} dr \\ &\quad = C \int_{\partial\Omega} |g_j(Q)|^2 \left\{ \int_0^\infty r^{\alpha-1} \left(\frac{|Q-Q_0|}{|Q-Q_0|+r} \right)^\lambda dr \right\} d\sigma(Q) \\ &\quad \le C \int_{\partial\Omega} |g_j(Q)|^2 |Q-Q_0|^\alpha \, d\sigma(Q). \end{aligned} \tag{4.33}$$

Finally, since $g_j \to g$ in $L^p(\partial\Omega)$ for some $p > 1$, we have $(\nabla u_j)^* \to (\nabla u)^*$ in $L^p(\partial\Omega)$ by the L^p estimate [DK]. Thus there exists a subsequence $\{u_{j_k}\}$ such that $(\nabla u_{j_k})^* \to (\nabla u)^*$ a.e. on $\partial\Omega$. This, together with (4.33) and Fatou's Lemma, gives the desired estimate (4.31). The proof is complete. □

Remark 4.34. It is not known whether the condition $\alpha > 3 - d - \varepsilon$ in Theorem 1.7 is sharp. However, condition $\alpha < d - 1$ is necessary even for smooth domains. Indeed, let Ω be the unit ball centered at the origin. The Neumann function for Ω is given by

$$N(X,Y) = c_d \left\{ \frac{1}{|X-Y|^{d-2}} + \frac{1}{|X-\overline{Y}|^{d-2}} \right\}, \tag{4.35}$$

where $\overline{Y} = Y/|Y|^2$. Suppose that the Neumann problem (1.4) is uniquely solvable (modulo constants) for every $g \in L^2(\partial\Omega, \omega)$ with $\int_{\partial\Omega} g\, d\sigma = 0$. It follows from the Banach open mapping theorem that

$$\|T_i(g)\|_{L^2(\partial\Omega,\omega)} \le C\, \|g\|_{L^2(\partial\Omega,\omega)}, \tag{4.36}$$

where

$$\begin{aligned} T_i(f)(P) &= \int_{\partial\Omega} \frac{\partial}{\partial P_i} \{N(P,Q)\}\, g(P)\, d\sigma(Q) \\ &= 2(2-d)c_d \int_{\partial\Omega} \frac{P_i - Q_i}{|P-Q|^d}\, g(Q)\, d\sigma(Q). \end{aligned} \tag{4.37}$$

By an argument similar to that in [St3, pp. 210-211], this implies that ω is an A_2 weight on $\partial\Omega$. In particular, if $\omega = \omega_\alpha$, then $1 - d < \alpha < d - 1$.

Remark 4.38. Theorem 1.7 also holds on the exterior domain $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$ if we impose the condition $u(X) = O(|X|^{2-d})$ as $|X| \to \infty$ in (1.4). In this case the mean zero condition on g is not needed.

5. The invertibilty of layer potentials on $L^2(\partial\Omega, \omega_\alpha)$

In this section we study the invertibilty of the layer potentials on $L^2(\partial\Omega, \omega_\alpha)$. We also give the proof of Theorem 1.5.

Let $\Gamma(X-Y) = \Gamma(X,Y)$ denote the fundamental solution for Δ on $\mathbb{R}^d$. For $f \in L^p(\partial\Omega)$ and $1 < p < \infty$, let

$$\mathcal{S}(f)(X) = -\int_{\partial\Omega} \Gamma(X-Q) f(Q)\, d\sigma(Q), \tag{5.1}$$

$$\mathcal{D}(f)(X) = -\int_{\partial\Omega} \frac{\partial}{\partial n(Q)}\big\{\Gamma(X-Q)\big\} f(Q)\, d\sigma(Q). \tag{5.2}$$

The functions $\mathcal{S}(f)$ and $\mathcal{D}(f)$ defined on $\mathbb{R}^d \setminus \partial\Omega$ are called the single layer potential and double layer potential of f, respectively. Let T_+ and T_- denote the normal derivatives of $\mathcal{S}(f)$ as a function in $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$, respectively. Then one has the jump relation $T_+ - T_- = I$. Moreover, $\mathcal{D}_+(f)|_{\partial\Omega} = T_-^*(f)$ and $\mathcal{D}_-(f)|_{\partial\Omega} = T_+^*(f)$, where $\mathcal{D}_\pm(f)|_{\partial\Omega}$ denotes the non-tangential limit of $\mathcal{D}(f)$ on $\partial\Omega$ taken from $\Omega_\pm$, and $T_\pm^*$ is the adjoint operator of $T_\pm$. The boundedness of $T_\pm$ on $L^p(\partial\Omega)$ is a consequence of the boundedness of the Cauchy integrals on Lipschitz curves [CMM]. See e.g [V1] for details.

Let $L^p_0(\partial\Omega) = \big\{f \in L^p(\partial\Omega) : \int_{\partial\Omega} f\, d\sigma = 0\big\}$. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that for $1 < p < 2+\varepsilon$, the operators $T_+ : L^p_0(\partial\Omega) \to L^p_0(\partial\Omega)$ and $T_- : L^p(\partial\Omega) \to L^p(\partial\Omega)$ are invertible. This was proved in [V1] for $p = 2$, and in [DK] for the optimal range $1 < p < 2 + \varepsilon$.

Since $T_\pm$ are Calderón-Zygmund operators on $\partial\Omega$ and $\omega_\alpha \in A_2(\partial\Omega)$ for $1-d < \alpha < d-1$ by (1.11), $T_\pm$ are bounded on $L^2(\partial\Omega, \omega_\alpha d\sigma)$. This follows directly from the general theory of weighted norm inequalities for singular integrals [CF]. Moreover, we have

$$\|(\nabla \mathcal{S}(f))^*\|_{L^2(\partial\Omega,\omega_\alpha)} + \|(\mathcal{D}(f))^*\|_{L^2(\partial\Omega,\omega_\alpha)} \le C\, \|f\|_{L^2(\partial\Omega,\omega_\alpha)}. \tag{5.3}$$

We should point out that for the power weight ω_α, the weighted norm inequalities for singular integrals were obtained earlier in [St1].

With Theorem 1.5, Theorem 1.7 and their counterparts in Ω_- (see Remarks 2.23 and 4.38) at our disposal, we are able to establish the invertibility of $T_\pm$ on $L^2(\partial\Omega, \omega_\alpha)$ for $3-d-\varepsilon < \alpha < d-1$.

Let $L^2_0(\partial\Omega, \omega_\alpha) = \big\{f \in L^2(\partial\Omega, \omega_\alpha) : \int_{\partial\Omega} f\, d\sigma = 0\big\}$.

Theorem 5.4. *There exists $\varepsilon = \varepsilon(\Omega) \in (0,2]$ such that for $3-d-\varepsilon < \alpha < d-1$, the operators $T_+ : L^2_0(\partial\Omega, \omega_\alpha) \to L^2_0(\partial\Omega, \omega_\alpha)$ and $T_- : L^2(\partial\Omega, \omega_\alpha) \to L^2(\partial\Omega, \omega_\alpha)$ are invertible. Moreover, if Ω is a C^1 domain, one may take $\varepsilon = 2$.*

Proof. We give the proof of the invertibility of T_+ on $L^2_0(\partial\Omega, \omega_\alpha)$. The invertibility of T_- on $L^2(\partial\Omega, \omega_\alpha)$ may be handled in the same manner.

