FIXED POINT INDEX IN SYMMETRIC PRODUCTS

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Abstract. Let $U$ be an open subset of a locally compact metric ANR $X$ and let $f : U \to X$ be a continuous map. In this paper we study the fixed point index of the map that $f$ induces in the $n$-symmetric product of $X$, $F_n(X)$. This index can detect the existence of periodic orbits of period $\leq n$ of $f$, and it can be used to obtain the Euler characteristic of the $n$-symmetric product of a manifold $X$, $\chi(F_n(X))$. We compute $\chi(F_n(X))$ for all orientable compact surfaces without boundary.

1. Introduction

Let $X$ be a locally compact metric ANR, $f : U \subset X \to X$ a semidynamical system and $K \subset U$ a compact isolated invariant set with respect to $f$. In this paper we construct the fixed point index of the map that $f$ induces in the spaces $F_n(X)$ of the non-empty finite subsets of $X$ with at most $n$ elements, endowed with the Hausdorff metric. These spaces were defined in 1931 by Borsuk and Ulam, in [3], with the name of $n$-symmetric product of $X$. They studied some topological properties which $X$ induces in $F_n(X)$ and topological properties of the space $F_n([0,1])$.

Our fixed point index detects the existence of periodic orbits of $f$ in $K$ of period less than or equal to $n$.

Let $2^X$ be the hyperspace of all non-empty compact subsets of $X$ endowed with the Hausdorff metric $d_H$, defined by

$$d_H(C,D) = \inf \{ \epsilon > 0 : C \subset B(D,\epsilon) \text{ and } D \subset B(C,\epsilon) \},$$

and let $C_n(X) \subset 2^X$ be the hyperspace of all non-empty compact subsets of $X$ having at most $n$ connected components. Our study will be harder than the analysis of the fixed point indices constructed in [23] for the hyperspaces $2^X$ and $C_n(X)$. The difficulties follow from the fact that the topological structure of $F_n(X)$ is more complicated than that of $2^X$ and $C_n(X)$.

In Section 2 we prove that our construction is consistent and we show the most important properties. We also compute the index for $K = \{ p \}$ a non-attracting and non-repelling fixed point of a local homeomorphism $f$ of $\mathbb{R}^2$.

If $f$ is an orientation-preserving local homeomorphism of the plane and $\{ p \}$ is a fixed point of $f$ that is an isolated invariant set which is not an attractor nor...
a repeller, Le Calvez and Yoccoz proved, in [14], that there exist integers $r, q \geq 1$ such that the fixed point index

$$i_{R^2}(f^k, p) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{N}, \\ 1 & \text{if } k \notin r\mathbb{N}. \end{cases}$$

In the above setting we will show that the fixed point index of the map that $f$ induces in $F_n(R^2)$ at $\{p\}$ is

$$\frac{\sum_{j=1}^{n} i_{R^2}(f^j, p)}{n}$$

for every $n \leq r$. We will give the proof of this result in Section 4.

In Section 3 we give techniques for computing the Euler characteristic of $F_n(X)$, for $X$ a finite dimensional manifold.

Sometimes the spaces $F_n(X)$ are topologically equivalent to convex subsets of a euclidean space. In this case our results have similarities with the computations given in [7].

Although many authors have considered the study of the topological structure of the spaces $F_n(X)$, the topological characterizations are exceptional. Borsuk and Ulam, in [3], proved that $F_n([0,1]) \simeq [0,1]^n$ for $n \leq 3$. Borsuk, in [2], claimed that $F_3(S^1) \simeq S^1 \times S^2$ but Bott, in [1], showed that $F_3(S^1) \simeq S^1$. Molski, in [18], saw that $F_2([0,1]^2) \simeq [0,1]^3$. In the same direction we have the work of Schori, [20], where there is a characterization of spaces of the type $F_n([0,1]^n)$, obtained by using suitable equivalence relations. Likewise we have some results about the topological properties of the symmetric products. In [3] it is proved that $dim(F_n([0,1])) = n$ for all $n$ and that $F_n([0,1])$ cannot be embedded in $\mathbb{R}^n$ for $n > 3$. Likewise, in [18], it is shown that $F_n([0,1]^2)$ and $F_2([0,1]^n)$ cannot be embedded in $\mathbb{R}^{2n}$ for $n \geq 3$. Wu, in [27], proved that, for $n$ odd, $F_n(S^1)$ has the homology of $S^n$ and, for $n$ even, $H^0(F_n(S^1)) = H^{n-1}(F_n(S^1)) = \mathbb{Z}$, and $H^i(F_n(S^1)) = 0$ if $i \neq 0, n - 1$. Schori, in [20], showed that for a 2-manifold $M$, $F_2(M)$ is a 4-manifold. In [12], Illanes saw that if $X$ is a locally connected normal space then $F_n(X)$ is unicoherent for $n \geq 3$. Macías proved in [15] that if $X$ is a continuum then $F_n(X)$ is unicoherent for $n \geq 3$. He proves that $H^1(F_n(X), \mathbb{Z}) = 0$.

In this paper we provide techniques which allow us to compute the Euler characteristics of the $n$-symmetric products of finite dimensional manifolds. Specifically, this is the aim of Section 3, where we make the explicit computation for $X$ an orientable compact surface without boundary.

In a final remark we suggest the possibility of using our techniques to study the dynamics of certain hyperbolic dynamical systems, such as the G-horseshoe.

2. Definitions and Preliminary Results

From now on, $X$ will denote a locally compact, metric ANR. Let $U \subset X$ be an open set. By a semidynamical system we mean a locally defined continuous map $f : U \to X$.

We say that a function $\sigma : \mathbb{Z} \to X$ is a solution to $f$ through $x$ in $N \subset U$ if $f(\sigma(i)) = \sigma(i + 1)$ for all $i \in \mathbb{Z}$, $\sigma(0) = x$ and $\sigma(i) \in N$ for all $i \in \mathbb{Z}$. The invariant part of $N$, $Inv(N, f)$, is defined as the set of all $x \in N$ that admit a solution to $f$ through $x$ in $N$. 
A compact set $K \subset U$ is invariant if $f(K) = K$. An invariant compact set $K$ is isolated with respect to $f$ if there exists a compact neighborhood $N$ of $K$ such that $Inv(N, f) = K$. The neighborhood $N$ is called an isolating neighborhood of $K$.

