FIXED POINT INDEX IN SYMMETRIC PRODUCTS

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Abstract. Let $U$ be an open subset of a locally compact metric ANR $X$ and let $f : U \to X$ be a continuous map. In this paper we study the fixed point index of the map that $f$ induces in the $n$-symmetric product of $X$, $F_n(X)$. This index can detect the existence of periodic orbits of period $\leq n$ of $f$, and it can be used to obtain the Euler characteristic of the $n$-symmetric product of a manifold $X$, $\chi(F_n(X))$. We compute $\chi(F_n(X))$ for all orientable compact surfaces without boundary.

1. Introduction

Let $X$ be a locally compact metric ANR, $f : U \subset X \to X$ a semidynamical system and $K \subset U$ a compact isolated invariant set with respect to $f$. In this paper we construct the fixed point index of the map that $f$ induces in the spaces $F_n(X)$ of the non-empty finite subsets of $X$ with at most $n$ elements, endowed with the Hausdorff metric. These spaces were defined in 1931 by Borsuk and Ulam, in [3], with the name of $n$-symmetric product of $X$. They studied some topological properties which $X$ induces in $F_n(X)$ and topological properties of the space $F_n([0,1])$.

Our fixed point index detects the existence of periodic orbits of $f$ in $K$ of period less than or equal to $n$.

Let $2^X$ be the hyperspace of all non-empty compact subsets of $X$ endowed with the Hausdorff metric $d_H$, defined by

$$d_H(C,D) = \inf\{\epsilon > 0 : C \subset B(D,\epsilon) \text{ and } D \subset B(C,\epsilon)\},$$

and let $C_n(X) \subset 2^X$ be the hyperspace of all non-empty compact subsets of $X$ having at most $n$ connected components. Our study will be harder than the analysis of the fixed point indices constructed in [23] for the hyperspaces $2^X$ and $C_n(X)$. The difficulties follow from the fact that the topological structure of $F_n(X)$ is more complicated than that of $2^X$ and $C_n(X)$.

In Section 2 we prove that our construction is consistent and we show the most important properties. We also compute the index for $K = \{p\}$ a non-attracting and non-repelling fixed point of a local homeomorphism $f$ of $\mathbb{R}^2$.

If $f$ is an orientation-preserving local homeomorphism of the plane and $\{p\}$ is a fixed point of $f$ that is an isolated invariant set which is not an attractor nor
a repeller, Le Calvez and Yoccoz proved, in \[14\], that there exist integers \( r, q \geq 1 \) such that the fixed point index

\[
i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 
1 - rq & \text{if } k \in r\mathbb{N}, \\
1 & \text{if } k \notin r\mathbb{N}.
\end{cases}
\]

In the above setting we will show that the fixed point index of the map that \( f \) induces in \( F_n(\mathbb{R}^2) \) at \( \{p\} \) is

\[
\frac{\sum_{j=1}^{n} i_{\mathbb{R}^2}(f^j, p)}{n}
\]

for every \( n \leq r \). We will give the proof of this result in Section 4.

In Section 3 we give techniques for computing the Euler characteristic of \( F_n(X) \), for \( X \) a finite dimensional manifold.

Sometimes the spaces \( F_n(X) \) are topologically equivalent to convex subsets of a euclidean space. In this case our results have similarities with the computations given in \[7\].

Although many authors have considered the study of the topological structure of the spaces \( F_n(X) \), the topological characterizations are exceptional. Borsuk and Ulam, in \[3\], proved that \( F_n([0,1]) \simeq [0,1]^n \) for \( n \leq 3 \). Borsuk, in \[2\], claimed that \( F_3(S^1) \simeq S^1 \times S^2 \) but Bott, in \[4\], showed that \( F_3(S^1) \simeq S^1 \). Molski, in \[18\], saw that \( F_2([0,1]^2) \simeq [0,1]^3 \). In the same direction we have the work of Schori, \[26\], where there is a characterization of spaces of the type \( F_n([0,1]^n) \), obtained by using suitable equivalence relations. Likewise we have some results about the topological properties of the symmetric products. In \[8\] it is proved that \( \dim(F_n([0,1])) = n \) for all \( n \) and that \( F_n([0,1]) \) cannot be embedded in \( \mathbb{R}^n \) for \( n > 3 \). Likewise, in \[15\], it is shown that \( F_n([0,1]^2) \) and \( F_2([0,1]^n) \) cannot be embedded in \( \mathbb{R}^{2n} \) for \( n \geq 3 \).

Wu, in \[27\], proved that, for \( n \) odd, \( F_{n}(S^1) \) has the homology of \( S^n \) and, for \( n \) even, \( H^0(F_n(S^1)) = H^{n-1}(F_n(S^1)) = \mathbb{Z} \) and \( H^i(F_n(S^1)) = 0 \) if \( i \neq 0, n - 1 \). Schori, in \[26\], showed that for a 2-manifold \( M \), \( F_2(M) \) is a 4-manifold. In \[12\], Illanes saw that if \( X \) is a locally connected normal space then \( F_n(X) \) is unicoherent for \( n \geq 3 \). Macías proved in \[15\] that if \( X \) is a continuum then \( F_n(X) \) is unicoherent for \( n \geq 3 \). He proves that \( \bar{H}^1(F_n(X), \mathbb{Z}) = 0 \).

In this paper we provide techniques which allow us to compute the Euler characteristics of the \( n \)-symmetric products of finite dimensional manifolds. Specifically, this is the aim of Section 3, where we make the explicit computation for \( X \) an orientable compact surface without boundary.

