TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 8, Pages 3201–3214
S 0002-9947(04)03571-8
Article electronically published on September 23, 2004

ABELIAN CATEGORIES, ALMOST SPLIT SEQUENCES, AND COMODULES

MARK KLEINER AND IDUN REITEN

Dedicated to the memory of Sheila Brenner

Abstract. The following are equivalent for a skeletally small abelian Hom-finite category over a field with enough injectives and each simple object being an epimorphic image of a projective object of finite length.

(a) Each indecomposable injective has a simple subobject.

(b) The category is equivalent to the category of socle-finitely copresented right comodules over a right semiperfect and right cocoherent coalgebra such that each simple right comodule is socle-finitely copresented.

(c) The category has left almost split sequences.

Introduction

The starting point for this work was the paper [RV] on the classification of noetherian hereditary abelian Ext-finite categories with Serre duality over an algebraically closed field k, and the paper [CKQ] on the investigation of existence of almost split sequences for comodules over coalgebras. In both papers representations of locally finite quivers appeared in a natural way. The category of finitely presented representations of a locally finite quiver with no infinite path ending at a vertex played an important role in [RV]. It is equivalent to the category of finitely presented modules over the path algebra of the quiver (although the path algebra Λ has no unity, every finitely generated projective Λ-module is a finite direct sum of modules $\Lambda\, e_x$, where e_x is the trivial path associated with a vertex x). The category of finitely copresented representations of a locally finite quiver with no infinite path starting at a vertex is equivalent to the category of socle-finitely copresented comodules over the path coalgebra of the quiver; the latter category was studied in [CKQ]. It was then natural to try to understand why similar conditions occurred and in particular understand why comodules appear in a natural way.

Received by the editors May 2, 2003 and, in revised form, November 17, 2003.

2000 *Mathematics Subject Classification.* Primary 18E10, 18E20, 16G10, 16G20, 16G30, 16G70, 16W30.

Key words and phrases. Abelian category, almost split sequence, semiperfect cocoherent coalgebra, comodule.

The main results were obtained when the first-named author visited Norwegian University of Science and Technology in November–December of 2001. He expresses his sincere gratitude to the members of the Department of Mathematical Sciences for their warm hospitality. The work was finished in February 2003, when the authors participated in the program in *Commutative Algebra* at the Mathematical Sciences Research Institute, Berkeley. The authors thank the members of the institute for their hospitality.

A natural setting motivated by the work in [CKQ] is the Hom-finite abelian k-categories $\mathfrak{C}$ with enough injectives, where the socle of each indecomposable injective object is simple, and each simple object has a projective cover of finite length. For example, these conditions are satisfied for the socle-finitely copresented comodules over the path coalgebras of the quivers described above, and the dual conditions for the finitely presented modules over the aforementioned path algebras of quivers. For the indicated path coalgebras and path algebras it is possible to pass from socle-finitely copresented comodules to finitely presented modules, or conversely, via the usual duality for finite-dimensional vector spaces. But this seems to be a special property of hereditary categories $\mathfrak{C}$.

Our results suggest that in the investigation of categories $\mathfrak{C}$ with the above conditions of arbitrary homological dimension, there is an essential difference whether one considers modules or comodules. We show that categories of comodules over coalgebras appear naturally in that investigation. In fact, each $\mathfrak{C}$ is equivalent to the category of socle-finitely copresented comodules over a right semiperfect and right cocoherent coalgebra such that the simple comodules are socle-finitely copresented. And actually, our properties of $\mathfrak{C}$ can be viewed as an abstract description of the latter categories.

Another main result is that the categories $\mathfrak{C}$ have left almost split sequences. Here we use the interpretation in terms of comodules, and rely heavily on results from [CKQ].

We now give a description of the content of the paper section by section. In Section 1 we give relevant preliminary material from [CKQ] and [RV], along with some improvements, and some first connections in the case of quivers. In Section 2 we give some useful results on categories, in particular a result of Freyd where being injective is preserved when passing to a functor category. In Section 3 we show that the categories $\mathfrak{C}$ are equivalent to categories of comodules, and in Section 4 we show that the categories $\mathfrak{C}$ have left almost split sequences, along with various equivalent statements.

The authors are grateful to Peter Freyd for the precise formulation and proof of Theorem 2.2 about injective representable functors, and to the referee for the helpful suggestions.

1. Preliminaries

This section begins with results on coalgebras and comodules needed in Section 3. In particular, we show how socle-finitely copresented comodules over the path coalgebra of a locally finite quiver with at most finitely many paths starting at each vertex arise in the theory of almost split sequences [CKQ]. We then show how finitely presented modules over the path algebra of a locally finite quiver with at most finitely many paths ending at each vertex arise in the theory of abelian categories with Serre duality [RV]. The section ends with a proof that the indicated categories of modules and comodules are opposite to one another, and with an explanation of how to interpret representations of quivers with relations as comodules. Based on that, we show in Section 4 how our results yield a description, obtained in [RV], of certain hereditary abelian categories with a right Serre functor.

Throughout the paper, k is a fixed field and $\mathrm{D} = \mathrm{Hom}_k(\ ,k)$. An additive category is a k-category if the morphism sets are k-spaces and the composition of morphisms is k-bilinear. A k-category is **Hom-finite** if the morphism spaces are

finite-dimensional over k. Let $\mathfrak{A}$ be a Hom-finite abelian k-category. The category $\mathfrak{A}$ is **Ext-finite** if $\dim_k \operatorname{Ext}^i(X,Y) < \infty$ for all i and $X, Y \in \mathfrak{A}$, and it is hereditary if $\operatorname{Ext}^i(X,Y) = 0$ for $i > 1$. We recall some notions from [ARS]. A morphism $f : A \to B$ in $\mathfrak{A}$ is left almost split if it is not a split monomorphism and, for every $h : A \to X$ in $\mathfrak{A}$ that is not a split monomorphism, there exists a $g : B \to X$ satisfying $h = gf$. The definition of a right almost split morphism is similar. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathfrak{A}$ is an almost split sequence if f is a left almost split morphism and g is a right almost split morphism. The category $\mathfrak{A}$ has left almost split sequences if for every indecomposable not injective $A \in \mathfrak{A}$ there exists an almost split sequence $0 \to A \to B \to C \to 0$, and for every indecomposable injective $I \in \mathfrak{A}$ there exists a left almost split morphism $I \to J$ in $\mathfrak{A}$. The definition of a category having right almost split sequences is similar. The category $\mathfrak{A}$ has almost split sequences if it has both left and right almost split sequences.