The $n$-symmetric product of $X$, $F_n(X)$, is the closed subspace of $2^X$, endowed with the Hausdorff metric, consisting of all non-empty subsets of $X$ with at most $n$ points.

A semidynamical system $f : U \rightarrow X$ induces in a natural way another one, $F_n(f) : F_n(U) \rightarrow F_n(X)$.

Let $K \subset U$ be a compact isolated invariant set and let $N$ be any isolating neighborhood of $K$. Consider an open set $W$ such that $K \subset W \subset N$. Take $F_n(f)|_{F_n(W)} : F_n(W) \rightarrow F_n(X)$. It is clear that $Fix(F_n(f)|_{F_n(W)}) \subset F_n(K)$; then $Fix(F_n(f)|_{F_n(W)})$ is a compact subset of $F_n(W)$. On the other hand, $F_n(f)|_{F_n(W)}$ is a compact map because it admits an obvious extension to $F_n(N)$.

The set $F_n(W)$ is an open subset of $F_n(X)$ and, since $X$ is an ANR, $F_n(X)$ is an ANR for all $n \in \mathbb{N}$ ([19]).

Then, $i_{F_n(X)}(F_n(f)|_{F_n(W)})$, the fixed point index of $F_n(f)|_{F_n(W)}$ in $F_n(W)$, is well defined. For information about the fixed point index theory, the reader is referred to [9], [20], [21], and [11].

It would be interesting to study the fixed point index in the so-called $n$-symmetric products, $SP_n(X)$, constructed as the quotient of $X^n$ by the action of the group of permutations of $n$ elements. Let us observe that $F_n(X) = SP_n(X)$ if $n \leq 2$. One can expect a better additive behavior of this fixed point index than in the case of $F_n(X)$. In this sense Masih and Rallis, in [16], [17] and [22], constructed certain indices for maps $X \rightarrow SP_n(X)$. For more information about these spaces and their relation with algebraic topology, see [1].

**Definition 1.** We define the fixed $n$-finite set index of the pair $(K, f)$ as

$$I^F_{X^n}(K, f) = i_{F_n(X)}(F_n(f)|_{F_n(W)}, F_n(W)).$$

The condition that $K$ be isolated is sufficient, but not necessary, to guarantee the consistency of this fixed point index.

**Remark 1.** From the excision property of the fixed point index we have that $I^F_{X^n}(K, f)$ does not depend on the choice of the isolating neighborhood $N$ of $K$ and the open set $W$.

**Remark 2.** The spaces $F_n(X)$ are not growth hyperspaces of $X$ (see [6]). A compactum $B$ can be locally connected and $F_n(B) \notin$ ANR. So the techniques of [23] for computing the fixed point index in hyperspaces will not be useful in the case of $F_n(X)$.

The main properties of our index follow immediately from the corresponding properties of the fixed point index. They are stated in the following propositions.

**Proposition 1** (Ważewski property). $I^F_{X^n}(K, f) \neq 0$ implies that

$$K \supset Fix(F_n(f)|_{F_n(W)}) \neq 0.$$

So there exists a periodic orbit of $f$ in $K$ of period $\leq n$.

**Proposition 2** (Particular cases of the additivity property). Let $K$ be a compact isolated invariant set. If $K$ is the disjoint union of two compact isolated invariant sets $K_1$ and $K_2$, then

$$I^F_{X^n}(K, f) = I^F_{X^n}(K_1, f) + I^F_{X^n}(K_2, f)$$
and
\[ I_{X}^{F_{n}}(K, f) = I_{X}^{F_{n}}(K_{1}, f) + I_{X}^{F_{n}}(K_{2}, f) + I_{X}^{F_{n}}(K_{1}, f)I_{X}^{F_{n}}(K_{2}, f). \]

The proof of the second equality follows from the fact that \( F_{2}(U_{1} \cup U_{2}) \) is homeomorphic to the disjoint union \( F_{2}(U_{1}) \vee F_{2}(U_{2}) \vee (F_{1}(U_{1}) \times F_{1}(U_{2})) \) for \( U_{1}, U_{2} \) disjoint open neighborhoods of \( K_{1} \) and \( K_{2} \) respectively.

**Proposition 3** (Commutativity property). Let \( X, Y \) be locally compact metric ANRs with \( U, V \) open subsets of \( X \) and \( Y \) respectively. Let
\[
\varphi : U \to Y, \\
\psi : V \to X
\]
be locally defined maps. Consider \( f = \psi \circ \varphi \) and \( g = \varphi \circ \psi \). If \( K \subset X \) is a compact isolated invariant set with respect to \( f \), then \( \varphi(K) \) is a compact isolated invariant set with respect to \( g \) and \( I_{X}^{F_{n}}(K, f) = I_{Y}^{F_{n}}(\varphi(K), g) \).

**Proposition 4** (Homotopy invariance property). Let \( f : U \times \Lambda \to X \) be a map such that \( U \) is an open subset of \( X \) and \( \Lambda \subset \mathbb{R} \) is a compact interval. Assume that \( N \) is an isolating neighborhood for each map \( f_{\lambda} : U \to X \). Then \( I_{X}^{F_{n}}(\text{Inv}(N, f_{\lambda}), f_{\lambda}) \) does not depend on \( \lambda \in \Lambda \).

Let us consider a local homeomorphism of the plane, \( f \), with \( K = \{ p \} \) a non-attracting and non-repelling fixed point. The next results allow us to relate the indices of the iterations of \( f \) and the corresponding indices in the symmetric product.

**Theorem 1** ([24]). Let \( f : U \subset \mathbb{R}^{2} \to \mathbb{R}^{2} \) be a local homeomorphism with \( p \in U \) a non-attracting and non-repelling fixed point of \( f \) such that \( \{ p \} \) is an isolated invariant set. Then there are a disc \( D \), containing a neighborhood \( V \) of \( p \), a finite subset \( \{ q_{1}, \ldots, q_{m} \} \subset D \) and a map \( \overline{f} : D \to D \) such that \( \overline{f}|_{V} = f|_{V} \), \( \overline{f}(\{ q_{1}, \ldots, q_{m} \}) \subset \{ q_{1}, \ldots, q_{m} \} \), and for every \( k \in \mathbb{N} \), \( \text{Fix}(\overline{f})^{k} \subset \{ p, q_{1}, \ldots, q_{m} \} \).