In a final remark we suggest the possibility of using our techniques to study the dynamics of certain hyperbolic dynamical systems, such as the G-horseshoe.

## 2. Definitions and Preliminary Results

From now on, \( X \) will denote a locally compact, metric ANR. Let \( U \subset X \) be an open set. By a semidynamical system we mean a locally defined continuous map \( f : U \to X \).

We say that a function \( \sigma : \mathbb{Z} \to X \) is a solution to \( f \) through \( x \) in \( N \subset U \) if \( f(\sigma(i)) = \sigma(i + 1) \) for all \( i \in \mathbb{Z} \), \( \sigma(0) = x \) and \( \sigma(i) \in N \) for all \( i \in \mathbb{Z} \). The invariant part of \( N \), \( \text{Inv}(N, f) \), is defined as the set of all \( x \in N \) that admit a solution to \( f \) through \( x \) in \( N \).
A compact set $K \subset U$ is invariant if $f(K) = K$. An invariant compact set $K$ is isolated with respect to $f$ if there exists a compact neighborhood $N$ of $K$ such that $Inv(N, f) = K$. The neighborhood $N$ is called an isolating neighborhood of $K$.

The $n$-symmetric product of $X$, $F_n(X)$, is the closed subspace of $2^X$, endowed with the Hausdorff metric, consisting of all non-empty subsets of $X$ with at most $n$ points.

A semidynamical system $f : U \rightarrow X$ induces in a natural way another one, $F_n(f) : F_n(U) \rightarrow F_n(X)$.

Let $K \subset U$ be a compact isolated invariant set and let $N$ be any isolating neighborhood of $K$. Consider an open set $W$ such that $K \subset W \subset N$. Take $F_n(f)|_{F_n(W)} : F_n(W) \rightarrow F_n(X)$. It is clear that $Fix(F_n(f)|_{F_n(W)}) \subset F_n(K)$; then $Fix(F_n(f)|_{F_n(W)})$ is a compact subset of $F_n(W)$. On the other hand, $F_n(f)|_{F_n(W)}$ is a compact map because it admits an obvious extension to $F_n(N)$.

The set $F_n(W)$ is an open subset of $F_n(X)$ and, since $X$ is an ANR, $F_n(X)$ is an ANR for all $n \in \mathbb{N}$ ([19]).

Then, $i_{F_n(X)}(F_n(f)|_{F_n(W)}, F_n(W))$, the fixed point index of $F_n(f)|_{F_n(W)}$ in $F_n(W)$, is well defined. For information about the fixed point index theory, the reader is referred to [9], [20], [21], and [11].

It would be interesting to study the fixed point index in the so-called $n$-symmetric products, $SP_n(X)$, constructed as the quotient of $X^n$ by the action of the group of permutations of $n$ elements. Let us observe that $F_n(X) = SP_n(X)$ if $n \leq 2$. One can expect a better additive behavior of this fixed point index than in the case of $F_n(X)$. In this sense Maslih and Rallis, in [16], [17] and [22], constructed certain indices for maps $X \rightarrow SP_n(X)$. For more information about these spaces and their relation with algebraic topology, see [1].

**Definition 1.** We define the fixed $n$-finite set index of the pair $(K, f)$ as

$$I_{X}^{F_n}(K, f) = i_{F_n(X)}(F_n(f)|_{F_n(W)}, F_n(W)).$$

The condition that $K$ be isolated is sufficient, but not necessary, to guarantee the consistency of this fixed point index.

**Remark 1.** From the excision property of the fixed point index we have that $I_{X}^{F_n}(K, f)$ does not depend on the choice of the isolating neighborhood $N$ of $K$ and the open set $W$.

**Remark 2.** The spaces $F_n(X)$ are not growth hyperspaces of $X$ (see [6]). A compactum $B$ can be locally connected and $F_n(B) \notin$ ANR. So the techniques of [23] for computing the fixed point index in hyperspaces will not be useful in the case of $F_n(X)$.

The main properties of our index follow immediately from the corresponding properties of the fixed point index. They are stated in the following propositions.

**Proposition 1** (Ważewski property). $I_{X}^{F_n}(K, f) \neq 0$ implies that $K \supset Fix(F_n(f)|_{F_n(W)}) \neq 0$.

So there exists a periodic orbit of $f$ in $K$ of period $\leq n$.

**Proposition 2** (Particular cases of the additivity property). Let $K$ be a compact isolated invariant set. If $K$ is the disjoint union of two compact isolated invariant sets $K_1$ and $K_2$, then

$$I_{X}^{F_i}(K, f) = I_{X}^{F_i}(K_1, f) + I_{X}^{F_i}(K_2, f)$$
and
\[ I_X^{F_2}(K, f) = I_X^{F_2}(K_1, f) + I_X^{F_2}(K_2, f) + I_X^{F_3}(K_1, f)I_X^{F_3}(K_2, f). \]

The proof of the second equality follows from the fact that \( F_2(U_1 \cup U_2) \) is homeomorphic to the disjoint union \( F_2(U_1) \cap F_2(U_2) \cap (F_1(U_1) \times F_1(U_2)) \) for \( U_1, U_2 \) disjoint open neighborhoods of \( K_1 \) and \( K_2 \) respectively.

**Proposition 3** (Commutativity property). Let \( X, Y \) be locally compact metric ANRs with \( U, V \) open subsets of \( X \) and \( Y \) respectively. Let
\[
\varphi : U \to Y, \\
\psi : V \to X
\]
be locally defined maps. Consider \( f = \psi \circ \varphi \) and \( g = \varphi \circ \psi \). If \( K \subset X \) is a compact isolated invariant set with respect to \( f \), then \( \varphi(K) \) is a compact isolated invariant set with respect to \( g \) and \( I_X^{F_n}(K, f) = I_Y^{F_n}(\varphi(K), g) \).