We now introduce the terminology on coalgebras and comodules, using freely the definitions, results, and notation of [Gr, Mo, T]. Denote by Γ a k-coalgebra with a comultiplication $\Gamma \to \Gamma \otimes \Gamma$ and a counit $\Gamma \to k$ where $\otimes = \otimes_k$. A right Γ-comodule M is given by a structure map $M \to M \otimes \Gamma$, and $\operatorname{Soc} M$ stands for the socle, i.e., the sum of simple subcomodules of M. Every finitely generated comodule is finite-dimensional. The category $\mathfrak{M}^\Gamma$ of right Γ-comodules is an abelian category with injective envelopes, where the indecomposable injectives are the injective envelopes of simple comodules. We identify the category of left Γ-comodules with $\mathfrak{M}^{\Gamma^{op}}$, where Γ^{op} is the opposite coalgebra of Γ. Unless otherwise stated, all comodules are right comodules. A comodule $M \in \mathfrak{M}^\Gamma$ is **socle-finite** if $\dim_k \operatorname{Soc} M < \infty$, and it is **socle-finitely copresented** if there exists an exact sequence $0 \to M \to I_0 \to I_1$ with I_0 and I_1 socle-finite and injective. In what follows, $\mathfrak{m}^\Gamma$ denotes the full subcategory of $\mathfrak{M}^\Gamma$ determined by the socle-finitely copresented comodules.

A coalgebra Γ is **basic** [CM] if a decomposition of the right comodule Γ as a direct sum of indecomposable comodules has no isomorphic direct summands. A coalgebra Γ is **hereditary** [NTZ] if every quotient of the right comodule Γ is injective. A coalgebra Γ is **right (left) semiperfect** [L] if every right (left) finite-dimensional comodule has a projective cover. Since a finitely generated comodule must be finite-dimensional, this definition is in line with the usual definition of a semiperfect ring. We will often use the following statement which is [L, Theorem 10].

Theorem 1.1 (Lin)**.** *The following are equivalent for a coalgebra* Γ.

(a) *The category* $\mathfrak{M}^\Gamma$ *has enough projectives.*
(b) *The injective envelope of every simple* Γ^{op}*-comodule is finite-dimensional.*
(c) *Every finite-dimensional* Γ*-comodule has a projective cover.*

Definition 1. A coalgebra Γ is **right cocoherent** if every socle-finite quotient of a socle-finite injective Γ-comodule is socle-finitely copresented. A coalgebra Γ is **right locally artinian** if every indecomposable injective Γ-comodule is artinian.

The notion of a right cocoherent coalgebra is dual to that of a right coherent ring as follows from the next statement.

Proposition 1.2. *A coalgebra* Γ *is right cocoherent if and only if* $\mathfrak{m}^\Gamma$ *is an abelian category.*

Proof. The proof is a slight modification of the argument dual to the proof of [SS, Proposition 4]. We leave it to the reader. □

We now give examples of right cocoherent coalgebras.

Proposition 1.3. *A coalgebra Γ is right cocoherent in each of the following cases.*

(a) Γ *is right locally artinian.*
(b) Γ *is left semiperfect.*
(c) Γ *is hereditary.*

Proof. (a) Any socle-finite injective is artinian because it is a finite direct sum of indecomposable injectives. Therefore every quotient of a socle-finite injective is artinian, which implies that every socle-finite comodule is socle-finitely copresented.

(b) By Theorem 1.1, every indecomposable injective Γ-comodule is finite-dimensional, hence, artinian. Now the statement follows from (a).

(c) Obvious. □

We need some facts from [T]. Let $X, Y \in \mathfrak{M}^\Gamma$ and assume that X is a **quasi-finite** comodule, i.e., that $\dim_k \operatorname{Hom}_\Gamma(F, X) < \infty$ for all finite-dimensional $F \in \mathfrak{M}^\Gamma$. Then the formula $\mathrm{h}_\Gamma(X, Y) = \varinjlim_\lambda \mathrm{D} \operatorname{Hom}_\Gamma(Y_\lambda, X)$, where $\{Y_\lambda\}$ is the set of finite-dimensional subcomodules of Y, defines an additive bifunctor $\mathrm{h}_\Gamma(\ ,\ )$ called the cohom functor. It is characterized by the property that $\mathrm{h}_\Gamma(X,\ )$ is a left adjoint of the functor $\_ \otimes X$. The cohom functor is used in the definition of the transpose, Tr, for comodules [CKQ, Definition 1, p. 6]. Let $M \in \mathfrak{M}^\Gamma$ be **quasi-finitely copresented**, i.e., assume that a minimal injective copresentation $0 \to M \to I_0 \xrightarrow{f} I_1$ of M has the property that I_0, I_1 are quasi-finite. Then $\operatorname{Tr} M \in \mathfrak{M}^{\Gamma^{op}}$ is a quasi-finite comodule determined from the exact sequence $0 \to \operatorname{Tr} M \to \mathrm{h}_\Gamma(I_1, \Gamma) \xrightarrow{\mathrm{h}_\Gamma(f,\Gamma)} \mathrm{h}_\Gamma(I_0, \Gamma)$.

Proposition 1.4. *Let Γ be a right semiperfect coalgebra.*

(a) *The full subcategory of $\mathfrak{M}^\Gamma$ determined by the socle-finite comodules is* Hom-*finite.*
(b) *If M is indecomposable, quasi-finitely copresented, and not injective, then $M \in \mathfrak{m}^\Gamma$ if and only if $\dim_k \operatorname{Tr} M < \infty$.*

Proof. (a) If $M, N \in \mathfrak{M}^\Gamma$ are socle-finite, then so are their injective envelopes $I(M), I(N)$. If $i : M \to I(M)$, $j : N \to I(N)$ are essential monomorphisms, then for any morphism $f : M \to N$ there exists a morphism $g : I(M) \to I(N)$ satisfying $jf = gi$. If $f_1, \dots, f_t \in \operatorname{Hom}_\Gamma(M, N)$ are linearly independent over k and $g_1, \dots, g_t \in \operatorname{Hom}_\Gamma(I(M), I(N))$ satisfy $jf_\alpha = g_\alpha i$, $\alpha = 1, \dots, t$, then $g_1, \dots, g_t$ are linearly independent because j is monic. It follows that $\dim_k \operatorname{Hom}_\Gamma(M, N) \le \dim_k \operatorname{Hom}_\Gamma(I(M), I(N))$ so it suffices to show that $\dim_k \operatorname{Hom}_\Gamma(I, J) < \infty$ for $I, J \in \mathfrak{M}^\Gamma$ indecomposable injective. By [CKQ, Proposition 3.1(c)], we have that $\operatorname{Hom}_\Gamma(I, J) \cong \operatorname{Hom}_{\Gamma^{op}}(\mathrm{h}_\Gamma(J, \Gamma), \mathrm{h}_\Gamma(I, \Gamma))$ and $\mathrm{h}_\Gamma(J, \Gamma)$, $\mathrm{h}_\Gamma(I, \Gamma)$ are indecomposable injective. Since every indecomposable injective in $\mathfrak{M}^{\Gamma^{op}}$ is finite-dimensional by Theorem 1.1, the proof is complete.