Moreover,
\( a \) (Le Calvez-Yoccoz, [14]). If \( f \) is orientation-preserving, then
\[
i_{\mathbb{R}^{2}}(f^{k}, p) = \begin{cases} 
1 - rq & \text{if } k \in r\mathbb{N}, \\
1 & \text{if } k \notin r\mathbb{N},
\end{cases}
\]
where \( k \in \mathbb{N} \), \( q \) is the number of periodic orbits of \( \overline{f} \) (excluding \( p \)) and \( r \) is their period.

\( b \) If \( f \) is orientation-reversing, then there are integers \( \delta \in \{ 0, 1, 2 \} \) and \( q \) such that
\[
i_{\mathbb{R}^{2}}(f^{k}, p) = \begin{cases} 
1 - \delta & \text{if } k \text{ is odd}, \\
1 - \delta - 2q & \text{if } k \text{ is even},
\end{cases}
\]
where \( q \) is the number of orbits of period \( 2 \) and \( \delta \) is the number of fixed points of \( \overline{f} \) in \( \{ q_{1}, \ldots, q_{m} \} \), and there is no other orbit of \( \overline{f} \) in \( \{ q_{1}, \ldots, q_{m} \} \).

If \( R \) is a finite set of \( r \) elements, let
\[ C_{s}^{r} = \text{Card}(\{ S \subset R : \text{Card}(S) = s \}). \]

A consequence of the above theorem is the following proposition. The reader can find its proof in Section 4.
Proposition 5. Let \( f : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \) be a homeomorphism, with \( p \in U \) a non-attracting and non-repelling fixed point of \( f \) such that \( \{p\} \) is an isolated invariant set.

a) If \( f \) is orientation-preserving, \( q \) is the number of periodic orbits of \( f \) in \( \{q_1, \ldots, q_m\} \) and \( r \) is their period, then for every \( n \in \mathbb{N} \)

\[
1 = \sum_{1 \leq j r \leq n} C^q_j + \sum_{0 \leq j r < n} C^q_j I^{F_n}_{D_{-jr}}(\{p\}, \mathcal{F}).
\]

b) If \( f \) is orientation-reversing, \( q \) the number of period-two orbits of \( f \) in \( \{q_1, \ldots, q_m\} \) and \( q' \leq 2 \) the number of fixed points of \( f \) in \( \{q_1, \ldots, q_m\} \), then for every \( n \in \mathbb{N} \)

\[
1 = \sum_{1 \leq j r + j' r \leq n} C^q_j C^q_{j'} + \sum_{0 \leq j r + j' r < n} C^q_j C^q_{j'} I^{F_n}_{D_{-j r - j' r}}(\{p\}, \mathcal{F}).
\]

Remark 3. In case a) of the above proposition, since \( \mathcal{F} \) is locally constant in \( \{q_1, \ldots, q_m\} \) (see [21]), we have

\[
I^{F_n}_{D_{r}}(\{p\}, \mathcal{F}) = \sum_{j=1}^{n} i_{\mathbb{Z}}(f^j, p) = \begin{cases} 1 & \text{ if } n < r, \\ 1 - q & \text{ if } n = r. \end{cases}
\]

Moreover, \( I^{F_n}_{D_{r+1}}(\{p\}, \mathcal{F}) = I^{F_n}_{D_{r}}(\{p\}, \mathcal{F}) \) for every \( n \in (kr, (k+1)r) \).

3. The Euler characteristic of the \( n \)-symmetric product of a manifold

The aim of this section is to develop techniques which allow us to compute the Euler characteristic of the \( n \)-symmetric product of a finite dimensional manifold \( X \). We will restrict ourselves to the case when \( X \) is an orientable, compact surface without boundary. This setting will provide us with techniques to study the general case.

If we choose an adequate dynamical system (homeomorphism) \( F : X \to X \) (\( F \simeq id \)), the Euler characteristic of \( F_n(X) \) is

\[
\chi(F_n(X)) = \Lambda(F_n(id)) = \Lambda(F_n(F)) = i_{F_n(X)}(F_n(F), F_n(X)),
\]

and, if \( F \) is such that the number of its periodic orbits of period \( \leq n \) is finite, by the additivity property, we only have to compute a finite number of indices \( i_{F_n(X)}(F_n(F), \bigcup_{j=1}^{r} \overline{p}_j) \) for \( \overline{p}_j \) periodic orbits of \( F \) of period \( p_j \) with \( \sum_{j=1}^{r} p_j \leq n \). The above fixed point indices, denoted by \( i_{n}(F, \bigcup_{j=1}^{r} \overline{p}_j) \), are defined in small enough neighborhoods, in \( F_n(X) \), of the isolated fixed points \( \bigcup_{j=1}^{r} \overline{p}_j \).

Note that if \( f : X \to X \) is a diffeomorphism of a manifold \( X \) of dimension \( m \) with \( p \) a hyperbolic fixed point for \( f \), then by the Grobman-Hartman theorem (see [10]) we can reduce the study of \( I^{F_n}_{X}(\{p\}, f) \) to the linear case \( I^{\cdot}_{\mathbb{R}^m}(\{0\}, Df(p)) \).

Let \( U \) be an open neighborhood of \( \{0\} \) in \( \mathbb{R}^m \) and let \( f : U \subset \mathbb{R}^m \to \mathbb{R}^m \) be a linear map. Assume that \( K = \{0\} \) is a compact isolated invariant set. The study of the index \( I^{\cdot}_{\mathbb{R}^m}(\{0\}, f) \) gives information which allows us to calculate \( \chi(F_n(X)) \) for a compact manifold \( X \).