**Proposition 4** (Homotopy invariance property). Let \( f : U \times \Lambda \to X \) be a map such that \( U \) is an open subset of \( X \) and \( \Lambda \subset \mathbb{R} \) is a compact interval. Assume that \( N \) is an isolating neighborhood for each map \( f_\lambda : U \to X \). Then \( I_X^{F_n}(\text{Inv}(N, f_\lambda), f_\lambda) \) does not depend on \( \lambda \in \Lambda \).

Let us consider a local homeomorphism of the plane, \( f \), with \( K = \{ p \} \) a non-attracting and non-repelling fixed point. The next results allow us to relate the indices of the iterations of \( f \) and the corresponding indices in the symmetric product.

**Theorem 1** ([24]). Let \( f : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \) be a local homeomorphism with \( p \in U \) a non-attracting and non-repelling fixed point of \( f \) such that \( \{ p \} \) is an isolated invariant set. Then there are a disc \( D \), containing a neighborhood \( V \) of \( p \), a finite subset \( \{ q_1, \ldots, q_m \} \subset D \) and a map \( \overline{f} : D \to D \) such that \( \overline{f}|_V = f|_V \), \( \overline{f}(\{ q_1, \ldots, q_m \}) \subset \{ q_1, \ldots, q_m \} \), and for every \( k \in \mathbb{N} \), \( \text{Fix}(\overline{f})^k \subset \{ p, q_1, \ldots, q_m \} \).

Moreover,

a) (Le Calvez-Yoccoz, [14]). If \( f \) is orientation-preserving, then
\[ i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{N}, \\ 1 & \text{if } k \notin r\mathbb{N}, \end{cases} \]
where \( k \in \mathbb{N} \), \( q \) is the number of periodic orbits of \( \overline{f} \) (excluding \( p \)) and \( r \) is their period.

b) If \( f \) is orientation-reversing, then there are integers \( \delta \in \{ 0, 1, 2 \} \) and \( q \) such that
\[ i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 1 - \delta & \text{if } k \text{ is odd,} \\ 1 - \delta - 2q & \text{if } k \text{ is even,} \end{cases} \]
where \( q \) is the number of orbits of period 2 and \( \delta \) is the number of fixed points of \( \overline{f} \) in \( \{ q_1, \ldots, q_m \} \), and there is no other orbit of \( \overline{f} \) in \( \{ q_1, \ldots, q_m \} \).

If \( R \) is a finite set of \( r \) elements, let
\[ C_s = \text{Card}(\{ S \subset R : \text{Card}(S) = s \}). \]
A consequence of the above theorem is the following proposition. The reader can find its proof in Section 4.
Let \( f : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \) be a homeomorphism, with \( p \in U \) a non-attracting and non-repelling fixed point of \( f \) such that \( \{p\} \) is an isolated invariant set.

a) \( f \) is orientation-preserving, \( q \) is the number of periodic orbits of \( f \) in \( \{q_1, \ldots, q_m\} \) and \( r \) is their period, then for every \( n \in \mathbb{N} \)

\[
1 = \sum_{1 \leq j r \leq n} C_j^q + \sum_{0 \leq j r < n} C_j^q I_D^{f_n^{q_j}}(\{p\}, \mathcal{F}).
\]

b) \( f \) is orientation-reversing, \( q \) the number of period-two orbits of \( f \) in \( \{q_1, \ldots, q_m\} \) and \( q' \leq 2 \) the number of fixed points of \( f \) in \( \{q_1, \ldots, q_m\} \), then for every \( n \in \mathbb{N} \)

\[
1 = \sum_{1 \leq 2 j + j' \leq n} C_j^q C_{j'}^{q'} + \sum_{0 \leq 2 j + j' < n} C_j^q C_{j'}^{q'} I_D^{f_n^{-(2j+j')}}(\{p\}, \mathcal{F}).
\]

Remark 3. In case a) of the above proposition, since \( \mathcal{F} \) is locally constant in \( \{q_1, \ldots, q_m\} \) (see [21]), we have

\[
I_D^{f_n}(\{p\}, \mathcal{F}) = \frac{\sum_{j=1}^n i_{\mathbb{Z}}(f^j, p)}{n} = \begin{cases} 1 & \text{if } n < r, \\ 1 - q & \text{if } n = r. \end{cases}
\]

Moreover, \( I_D^{f_n}(\{p\}, \mathcal{F}) = I_D^{f_n+1}(\{p\}, \mathcal{F}) \) for every \( n \in (kr, (k+1)r) \).

3. THE EULER CHARACTERISTIC OF THE \( n \)-SYMMETRIC PRODUCT OF A MANIFOLD

The aim of this section is to develop techniques which allow us to compute the Euler characteristic of the \( n \)-symmetric product of a finite dimensional manifold \( X \). We will restrict ourselves to the case when \( X \) is an orientable, compact surface without boundary. This setting will provide us with techniques to study the general case.