First, we note that since $L^2_0(\partial\Omega, \omega_\alpha) \subset L^p_0(\partial\Omega)$ for some $p \in (1,2)$ and T_+ is one-to-one on $L^p_0(\partial\Omega)$, T_+ must be one-to-one on $L^2_0(\partial\Omega, \omega_\alpha)$.

To show $T_+ : L^2_0(\partial\Omega, \omega_\alpha) \to L^2_0(\partial\Omega, \omega_\alpha)$ is onto, we let $f \in L^2_0(\partial\Omega, \omega_\alpha)$ and $u = \mathcal{S}(f)$. Using the jump relation and Remark 2.23 for the regularity problem in

Ω_-, we have

$$
\begin{aligned}
\|f\|_{L^2(\partial\Omega,\omega_\alpha)} &\le \|T_+(f)\|_{L^2(\partial\Omega,\omega_\alpha)} + \|T_-(f)\|_{L^2(\partial\Omega,\omega_\alpha)} \\
&\le \|T_+(f)\|_{L^2(\partial\Omega,\omega_\alpha)} + C\left\{\|\nabla_t \mathcal{S}(f)\|_{L^2(\partial\Omega,\omega_\alpha)} + \|\mathcal{S}(f)\|_{L^2(\partial\Omega,\omega_\alpha)}\right\} \\
&\le \|T_+(f)\|_{L^2(\partial\Omega,\omega_\alpha)} + C\left\{\|\nabla_t \mathcal{S}(f)\|_{L^2(\partial\Omega,\omega_\alpha)} + \left|\int_{\partial\Omega} \mathcal{S}(f)\, d\sigma\right|\right\},
\end{aligned} \tag{5.5}
$$

where we have used Poincaré's inequality in the last inequality. It follows from Theorem 1.5 that

$$
\|f\|_{L^2(\partial\Omega,\omega_\alpha)} \le C\left\{\|T_+(f)\|_{L^2(\partial\Omega,\omega_\alpha)} + \left|\int_{\partial\Omega} \mathcal{S}(f)\, d\sigma\right|\right\}. \tag{5.6}
$$

This implies that the range of $T_+ : L^2_0(\partial\Omega,\omega_\alpha) \to L^2_0(\partial\Omega,\omega_\alpha)$ is closed.

If $0 \le \alpha < d-1$, the range of $T_+ : L^2_0(\partial\Omega,\omega_\alpha) \to L^2_0(\partial\Omega,\omega_\alpha)$ is also dense. This is because $L^2_0(\partial\Omega)$ is dense in $L^2_0(\partial\Omega,\omega_\alpha)$ and T_+ is invertible on $L^2_0(\partial\Omega)$. It follows that $T_+ : L^2_0(\partial\Omega,\omega_\alpha) \to L^2_0(\partial\Omega,\omega_\alpha)$ is onto, and hence invertible.

Finally suppose $3-d-\varepsilon_1 < \alpha < 0$, where ε_1 is the smaller of the two ε's in Lemma 4.18 and Remark 2.23. Note that we may take $\varepsilon_1 = 2$ if Ω is a C^1 domain. Let $g \in L^2_0(\partial\Omega,\omega_\alpha)$. Since $L^2_0(\partial\Omega,\omega_\alpha) \subset L^2_0(\partial\Omega)$, there exists $f \in L^2_0(\partial\Omega)$ such that $T_+(f) = g$. Let $u = \mathcal{S}(f)$. It follows from Lemma 4.18 that $\nabla_t u \in L^2_0(\partial\Omega,\omega_\alpha)$. By Remark 2.23, $(\nabla u)^*_e \in L^2_0(\partial\Omega,\omega_\alpha)$. It follows that $T_-(f) \in L^2_0(\partial\Omega,\omega_\alpha)$. Thus $f = T_+(f) - T_-(f) \in L^2_0(\partial\Omega,\omega_\alpha)$. This shows that $T_+ : L^2_0(\partial\Omega,\omega_\alpha) \to L^2_0(\partial\Omega,\omega_\alpha)$ is onto, and hence invertible. The proof is now complete. □

We end this section with

Proof of Theorem 1.5. Let $f \in L^2(\partial\Omega,\omega_\alpha)$ and $u = \mathcal{D}(f)$ be the double layer potential of f. Then $u_+ = T^*_-(f)$ on $\partial\Omega$. By Theorem 5.4, T_- is invertible on $L^2(\partial\Omega,\omega_\alpha)$ for $3-d-\varepsilon < \alpha < d-1$ for some $\varepsilon \in (0,2]$. It follows from duality that T^*_- is invertible on $L^2(\partial\Omega,\omega_\alpha)$ for $1-d < \alpha < d-3+\varepsilon$. This gives the existence of solutions as well as estimate (1.6). Note that we may take $\varepsilon = 2$ if Ω is a C^1 domain.

For $1-d < \alpha \le 0$, the uniqueness follows from the uniqueness of the Dirichlet problem with data in $L^2(\partial\Omega)$ [D2], since $L^2(\partial\Omega,\omega_\alpha d\sigma) \subset L^2(\partial\Omega)$.

To show the uniqueness for the remaining case $0 < \alpha < d-3+\varepsilon$, we suppose that u is a harmonic function in Ω such that $(u)^* \in L^2(\partial\Omega,\omega_\alpha)$ and $u = 0$ on $\partial\Omega$. We may assume that $Q_0 = (0,0)$ and

$$
\Omega \cap B(Q_0,r_0) = \left\{(X',x_d) \in \mathbb{R}^d : \, x_d > \psi(X')\right\} \cap B(Q_0,r_0) \tag{5.7}
$$

for some $r_0 > 0$ and some Lipschitz function ψ with $\psi(0) = 0$. Consider the inverted cone

$$
\gamma_j = \left\{(X',x_d) : \; x_d < -K|X'| + \frac{1}{j}\right\}, \tag{5.8}
$$

where $K = 2\|\nabla\psi\|_\infty + 1$. Let $\Omega_j = \Omega \setminus (\gamma_j \cap B(Q_0,r_0))$ for j large. Then Ω_j is a bounded Lipschitz domain with connected boundary and uniform Lipschitz character. Let $G_j(X,Y)$ denote the Green's function for the Laplacian on Ω_j. Fix $X \in \Omega$. Since the non-tangential maximal function of u with respect to the domain Ω_j is in $L^2(\partial\Omega_j)$, we have

$$
u(X) = \int_{\partial\Omega_j} \frac{\partial G_j^X}{\partial n}\, u(Q)\, d\sigma(Q) \tag{5.9}
$$

for j sufficiently large, where $G_j^X(\cdot) = G_j(X,\cdot)$. It follows that

$$
\begin{aligned}
|u(X)| &\le \int_{\partial\Omega_j} |\nabla G_j^X(Q)|\,|u(Q)|\,d\sigma(Q) \\
&\le \left\{\int_{\partial\Omega_j} |\nabla G_j^X(Q)|^2 |Q-Q_j|^{-\alpha}\,d\sigma(Q)\right\}^{\frac{1}{2}} \left\{\int_{\partial\Omega_j} |u(Q)|^2 |Q-Q_j|^{\alpha}\,d\sigma(Q)\right\}^{\frac{1}{2}},
\end{aligned} \tag{5.10}
$$

where $Q_j = (0, \frac{1}{j})$.