Let us denote by \( D(\lambda_1, \ldots, \lambda_m) \) the diagonal \( m \times m \) matrix with \( \lambda_1, \ldots, \lambda_m \) on the diagonal.
The only linear cases which we will need here are given in the next proposition:

**Proposition 6.**

\[ I_{\mathbb{R}^m}^F(\{0\}, D(0, \ldots, 0)) = 1 \]

and

\[ I_{\mathbb{R}^m}^F(\{0\}, D(2, \ldots, 2)) = \begin{cases} 
1 & \text{if } m \text{ is even,} \\
-1 & \text{if } m \text{ and } n \text{ are odd,} \\
0 & \text{if } m \text{ is odd and } n \text{ is even.}
\end{cases} \]

The first equality is trivial, and the second one is proved in the Appendix.

The next theorem provides a complete study of \( I_{\mathbb{R}^m}^F(\{0\}, f) \) for a linear map and 0 a hyperbolic fixed point. We give an outline of the proof in the Appendix (see [25] for a complete proof).

This result is useful if one wants to study the Euler characteristic of the symmetric product of a manifold of dimension \( n > 2 \).

**Theorem 2.** Let \( f : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^m \) be a linear map with \( K = \{0\} \) a compact isolated invariant set. Consider the set of the real eigenvalues (repeated) which have modulus greater than 1, \( \{\lambda_1, \ldots, \lambda_r\} \).

Let \( r_2 \) be the number of eigenvalues greater than 1, and \( r_{-2} \) the number of eigenvalues smaller than \(-1\). Of course \( r = r_2 + r_{-2} \). Then,

\[ I_{\mathbb{R}^m}^F(\{0\}, f) = \begin{cases} 
\text{if } r_2 \text{ is odd and } r_{-2} \text{ is even,} \\
I_{\mathbb{R}^m}^F(\{0\}, D(2)) = \begin{cases} 
0 & \text{if } n \text{ is even,} \\
-1 & \text{if } n \text{ is odd;}
\end{cases} \\
\text{if } r_2 \text{ is even and } r_{-2} \text{ is odd,} \\
I_{\mathbb{R}^m}^F(\{0\}, D(-2)) = \begin{cases} 
0 & \text{if } n \text{ is even,} \\
(-1)^k & \text{if } n = 2k + 1;
\end{cases} \\
\text{if } r_2 \text{ is odd and } r_{-2} \text{ is odd,} \\
I_{\mathbb{R}^m}^F(\{0\}, D(-2)) = \begin{cases} 
1 & \text{if } n \text{ is even,} \\
-1 & \text{if } n \text{ is odd;}
\end{cases} \\
\text{if } r_2 \text{ is even and } r_{-2} \text{ is even,} \\
I_{\mathbb{R}^m}^F(\{0\}, D(0, \ldots, 0)) = 1.
\end{cases} \]

From now on, we study \( \chi(F_n(X)) \) for \( X \) an orientable, compact surface without boundary.

In the next proposition we compute \( \chi(F_n(S^k)) \).

**Proposition 7.** The Euler characteristic \( \chi(F_n(S^k)) \) of the \( n \)-symmetric products of \( S^k \) is

\[ \chi(F_n(S^{2k + 1})) = 0 \]

for all \( n \in \mathbb{N} \), and

\[ \chi(F_n(S^{2k})) = \begin{cases} 
2 & \text{if } n = 1, \\
3 & \text{if } n \geq 2.
\end{cases} \]

**Proof.** Consider the dynamical system \( J : S^k \rightarrow S^k \), shown in Figure 1.

We have \( J \simeq id \), and there are two hyperbolic fixed points, a repeller \( p \) and an attractor \( q \).

We have

\[ \chi(F_n(S^k)) = \Lambda(F_n(J)) = I_{\mathbb{R}^m}^F(S^k, J) = i_n(J, p) + i_n(J, q) + i_n(J, \{p, q\}). \]
Let us consider a small enough open neighborhood of \( q, U_n(q), \) in \( F_n(S^k) \). Since \( q \) is an attractor, we can construct a homotopy \( H : cl(U_n(q)) × I → F_n(S^k) \) such that \( H_0 = F_n(J) \) and \( H_1 = q \) and \( H(\bar{x}, t) \neq \bar{x} \quad \forall (\bar{x}, t) ∈ \partial(U_n(q)) × I. \) Then, by the homotopy property of the fixed point index, it is obvious that

\[
i_n(J, q) = i_{F_n(S^k)}(H_0, U_n(q)) = i_{F_n(S^k)}(H_1, U_n(q)) = 1.
\]

The next step is to prove that \( i_n(J, \{p, q\}) = i_{n-1}(J, p) \). Given the open balls \( V_1 = B(p, \varepsilon) \) and \( V_2 = B(q, \varepsilon) \), we define the open neighborhood of the point \( \{p, q\} \in F_n(S^k), \)

\[
U_n(\{p, q\}) = \{\bar{x} ∈ F_n(S^k) : \bar{x} ⊂ \bigcup V_i \text{ and } \bar{x} ∩ V_i \neq ∅ \text{ for all } i = 1, 2\}.
\]

Given \( \bar{x} = \{x_1, \ldots, x_s, x_{s+1}, \ldots, x_t\} \in U_n(\{p, q\}) \), with \( \{x_1, \ldots, x_s\} ⊂ V_1 \) and \( \{x_{s+1}, \ldots, x_t\} ⊂ V_2 \), we define the continuous map

\[
F : U_n(\{p, q\}) → F_{n-1}(S^k)
\]
as \( F(\{x_1, \ldots, x_t\}) = \{J(x_1), \ldots, J(x_s)\}. \)

In the same way we consider the continuous map

\[
G : U_{n-1}(p) → F_n(S^k)
\]
defined as \( G(\{x_1, \ldots, x_t\}) = \{x_1, \ldots, x_t, q\}. \)

Now, we take the compositions \( F ∘ G : U_{n-1}(p) → F_{n-1}(S^k) \) and \( G ∘ F : F^{-1}(U_{n-1}(p)) → F_n(S^k) \). It is obvious that \( F ∘ G = F_{n-1}(J) \). On the other hand, \( (G ∘ F)(\{x_1, \ldots, x_t\}) = \{J(x_1), \ldots, J(x_s), q\} \), and it is not difficult to construct a homotopy

\[
H : cl(F^{-1}(U_{n-1}(p))) × I → F_n(S^k)
\]
such that \( H_0 = F_n(J) \) and \( H_1 = G ∘ F \), with

\[
H(\bar{x}, t) \neq \bar{x} \quad \forall (\bar{x}, t) ∈ \partial(F^{-1}(U_{n-1}(p))) × I.
\]