If we choose an adequate dynamical system (homeomorphism) \( F : X \to X \) \( (F \simeq id) \), the Euler characteristic of \( F_n(X) \) is

\[
\chi(F_n(X)) = \Lambda(F_n(id)) = \Lambda(F_n(F)) = i_{F_n(X)}(F_n(F), F_n(X)),
\]

and, if \( F \) is such that the number of its periodic orbits of period \( \leq n \) is finite, by the additivity property, we only have to compute a finite number of indices \( i_{F_n(X)}(F_n(F), \bigcup_{j=1}^r \overline{p_j}) \) for \( \overline{p_j} \) periodic orbits of \( F \) of period \( p_j \) with \( \sum_{j=1}^r p_j \leq n \). The above fixed point indices, denoted by \( i_n(F, \bigcup_{j=1}^r \overline{p_j}) \), are defined in small enough neighborhoods, in \( F_n(X) \), of the isolated fixed points \( \bigcup_{j=1}^r \overline{p_j} \).

Note that if \( f : X \to X \) is a diffeomorphism of a manifold \( X \) of dimension \( m \) with \( p \) a hyperbolic fixed point for \( f \), then by the Grobman-Hartman theorem (see [10]) we can reduce the study of \( I_X^F(\{p\}, f) \) to the linear case \( I_{R^m}^F(\{0\}, Df(p)) \).

Let \( U \) be an open neighborhood of \( \{0\} \) in \( \mathbb{R}^m \) and let \( f : U \subset \mathbb{R}^m \to \mathbb{R}^m \) be a linear map. Assume that \( K = \{0\} \) is a compact isolated invariant set. The study of the index \( I_{R^m}^F(\{0\}, f) \) gives information which allows us to calculate \( \chi(F_n(X)) \) for a compact manifold \( X \).

Let us denote by \( D(\lambda_1, \ldots, \lambda_m) \) the diagonal \( m \times m \) matrix with \( \lambda_1, \ldots, \lambda_m \) on the diagonal.
The only linear cases which we will need here are given in the next proposition:

**Proposition 6.**

\[
I^{F_n}_{\mathbb{R}^m}(\{0\}, D(0, \ldots, 0)) = 1
\]

and

\[
I^{F_n}_{\mathbb{R}^m}(\{0\}, D(2, \ldots, 2)) = \begin{cases} 
1 & \text{if } m \text{ is even}, \\
-1 & \text{if } m \text{ and } n \text{ are odd}, \\
0 & \text{if } m \text{ is odd and } n \text{ is even}.
\end{cases}
\]

The first equality is trivial, and the second one is proved in the Appendix.

The next theorem provides a complete study of \(I^{F_n}_{\mathbb{R}^m}(\{0\}, f)\) for a linear map and 0 a hyperbolic fixed point. We give an outline of the proof in the Appendix (see [25] for a complete proof).

**Theorem 2.** Let \(f : U \subset \mathbb{R}^m \to \mathbb{R}^m\) be a linear map with \(K = \{0\}\) a compact isolated invariant set. Consider the set of the real eigenvalues (repeated) which have modulus greater than 1, \(\{\lambda_1, \ldots, \lambda_r\}\).

Let \(r_2\) be the number of eigenvalues greater than 1, and \(r_{-2}\) the number of eigenvalues smaller than \(-1\). Of course \(r = r_2 + r_{-2}\). Then,

\[
I^{F_n}_{\mathbb{R}^m}(\{0\}, f) = \begin{cases} 
\text{if } r_2 \text{ is odd and } r_{-2} \text{ is even}, \\
I^{F_n}_{\mathbb{R}^m}(\{0\}, D(2)) = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
-1 & \text{if } n \text{ is odd};
\end{cases} \\
\text{if } r_2 \text{ is even and } r_{-2} \text{ is odd}, \\
I^{F_n}_{\mathbb{R}^m}(\{0\}, D(-2)) = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
(-1)^k & \text{if } n = 2k + 1;
\end{cases} \\
\text{if } r_2 \text{ is odd and } r_{-2} \text{ is odd}, \\
I^{F_n}_{\mathbb{R}^m}(\{0\}, D(2, -2)) = \begin{cases} 
1 & \text{if } n \text{ is even}, \\
-1 & \text{if } n \text{ is odd};
\end{cases} \\
\text{if } r_2 \text{ is even and } r_{-2} \text{ is even}, \\
I^{F_n}_{\mathbb{R}^m}(\{0\}, D(0, \ldots, 0)) = 1.
\end{cases}
\]

From now on, we study \(\chi(F_n(X))\) for \(X\) an orientable, compact surface without boundary.

In the next proposition we compute \(\chi(F_n(S^k))\).

**Proposition 7.** The Euler characteristic \(\chi(F_n(S^k))\) of the \(n\)-symmetric products of \(S^k\) is

\[
\chi(F_n(S^{2k+1})) = 0
\]

for all \(n \in \mathbb{N}\), and

\[
\chi(F_n(S^{2k})) = \begin{cases} 
2 & \text{if } n = 1, \\
3 & \text{if } n \geq 2.
\end{cases}
\]

**Proof.** Consider the dynamical system \(J : S^k \to S^k\), shown in Figure 1.

We have \(J \simeq id\), and there are two hyperbolic fixed points, a repeller \(p\) and an attractor \(q\).