Finally we note that $G_j(X,Y)$ may be written as $\Gamma(X-Y) - W_j^X(Y)$, where W_j^X is the harmonic function in Ω_j such that $(\nabla W_j^X)^* \in L^2(\partial\Omega_j)$ and $W_j^X(Q) = \Gamma(X-Q)$ on $\partial\Omega_j$. By Theorem 1.9 (for the case $3-d-\varepsilon < \alpha < d-1$ which we have proved), we have

$$
\begin{aligned}
&\int_{\partial\Omega_j} |\nabla G_j^X(Q)|^2 |Q-Q_j|^{-\alpha}\,d\sigma(Q) \\
&\qquad \le C \int_{\partial\Omega_j} |\nabla\Gamma(X-Q)|^2 |Q-Q_j|^{-\alpha}\,d\sigma(Q) \le C_X,
\end{aligned} \tag{5.11}
$$

where C_X is independent of j. It follows from (5.10)-(5.11) that

$$
\begin{aligned}
|u(X)|^2 &\le C_X \int_{\partial\Omega_j} |u(Q)|^2 |Q-Q_j|^{\alpha}\,d\sigma(Q) \\
&\le C_X \int_{B(0,\frac{c}{j})\cap\partial\Omega} |(u)^*|^2\,|Q|^{\alpha}\,d\sigma(Q) \to 0
\end{aligned}
$$

as $j \to \infty$. Thus $u(X) = 0$. The uniqueness is proved. □

Remark 5.12. The condition $\alpha < d-3+\varepsilon$ in Theorem 1.5 is sharp. To see this, we recall the Lipschitz domain Ω and the harmonic function $u_\lambda = r^\lambda v_\lambda(\theta)$ constructed in Remark 2.22. We may assume $v_\lambda(\theta) \ge 0$ since it has a constant sign. Let $G^X(Y) = G(X,Y)$ be the Green's function on Ω. Fix $Z \in \Omega$. Since $u_\lambda \ge 0$ in Ω and vanishes on $B(0,1)\cap\partial\Omega$, we may use the comparison principle (see e.g. [K2, p. 10]) to show that $|\nabla G^Z(Q)| \ge c\,|\nabla u_\lambda(Q)| \sim |Q|^{\lambda-1}$ for $Q \in B(0,1/2)\cap\partial\Omega$. It follows that $|\frac{\partial G^Z}{\partial n}| \notin L^2(\partial\Omega,\omega_\alpha)$ if $\alpha \le 3-d-2\lambda$.

Now suppose that the Dirichlet problem (1.3) is uniquely solvable for every $f \in L^2(\partial\Omega,\omega_\alpha)$. Let $f \in C(\partial\Omega)$ and u_f be the solution of the classical Dirichlet problem with data f. Then

$$
|u_f(Z)| \le C \int_{\partial\Omega} |(u_f)^*|^2\,\omega_\alpha\,d\sigma \le C \int_{\partial\Omega} |f|^2\,\omega_\alpha\,d\sigma. \tag{5.13}
$$

By duality, this implies that $|\frac{\partial G^Z}{\partial n}| \in L^2(\partial\Omega,\omega_{-\alpha})$. It follows that the Dirichlet problem (1.3) is not uniquely solvable for $\alpha \ge d-3+2\lambda$.

6. The proof of Theorem 1.9 for the case $d-1 \le \alpha < d-1+\varepsilon$

In this section we complete the proof of Theorem 1.9 by finishing the remaining case $d-1 \le \alpha < d-1+\varepsilon$. Recall that for $f \in L^2(\partial\Omega)$, u_f denotes the unique harmonic function in Ω such that $(u)^* \in L^2(\partial\Omega)$ and $u = f$ on $\partial\Omega$.

Lemma 6.1. *There exists $\varepsilon \in (0,2]$ such that if f and $|\nabla_t f| \in L^2(\partial\Omega)$, then*

$$\int_{\substack{Q\in\partial\Omega\\ |Q-Q_0|>r}} |(\nabla u_f)^*|^2 d\sigma \leq C_\lambda \int_{\partial\Omega} \left(\frac{|Q-Q_0|}{|Q-Q_0|+r}\right)^\lambda \left(|\nabla_t f(Q)|^2 + \frac{|f(Q)|^2}{|Q-Q_0|^2}\right) d\sigma(Q) \tag{6.2}$$

for any $Q_0 \in \partial\Omega$ and $r>0$, where $\lambda \in [d-1, d-1+\varepsilon)$. If Ω is a C^1 domain, one may take $\varepsilon = 2$.

Proof. Assume $Q_0 = 0$. We proceed as in the proof of Lemma 2.1. Let $f = g+h$, where $g = f\varphi$, $h = f(1-\varphi)$, and φ is a function in $C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on Δ_r, $\varphi \equiv 0$ on $\partial\Omega \setminus \Delta_{2r}$ and $|\nabla\varphi| \leq C/r$. Then $u = u_g + u_h$. By the L^2 regularity estimate,

$$\int_{\partial\Omega} |(\nabla u_h)^*|^2 \, d\sigma \leq C \int_{\partial\Omega} |\nabla_t h|^2 \, d\sigma \leq C \int_{\partial\Omega\setminus\Delta_r} |\nabla_t f|^2 \, d\sigma + \frac{C}{r^2} \int_{\Delta_{2r}\setminus\Delta_r} |f|^2 \, d\sigma. \tag{6.3}$$

To estimate $(\nabla u_g)^*$ on $\partial\Omega \setminus \Delta_{cr}$, we note that $g = 0$ on $\partial\Omega \setminus \Delta_{2r}$. Let $E_j = \Delta_{2^j r} \setminus \Delta_{2^{j-1} r}$ where $j \geq 5$. As in the proof of Lemma 4.2, we have $(\nabla u_g)^*(Q) \leq M_1(\nabla u_g)(Q) + M_2(\nabla u_g)(Q)$ for $Q \in E_j$, where M_1, M_2 are defined in (4.8)-(4.9). By an argument similar to (4.12)-(4.13), we may show that

$$\int_{E_j} |M_1(\nabla u_g)|^2 \, d\sigma \leq \frac{C}{(2^j r)^3} \int_{D_{2^{j+2}r}\setminus D_{2^{j-3}r}} |u_g(X)|^2 \, dX. \tag{6.4}$$

Note that for $X \in D_{2^{j+2}r} \setminus D_{2^{j-3}r}$, we have

$$\begin{aligned} |u_g(X)| &\leq \int_{\Delta_{2r}} |\nabla G^X(Q)| \, |f(Q)| \, d\sigma(Q) \\ &\leq \left\{\int_{\Delta_{2r}} |\nabla G^X(Q)|^2 \, |Q|^{2-\lambda} d\sigma(Q)\right\}^{1/2} \left\{\int_{\Delta_{2r}} |f(Q)|^2 |Q|^{\lambda-2} d\sigma(Q)\right\}^{1/2}, \end{aligned} \tag{6.5}$$

where $G^X(Y) = G(X,Y)$ denotes the Green's function on Ω.