Then, using the commutativity and the homotopy properties of the fixed point index, we have that

\[
i_n(J, \{p, q\}) = i_{F_n(S^k)}(F_n(J), U_n(\{p, q\})) = i_{F_n(S^k)}(G ∘ F, F^{-1}(U_{n-1}(p)))
\]

\[
= i_{F_{n-1}(S^k)}(F ∘ G, U_{n-1}(p))) = i_{F_{n-1}(S^k)}(F_{n-1}(J), U_{n-1}(p))) = i_{n-1}(J, p).
\]
Then $\chi(F_n(S^k)) = i_n(J,p) + 1 + i_{n-1}(J,p)$, and from Proposition 6 and the Grobman-Hartman theorem, we have

\[ i_n(J,p) = \begin{cases} 
1 & \text{if } k \text{ is even}, \\
-1 & \text{if } k \text{ and } n \text{ are odd}, \\
0 & \text{if } k \text{ is odd and } n \text{ is even}.
\end{cases} \]

Now the result follows automatically. \qed

**Remark 4.** Let us notice that we can construct a map $F_k : S^{2k+1} \to S^{2k+1}$ homotopic to the identity without periodic points. The equality $\chi(F_n(S^{2k+1})) = 0$ follows from this fact. We define $F_k$ as the restriction to $S^{2k+1} \subset C^{k+1}$ of the map $(z_1, \ldots, z_{k+1}) \mapsto (e^{2i\pi\alpha}z_1, \ldots, e^{2i\pi\alpha}z_{k+1})$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

We can also use this map to compute $\chi(F_n(S^{2k+2}))$. In fact, let us consider $g : [-1,1] \to [-1,1]$ with $g(x) = 2x$ if $|x| \leq 1/2$, and $g(x) = \frac{x}{|x|}$ if $1/2 \leq |x| \leq 1$. The map

\[ F_k \times g : S^{2k+1} \times [-1,1] \to S^{2k+1} \times [-1,1] \]

defines a continuous map on the sphere $S^{2k+2}$ obtained by identifying each sphere $S^{2k+1} \times \{\epsilon\}$, $\epsilon \in \{-1,1\}$, to a point. The only periodic orbits are the two fixed points, where the map is locally constant.

The same ideas can be applied to the torus $T$, to prove that $\chi(F_n(T)) = 0$.

Let us compute the Euler characteristic of the $n$-symmetric product of the compact oriented surfaces of genus $k$, $\chi(F_n(M_k))$.

**Proposition 8.** The Euler characteristic of $F_n(M_k)$, with $k \geq 2$, is

\[ \chi(F_n(M_k)) = \sum_{j=1}^{n} (-1)^j C_{2k-3+j}. \]

If $k = 1$, then $\chi(F_n(T)) = 0$.

**Proof.** Let us consider the dynamical system $J : M_k \to M_k$ shown in Figure 2.
We have that \( J \simeq id \), with two fixed points \( p \) and \( q \). The point \( q \) is a source and the map \( J|_{M_k \setminus \{q\}} \) is conjugated to the product
\[
L_{2k} \times f : Y_{2k} \times [0,1) \to Y_{2k} \times [0,1),
\]
where \( f(x) = x^2 \) and \( L_{2k} : Y_{2k} \to Y_{2k} \) is the dynamical system defined on the pointed union of \( 2k \) loops \( Y_{2k} \) shown in Figure 3.

The Euler characteristic of \( F_n(M_k) \) is
\[
\chi(F_n(M_k)) = \chi(F_n(J)) = \int_{M_k} \chi(M_k, J) = i_n(J, p) + i_n(J, q) + i_n(J, \{p, q\}).
\]

By Proposition 6 and the Grobman-Hartman theorem, we have \( i_n(J, q) = 1 \).

On the other hand, let us see that \( i_n(J, \{p, q\}) = i_{n-1}(J, p) \). We consider the continuous maps
\[
F : U_n(\{p, q\}) \to F_{n-1}(M_k) \quad \text{and} \quad G : U_{n-1}(p) \to F_n(M_k)
\]
defined as in the proof of Proposition 7, and a homotopy
\[
H : cl(F^{-1}(U_{n-1}(p))) \times I \to F_n(M_k)
\]
such that \( H_0 = F_n(J) \) and \( H_1 = G \circ F \), with
\[
H(\bar{x}, t) \neq \bar{x} \quad \text{for all } (\bar{x}, t) \in \partial(F^{-1}(U_{n-1}(p))) \times I
\]
(for a construction of the homotopy \( H \), see the proof of Proposition 6 in the Appendix).

From the commutativity and the homotopy invariance properties of the fixed point index, we have \( i_n(J, \{p, q\}) = i_{n-1}(J, p) \), and therefore
\[
\chi(F_n(M_k)) = i_n(J, p) + 1 + i_{n-1}(J, p).
\]

It only remains to compute \( i_n(J, p) \). Since \( J|_{M_k \setminus \{q\}} \) is conjugated to \( L_{2k} \times f \), then \( i_n(J, p) = i_n(L_{2k} \times f, (p', 0)) = i_n(L_{2k}, p') \). The last equality follows from the homotopy and commutativity properties of the fixed point index.

Let us define the dynamical systems \( H_k, H_k' : Z_k \to Z_k \) with \( Z_k \) the union of \( k \) arcs connected by the endpoints (see Figure 4).

Given a fixed point \( \alpha \) of \( F_n(H_k) \), we denote \( i_{F_n(Z_k)}(F_n(H_k), \alpha) = i_n(H_k, \alpha) \).

Let us prove that \( i_n(L_{2k}, p') = i_n(H_{2k}, p) \). Given the map \( g : [0,1] \to [0,1] \) with
\[
g(x) = 2x \text{ if } |x| \leq 1/2, \quad \text{and} \quad g(x) = \frac{x}{|x|} \quad \text{if } 1/2 \leq |x| \leq 1,
\]
the restriction of \( L_{2k} \) to
We can consider the dynamical system $L_{2k} : Y_{2k} \rightarrow Y_{2k}$ as a identification in $H_{2k} : Z_{2k} \rightarrow Z_{2k}$ of the points $p$ and $q$ to a point $p'$. If $x \in Z_{2k}$, we call $[x] \in Y_{2k}$ the corresponding point obtained by the identification.