We have

\[
\chi(F_n(S^k)) = \Lambda(F_n(J)) = I_{S^k}^{F_n}(S^k, J) = i_n(J, p) + i_n(J, q) + i_n(J, \{p, q\}).
\]
Let us consider a small enough open neighborhood of \( q \), \( U_n(q) \), in \( F_n(S^k) \). Since \( q \) is an attractor, we can construct a homotopy \( H : cl(U_n(q)) \times I \to F_n(S^k) \) such that \( H_0 = F_n(J) \) and \( H_1 \equiv q \) and \( H(\bar{x}, t) \neq \bar{x} \quad \forall (\bar{x}, t) \in \partial(U_n(q)) \times I \). Then, by the homotopy property of the fixed point index, it is obvious that

\[
i_n(J, q) = i_{F_n(S^k)}(H_0, U_n(q)) = i_{F_n(S^k)}(H_1, U_n(q)) = 1.
\]

The next step is to prove that \( i_n(J, \{ p, q \}) = i_{n-1}(J, p) \). Given the open balls \( V_1 = B(p, \epsilon) \) and \( V_2 = B(q, \epsilon) \), we define the open neighborhood of the point \( \{ p, q \} \in F_n(S^k) \),

\[
U_n(\{ p, q \}) = \{ \bar{x} \in F_n(S^k) : \bar{x} \subset \bigcup V_i \text{ and } \bar{x} \cap V_i \neq \emptyset \text{ for all } i = 1, 2 \}.
\]

Given \( \bar{x} = \{ x_1, \ldots, x_s, x_{s+1}, \ldots, x_t \} \in U_n(\{ p, q \}) \), with \( \{ x_1, \ldots, x_s \} \subset V_1 \) and \( \{ x_{s+1}, \ldots, x_t \} \subset V_2 \), we define the continuous map

\[
F : U_n(\{ p, q \}) \to F_{n-1}(S^k)
\]
as \( F(\{ x_1, \ldots, x_t \}) = \{ J(x_1), \ldots, J(x_s) \} \).

In the same way we consider the continuous map

\[
G : U_{n-1}(p) \to F_n(S^k)
\]
defined as \( G(\{ x_1, \ldots, x_t \}) = \{ x_1, \ldots, x_t, q \} \).

Now, we take the compositions \( F \circ G : U_{n-1}(p) \to F_{n-1}(S^k) \) and \( G \circ F : F^{-1}(U_{n-1}(p)) \to F_n(S^k) \). It is obvious that \( F \circ G = F_{n-1}(J) \). On the other hand,

\[
(G \circ F)(\{ x_1, \ldots, x_t \}) = \{ J(x_1), \ldots, J(x_s), q \},
\]
and it is not difficult to construct a homotopy

\[
H : cl(F^{-1}(U_{n-1}(p))) \times I \to F_n(S^k)
\]
such that \( H_0 = F_n(J) \) and \( H_1 = G \circ F \), with

\[
H(\bar{x}, t) \neq \bar{x} \quad \forall (\bar{x}, t) \in \partial(F^{-1}(U_{n-1}(p))) \times I.
\]

Then, using the commutativity and the homotopy properties of the fixed point index, we have that

\[
i_n(J, \{ p, q \}) = i_{F_n(S^k)}(F_n(J), U_n(\{ p, q \})) = i_{F_n(S^k)}(G \circ F, F^{-1}(U_{n-1}(p)))
\]

\[
= i_{F_{n-1}(S^k)}(F \circ G, U_{n-1}(p)) = i_{F_{n-1}(S^k)}(F_{n-1}(J), U_{n-1}(p)) = i_{n-1}(J, p).
\]
Then \( \chi(F_n(S^k)) = i_n(J, p) + 1 + i_{n-1}(J, p) \), and from Proposition 6 and the Grobman-Hartman theorem, we have

\[
i_n(J, p) = \begin{cases} 
1 & \text{if } k \text{ is even,} \\
-1 & \text{if } k \text{ and } n \text{ are odd,} \\
0 & \text{if } k \text{ is odd and } n \text{ is even.}
\end{cases}
\]

Now the result follows automatically. \( \square \)

**Remark 4.** Let us notice that we can construct a map \( F_k : S^{2k+1} \to S^{2k+1} \) homotopic to the identity without periodic points. The equality \( \chi(F_n(S^{2k+1})) = 0 \) follows from this fact. We define \( F_k \) as the restriction to \( S^{2k+1} \subset C^{k+1} \) of the map \((z_1, \ldots, z_{k+1}) \mapsto (e^{2i\pi\alpha}z_1, \ldots, e^{2i\pi\alpha}z_{k+1})\), where \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \).

We can also use this map to compute \( \chi(F_n(S^{2k+2})) \). In fact, let us consider \( g : [-1, 1] \to [-1, 1] \) with \( g(x) = 2x \) if \(|x| \leq 1/2\), and \( g(x) = \frac{x}{|x|} \) if \( 1/2 \leq |x| \leq 1 \). The map

\[
F_k \times g : S^{2k+1} \times [-1, 1] \to S^{2k+1} \times [-1, 1]
\]

defines a continuous map on the sphere \( S^{2k+2} \) obtained by identifying each sphere \( S^{2k+1} \times \{\epsilon\} \), \( \epsilon \in \{-1, 1\} \), to a point. The only periodic orbits are the two fixed points, where the map is locally constant.

The same ideas can be applied to the torus \( T \), to prove that \( \chi(F_n(T)) = 0 \).

Let us compute the Euler characteristic of the \( n \)-symmetric product of the compact oriented surfaces of genus \( k \), \( \chi(F_n(M_k)) \).

**Proposition 8.** The Euler characteristic of \( F_n(M_k) \), with \( k \geq 2 \), is

\[
\chi(F_n(M_k)) = \sum_{j=1}^{n} (-1)^{j} C_{2k-3+j}^{j}.
\]

If \( k = 1 \), then \( \chi(F_n(T)) = 0 \).