(b) By [CKQ, Propositions 3.1(c) and 3.2(c)], we have that $M \in \mathfrak{m}^\Gamma$ if and only if $\operatorname{Tr} M \in \mathfrak{m}^{\Gamma^{op}}$. Since every indecomposable injective in $\mathfrak{M}^{\Gamma^{op}}$ is finite-dimensional, $\operatorname{Tr} M \in \mathfrak{m}^{\Gamma^{op}}$ if and only if $\dim_k \operatorname{Tr} M < \infty$. □

Parts (b) and (c) of the next statement are [CKQ, Theorem 5.1 and Corollary 5.3].

Theorem 1.5. *Let Γ be a right semiperfect coalgebra.*

(a) *If $M \in \mathfrak{m}^\Gamma$ is indecomposable not injective, there exists an almost split sequence $0 \to M \to E \to \operatorname{DTr} M \to 0$ in $\mathfrak{M}^\Gamma$ with $\dim_k \operatorname{DTr} M < \infty$.*

(b) *If $N \in \mathfrak{M}^\Gamma$ is indecomposable not projective and $\dim_k N < \infty$, there exists an almost split sequence $0 \to \operatorname{TrD} N \to F \to N \to 0$ in $\mathfrak{M}^\Gamma$ with $\operatorname{TrD} N \in \mathfrak{m}^\Gamma$.*

(c) *The category $\mathfrak{m}^\Gamma$ contains all simple comodules in $\mathfrak{M}^\Gamma$ if and only if it contains all finite-dimensional comodules in $\mathfrak{M}^\Gamma$.*

Proof. (a) This follows immediately from Proposition 1.4(b) and [CKQ, Corollary 4.3(a)]. □

We now describe right semiperfect coalgebras Γ for which $\mathfrak{m}^\Gamma$ is abelian and has left almost split sequences or almost split sequences.

Corollary 1.6. (a) *Let Γ be a right semiperfect and right cocoherent coalgebra such that $\mathfrak{m}^\Gamma$ contains all simple comodules.*

(i) *The category $\mathfrak{m}^\Gamma$ is a* Hom*-finite abelian category with enough injectives and left almost split sequences, where the socle of each indecomposable injective comodule is simple, and each simple comodule has a projective cover of finite length.*

(ii) *The category $\mathfrak{m}^\Gamma$ has almost split sequences if and only if Γ is left semiperfect.*

(b) *If Γ is left semiperfect, it is right cocoherent and $\mathfrak{m}^\Gamma$ contains all simple comodules.*

Proof. (a) According to [Gr, 1.5g(i)], a Γ-comodule is indecomposable injective if and only if it is an injective envelope of a simple comodule. Therefore part (i) follows from Proposition 1.2 and parts (a) and (c) of Theorem 1.5, using Theorem 1.1. To prove the necessity of (ii), we assume that $\mathfrak{m}^\Gamma$ has almost split sequences and show that every indecomposable $N \in \mathfrak{m}^\Gamma$ is finite-dimensional over k. Then, in particular, every indecomposable injective is finite-dimensional, so Theorem 1.1 says that Γ is left semiperfect. If N is not projective, there exists an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathfrak{m}^\Gamma$ where L is not injective in $\mathfrak{m}^\Gamma$ and, hence, in $\mathfrak{M}^\Gamma$. By Theorem 1.5(a), there exists an almost split sequence $0 \to L \to E \to \operatorname{DTr} L \to 0$ in $\mathfrak{M}^\Gamma$ with $\dim_k \operatorname{DTr} L < \infty$. Since the terms of the latter sequence are in $\mathfrak{m}^\Gamma$ because $\mathfrak{m}^\Gamma$ is extension closed, we get $N \cong \operatorname{DTr} L$ by the uniqueness of an almost split sequence. If N is projective, then [RV, Lemma I.3.1], which holds in any abelian category, says that N has a unique maximal subcomodule $\operatorname{rad} N$. By (i), there exists a projective cover $P \to N/\operatorname{rad} N$ with P finite-dimensional, so we must have $N \cong P$. To prove the sufficiency of (ii), assume that Γ is left semiperfect. Then every indecomposable injective right comodule is finite-dimensional, whence so is every comodule in $\mathfrak{m}^\Gamma$. It then follows by Theorem 1.5(b) that $\mathfrak{m}^\Gamma$ has right almost split sequences, and hence almost split sequences.

(b) This follows from the above and Proposition 1.3(b). □

The next statement says how to verify the assumptions of Corollary 1.6 when Γ is hereditary and k is algebraically closed. Recall that every hereditary coalgebra

is right cocoherent by Proposition 1.3(c). For the definition of the path coalgebra of a quiver, see [CM, Definition 4.1]. A quiver is locally finite if at most finitely many arrows start or end at each vertex.

Proposition 1.7. *Let Γ be a basic hereditary coalgebra over an algebraically closed field k.*

(a) *The coalgebra Γ is right semiperfect and $\mathfrak{m}^\Gamma$ contains all simple comodules if and only if Γ is isomorphic to the path coalgebra of a locally finite quiver in which at most finitely many paths start at each vertex.*
(b) *The coalgebra Γ is left and right semiperfect if and only if Γ is isomorphic to the path coalgebra of a quiver in which at most finitely many paths start or end at each vertex.*

Proof. By [C, Corollary 2 and Theorem 4], a coalgebra Γ over an algebraically closed field is basic and hereditary if and only if it is isomorphic to the path coalgebra of a quiver. Therefore we may assume that Γ is a path coalgebra. Then (a) is just [CKQ, Corollary 6.3], and (b) is a direct consequence of (a). □

If Q is a quiver whose path coalgebra is isomorphic to the coalgebra Γ of part (a) or (b) of Proposition 1.7, then, by [CKQ, Proposition 6.1], the category $\mathfrak{M}^\Gamma$ is equivalent to the category $\operatorname{Rep} Q$ of all representations of Q, and the relevant equivalence of categories carries $\mathfrak{m}^\Gamma$ to the full subcategory of $\operatorname{Rep} Q$ determined by certain representations with a finite-dimensional k-space associated to each vertex.