Given a small enough neighborhood $U_n(p')$ of $p'$ in $F_n(Y_{2k})$, let
\[
\bar{x} = \{[x_1], \ldots, [x_r], [x_{r+1}], \ldots, [x_s]\} \in U_n(p')
\]
with $\{[x_1], \ldots, [x_r]\}$ the points of $\bar{x}$ contained in the local repelling part of $p'$ in $Y_{2k}$.

Then let us consider the map $F : U_n(p') \subset F_n(Y_{2k}) \rightarrow Z_{2k}$ defined as
\[
F(\{[x_1], \ldots, [x_r], [x_{r+1}], \ldots, [x_s]\}) = \{H_{2k}(x_1), \ldots, H_{2k}(x_r), p\}
\]
If $r = s$, the point $p$ does not appear in the image of $F$.

Let $G : U_n(p) \subset F_n(Z_{2k}) \rightarrow F_n(Y_{2k})$ be the map defined as
\[
G(\{x_1, \ldots, x_r\}) = \{[x_1], \ldots, [x_r]\}
\]
By the commutativity property of the fixed point index applied to $F$ and $G$ we obtain that $i_n(L_{2k}, p') = i_n(H_{2k}, p)$. Therefore
\[
i_n(J, p) = i_n(H_{2k}, p),
\]
and we only have to compute $i_n(H_{2k}, p)$.

If $n \geq 2$, we have
\[
I_{Z_k}^{F_n}(Z_k, H_k) = i_n(H_k, p) + i_n(H_k, q) + i_n(H_k, \{p, q\}) + i_n(H_k, p) + 1 + i_{n-1}(H_k, p).
\]

The equality $i_n(H_k, q) = 1$ is a consequence of the fact that $q$ is an attractor,

and $i_n(H_k, \{p, q\}) = i_{n-1}(H_k, p)$ follows again from the homotopy invariance and

the commutativity properties of the fixed point index.

Using similar arguments it is easy to see that
\[
I_{Z_k}^{F_n}(Z_k, H'_k) = i_n(H_{k-1}, p).
\]
Since $H_k \simeq H_k$, then $I_{Z_k}^F(Z_k, H_k) = I_{Z_k}^F(Z_k, H_k)$ and
\[(1) \quad i_n(H_{k-1}, p) = i_n(H_k, p) + 1 + i_{n-1}(H_k, p).
\]
This formula allows us to compute $i_n(H_k, p)$ in a recurrent way (it is easy to see that $i_n(H, p) = 0$ for all $n$). Our aim is to obtain $i_n(H_k, p)$ in an explicit expression by an induction argument.

Let us prove that $i_n(H_k, p) - i_{n-1}(H_k, p) = (-1)^n C_n^{k+n-2}$.

Let $n = 2$ and $k = 1$. Then, since $i_n(H_1, p) = 0$, we have $i_2(H_1, p) - i_1(H_1, p) = (-1)^2 C_2^1 = 0$. Let us suppose that
\[i_n(H_k, p) - i_{n-1}(H_k, p) = (-1)^n C_n^{k+n-2}
\]
for all $n \geq 2, k \geq 1$ with $n + k \leq m_0$, and consider $n, k$ with $n + k = m_0 + 1$. Then, using (1), we have
\[i_n(H_k, p) - i_{n-1}(H_k, p) = i_n(H_k, p) + i_{n-1}(H_k, p) + 1,
\]
and
\[i_n(H_k, p) = i_{n-1}(H_k, p) - i_{n-2}(H_k, p) - 1.
\]
It follows that
\[i_n(H_k, p) - i_{n-1}(H_k, p)
\]
\[= i_n(H_{k-1}, p) - i_{n-1}(H_{k-1}, p) + i_{n-2}(H_k, p) - i_{n-1}(H_k, p)
\]
\[= (-1)^n C_n^{k+n-3} + (-1)^n C_{n-1}^{k+n-3} = (-1)^n C_n^{k+n-2},
\]
and the result is proved.

In the same way, it follows that $i_1(H_k, p) = -(k - 1)$, and then
\[i_n(H_k, p) = \sum_{j=1}^{n} (-1)^j C_j^{k-2+j}
\]
and
\[
\chi(F_n(M_k)) = i_n(H_{2k}, p) + i_{n-1}(H_{2k}, p) + 1
\]
\[= i_n(H_{2k-1}, p) = \sum_{j=1}^{n} (-1)^j C_j^{2k-3+j}.
\]

**Remark 5.** Given a manifold $X$ and a continuous map $F : X \to X$, we can obtain, under certain conditions of hyperbolicity, information about the dynamics of $F$ by studying the fixed point indices $I_{Z}^{F}(\text{Inv}(X, F), F)$. Certainly, there are other techniques which allow us to study this, but it seemed interesting for us to present this alternative method.

**Example.** Dynamics of the G-horseshoe. If we want to study the periodic orbits of the G-horseshoe with our techniques, let us consider the dynamical system $F : C \to C$ given by the extended G-horseshoe of Figure 5. We are interested in detecting the periodic orbits of $F$ on $I^2$ (the unique periodic orbit out of $I^2$ is a fixed point). Let us consider the continuous map $g : \Pi \circ F|_{S^1} : S^1 \to S^1$ defined as the composition of $F|_{S^1}$ with the projection $\Pi : C \to S^1$, where $S^1$ is the interior circle of $C$. It is not difficult to see, by the homotopy invariance and the commutativity properties of the fixed point index, that
\[(2) \quad I^{F(g)}_{Z}((\text{Inv}(I^2, F), F) = I^{F}_Z((\text{Inv}(I, g), g)).
\]
Let us observe that $g | I$ is an expansion, and $F | I^2$ is a contraction in the vertical direction and an expansion in the horizontal one. Then, given $n$ fixed, the number of periodic orbits of period $\leq n$ for $g$ and $F$ is finite, and the fixed points of $F_n(g)$ and $F_n(F)$ are isolated.