**Proof.** Let us consider the dynamical system \( J : M_k \to M_k \) shown in Figure 2.
We have that \( J \simeq id \), with two fixed points \( p \) and \( q \). The point \( q \) is a source and the map \( J|_{M_k \setminus \{q\}} \) is conjugated to the product
\[
L_{2k} \times f : Y_{2k} \times [0,1) \to Y_{2k} \times [0,1),
\]
where \( f(x) = x^2 \) and \( L_{2k} : Y_{2k} \to Y_{2k} \) is the dynamical system defined on the pointed union of \( 2k \) loops \( Y_{2k} \) shown in Figure 3.

The Euler characteristic of \( F_n(M_k) \) is
\[
\chi(F_n(M_k)) = \Lambda(F_n(J)) = \Gamma_{M_k}(M_k, J) = i_n(J, p) + i_n(J, q) + i_n(J, \{p, q\}).
\]

By Proposition 6 and the Grobman-Hartman theorem, we have \( i_n(J, q) = 1 \).

On the other hand, let us see that \( i_n(J, \{p, q\}) = i_{n-1}(J, p) \). We consider the continuous maps
\[
F : U_n(\{p, q\}) \to F_{n-1}(M_k) \text{ and } G : U_{n-1}(p) \to F_n(M_k)
\]
defined as in the proof of Proposition 7, and a homotopy
\[
H : cl(F^{-1}(U_{n-1}(p))) \times I \to F_n(M_k)
\]
such that \( H_0 = F_n(J) \) and \( H_1 = G \circ F \), with
\[
H(\bar{x}, t) \neq \bar{x} \text{ for all } (\bar{x}, t) \in \partial(F^{-1}(U_{n-1}(p))) \times I
\]
(for a construction of the homotopy \( H \), see the proof of Proposition 6 in the Appendix).

From the commutativity and the homotopy invariance properties of the fixed point index, we have \( i_n(J, \{p, q\}) = i_{n-1}(J, p) \), and therefore
\[
\chi(F_n(M_k)) = i_n(J, p) + 1 + i_{n-1}(J, p).
\]

It only remains to compute \( i_n(J, p) \). Since \( J|_{M_k \setminus \{q\}} \) is conjugated to \( L_{2k} \times f \), then \( i_n(J, p) = i_n(L_{2k} \times f, (p', 0)) = i_n(L_{2k}, p') \). The last equality follows from the homotopy and commutativity properties of the fixed point index.

Let us define the dynamical systems \( H_k, H'_k : Z_k \to Z_k \) with \( Z_k \) the union of \( k \) arcs connected by the endpoints (see Figure 4).

Given a fixed point \( \bar{a} \) of \( F_n(H_k) \), we denote \( i_{F_n(Z_k)}(F_n(H_k), \bar{a}) = i_n(H_k, \bar{a}) \).

Let us prove that \( i_n(L_{2k}, p') = i_n(H_{2k}, p) \). Given the map \( g : [0,1] \to [0,1] \) with \( g(x) = 2x \) if \( |x| \leq 1/2 \), and \( g(x) = \frac{x}{|x|} \) if \( 1/2 \leq |x| \leq 1 \), the restriction of \( L_{2k} \) to

\[\text{Figure 3.}\]
each loop can be considered as a map of the type
\[ g : [0, 1]/(0 \equiv 1) \to [0, 1]/(0 \equiv 1). \]

We can consider the dynamical system \( L_{2k} : Y_{2k} \to Y_{2k} \) as a identification in \( H_{2k} : Z_{2k} \to Z_{2k} \) of the points \( p \) and \( q \) to a point \( p' \). If \( x \in Z_{2k} \), we call \( [x] \in Y_{2k} \) the corresponding point obtained by the identification.

Given a small enough neighborhood \( U_n(p') \) of \( p' \) in \( F_n(Y_{2k}) \), let
\[ \bar{x} = \{[x_1], \ldots, [x_r], [x_{r+1}], \ldots, [x_s]\} \in U_n(p') \]
with \( \{[x_1], \ldots, [x_r]\} \) the points of \( \bar{x} \) contained in the local repelling part of \( p' \) in \( Y_{2k} \).

Then let us consider the map \( F : U_n(p') \subset F_n(Y_{2k}) \to Z_{2k} \) defined as
\[ F(\{[x_1], \ldots, [x_r], [x_{r+1}], \ldots, [x_s]\}) = \{H_{2k}(x_1), \ldots, H_{2k}(x_r), p\} \]
If \( r = s \), the point \( p \) does not appear in the image of \( F \).

Let \( G : U_n(p) \subset F_n(Z_{2k}) \to F_n(Y_{2k}) \) be the map defined as
\[ G(\{x_1, \ldots, x_r\}) = \{[x_1], \ldots, [x_r]\} \]
By the commutativity property of the fixed point index applied to \( F \) and \( G \) we obtain that \( i_n(L_{2k}, p') = i_n(H_{2k}, p) \). Therefore
\[ i_n(J, p) = i_n(H_{2k}, p), \]
and we only have to compute \( i_n(H_{2k}, p) \).

If \( n \geq 2 \), we have
\[ I_{Z_{2k}}^{F_n}(Z_k, H_k) = i_n(H_k, p) + i_n(H_k, q) + i_n(H_k, \{p, q\}) \]
\[ = i_n(H_k, p) + 1 + i_{n-1}(H_k, p). \]

The equality \( i_n(H_k, q) = 1 \) is a consequence of the fact that \( q \) is an attractor, and \( i_n(H_k, \{p, q\}) = i_{n-1}(H_k, p) \) follows again from the homotopy invariance and the commutativity properties of the fixed point index.