We now apply Theorem 1.9 for the case $3-d-\varepsilon < \alpha \leq 0$ to the harmonic function $G^X(\cdot)$ on Lipchitz domain D_{tr} with $t \in (2, 5/2)$. This gives

$$\begin{aligned} \int_{\Delta_{2r}} |\nabla G^X(Q)|^2 |Q|^{2-\lambda} \, d\sigma(Q) &\leq C \int_{\Omega\cap\partial D_{tr}} |\nabla G^X(Q)|^2 |Q|^{2-\lambda} \, d\sigma(Q) \\ &\leq C \, r^{2-\lambda} \int_{\Omega\cap\partial D_{tr}} |\nabla G^X(Q)|^2 \, d\sigma(Q) \end{aligned} \tag{6.6}$$

for $d-1 \leq \lambda < d-1+\varepsilon$, where $\varepsilon = \varepsilon(\Omega) > 0$. From (6.6), by a familiar integration argument and the Cacciopoli inequality, we obtain

$$\begin{aligned} \int_{\Delta_{2r}} |\nabla G^X(Q)|^2 |Q|^{2-\lambda} \, d\sigma(Q) &\leq C \, r^{-1-\lambda} \int_{D_{3r}} |G^X(Y)|^2 \, dY \\ &\leq \frac{C \, r^{-1-\lambda+d+2\eta}}{(2^j r)^{2d-4+2\eta}}, \end{aligned} \tag{6.7}$$

where $\eta \in (0,1)$. In view of (6.4)-(6.5) and (6.7), we have proved that

$$\int_{E_j} |M_1(\nabla u_g)|^2 \, d\sigma \leq \frac{C \, r^{-\lambda}}{(2^j)^{d-1+2\eta}} \int_{\Delta_{2r}} |f(Q)|^2 \, |Q|^{\lambda-2} \, d\sigma(Q). \tag{6.8}$$

We point out that the same estimate can also be proved for $M_2(\nabla u_g)$ on E_j, using the interior estimates and (6.5)-(6.7). Thus, by summation, we have

$$\int_{\partial\Omega\setminus\Delta_{cr}} |(\nabla u_g)^*|^2\, d\sigma \le \frac{C}{r^\lambda}\int_{\Delta_{2r}} |f(Q)|^2\, |Q|^{\lambda-2}\, d\sigma(Q) \tag{6.9}$$

for $d-1 \le \lambda < d-1+\varepsilon$. It is not hard to see that the desired estimate (6.2) follows from (6.3) and (6.9).

Finally we note that ε depends on the Lipschitz character of the domain D_τ. If Ω is a C^1 domain, one may take $\varepsilon = 2$. To see this, it suffices to observe that in place of D_τ, we may construct a continuum of C^1-domains Ω_τ such that $\Omega_\tau \subset D_{c\tau}$ and $\Delta_\tau \subset \partial\Omega_\tau \cap \partial\Omega$. This completes the proof of Lemma 6.1. □

Lemma 6.10. *Let $\varepsilon \in (0,2]$ be given in Lemma 6.1. Suppose f, $|\nabla_t f| \in L^2(\partial\Omega)$. Then*

$$\int_{\partial\Omega} |(\nabla u_f)^*|^2\, \omega_\alpha\, d\sigma \le C\int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma \tag{6.11}$$

for any $d-1 \le \alpha < d-1+\varepsilon$. If Ω is a C^1 domain, one may take $\varepsilon = 2$.

Proof. Let $\alpha \in [d-1, d-1+\varepsilon)$. Choose any $\lambda \in (\alpha, d-1+\varepsilon)$. We multiply both sides of (6.2) by $r^{\alpha-1}$ and integrate the resulting inequality in $r \in (0,\infty)$. This gives

$$\int_{\partial\Omega} |(\nabla u_f)^*|^2\, \omega_\alpha\, d\sigma \le C\int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma + C\int_{\partial\Omega} |f|^2\, \omega_{\alpha-2}\, d\sigma. \tag{6.12}$$

By Hardy's inequality, one may show that

$$\int_{\partial\Omega} |f|^2\, \omega_{\alpha-2}\, d\sigma \le C\int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma + C\int_{\partial\Omega\setminus B(Q_0,c_0)} |f|^2\, d\sigma. \tag{6.13}$$

It follows that

$$\int_{\partial\Omega} |(\nabla u_f)^*|^2\, \omega_\alpha\, d\sigma \le C\int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma + C\int_{\partial\Omega\setminus B(Q_0,c_0)} |f-k|^2\, d\sigma \tag{6.14}$$

for any constant $k \in \mathbb{R}$. Finally, let $k = f_{\partial\Omega\setminus B(Q_0,c_0)}$ be the average of f over $\partial\Omega \setminus B(Q_0, c_0)$. By Poincaré's inequality, we have

$$\begin{aligned}\int_{\partial\Omega} |(\nabla u_f)^*|^2\, \omega_\alpha\, d\sigma &\le C\int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma + C\int_{\partial\Omega\setminus B(Q_0,c_0)} |\nabla_t f|^2\, d\sigma\\ &\le C\int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma.\end{aligned} \tag{6.15}$$

The proof is finished. □

The following lemma will be used to prove the uniqueness.

Lemma 6.16. *Let u be a harmonic function in Ω. Suppose $\alpha > 3-d$. Then*

$$\int_{\partial\Omega} |(u)^*|^2\, \omega_{\alpha-2}\, d\sigma \le C\int_{\partial\Omega} |(\nabla u)^*|^2\, \omega_\alpha\, d\sigma + C\sup_{X\in K} |u(X)|^2, \tag{6.17}$$

where K is a compact subset of Ω.

Proof. By approximating Ω with a sequence of smooth domains, we may assume that $u \in C^1(\overline{\Omega})$. Let $p \in (1, \infty)$ and $q = (2p-1)/(2p-2)$. It follows from the proof of Lemma 2.2 in [B] that

$$(u)^* \le C\, \{S(u)\}^{1/p} \left\{ I_{1/q}(|(\nabla u)^*|^{1/q}) \right\}^{1-\frac{1}{2p}} + C \sup_K |u|, \tag{6.18}$$

where $S(u)$ denotes the square function of u (see (7.17)) and $I_{1/q}$ is the fractional integral operator of order $1/q$ [St2]. It follows from Hölder inequality that

(6.19)

$$\begin{aligned} &\int_{\partial\Omega} |(u)^*|^2 \omega_{\alpha-2}\, d\sigma \\ &\le C \left\{ \int_{\partial\Omega} |S(u)|^2\, \omega_{\alpha-2}\, d\sigma \right\}^{\frac{1}{p}} \left\{ \int_{\partial\Omega} \left\{ I_{\frac{1}{q}}(|(\nabla u)^*|^{\frac{1}{q}}) \right\}^{2q} \omega_{\alpha-2}\, d\sigma \right\}^{\frac{1}{p'}} + C \sup_K |u|^2, \end{aligned}$$

where $p' = p/(p-1)$. This, together with Dahlberg's square function estimates [D3] and Hölder inequality with an ε, implies that

$$\int_{\partial\Omega} |(u)^*|^2\, \omega_{\alpha-2}\, d\sigma \le C \int_{\partial\Omega} \left\{ I_{\frac{1}{q}}(|(\nabla u)^*|^{\frac{1}{q}}) \right\}^{2q} \omega_{\alpha-2}\, d\sigma + C \sup_K |u|^2. \tag{6.20}$$