It is not hard to prove that if $\bar{\alpha}$ is a periodic orbit of period $n$ of $F | I^2$, then $i_{F_n(C)}(F_n(F), \bar{\alpha}) = -1$ (the same fact occurs with $g | I$). Then we can see, using (2) and an induction argument, that the number of periodic orbits of all periods of $F | I^2$ is the same as in the case of $g | I$.

By the commutativity and the homotopy properties of the fixed point index,

$$I_{C}^{F_n}(Inv(C, F), F) = I_{S^1}^{F_n}(Inv(S^1, g), g) = I_{S^1}^{F_n}(Inv(S^1, f), f),$$

where $f : S^1 \to S^1$ is the doubling angle map. A careful observation of $f$ and $g$ allows us to see that, although $f$ has one fixed point and $g | I$ has two (repelling) fixed points, the remaining periodic orbits are the same in both dynamical systems.

Since the set $\{x \in S^1 : F^n(x) = x\}$ has $2^n - 1$ points, then the set $\{x \in I^2 : F^n(x) = x\}$ has $2^n$ points. So, we have a characterization of the periodic orbits of the G-horseshoe.

4. Appendix. Proofs

Proof of Proposition 5. Let us see the proof of a) (the proof of b) is analogous). Since $D$ is an AR, $1 = I_{D}^{F_{n}}(D, \mathcal{I})$.

Let us consider the point $\bar{\alpha}(\mathcal{I}) = \overline{\alpha_1} \cup \cdots \cup \overline{\alpha_l} \in F_n(D)$ with $\overline{\alpha_i} = \{q_{i_1}, \ldots, q_{i_r}\}$ a periodic orbit of $\mathcal{I}$ in $\{q_{1}, \ldots, q_{m}\}$ for all $i = 1, \ldots, l$

$\text{Per}(\mathcal{I})$ is the set of periodic orbits of $\mathcal{I}$ in $\{q_{1}, \ldots, q_{m}\}$. Let us denote

$$i_{F_n(D)}(F_n(\mathcal{I}), \overline{\alpha(j)}) = i_n(\mathcal{I}, \overline{\alpha(j)}).$$
From the additivity property of the fixed point index for ANRs, we have
\[ 1 = I^{F_n}_{D}(D, \overline{f}) = \sum_{\overline{\alpha(j)} \in \text{Per} (\overline{f}), jr \leq n} i_n(\overline{f}, \overline{\alpha(j)}) \]
\[ + \sum_{\overline{\alpha(j)} \in \text{Per} (\overline{f}), jr < n} i_n(\overline{f}; p \cup \overline{\alpha(j)}) + i_n(\overline{f}, p). \]

Since \( \overline{f} \) is locally constant in each \( q_i \), see [24], we have
\[ i_n(\overline{f}, \overline{\alpha(j)}) = 1 \]
for all \( \overline{\alpha(j)} \subset \text{Per} (\overline{f}), jr \leq n. \)

Let \( \overline{\alpha(j)} \) be fixed with \( jr < n \). We prove that
\[ i_n(\overline{f}, p \cup \overline{\alpha(j)}) = i_{n-jr}(\overline{f}, p). \]

Let \( U_n(p \cup \overline{\alpha(j)}) \) be a small enough neighborhood in \( F_n(D) \) of the point \( p \cup \overline{\alpha(j)} \), and let \( \bar{x} \in U_n(p \cup \overline{\alpha(j)}) \) with \( \bar{x}_p = \bar{x} \cap B(p, \epsilon) \) for \( \epsilon \) small enough. The set \( \bar{x}_p = \{x_1, \ldots, x_l\} \) is such that \( 1 \leq l \leq n - jr \).

Let \( F : U_n(p \cup \overline{\alpha(j)}) \to F_{n-jr}(D) \) and \( G : U_{n-jr}(p) \to F_n(D) \) be the continuous maps
\[ F(\bar{x}) = \{\overline{f}(x_1), \ldots, \overline{f}(x_l)\}, \quad G(\bar{x}) = \overline{\alpha(j)} \cup \bar{x}. \]
The map \( F \circ G : U_{n-jr}(p) \to F_{n-jr}(D) \) is such that
\[ (F \circ G)(\bar{x}) = F_{n-jr}(\overline{f})(\bar{x}). \]

On the other hand, since \( \overline{f} \) is locally constant in each \( q_i \in \{q_1, \ldots, q_m\} \), the map \( G \circ F : F^{-1}(U_{n-jr}(p)) \to F_n(D) \) is such that
\[ (G \circ F)(\bar{x}) = F_n(\overline{f})(\bar{x}). \]

From the commutativity property of the fixed point index for ANRs we have the equality
\[ i_n(\overline{f}, p \cup \overline{\alpha(j)}) = i_{n-jr}(\overline{f}, p). \]

The proof of case a) is finished.

Proof of Proposition 6. Let us see that \( I^{F_n}_{\mathbb{R}^m}([0], 2I) = 1 \) for \( m = 2 \) (the case of \( m \) even will be analogous).

Let \( U_0 = B(0, 1) \) be an open neighborhood of \( \{0\} \) and let \( H : F_n(\text{cl}(U_0)) \times I \to F_n(\mathbb{R}^2) \) be the homotopy
\[ H(\{x_1, \ldots, x_r\}, t) = \begin{cases} \{A(t)(2x_1), \ldots, A(t)(2x_r)\} & \text{if } t \in [0, 1/2], \\ \{2(1-t)A(1/2)(2x_1), \ldots, 2(1-t)A(1/2)(2x_r)\} & \text{if } t \in [1/2, 1], \end{cases} \]
with
\[ A(t) = \begin{pmatrix} \cos(\frac{2\pi}{n+1}2t) & \sin(\frac{2\pi}{n+1}2t) \\ -\sin(\frac{2\pi}{n+1}2t) & \cos(\frac{2\pi}{n+1}2t) \end{pmatrix}. \]

We consider \( x_i \neq x_j \) if \( i \neq j \). It is obvious that \( r \leq n \).