Our goal now is to prove the statement that motivated this work: if Γ is the path coalgebra of a locally finite quiver with at most finitely many paths starting at each vertex, then $\mathrm{D} = \operatorname{Hom}_k(\ ,k)$ induces a duality between the category $\mathfrak{m}^\Gamma$ and the category of finitely presented representations of the opposite quiver. We begin by recalling the results of [RV] showing that the latter category arises naturally in the study of hereditary abelian categories with Serre duality. In Section 4 we will give another explanation of this fact, based on the indicated duality D and our characterization of certain abelian Hom-finite k-categories as categories of comodules.

An Ext-finite abelian k-category $\mathfrak{B}$ has a right Serre functor [RV] if there exists an additive endofunctor $F : D^b(\mathfrak{B}) \to D^b(\mathfrak{B})$ of the bounded derived category together with isomorphisms $\operatorname{Hom}(A,B) \to \mathrm{D}\operatorname{Hom}(B,FA)$ natural in $A,B \in D^b(\mathfrak{B})$. We quote [RV, p. 311, lines 13 through 16] to show how to construct an Ext-finite hereditary abelian k-category with a right Serre functor from a quiver.

Let Q be a quiver with the following two properties.

(P1) The quiver is locally finite.
(P2) There is no infinite path of the form $x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \dots$.

Then the category $\operatorname{rep} Q$ of finitely presented representations of Q over k is an Ext-finite hereditary abelian k-category with enough projectives and a right Serre functor F such that $F(P)$ is injective of finite length for each projective object P.

The latter properties determine the category $\operatorname{rep} Q$ up to equivalence, which follows from the results of [RV].

Theorem 1.8. *Let $\mathfrak{B}$ be an* Ext*-finite hereditary abelian category over an algebraically closed field k with enough projectives and a right Serre functor F such that $F(P)$ is injective of finite length for each projective object P. Then $\mathfrak{B}$ is equivalent to* $\operatorname{rep} Q$ *for some quiver Q satisfying* (P1) *and* (P2).

Proof. By [RV, Theorem II.1.3], there exists a full faithful exact embedding of $\mathfrak{B}$ into an Ext-finite abelian hereditary k-category $\mathfrak{C}$ with a Serre functor that extends F. Moreover, the projectives in $\mathfrak{C}$ are the projectives in $\mathfrak{B}$, and the injectives in $\mathfrak{C}$ are of the form $F(P)$, where $P \in \mathfrak{B}$ is projective. Since $\mathfrak{B}$ has enough projectives and the embedding is exact, $\mathfrak{B}$ is equivalent to the full subcategory of $\mathfrak{C}$ determined by the quotients of projectives in $\mathfrak{C}$. By [RV, Proposition II.2.3], $\mathfrak{B}$ is equivalent to $\operatorname{rep} Q$, where Q is a quiver satisfying (P1) and (P2). □

Our next goal is to show that if a quiver satisfies (P1) and (P2), then the functor D takes finitely presented representations of the quiver to socle-finitely copresented comodules over the path coalgebra of the opposite quiver, and conversely.

Proposition 1.9. (a) *A quiver Q is locally finite with at most finitely many paths starting at each vertex if and only if the opposite quiver Q^{op} satisfies* (P1) *and* (P2).

(b) *Let Γ be the path coalgebra over k of a quiver Q satisfying the conditions of* (a). *Then $\mathrm{D} : \mathfrak{m}^\Gamma \to \operatorname{rep} Q^{op}$ is a duality.*

Proof. (a) Part 1 of [RV, Lemma II.1.1] says that if a quiver satisfies (P1) and (P2), it also satisfies the following.

(P2a) At most finitely many paths end at each vertex.

It is straightforward to check that (P1) and (P2) hold if and only if so do (P1) and (P2a), and that Q^{op} satisfies (P1) and (P2a) if and only if Q satisfies the stated conditions.

(b) By [CKQ, Propositions 6.1 and 6.2], the comodules in $\mathfrak{m}^\Gamma$ can be viewed as certain representations of Q over k with a finite-dimensional k-space associated to each vertex, and an indecomposable injective Γ-comodule identifies with a representation (V, f) that corresponds to a vertex u of Q as follows. For each vertex x in Q, $V(x)$ is the k-span of all paths starting at x and ending at u and, for each arrow $\alpha : x \to y$, the linear transformation $f_\alpha : V(x) \to V(y)$ sends each path $\alpha_1 \dots \alpha_{s-1}\alpha_s : x \to u$ to $\alpha_1 \dots \alpha_{s-1}$ if $\alpha = \alpha_s$, and to 0 if $\alpha \neq \alpha_s$. Applying the duality D to the k-spaces and linear transformations of (V, f), we obtain the indecomposable projective representation of Q^{op} that corresponds to the vertex u. Since D is a contravariant exact endofunctor of the category of finite-dimensional k-spaces, we obtain a duality $\mathrm{D} : \mathfrak{m}^\Gamma \to \operatorname{rep} Q^{op}$. □

We conclude this section by recalling how representations of quivers with relations can be realized as comodules. A small Hom-finite k-category Λ is **left locally bounded** [CKQ, p. 16] if distinct objects of Λ are not isomorphic and, for all $x \in \Lambda$, we have $\operatorname{Hom}(x,x)/\operatorname{rad}\operatorname{Hom}(x,x) \cong k$ and $\sum_{y\in\Lambda} \dim_k \operatorname{Hom}(x,y) < \infty$. Given a quiver Q, denote by kQ the path category and by kQ^{+2} the span of all paths of length ≥ 2. The next statement is [CKQ, Proposition 7.1].

Proposition 1.10. *A k-category Λ is left locally bounded if and only if $\Lambda \cong kQ/I$ for some quiver Q and a two-sided ideal I of kQ with $I \subset kQ^{+2}$, where for each vertex x of Q the following hold.*

(a) *At most finitely many arrows start at x.*

(b) *There is an integer $N_x > 0$ such that I contains each path of length $\geq N_x$ starting at x.*

If $\Lambda = kQ/I$ is a left locally bounded category, then a Λ-module is a representation (V, f) of the quiver Q with relations determined by the ideal I; see [ARS, Section III.1]. A Λ-module (V, f) is finite-dimensional if $\sum_{x\in\Lambda} \dim_k V(x) < \infty$. Now let Γ be the path coalgebra of Q. Identifying a path p in Q with the element ϵ_p of the convolution algebra $\mathrm{D}\,\Gamma$ satisfying $\langle \epsilon_p, p\rangle = 1$ and $\langle \epsilon_p, q\rangle = 0$ for all paths $q \neq p$, we obtain an embedding of kQ (viewed as an algebra without identity) into $\mathrm{D}\,\Gamma$. The next statement is [CKQ, Propositions 7.2 and 7.3].