Finally, we choose $p \in (1, \infty)$ so that $p' > \alpha/(d-1)$. The desired estimate (6.17) follows from the weighted norm inequalities for the fractional integral operator $I_{1/q}$ (see e.g. [SW]). □

We are now in a position to give

Proof of Theorem 1.9 *for the case* $d-1 \le \alpha < d-1+\varepsilon$. We begin with the uniqueness. Suppose that $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$ and $(\nabla u)^* \in L^2(\partial\Omega, \omega_\alpha)$ for some $d-1 \le \alpha < d-1+\varepsilon$, where ε is the smaller of the two ε's in Theorem 1.5 and Lemma 6.10. By Lemma 6.16, this implies that $(u)^* \in L^2(\partial\Omega, \omega_{\alpha-2})$. Since $\alpha - 2 \in [d-3, d-3+\varepsilon)$, by the uniqueness in Theorem 1.5, we have $u \equiv 0$ in Ω.

To show the existence, we let $f \in L^2_1(\partial\Omega, \omega_\alpha)$ for some $\alpha \in [d-1, d-1+\varepsilon)$. By (6.13), $f \in L^2(\partial\Omega, \omega_{\alpha-2})$. It follows from Theorem 1.5 that there exists a harmonic function u on Ω such that $(u)^* \in L^2(\partial\Omega, \omega_{\alpha-2})$ and $u = f$ on $\partial\Omega$. Also, since $L^2_1(\partial\Omega)$ is dense in $L^2_1(\partial\Omega, \omega_\alpha)$, there exists a sequence of functions $\{f_j\}$ in $L^2_1(\partial\Omega)$ such that $f_j \to f$ in $L^2_1(\partial\Omega, \omega_\alpha)$. It follows from estimate (1.6) that $u_{f_j} \to u$ uniformly in any compact subset of Ω. By Lemma 6.10, this implies

$$\int_{\partial\Omega} |(\nabla u)^*_\delta|^2\, \omega_\alpha\, d\sigma \le C \int_{\partial\Omega} |\nabla_t f|^2\, \omega_\alpha\, d\sigma, \tag{6.21}$$

where $(\nabla u)^*_\delta(Q) = \sup\{|\nabla u(X)| :\ X \in \Omega \text{ and } \delta < |X-Q| < 2\,\mathrm{dist}(X, \partial\Omega)\}$. The desired estimate follows from (6.21) by the monotone convergence theorem. The proof is finished. □

7. Estimates with A_p weights

In this section we establish two weighted estimates with general A_p weights for the Dirichlet and regularity problems in Lipschitz domains.

Recall that for $1 < p < \infty$, a non-negative, locally integrable function ω on $\partial\Omega$ is called an A_p weight if

$$\frac{1}{r^{d-1}}\int_{B(Q,r)\cap\partial\Omega}\omega\,d\sigma\cdot\left(\frac{1}{r^{d-1}}\int_{B(Q,r)\cap\partial\Omega}\omega^{-\frac{1}{p-1}}\,d\sigma\right)^{p-1}\le A<\infty \tag{7.1}$$

for any $Q\in\partial\Omega$ and $0<r<\operatorname{diam}(\partial\Omega)$. The smallest constant A for which (7.1) holds is called the A_p bound of ω. We will write $\omega\in A_p(\partial\Omega)$ if ω is an A_p weight on $\partial\Omega$. It is well known that $\omega\in A_p(\partial\Omega)$ implies that $\omega\in A_q(\partial\Omega)$ for some $q<p$, where q depends on the A_p bound of ω. We refer the reader to [St3] for the elegant theory of A_p weights as well as the definitions of A_1 and A_∞ weights.

Theorem 7.2. *There exists $\delta\in(0,1]$ such that if $f\in L^2(\partial\Omega,\omega)$ and $\omega\in A_p(\partial\Omega)$ with $p=1+\delta$, then there exists a unique harmonic function u in Ω such that $(u)^*\in L^2(\partial\Omega,\omega)$ and $u=f$ on $\partial\Omega$. Moreover, we have*

$$\|(u)^*\|_{L^2(\partial\Omega,\omega)}\le C\,\|f\|_{L^2(\partial\Omega,\omega)}, \tag{7.3}$$

where C depends on the A_p bound of ω. If Ω is a C^1 domain, one may take $\delta=1$.

Proof. It follows from [D2] that there exists $\varepsilon\in(0,1]$ such that given any $f\in L^q(\partial\Omega)$ with $2-\varepsilon<q\le\infty$, there exists a unique harmonic function u such that $(u)^*\in L^q(\partial\Omega)$ and $u=f$ on $\partial\Omega$. Furthermore, the solution u satisfies

$$(u)^*\le C_s\big\{M(|f|^s)\big\}^{1/s}\quad\text{for any } 2-\varepsilon<s<\infty, \tag{7.4}$$

where M denotes the Hardy-Littlewood maximal operator on $\partial\Omega$. In the C^1 case, one may take $\varepsilon=1$.

We let $\delta=\varepsilon/(2-\varepsilon)$. Recall that $\omega\in A_{1+\delta}(\partial\Omega)$ implies that $\omega\in A_p(\partial\Omega)$ for some $1<p<1+\delta$. Also, by Hölder inequality, $L^2(\partial\Omega,\omega)\subset L^q(\partial\Omega)$ if $1<q<2$ and ω is an $A_{2/q}(\partial\Omega)$ weight. Thus if ω is an $A_{1+\delta}$ weight on $\partial\Omega$, then $L^2(\partial\Omega,\omega)\subset L^q(\partial\Omega)$ for some $2-\varepsilon<q<2$. The uniqueness in the theorem follows from [D2].

To show the existence as well as estimate (7.3), we let $f\in L^2(\partial\Omega,\omega)$, where $\omega\in A_p(\partial\Omega)$ for some $1<p<1+\delta$. Because $L^2(\partial\Omega,\omega)\subset L^q(\partial\Omega)$ for $q=2/p\in(2-\varepsilon,2)$, there exists a harmonic function satisfying $u=f$ on $\partial\Omega$ and estimate (7.4) for $[2/(\delta+1)]<s<\infty$. Let $s=\frac{2}{p}$. Since $s>2/(1+\delta)$, we have

$$\begin{aligned}\int_{\partial\Omega}|(u)^*|^2\,\omega\,d\sigma&\le C\int_{\partial\Omega}\big\{M(|f|^s)\big\}^{\frac{2}{s}}\,\omega\,d\sigma\\&\le C\int_{\partial\Omega}|f|^2\,\omega\,d\sigma,\end{aligned} \tag{7.5}$$

where we have used the weighted norm inequality for the operator M in the second inequality [M]. The proof is complete. □

The rest of this section is devoted to the proof of the following theorem on the regularity problem.