Using similar arguments it is easy to see that
\[ I_{Z_{2k}}^{F_n}(Z_k, H'_k) = i_n(H_{k-1}, p). \]
Since $H_k \simeq H'_k$, then $\iota^E_n(Z_k, H_k) = \iota^E_n(Z_k, H'_k)$ and
\[
(1) \quad i_n(H_{k-1}, p) = i_n(H_k, p) + 1 + i_{n-1}(H_k, p).
\]
This formula allows us to compute $i_n(H_k, p)$ in a recurrent way (it is easy to see that $i_n(H_1, p) = 0$ for all $n$). Our aim is to obtain $i_n(H_k, p)$ in an explicit expression by an induction argument.

Let us prove that $i_n(H_k, p) - i_{n-1}(H_k, p) = (-1)^nC^{k+n-2}_n$.

Let $n = 2$ and $k = 1$. Then, since $i_n(H_1, p) = 0$, we have $i_2(H_1, p) - i_1(H_1, p) = (-1)^2C^1_1 = 0$. Let us suppose that
\[
i_n(H_k, p) - i_{n-1}(H_k, p) = (-1)^nC^{k+n-2}_n
\]
for all $n \geq 2, k \geq 1$ with $n+k \leq m_0$, and consider $n, k$ with $n+k = m_0 + 1$. Then, using (1), we have
\[
i_n(H_{k-1}, p) = i_n(H_k, p) + i_{n-1}(H_k, p) + 1,
\]
\[-i_{n-1}(H_{k-1}, p) = -i_{n-1}(H_k, p) - i_{n-2}(H_k, p) - 1.
\]
It follows that
\[
i_n(H_k, p) - i_{n-1}(H_k, p)
\]= i_n(H_{k-1}, p) - i_{n-1}(H_{k-1}, p) + i_{n-2}(H_k, p) - i_{n-1}(H_k, p)
\]= (-1)^nC^{k+n-3}_n + (-1)^nC^{k+n-3}_{n-1} = (-1)^nC^{k+n-2}_n,
\]
and the result is proved.

In the same way, it follows that $i_1(H_k, p) = -(k-1)$, and then
\[
i_n(H_k, p) = \sum_{j=1}^{n}(-1)^jC^{k-2+j}_j
\]
and
\[
\chi(F_n(M_k)) = i_n(H_{2k}, p) + i_{n-1}(H_{2k}, p) + 1
\]= i_n(H_{2k-1}, p) = \sum_{j=1}^{n}(-1)^jC^{2k-3+j}_j.
\]

\[\square\]

**Remark 5.** Given a manifold $X$ and a continuous map $F : X \to X$, we can obtain, under certain conditions of hyperbolicity, information about the dynamics of $F$ by studying the fixed point indices $\lambda^E_n(\Inv(X, F), F)$. Certainly, there are other techniques which allow us to study this, but it seemed interesting for us to present this alternative method.

**Example.** Dynamics of the G-horseshoe. If we want to study the periodic orbits of the G-horseshoe with our techniques, let us consider the dynamical system $F : C \to C$ given by the extended G-horseshoe of Figure 5. We are interested in detecting the periodic orbits of $F$ on $I^2$ (the unique periodic orbit out of $I^2$ is a fixed point). Let us consider the continuous map $g : \Pi \circ F|S^1 : S^1 \to S^1$ defined as the composition of $F|S^1$ with the projection $\Pi : C \to S^1$, where $S^1$ is the interior circle of $C$. It is not difficult to see, by the homotopy invariance and the commutativity properties of the fixed point index, that
\[
(2) \quad \lambda^E_n(\Inv(I^2, F), F) = \lambda^E_n(\Inv(I, g), g).
\]
Let us observe that \( g|_I \) is an expansion, and \( F|_{I^2} \) is a contraction in the vertical direction and an expansion in the horizontal one. Then, given \( n \) fixed, the number of periodic orbits of period \( \leq n \) for \( g \) and \( F \) is finite, and the fixed points of \( F_n(g) \) and \( F_n(F) \) are isolated.

It is not hard to prove that if \( \bar{\alpha} \) is a periodic orbit of period \( n \) of \( F|_{I^2} \), then \( i_{F_n(C)}(F_n(F), \bar{\alpha}) = -1 \) (the same fact occurs with \( g|_I \)). Then we can see, using (2) and an induction argument, that the number of periodic orbits of all periods of \( F|_{I^2} \) is the same as in the case of \( g|_I \).

By the commutativity and the homotopy properties of the fixed point index,

\[
I^{F_n}_C(\text{Inv}(C, F), F) = I^{F_n}_{S^1}(\text{Inv}(S^1, g), g) = I^{F_n}_{S^1}(\text{Inv}(S^1, f), f),
\]

where \( f : S^1 \to S^1 \) is the doubling angle map. A careful observation of \( f \) and \( g \) allows us to see that, although \( f \) has one fixed point and \( g|_I \) has two (repelling) fixed points, the remaining periodic orbits are the same in both dynamical systems.

Since the set \( \{ x \in S^1 : f^n(x) = x \} \) has \( 2^n - 1 \) points, then the set \( \{ x \in I^2 : F^n(x) = x \} \) has \( 2^n \) points. So, we have a characterization of the periodic orbits of the G-horseshoe.

4. Appendix. Proofs

Proof of Proposition 5. Let us see the proof of a) (the proof of b) is analogous).
Since \( D \) is an AR, \( 1 = I^{F_n}_D(D, \mathcal{J}) \).