Proposition 1.11. *Let $\Lambda = kQ/I$ be a left locally bounded category and let Γ be the path coalgebra of the quiver Q.*

(a) *The set $I^\perp = \{\gamma \in \Gamma | \langle I, \gamma\rangle = 0\}$ is a right semiperfect subcoalgebra of Γ.*
(b) *The category of (finite-dimensional) Λ-modules is equivalent to the category of (finite-dimensional) $I^\perp$-comodules.*
(c) *The category $\mathfrak{m}^{I^\perp}$ contains all simple comodules in $\mathfrak{M}^{I^\perp}$ if and only if Q is locally finite.*
(d) *If Q is locally finite, then $\mathfrak{m}^{I^\perp}$ and the category of socle-finitely copresented Λ-modules have left almost split sequences.*

2. Categories and functors

This section contains two results on categories and functors needed in Section 3. The theorem on injective representable functors due to Freyd is of independent interest.

Lemma 2.1. *Let $\mathfrak{A}$ be a category with inverse images and let $(A_i)_{i\in I}$ be a family of subobjects of $A \in \mathfrak{A}$.*

(a) *$A' = \bigcup_i A_i$ if and only if A' is the smallest subobject of A with $A_i \leq A'$, $i \in I$.*
(b) *If no proper subobject Y of A satisfies $A_i \leq Y$ for all $i \in I$, then $A = \bigcup_i A_i$.*

Proof. (a) The necessity is clear. For the sufficiency, let $f : A \to B$ be a morphism and let B' be a subobject of B such that, for all $i \in I$, f carries A_i into B'. We obtain a commutative diagram and a pullback diagram, respectively,

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "\kappa_i"'] & B' \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f^{-1}(B') \arrow[r] \arrow[d, "\kappa"'] & B' \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd},$$

where κ is a monomorphism. Then $\kappa_i = \kappa\phi_i$ for a unique $\phi_i : A_i \to f^{-1}(B')$ whence $A_i \leq f^{-1}(B')$. By assumption, $A' \leq f^{-1}(B')$ so that f carries A' into B'.

(b) This follows immediately from (a). □

The next statement deals with injective representable functors. Recall [Mi, Section I.15] that an exact category is a category with kernels and cokernels in which every monomorphism is a kernel, every epimorphism is a cokernel, and every morphism is a composition of a monomorphism and an epimorphism. Given an additive category $\mathfrak{A}$, we denote by $(\mathfrak{A}^{op}, \mathbf{Ab})$ the category of contravariant additive functors from $\mathfrak{A}$ into the category of abelian groups. If $X \in \mathfrak{A}$, then $(\ , X)$ stands for the representable functor determined by X.

Theorem 2.2 (Freyd). *If I is an injective object of an exact additive Hom-finite k-category $\mathfrak{A}$, then $(\ ,I)$ is an injective object of $(\mathfrak{A}^{op}, \mathbf{Ab})$.*

Proof. Recall that a functor $T \in (\mathfrak{A}^{op}, \mathbf{Ab})$ is finitely generated if there exists an epimorphism $(\ ,B) \to T$ for some $B \in \mathfrak{A}$. By [Gro, Lemma 1 of Theorem 1.10.1] it suffices to show that if $T \subset (\ ,C)$, for some $C \in \mathfrak{A}$, and $g : T \to (\ ,I)$ is a morphism in $(\mathfrak{A}^{op}, \mathbf{Ab})$, then there exists a morphism $h : C \to I$ satisfying $g = (\ ,h)|T$. If T is finitely generated, this holds, for there exists an epimorphism $f : (\ ,B) \to T$, for some $B \in \mathfrak{A}$, which gives rise to an exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C$ in $\mathfrak{A}$ with $wu = 0$ where w is determined by $(\ ,w) = gf$. Then any morphism h satisfying $w = hv$ has the desired property. If T is arbitrary, for every finitely generated $F \subset T$ there exists a non-empty subset $S_F \subset (C,I)$ such that $(\ ,h)|F = g|F$ whenever $h \in S_F$. If $h_1, \dots, h_m \in S_F$ and $r_1, \dots, r_m \in k$ satisfy $r_1 + \cdots + r_m = 1$, then $(\ ,h_i)|F = g|F$ for $0 < i \le m$, whence $(\ ,r_1h_1 + \cdots + r_mh_m)|F = g|F$ and $r_1h_1 + \cdots + r_mh_m \in S_F$. Thus each S_F is an **affine subset** of the finite-dimensional k-space (C,I).

Recall that a subset S of a vector space V is affine if it is closed under the barycentric linear combinations, i.e., if $x_1, \dots, x_m \in S$ and $r_1, \dots, r_m \in k$ satisfy $r_1 + \cdots + r_m = 1$, then $r_1x_1 + \cdots + r_mx_m \in S$. Equivalently, S is affine if it is closed under the linear combinations of the form $rx + (1-r)y$ if k consists of more than two elements, and of the form $x + y + z$ if k is the field with two elements. For $x \in S$ the translate $S - x$ is a subspace of V that does not depend on x. By definition, $\dim_k S = \dim_k(S - x)$.

If $G \subset T$ is finitely generated and $F \subset G$, then $S_G \subset S_F$, so if we choose F with S_F of minimal dimension, we must have $S_G = S_F$. Let $u \in T(X)$ for some $X \in \mathfrak{A}$, then there exists a finitely generated G such that $F \subset G \subset T$ and $u \in G(X)$. For all $h \in S_F$ we have $(\ ,h)|G = g|G$, so that $(X,h)(u) = g_X(u)$. Hence $(\ ,h)|T = g$. $\square$

3. Embedding into a category of comodules

In this section we fix a skeletally small Hom-finite abelian k-category $\mathfrak{C}$ with enough injectives in which every indecomposable injective has a simple subobject and for every simple object S there exists an epimorphism $P \to S$ with P projective of finite length. Our goal is to show that $\mathfrak{C}$ is equivalent to a certain category of comodules.