Theorem 7.6. *There exists $\delta\in(0,1]$ such that if f, $|\nabla_t f|\in L^2(\partial\Omega,\frac{1}{\omega})$ and $\omega\in A_p(\partial\Omega)$ with $p=1+\delta$, then there exists a unique harmonic function u on Ω satisfying $(\nabla u)^*\in L^2(\partial\Omega,\frac{1}{\omega})$ and $u=f$ on $\partial\Omega$. Moreover, we have*

$$\|(\nabla u)^*\|_{L^2(\partial\Omega,\frac{1}{\omega})}\le C\,\|\nabla_t f\|_{L^2(\partial\Omega,\frac{1}{\omega})}, \tag{7.7}$$

where C depends on the A_p bound of ω. If Ω is a C^1 domain, one may take $\delta=1$.

To prove Theorem 7.6, we follow an approach found in [V1], where it was used by Verchota to establish the solvability of the L^p regularity problem for $1 < p < 2$. We will first prove Theorem 7.6 for the starshaped Lipschitz domains.

Lemma 7.8. *Let Ω be a starshaped Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$. Let $\delta \in (0,1]$ be given by Theorem 7.2. Then if $\Delta u = 0$ in Ω and $(\nabla u)^* \in L^2(\partial\Omega, \frac{1}{\omega})$ for some $\omega \in A_{1+\delta}(\partial\Omega)$, we have*

$$\|\nabla u\|_{L^2(\partial\Omega,\frac{1}{\omega})} \leq C\,\|\nabla_t u\|_{L^2(\partial\Omega,\frac{1}{\omega})}. \tag{7.9}$$

Proof. Recall that $\omega \in A_{1+\delta}(\partial\Omega)$ implies that $\omega \in L^q(\partial\Omega)$ for some $q > 1$. It follows from Hölder inequality that $L^2(\partial\Omega, \frac{1}{\omega}) \subset L^p(\partial\Omega)$ for some $p > 1$. Consequently, ∇u has non-tangential limit a.e. on $\partial\Omega$ if $(\nabla u)^* \in L^2(\partial\Omega, \frac{1}{\omega})$.

We may assume that Ω is starshaped with respect to the origin and $u(0) = 0$. Let g be a Lipschitz continuous function on $\partial\Omega$ and v its harmonic extension on Ω. We shall use a radial version of the conjugate harmonic system for v, introduced in [V2] (also see [PV]). Let

$$H(X) = \int_0^1 v(rX)\,\frac{dr}{r}, \quad \text{for } X \in \Omega. \tag{7.10}$$

Then $\Delta H = 0$ in Ω and $X \cdot \nabla H(X) = v(X)$. We claim that

$$\|(\nabla H)^*\|_{L^2(\partial\Omega,\omega)} \leq C\|g\|_{L^2(\partial\Omega,\omega)}. \tag{7.11}$$

Assume (7.11) for a moment; we give the proof of estimate (7.9).

It follows from integration by parts that

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial u}{\partial n}\, g\, d\sigma &= \int_{\partial\Omega} u \frac{\partial v}{\partial n}\, d\sigma \\ &= \int_{\partial\Omega} u \frac{\partial H}{\partial n}\, d\sigma + \int_{\partial\Omega} u\, x_j \frac{\partial}{\partial t_{ij}} \left(\frac{\partial H}{\partial x_i}\right) d\sigma \\ &= (2-d)\int_{\partial\Omega} u\, \frac{\partial H}{\partial n}\, d\sigma - \int_{\partial\Omega} x_j \frac{\partial u}{\partial t_{ij}}\, \frac{\partial H}{\partial x_i}\, d\sigma,\end{aligned} \tag{7.12}$$

where $\frac{\partial}{\partial t_{ij}} = n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i}$ is a tangential derivative, and we have used the convention that repeated indices are summed from 1 to d. We point out that with estimate (7.11), we may justify the integration by parts argument in (7.12) easily by approximating Ω from inside with a sequence of starshaped smooth domains (see e.g. [JK]). It follows from (7.12), Cauchy inequality and (7.11) that

$$\left|\int_{\partial\Omega} \frac{\partial u}{\partial n}\, g\, d\sigma\right| \leq C\,\left\{\|\nabla_t u\|_{L^2(\partial\Omega,\frac{1}{\omega})} + \|u - u_{\partial\Omega}\|_{L^2(\partial\Omega,\frac{1}{\omega})}\right\} \|g\|_{L^2(\partial\Omega,\omega)}, \tag{7.13}$$

where $u_{\partial\Omega} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u\, d\sigma$. Since $\omega \in A_2(\partial\Omega)$, we have $\frac{1}{\omega} \in A_2(\partial\Omega)$. By [FKS], we have

$$\|u - u_{\partial\Omega}\|_{L^2(\partial\Omega,\frac{1}{\omega})} \leq C\,\|\nabla_t u\|_{L^2(\partial\Omega,\frac{1}{\omega})}. \tag{7.14}$$

Hence, by duality, estimate (7.13) gives

$$\|\frac{\partial u}{\partial n}\|_{L^2(\partial\Omega,\frac{1}{\omega})} \leq C\,\|\nabla_t u\|_{L^2(\partial\Omega,\frac{1}{\omega})}. \tag{7.15}$$

Estimate (7.9) now follows.

Finally we need to prove the claim (7.11). The proof is similar to that of the L^p case in [V2]. Indeed, since $\Delta H = 0$ in Ω and $\omega \in A_\infty(\partial\Omega)$, by the square function estimates in [D3],

$$\|(\nabla H)^*\|_{L^2(\partial\Omega,\omega)} \le C\|S(\nabla H)\|_{L^2(\partial\Omega,\omega)}, \tag{7.16}$$

where $S(h)$ is the square function defined by

$$S(h)(Q) = \left\{ \int_{\gamma(Q)} |\nabla h(X)|^2\, |X-Q|^{2-d}\, dX \right\}^{1/2} \tag{7.17}$$

and $\gamma(Q) = \{X \in \Omega :\ \text{dist}(X, \partial\Omega) < 2|X-Q|\}$. Let $\widetilde{S}(h)$ denote a square function defined similarly using $\widetilde{\gamma}(Q) = \{X \in \Omega :\ \text{dist}(X, \partial\Omega) < 4|X-Q|\}$. Then an argument similar to that in [St2, pp. 213-216] shows that

$$S\left(\frac{\partial H}{\partial x_i}\right)(Q) \le C\left\{ \widetilde{S}(v)(Q) + \widetilde{S}(H)(Q) + \sup_{X\in K} |H(X)| \right\}, \tag{7.18}$$

where K is some compact set in Ω. Again, by the square function estimates in [D3], we obtain

$$\begin{aligned} \|(\nabla H)^*\|_{L^2(\partial\Omega,\omega)} &\le C\left\{ \|(v)^*\|_{L^2(\partial\Omega,\omega)} + \|(H)^*\|_{L^2(\partial\Omega,\omega)} \right\} \\ &\le C\, \|(v)^*\|_{L^2(\partial\Omega,\omega)} \\ &\le C\, \|g\|_{L^2(\partial\Omega,\omega)}, \end{aligned}$$

where we have used Theorem 7.2 in the last inequality. This completes the proof. □

The following lemma for the exterior domain $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$ may be proved by the same argument as in the proof of Lemma 7.8. We remark that in place of (7.10), one should use

$$H(X) = -\int_1^\infty v(rX)\, \frac{dr}{r}. \tag{7.19}$$

Lemma 7.20. *Let Ω be a starshaped Lipschitz domain. Then there exists $\delta \in (0,1]$ such that if $\Delta u = 0$ in Ω_-, $(\nabla u)^*_e \in L^2(\partial\Omega, \frac{1}{\omega})$ for some $\omega \in A_{1+\delta}(\partial\Omega)$, and $|u(X)| = O(|X|^{2-d})$ as $|X| \to \infty$, then*

$$\|\nabla u\|_{L^2(\partial\Omega,\frac{1}{\omega})} \le C\left\{ \|\nabla_t u\|_{L^2(\partial\Omega,\frac{1}{\omega})} + \|u\|_{L^2(\partial\Omega,\frac{1}{\omega})} \right\}. \tag{7.21}$$

If Ω is a starshaped C^1 domain, one may take $\delta = 1$.