The continuity of \( H \) is clear, and it is not hard to see that \( H(\bar{x}, t) \neq \bar{x} \) for all \( (\bar{x}, t) \in \partial(F_n(U_0)) \times I \). Since \( H_0 = F_n(2I) \) and \( H_1 = F_n(D(0, 0)) \), we have proved the result for \( m = 2 \).
Let us see that
\[ I_{\mathbb{R}^m}^F_n(\{0\}, 2Id) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ -1 & \text{if } n \text{ is odd}, \end{cases} \]
for \( m \) odd. We will prove the result for \( m = 1 \) (the general case is easy to obtain by combining the cases \( m = 1 \) and \( m \) even).

Let us consider the map \( g : J \to J \) with \( g(x) = x^{1/3} \) and \( J = [-1, 1] \). The only periodic orbits are the fixed points \( \{-1, 0, 1\} \).

Since \( F_n(J) \) is an absolute retract, we have
\[ \chi(F_n^J(J, g)) = \chi(F_n(id)) = 1. \]

Let us denote \( i_{F_n^J}(F_n(g), \bar{\alpha}) = i_n(g, \bar{\alpha}) \) for \( \bar{\alpha} \in Fix(F_n(g)) \). Then
\[ 1 = I_{F_n^J}^F(J, g) = \sum_{\bar{\alpha} \subset \{-1,0,1\}} i_n(g, \bar{\alpha}). \]

Using the commutativity and the homotopy invariance properties of the fixed point index as in the proof of Proposition 7, it is not difficult to see that
\[ i_n(g, 1) = i_n(g, -1) = i_n(g, \{-1, 1\}) = 1, \]
\[ i_n(g, \{-1, 0\}) = i_n(g, \{0, 1\}) = i_{n-1}(g, 0), \]
and
\[ i_n(g, \{-1, 0, 1\}) = i_{n-2}(g, 0). \]

Then, for \( n > 2 \),
\[ 1 = I_{F_n^J}^F(J, g) = i_n(g, 0) + 2i_{n-1}(g, 0) + i_{n-2}(g, 0) + 3. \]

Since \( I_{\mathbb{R}^2}^F(\{0\}, 2Id) = i_n(g, 0) \), by an induction argument on the last formula we finish the proof. \( \square \)

**Proof of Theorem 2.** Since \( \{0\} \) is an isolated invariant set, the eigenvalues \( \{\lambda_1, \ldots, \lambda_m\} \) of \( f \) have modulus different from 1. The first equality of the theorem, which reduces the study of the fixed point index in \( \mathbb{R}^m \) to the cases of \( \mathbb{R} \) and \( \mathbb{R}^2 \), it is easy to prove by using the techniques employed in the proof of Proposition 6.

On the other hand, the computation of \( I_{\mathbb{R}^2}^F(\{0\}, D(2)) \) follows from studying the dynamical system \( g_1 : J \to J \) defined as \( g_1(x) = x^{1/3} \) with \( J = [-1, 1] \). In fact,
\[ 1 = \chi(F_n(J)) = I_{g_1}^F(J, g_1) = \sum_{\bar{\alpha} \subset \{-1,0,1\}} i_n(g_1, \bar{\alpha}), \]
and we compute \( I_{\mathbb{R}^2}^F(\{0\}, D(2)) = i_n(g_1, 0) \) from the above equality. In an analogous way we have \( I_{\mathbb{R}^2}^F(\{0\}, D(-2)) \) with \( g_2(x) = -x^{1/3} \).

It only remains to compute \( I_{\mathbb{R}^2}^F(\{0\}, D(2,-2)) \). Let us consider the dynamical systems \( F = s \circ f : S^2 \to S^2 \) and \( G = s \circ g : S^2 \to S^2 \), where \( s : S^2 \to S^2 \) is a symmetry with respect to the plane \( \{z = 0\} \) and \( f, g : S^2 \to S^2 \) are the dynamical systems shown in Figure 6.

For the dynamical system given by \( F \), the fixed point \( p \) is of type \( D(2,-2) \) and \( q \) is an attractor. The fixed points \( a \) and \( b \) of \( G \) are of type \( D(2, -1/2) \) and \( D(-2, 1/2) \) respectively. The pairs \( \{c_1, c_2\} \) and \( \{d_1, d_2\} \) are attracting periodic orbits of period 2.
We have $F \simeq G$. Therefore, if $n \geq 2$,
\[
I_{S^2}^F(S^2, G) = I_{S^2}^F(S^2, F) = i_n(F, p) + i_n(F, q) + i_n(F, \{p, q\})
\]
\[
= i_n(F, p) + 1 + i_{n-1}(F, p).
\]

Let us prove the equality $I_{S^2}^F(S^2, G) = 1$. By the additivity property of the fixed point index,
\[
I_{S^2}^F(S^2, G) = \sum_{\bar{\alpha} \subset \{a, b, \{c_1, c_2\}, \{d_1, d_2\}\}} i_n(G, \bar{\alpha}).
\]

We have that $i_n(G, a) = I_{R^2}^F(\{0\}, D(2))$ and $i_n(G, b) = I_{R^2}^F(\{0\}, D(-2))$. The only difficulty is to compute $i_n(G, \{a, b\})$.

Let $J_1 = J_2 = [-1, 1]$. We denote by $X = J_1 \lor J_2$ the disjoint union of the intervals. Let us consider the map $h : X \to X$, defined as $h(x) = x^{1/3}$ if $x \in J_1$ and $h(x) = -x^{1/3}$ if $x \in J_2$.

Since $\chi(F_1(X)) = I_X^F(X, h) = 2$ and $\chi(F_n(X)) = I_X^F(X, h) = 3$ if $n > 1$, we can prove that
\[
i_n(h, \{0, 0\}) = \begin{cases} 
-1 & \text{if } n = 4k + 2, \\
1 & \text{if } n = 4k + 3, \\
0 & \text{otherwise},
\end{cases}
\]
for $k \in \mathbb{N}$.

Since $i_n(h, \{0, 0\}) = i_n(G, \{a, b\})$, the equality $I_{S^2}^F(S^2, G) = 1$ follows from (3).

Then $i_n(F, p) + i_{n-1}(F, p) = 0$. Since $i_1(F, p) = -1$, we obtain the value of $I_{R^2}^F(\{0\}, D(2, -2)) = i_n(F, p)$, and the proof is finished. \(\square\)

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