Proposition 3.1. *Let $X \in \mathfrak{C}$ and let $(X_j)_{j \in J}$ be the family of subobjects of X of finite length. Then $X = \varinjlim X_j$.*

Proof. We show first that $X = \bigcup X_j$. Let $Y \subset X$ and $X_j \subset Y$ for all $j \in J$. In view of Lemma 2.1(b) it suffices to show that $Y = X$. If not, Y is a proper subobject so X/Y has a simple subobject S. By assumption, there exists an epimorphism $P \to S$ with P projective of finite length. Let $\phi : P \to X/Y$ be the composition of $P \to S$ and the inclusion $S \to X/Y$, let $\pi : X \to X/Y$ be a cokernel of the inclusion $Y \to X$, and choose $\psi : P \to X$ satisfying $\phi = \pi\psi$. Then $\operatorname{Im}\psi$ is of finite length and $\operatorname{Im}\psi \not\subset Y$, a contradiction.

Let $\kappa_j : X_j \to X$ and $\kappa_{ji} : X_i \to X_j$ be monomorphisms satisfying $\kappa_j\kappa_{ji} = \kappa_i$, $i,j \in J$. For $Y \in \mathfrak{C}$ let $\lambda_j : X_j \to Y$ be a family of morphisms satisfying $\lambda_j\kappa_{ji} = \lambda_i$. Passing to the representable functors we obtain cocompatible families $(\ ,\kappa_j) : (\ ,X_j) \to (\ ,X)$ and $(\ ,\lambda_j) : (\ ,X_j) \to (\ ,Y)$ in $(\mathfrak{C}^{op}, \mathbf{Ab})$. By [Mi, proof of Theorem IV.2.6], $(\mathfrak{C}^{op}, \mathbf{Ab})$ is a cocomplete abelian category with exact

inductive limits so [Mi, Proposition III.1.2] implies the existence of $\varinjlim(\ ,X_j) = \bigcup(\ ,X_j)$. If $\mu_j : (\ ,X_j) \to \varinjlim(\ ,X_j)$ are canonical morphisms, there exist unique morphisms $i : \varinjlim(\ ,X_j) \to (\ ,X)$ and $g : \varinjlim(\ ,X_j) \to (\ ,Y)$ satisfying $(\ ,\kappa_j) = i\mu_j$ and $(\ ,\lambda_j) = g\mu_j$, respectively, for all j. We note that i is monic. Choosing a monomorphism $Y \to I$ in $\mathfrak{C}$ with I injective and using Theorem 2.2 we obtain a commutative diagram

$$\begin{array}{ccc} \varinjlim(\ ,X_j) & \xrightarrow{\ i\ } & (\ ,X) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (\ ,h)} \\ (\ ,Y) & \longrightarrow & (\ ,I) \end{array}$$

for some $h : X \to I$. For all j, the commutativity of the diagram implies that h carries X_j to the subobject Y of I. Since $X = \bigcup X_j$, then h carries X to Y, i.e., we obtain a morphism $f : X \to Y$ satisfying $\lambda_j = f\kappa_j$ for all j. The uniqueness of f follows from [Mi, Proposition I.9.1]. □

If $\mathfrak{A}$ is an abelian category we denote by $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ [Mi, p. 150] the category of left exact additive functors $\mathfrak{A} \to \mathbf{Ab}$; in [Ga, p. 348] the notation is $\mathrm{Sex}(\mathfrak{A}, \mathbf{Ab})$. If $\mathfrak{a}$ is a subcategory of $\mathfrak{A}$ we denote by $(\ ,X)|\mathfrak{a}$ the restriction to $\mathfrak{a}$ of the functor $(\ ,X)$. An object $A \in \mathfrak{A}$ is **finitely copresented** if it has an injective copresentation $0 \to A \to I_0 \to I_1$ in which both injectives, I_0 and I_1, are finite direct sums of indecomposables. Recall [Ga, p. 356] that an abelian category is **locally finite** if it is a category with exact inductive limits and a family of generators of finite length.

Theorem 3.2. *Let $\mathfrak{C}$ be a skeletally small* Hom*-finite abelian k-category with enough injectives in which every indecomposable injective has a simple subobject and for every simple object S there exists an epimorphism $P \to S$ with P projective of finite length.*

(a) *The full subcategory $\mathfrak{c}$ of $\mathfrak{C}$ determined by the objects of finite length is an abelian k-category with enough projectives.*
(b) *The category $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ is a locally finite k-category with enough projectives.*
(c) *The functor $\Phi : \mathfrak{C} \to \mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ given by $\Phi X = (\ ,X)|\mathfrak{c},\ X \in \mathfrak{C}$, is a full exact embedding and* $\mathrm{Im}\,\Phi$ *is equivalent to the full subcategory of $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ determined by the finitely copresented functors.*
(d) *The category $\mathfrak{C}$ is equivalent to $\mathfrak{m}^\Gamma$, where Γ is a right cocoherent and right semiperfect k-coalgebra such that $\mathfrak{m}^\Gamma$ contains all simple comodules.*
(e) *In the setting of (d), $\mathfrak{C}$ is hereditary if and only if Γ is hereditary.*

Proof. (a) This is a direct consequence of the definitions.

(b) By [Ga, Proposition II.2.5], $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ is an abelian category with exact inductive limits. Since $\mathfrak{c}^{op}$ is an artinian category, every subfunctor of a representable functor is representable [Ga, p. 356, paragraph 5] so every representable functor is of finite length. Using [Ga, Proposition II.3.6] we conclude that $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ has a family of generators of finite length. Since $\mathfrak{c}^{op}$ has enough injectives by (a), it follows from [Mi, Exercise 17, p. 159] that $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ has enough projectives.

(c) Since the functor $\mathfrak{C} \to \mathcal{L}(\mathfrak{C}^{op}, \mathbf{Ab})$ sending each $X \in \mathfrak{C}$ to $(\ ,X)$ is exact by [Ga, Proposition II.3.6], so is Φ. If $(X_j)_{j\in J}$ is the family of subobjects of X of finite length, then $X = \varinjlim X_j = \bigcup X_j$ by Proposition 3.1.

We claim that $\bigcup(M, X_j) = (M, \cup X_j) = (M, X)$ for $M \in \mathfrak{c}$. Note that neither the statement nor the proof of [Ga, Lemma II.4.2] apply here because $\mathfrak{C}$ need not have exact inductive limits. Since $\bigcup(M, X_j) \subset (M, X)$ we let $h \in (M, X)$ and show that $h \in \bigcup(M, X_j)$. Since M is of finite length, so is $\operatorname{Im} h$ whence $\operatorname{Im} h = X_i$ for some $i \in J$. Hence $h = \kappa_i g$ where $\kappa_i : X_i \to X$ is the inclusion and $g : M \to X_i$ a morphism, i.e., $h \in \bigcup(M, X_j)$.