Lemma 7.22. *Theorem* 7.6 *holds if in addition, we assume that Ω is starshaped.*

Proof. Suppose that Ω is a starshaped Lipschitz domain. Since $L^2(\partial\Omega, \frac{1}{\omega}) \subset L^q(\partial\Omega)$ for some $q > 1$ if $\omega \in A_p(\partial\Omega)$ for some $p > 1$, the uniqueness part of the theorem follows from the uniqueness of the L^q regularity problem.

Note that if $\omega \in A_2(\partial\Omega)$ and $\mathcal{S}(f)$ denotes the single layer potential of f, then

$$\|(\nabla \mathcal{S}(f))^*\|_{L^2(\partial\Omega,\frac{1}{\omega})} \le C\, \|f\|_{L^2(\partial\Omega,\frac{1}{\omega})}. \tag{7.23}$$

Thus, to establish the existence and estimate (7.7), it suffices to show that $\mathcal{S} : L^2(\partial\Omega, \frac{1}{\omega}) \to L^2_1(\partial\Omega, \frac{1}{\omega})$ is invertible if $\omega \in A_{1+\delta_1}(\partial\Omega)$ and $\delta_1 = \delta_1(\Omega) > 0$ is small.

To this end, we use Lemmas 7.8 and 7.20 to obtain

$$\begin{aligned} \|f\|_{L^2(\partial\Omega,\frac{1}{\omega})} &\le \|T_+(f)\|_{L^2(\partial\Omega,\frac{1}{\omega})} + \|T_-(f)\|_{L^2(\partial\Omega,\frac{1}{\omega})} \\ &\le C\left\{\|\nabla_t \mathcal{S}(f)\|_{L^2(\partial\Omega,\frac{1}{\omega})} + \|\mathcal{S}(f)\|_{L^2(\partial\Omega,\frac{1}{\omega})}\right\} \end{aligned} \tag{7.24}$$

if $\omega \in A_{1+\delta_1}(\partial\Omega)$ and $\delta_1 > 0$ is the smaller number of the two δ's in Lemmas 7.8 and 7.20. This shows that the operator $\mathcal{S} : L^2(\partial\Omega, \frac{1}{\omega}) \to L^2_1(\partial\Omega, \frac{1}{\omega})$ is one-to-one and the range is closed.

Finally, since $S : L^q(\partial\Omega) \to L^q_1(\partial\Omega)$ is invertible for some $q > 2$ [V1], [DK] and $L^q(\partial\Omega) \subset L^2(\partial\Omega, \frac{1}{\omega})$ if $\omega \in A_{2(1-\frac{1}{q})}$, we conclude that the range of $\mathcal{S} : L^2(\partial\Omega, \frac{1}{\omega}) \to L^2_1(\partial\Omega, \frac{1}{\omega})$ is dense if $\omega \in A_{1+\delta_1}$ and $\delta_1 \le 1 - (2/q)$. The invertibility of $\mathcal{S}$ on $L^2(\partial\Omega, \frac{1}{\omega})$ now follows. Note that if Ω is a C^1 domain, then $\mathcal{S} : L^q(\partial\Omega) \to L^q_1(\partial\Omega)$ is invertible for any $1 < q < \infty$ [FJR]. An inspection of the proof given above for the Lipschitz domains shows clearly that one may take $\delta_1 = 1$ in the C^1 case. This completes the proof. □

We are now in a position to give

Proof of Theorem 7.6. As in the proof of Lemma 7.22, the uniqueness part of the theorem follows from the uniqueness of the L^p regularity problem for $1 < p < 2$. For the existence and estimate (7.7), we only need to show that there exists $\delta \in (0, 1]$ such that $\mathcal{S} : L^2(\partial\Omega, \frac{1}{\omega}) \to L^2_1(\partial\Omega, \frac{1}{\omega})$ is invertible if $\omega \in A_{1+\delta}(\partial\Omega)$. To this end, we fix $q > 2$ so that $\mathcal{S} : L^q(\partial\Omega) \to L^q_1(\partial\Omega)$ is invertible. It suffices to show that there exists $0 < \delta = \delta(\Omega) \le 1 - (2/q)$ so that

$$\|f\|_{L^2(\partial\Omega,\frac{1}{\omega})} \le C_\omega \|\mathcal{S}(f)\|_{L^2_1(\partial\Omega,\frac{1}{\omega})} \tag{7.25}$$

for all $f \in L^q(\partial\Omega)$ and $\omega \in A_{1+\delta}(\partial\Omega)$. This is because $L^q_1(\partial\Omega)$ is dense in $L^2_1(\partial\Omega, \frac{1}{\omega})$ if $0 < \delta \le 1 - (2/q)$.

Let $f \in L^q(\partial\Omega)$ and $u_\pm = \mathcal{S}(f)$ on $\Omega_\pm$. Note that, by the jump relation, (7.25) follows from

$$\|\frac{\partial u_\pm}{\partial n}\|_{L^2(\partial\Omega,\frac{1}{\omega})} \le C\left\{\|\nabla_t u\|_{L^2(\partial\Omega,\frac{1}{\omega})} + \|u\|_{L^2(\partial\Omega,\frac{1}{\omega})}\right\}. \tag{7.26}$$

We wil give the proof of (7.26) for u_+. The proof for u_- may be carried out in the same manner.

Let $g = \mathcal{S}(f)$ on $\partial\Omega$. By a partition of unity, we may assume that $\operatorname{supp} g \subset B(Q_0, r_0) \cap \partial\Omega$ and $r_0 = r_0(\Omega) > 0$ is small so that, after a possible rotation of the coordinate system,

$$B(Q_0, r_0) \cap \Omega = B(Q_0, r_0) \cap \left\{(X', x_d) \in \mathbb{R}^d : \ x_d > \psi(X')\right\}, \tag{7.27}$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz continuous. We may also assume that $Q_0 = 0$. Let $\Delta_r = \left\{(X', x_d) \in \mathbb{R}^d : |X'| < r\right\}$ and

$$\Omega_r = \left\{(X', x_d) \in \mathbb{R}^d : \ |X'| < r, \ r < \psi(X') < B\, r\right\}, \tag{7.28}$$

where the constant $B = B(\|\nabla\psi\|_\infty) > 0$ is chosen so that Ω_r is a star-like Lipschitz domain for any $r > 0$. We assume that r_0 is sufficiently small such that $\Omega_r \subset \Omega$ and $\partial\Omega_r \cap \partial\Omega = \Delta_r$ for all $0 < r < 8r_0$.