In view of (b) we now have that $(\ ,X)|\mathfrak{c} = \bigcup(\ ,X_j)|\mathfrak{c} = \varinjlim(\ ,X_j)|\mathfrak{c}$. Hence for $Y \in \mathfrak{C}$ we have

$$(\Phi X, \Phi Y) = ((\ ,X)|\mathfrak{c}, (\ ,Y)|\mathfrak{c}) = (\varinjlim(\ ,X_j)|\mathfrak{c}, (\ ,Y)|\mathfrak{c}) \cong \varprojlim((\ ,X_j)|\mathfrak{c}, (\ ,Y)|\mathfrak{c}).$$

Since X_j is of finite length, $(\ ,X_j)|\mathfrak{c}$ is a representable functor $\mathfrak{c}^{op} \to \mathbf{Ab}$. Using Yoneda's lemma and Proposition 3.1, we get

$$(\Phi X, \Phi Y) \cong \varprojlim(X_j, Y) \cong (\varinjlim X_j, Y) \cong (X, Y),$$

i.e., Φ is a full embedding.

We claim that a functor in $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ is simple (indecomposable injective) if and only if it is isomorphic to the image of a simple (indecomposable injective) object under Φ. For simple functors, this follows from the exactness of Φ and the fact that every subfunctor and every quotient of a representable functor is representable [Ga, p. 356, paragraph 5]. We now proceed to justify our claim for indecomposable injective functors. Since $\mathfrak{C}$ is Hom-finite and Φ is fully faithful, the endomorphism algebra of any simple functor is finite-dimensional over k. Since $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ is locally finite by (b), [T, Theorem 5.1] says that $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ is k-linearly equivalent to $\mathfrak{M}^{\Gamma}$ for some k-coalgebra Γ. If $I \in \mathfrak{C}$ is indecomposable injective, then $\Phi I = (\ ,I)|\mathfrak{c}$ is indecomposable injective by either Theorem 2.2 or [Ga, Lemma II.4.4], the latter saying that a (left exact) functor is injective if and only if it is exact. Consider now an indecomposable injective functor F and the indecomposable injective comodule that corresponds to it under the k-linear equivalence mentioned above. By [Gr, 1.5g(i)], an injective comodule is indecomposable if and only if it is an injective envelope of a simple comodule. Hence F is an injective envelope of ΦS for some simple $S \in \mathfrak{C}$. Let $i : S \to I$ be a monomorphism in $\mathfrak{C}$ with I indecomposable injective. Since Φ is exact, $\Phi i : \Phi S \to \Phi I$ is monic, and the indecomposability of ΦI implies $F \cong \Phi I$.

We now show that $\operatorname{Im}\Phi$ is equivalent to the full subcategory of $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ determined by the finitely copresented functors. For $X \in \mathfrak{C}$ we have an exact sequence $0 \to X \to I_0 \to I_1$ where I_0, I_1 are finite direct sums of indecomposable injectives. By what we have just proved, the sequence $0 \to \Phi X \to \Phi I_0 \to \Phi I_1$ is exact and $\Phi I_0, \Phi I_1$ are finite direct sums of indecomposable injectives, so ΦX is finitely copresented. Conversely, if F is a finitely copresented functor, we have an exact sequence $0 \to F \to (\ ,I_0)|\mathfrak{c} \to (\ ,I_1)|\mathfrak{c}$ where I_0 and I_1 are finite direct sums of indecomposable injectives in $\mathfrak{C}$. This exact sequence is isomorphic to

$$0 \to (\ ,\operatorname{Ker} f)|\mathfrak{c} \to (\ ,I_0)|\mathfrak{c} \xrightarrow{(\ ,f)} (\ ,I_1)|\mathfrak{c},$$

for some morphism $f : I_0 \to I_1$, whence $F \cong \Phi \operatorname{Ker} f$.

(d) In view of the preceding considerations, we note that since $\mathcal{L}(\mathfrak{c}^{op}, \mathbf{Ab})$ has enough projectives by (b), Γ is right semiperfect according to [L, Theorem p. 358]; and since $\mathfrak{m}^\Gamma$ must be an abelian category, Γ is right cocoherent.

(e) Using the properties of the functor Φ established in (c), we note that $\mathfrak{C}$ is hereditary if and only if an epimorphic image of an indecomposable injective object is injective, if and only if an epimorphic image of an indecomposable injective comodule is injective, if and only if Γ is hereditary [NTZ, Theorem 4]. By Proposition 1.3(c), a hereditary coalgebra is cocoherent. □

4. Main results

According to Theorem 3.2(d), the abelian category $\mathfrak{C}$ studied in Section 3 is equivalent to the category $\mathfrak{m}^\Gamma$ for a certain coalgebra Γ. To get more information about $\mathfrak{C}$ we use the results on the existence of almost split sequences for comodules presented in Section 1.

Theorem 4.1. *The following are equivalent for a skeletally small abelian* Hom-*finite k-category $\mathfrak{C}$ with enough injectives, where for each simple object S there is an epimorphism $P \to S$ with P projective of finite length.*

(a) *Each indecomposable injective has a simple subobject.*
(b) *The category $\mathfrak{C}$ is equivalent to the category $\mathfrak{m}^\Gamma$, where Γ is a right semiperfect and right cocoherent k-coalgebra such that $\mathfrak{m}^\Gamma$ contains all simple Γ-comodules.*
(c) *The category $\mathfrak{C}$ has left almost split sequences.*

Proof. (a)⇒(b) This is Theorem 3.2(d).

(b)⇒(c) This is Corollary 1.6(a).

(c)⇒(a) By [RV, Lemma I.3.1], every indecomposable injective has a unique simple subobject. □

Theorem 4.2. *The following are equivalent for a skeletally small abelian* Hom-*finite k-category $\mathfrak{C}$ with enough injectives, where for each simple object S there is an epimorphism $P \to S$ with P projective of finite length.*

(a) *Each indecomposable injective is of finite length.*
(b) *The category $\mathfrak{C}$ is equivalent to the category $\mathfrak{m}^\Gamma$ where Γ is a left and right semiperfect k-coalgebra.*
(c) *The category $\mathfrak{C}$ has almost split sequences.*

Proof. (a)⇒(b) This is a direct consequence of the implication (a)⇒(b) of Theorem 4.1 together with Theorem 1.1.

(b)⇒(a) Obvious.

(b)⇔(c) This follows immediately from parts (ii) and (b) of Corollary 1.6. □

In the remainder of this section we use our results to obtain a description of a skeletally small abelian Hom-finite k-category $\mathfrak{B}$ with enough projectives, where for each simple object S there is a monomorphism $S \to I$ with I injective of finite length. Such a category is opposite to the category $\mathfrak{C}$ of Section 3. When $\mathfrak{B}$ is hereditary, it is studied in [RV, Chapter II].