We first estimate ∇u_+ on Δ_{3r_0}. To this end, we apply Lemma 7.22 on Ω_r with weight

$$\widetilde{\omega} = \begin{cases} \omega & \text{on } \Delta_r, \\ \omega(\partial\Omega) & \text{on } \partial\Omega_r \cap \Omega. \end{cases} \tag{7.29}$$

It follows that

$$\int_{\Delta_r} |\nabla u_+|^2 \frac{d\sigma}{\omega} \le C \left\{ \int_{\Delta_r} |\nabla_t u|^2 \frac{d\sigma}{\omega} + \frac{1}{\omega(\partial\Omega)} \int_{\partial\Omega_r \cap \Omega} |\nabla u|^2 \, d\sigma \right\}. \tag{7.30}$$

Integrating both sides of (7.30) in $r \in (3r_0, 4r_0)$, we obtain

$$\begin{aligned} \int_{\Delta_{3r_0}} |\nabla u_+|^2 \frac{d\sigma}{\omega} &\le C \left\{ \int_{\partial\Omega} |\nabla_t g|^2 \frac{d\sigma}{\omega} + \frac{1}{\omega(\partial\Omega)} \int_{\Omega_{4r_0} \setminus \Omega_{3r_0}} |\nabla u|^2 \, dX \right\} \\ &\le C \left\{ \int_{\partial\Omega} |\nabla_t g|^2 \frac{d\sigma}{\omega} + \frac{1}{\omega(\partial\Omega)} \int_{\Omega \setminus \Omega_{2r_0}} |u|^2 \, dX \right\}, \end{aligned} \tag{7.31}$$

where the second inequality follows from the Cacciopoli's inequality, since $u = 0$ on $\partial\Omega \setminus \Delta_{r_0}$. Note that by the boundary L^∞ estimates for harmonic functions and the L^p estimate for the regularity problem, we have

$$\begin{aligned} \left\{ \int_{\Omega \setminus \Omega_{2r_0}} |u|^2 \, dX \right\}^{1/2} &\le C \int_{\Omega \setminus \Omega_{r_0}} |u| \, dX \le C \left\{ \int_{\partial\Omega} |(\nabla u)^*|^s d\sigma \right\}^{1/s} \\ &\le C_s \left\{ \int_{\partial\Omega} |\nabla_t g|^s \, d\sigma \right\}^{1/s} \\ &\le C_s \left\{ \int_{\partial\Omega} |\nabla_t g|^2 \frac{d\sigma}{\omega} \right\}^{1/2} \left\{ \int_{\partial\Omega} \omega^{\frac{s}{2-s}} \, d\sigma \right\}^{\frac{2-s}{2s}}, \end{aligned} \tag{7.32}$$

where $s \in (1,2)$. Since $\omega \in A_{1+\delta}(\partial\Omega)$, it satisfies a reverse Hölder inequality [CF]. There exists $s \in (1,2)$, which depends only on the $A_{1+\delta}$ bound of ω, such that $\|\omega\|_{L^{\frac{s}{2-s}}(\partial\Omega)} \le C\,\omega(\partial\Omega)$. In view of (7.31)-(7.32), this gives

$$\int_{\Delta_{3r_0}} |\nabla u_+|^2 \frac{d\sigma}{\omega} \le C \int_{\partial\Omega} |\nabla_t g|^2 \frac{d\sigma}{\omega}. \tag{7.33}$$

Next, to estimate ∇u_+ on $\partial\Omega \setminus \Delta_{3r_0}$, we choose $p \in (2, q)$ so that the L^p regularity problem is solvable on $\Omega \setminus \Omega_r$ for all $r \in (2r_0, 3r_0)$. Assume that $\delta \le 1 - (2/p)$. It follows from Hölder inequality that

$$\begin{aligned} \left\{ \int_{\partial\Omega \setminus \Delta_{3r_0}} |\nabla u_+|^2 \frac{d\sigma}{\omega} \right\}^{1/2} &\le C \, \|\frac{1}{\omega}\|^{1/2}_{L^{\frac{p}{p-2}}(\partial\Omega)} \left\{ \int_{\partial\Omega \setminus \Delta_{3r_0}} |\nabla u_+|^p \, d\sigma \right\}^{1/p} \\ &\le \frac{C}{[\omega(\partial\Omega)]^{1/2}} \left\{ \int_{\Omega \cap \partial\Omega_r} |\nabla u_+|^p \, d\sigma \right\}^{1/p}. \end{aligned} \tag{7.34}$$

From this, we may integrate (7.34) in $r \in ((5/2)r_0, 3r_0)$ to show that the left-hand side of (7.34) is bounded by

$$\begin{aligned}(7.35)\quad \frac{C}{[\omega(\partial\Omega)]^{1/2}} \left\{ \int_{\Omega \setminus \Omega_{\frac{5}{2}r_0}} |\nabla u|^p dX \right\}^{1/p} &\le \frac{C}{[\omega(\partial\Omega)]^{1/2}} \left\{ \int_{\Omega \setminus \Omega_{\frac{9}{4}r_0}} |\nabla u|^2 dX \right\}^{1/2} \\ &\le \frac{C}{[\omega(\partial\Omega)]^{1/2}} \left\{ \int_{\Omega \setminus \Omega_{2r_0}} |\nabla u|^2 dX \right\}^{1/2}.\end{aligned}$$

We point out that the first inequality in (7.35) follows by a technique of Dahlberg and Kenig (see [FS]). In view of (7.34)-(7.35) and (7.32), we have proved that

$$(7.36)\qquad \int_{\partial\Omega \setminus \Delta_{3r_0}} |\nabla u_+|^2 \frac{d\sigma}{\omega} \le C \int_{\partial\Omega} |\nabla_t g|^2 \frac{d\sigma}{\omega}.$$

This, together with (7.33), gives the desired estimate (7.26) for u_+.

Finally in the case that Ω is a C^1 domain, we use the fact that $\mathcal{S} : L^p(\partial\Omega) \to L^p_1(\partial\Omega)$ is invertible and the L^p regularity problem is uniquely solvable for all $p \in (1, \infty)$ [FJR]. However, to estimate $|\nabla u_+|$ on Δ_{3r_0}, in the place of Ω_r in (7.28), we need to construct a continuum of starshaped C^1 domains $\widetilde{\Omega}_r$ such that $\widetilde{\Omega}_r \subset \Omega$ and $\Delta_r \subset \partial\widetilde{\Omega}_r \cap \partial\Omega$ for $r \in (2r_0, 8r_0)$. On the other hand, for the estimate of $|\nabla u_+|$ on $\partial\Omega \setminus \Delta_{3r_0}$, we should construct a continuum of C^1 domains G_r with connected boundaries such that $G_r \subset \Omega$, $\partial\Omega \setminus \Delta_{3r_0} \subset \partial G_r \cap \partial\Omega$, and $\text{dist}(\partial G_r, \Delta_{2r_0}) \ge c > 0$. With these observations, the same argument as in the Lipschitz case shows that one may take δ to be any number in $(0, 1)$. Since $\omega \in A_2(\partial\Omega)$ implies that $\omega \in A_p(\partial\Omega)$ for some $p \in (1, 2)$, it follows that one may take $\delta = 1$. This completes the proof. □

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