Corollary 4.3. *The following are equivalent for a skeletally small abelian* Hom-*finite k-category $\mathfrak{B}$ with enough projectives, where for each simple object S there is a monomorphism $S \to I$ with I injective of finite length.*

(a) *Each indecomposable projective has a simple factor object.*
(b) *The category* $\mathfrak{B}$ *is equivalent to* $(\mathfrak{m}^\Gamma)^{op}$*, where* Γ *is a right semiperfect and right cocoherent* k*-coalgebra such that* $\mathfrak{m}^\Gamma$ *contains all simple comodules.*
(c) *The category* $\mathfrak{B}$ *has right almost split sequences.*

Proof. This follows from Theorem 4.1 by the categorical duality. □

Corollary 4.4. *The following are equivalent for a skeletally small abelian* Hom*-finite* k*-category* $\mathfrak{B}$ *with enough projectives, where for each simple object* S *there is a monomorphism* $S \to I$ *with* I *injective of finite length.*

(a) *Each indecomposable projective is of finite length.*
(b) *The category* $\mathfrak{B}$ *is equivalent to the category* $\mathfrak{m}^\Gamma$ *where* Γ *is a left and right semiperfect* k*-coalgebra.*
(c) *The category* $\mathfrak{B}$ *has almost split sequences.*

Proof. Applying the categorical duality to Theorem 4.2, we see that each of (a) and (c) holds if and only if $\mathfrak{B}$ is equivalent to $(\mathfrak{m}^\Gamma)^{op}$ where Γ is a left and right semiperfect coalgebra. Clearly Γ is left and right semiperfect if and only if so is Γ^{op}. If the latter is the case, Theorem 1.1 implies that $\mathfrak{m}^\Gamma$ ($\mathfrak{m}^{\Gamma^{op}}$) coincides with the category of finite-dimensional Γ-comodules (Γ^{op}-comodules). Since [Gr, (1.2g), Remark] says that D is a duality between the categories of finite-dimensional Γ- and Γ^{op}-comodules, D is a duality between $\mathfrak{m}^\Gamma$ and $\mathfrak{m}^{\Gamma^{op}}$. Therefore the categories $(\mathfrak{m}^\Gamma)^{op}$ and $\mathfrak{m}^{\Gamma^{op}}$ are equivalent, which finishes the proof. □

Suppose now that the category $\mathfrak{B}$ is hereditary and the field k is algebraically closed. We show how our results imply Theorem 1.8, which is proved in [RV].

Proposition 4.5. *A skeletally small hereditary abelian* Hom*-finite category* $\mathfrak{B}$ *over an algebraically closed field with enough projectives has the property that each indecomposable projective has a simple factor object and, for each simple object* S*, there is a monomorphism* $S \to I$ *with* I *injective of finite length if and only if* $\mathfrak{B}$ *is equivalent to the category* $\operatorname{rep} Q$ *for a locally finite quiver* Q *in which at most finitely many paths end at each vertex.*

Proof. For the necessity, suppose that $\mathfrak{B}$ satisfies the indicated conditions. By Corollary 4.3, it is equivalent to $(\mathfrak{m}^\Gamma)^{op}$ where Γ is a right semiperfect basic hereditary k-coalgebra such that $\mathfrak{m}^\Gamma$ contains all simple comodules (remember, a hereditary coalgebra is right cocoherent by Proposition 1.3(c)). By Proposition 1.7(a), Γ is isomorphic to the path coalgebra of a locally finite quiver Q in which at most finitely many paths start at each vertex, so we may assume that Γ is the path coalgebra. By Proposition 1.9(a), the opposite quiver Q^{op} is a locally finite quiver in which at most finitely many paths end at each vertex. By Proposition 1.9(b), $\mathrm{D} : \mathfrak{m}^\Gamma \to \operatorname{rep} Q^{op}$ is a duality, so $\mathfrak{B}$ is equivalent to $\operatorname{rep} Q^{op}$. Reversing the argument gives the sufficiency, which we leave to the reader. □

It remains to note that a category $\mathfrak{B}$ satisfying the hypotheses of Theorem 1.8 also satisfies the hypotheses of the necessity of Proposition 4.5. Indeed, an Ext-finite category is Hom-finite, and if $\mathfrak{B}$ has enough projectives and a right Serre functor F such that $F(P)$ is injective of finite length for each projective P, then [RV, Corollary I.3.4, part 3] implies that each indecomposable projective has a simple factor object and, for each simple object S, there is a monomorphism $S \to I$ with I injective of finite length.

References

[ARS] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, New York, 1994. MR96c:16015

[C] W. Chin, *Hereditary and path coalgebras*, Comm. Algebra **30** (2002), 1829-1831.

[CM] W. Chin and S. Montgomery, *Basic coalgebras*, AMS/IP Studies in Advanced Mathematics, vol. **4**, Amer. Math. Soc., 1997, pp. 41-47. MR99c:16037

[CKQ] W. Chin, M. Kleiner, and D. Quinn, *Almost split sequences for comodules*, J. Algebra **249** (2002), 1-19. MR2003e:16045

[Ga] P. Gabriel, *Des catégories abéliannes*, Bull. Soc. Math. France **90** (1962), 323-448. MR38:1144

[Gr] J. A. Green, *Locally finite representations*, J. Algebra **41** (1976), 131-171. MR54:348

[Gro] A. Grothendieck, *Sur quelques points d'algébre homologique*, Tohôku Math. J., Série 2, **9** (1957), 119-221. MR21:1328

[L] I-P. Lin, *Semiperfect coalgebras*, J. Algebra **49** (1977), 357-373. MR58:16749

[Mi] B. Mitchell, *Theory of Categories*, Academic Press, New York and London, 1965. MR34:2647

[Mo] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Lecture Notes **82**, Amer. Math. Soc., 1993. MR94i:16019

[NTZ] Nastasescu, B. Torrecillas, Y. H. Zhang, *Hereditary coalgebras*, Comm. Algebra **24** (1996), 1521-1528. MR96m:16057

[SS] S. A. Sikko and S. O. Smalø, *Coherent rings and homologically finite subcategories*, Math. Scand. 77 (1995), no. 2, 175–183. MR97e:16011

[T] M. Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Univ. Tokyo **24** (1977), 629-644. MR57:12646

[RV] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), 295-366. MR2003a:18011

Department of Mathematics, Syracuse University, Syracuse, New York 13244-1150
E-mail address: mkleiner@syr.edu

Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway
E-mail address: idunr@math.ntnu.no