Let us consider the point \( \bar{\alpha}(l) = \bar{\alpha}_1 \cup \cdots \cup \bar{\alpha}_t \in F_n(D) \) with \( \bar{\alpha}_i = \{ q_{i_1}, \ldots, q_{i_r} \} \) a periodic orbit of \( \mathcal{J} \) in \( \{ q_1, \ldots, q_m \} \) for all \( i = 1, \ldots, t \).

\( \text{Per}(\mathcal{J}) \) is the set of periodic orbits of \( \mathcal{J} \) in \( \{ q_1, \ldots, q_m \} \). Let us denote

\[
i_{F_n(D)}(F_n(\mathcal{J}), \bar{\alpha}(j)) = i_n(\mathcal{J}, \bar{\alpha}(j)).
\]
From the additivity property of the fixed point index for ANRs, we have

\[ 1 = I_{D}^{F_{n}}(D, \overline{f}) = \sum_{\alpha(j) \in \text{Per}(\overline{f}), j \leq n} i_{n}(\overline{f}, \alpha(j)) \]

\[ + \sum_{\alpha(j) \in \text{Per}(\overline{f}), j < n} i_{n}(\overline{f}, p \cup \alpha(j)) + i_{n}(\overline{f}, p). \]

Since \( \overline{f} \) is locally constant in each \( q_{i} \), see [24], we have

\[ i_{n}(\overline{f}, \alpha(j)) = 1 \]

for all \( \alpha(j) \subset \text{Per}(\overline{f}), j \leq n \).

Let \( \alpha(j) \) be fixed with \( j r < n \). We prove that

\[ i_{n}(\overline{f}, p \cup \alpha(j)) = i_{n-jr}(\overline{f}, p). \]

Let \( U_{n}(p \cup \alpha(j)) \) be a small enough neighborhood in \( F_{n}(D) \) of the point \( p \cup \alpha(j) \), and let \( \bar{x} \in U_{n}(p \cup \alpha(j)) \) with \( \bar{x}_{p} = \bar{x} \cap B(p, \epsilon) \) for \( \epsilon \) small enough. The set \( \bar{x}_{p} = \{x_{1}, \ldots, x_{l}\} \) is such that \( 1 \leq l \leq n - jr \).

Let \( F : U_{n}(p \cup \alpha(j)) \to F_{n-jr}(D) \) and \( G : U_{n-jr}(p) \to F_{n}(D) \) be the continuous maps

\[ F(\bar{x}) = (\overline{f}(x_{1}), \ldots, \overline{f}(x_{l})), \quad G(\bar{x}) = \overline{\alpha(j)} \cup \bar{x}. \]

The map \( F \circ G : U_{n-jr}(p) \to F_{n-jr}(D) \) is such that

\[ (F \circ G)(\bar{x}) = F_{n-jr}(\overline{f})(\bar{x}). \]

On the other hand, since \( \overline{f} \) is locally constant in each \( q_{i} \in \{q_{1}, \ldots, q_{m}\} \), the map

\[ G \circ F : F^{-1}(U_{n-jr}(p)) \to F_{n}(D) \]

is such that

\[ (G \circ F)(\bar{x}) = F_{n}(\overline{f})(\bar{x}). \]

From the commutativity property of the fixed point index for ANRs we have the equality

\[ i_{n}(\overline{f}, p \cup \alpha(j)) = i_{n-jr}(\overline{f}, p). \]

The proof of case a) is finished. \( \square \)

**Proof of Proposition 6.** Let us see that \( I_{x_{m}}^{F_{n}}(\{0\}, 2Id) = 1 \) for \( m = 2 \) (the case of \( m \) even will be analogous).

Let \( U_{0} = B(0, 1) \) be an open neighborhood of \( \{0\} \) and let \( H : F_{n}(\text{cl}(U_{0})) \times I \to F_{n}(\mathbb{R}^{2}) \) be the homotopy

\[ H(\{x_{1}, \ldots, x_{r}\}, t) \]

\[ = \begin{cases} \{ A(t)(2x_{1}), \ldots, A(t)(2x_{r}) \} & \text{if } t \in [0, 1/2], \\ \{ 2(1-t)A(1/2)(2x_{1}), \ldots, 2(1-t)A(1/2)(2x_{r}) \} & \text{if } t \in [1/2, 1], \end{cases} \]

with

\[ A(t) = \begin{pmatrix} \cos\left(\frac{2\pi}{n+1}2t\right) & \sin\left(\frac{2\pi}{n+1}2t\right) \\ -\sin\left(\frac{2\pi}{n+1}2t\right) & \cos\left(\frac{2\pi}{n+1}2t\right) \end{pmatrix}. \]

We consider \( x_{i} \neq x_{j} \) if \( i \neq j \). It is obvious that \( r \leq n \).

The continuity of \( H \) is clear, and it is not hard to see that \( H(\bar{x}, t) \neq \bar{x} \) for all \( (\bar{x}, t) \in \partial(F_{n}(U_{0})) \times I \). Since \( H_{0} = F_{n}(2Id) \) and \( H_{1} = F_{n}(D(0, 0)) \), we have proved the result for \( m = 2 \).
Let us see that
\[ I_{\mathbb{R}^m}^{F_n} \{0\}, 2Id) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ -1 & \text{if } n \text{ is odd}, \end{cases} \]
for \textit{m} odd. We will prove the result for \textit{m} = 1 (the general case is easy to obtain by combining the cases \textit{m} = 1 and \textit{m} even).

Let us consider the map \( g : J \to J \) with \( g(x) = x^{1/3} \) and \( J = [-1, 1] \). The only periodic orbits are the fixed points \{-1, 0, 1\}.

Since \( F_n(J) \) is an absolute retract, we have
\[ I_{\mathbb{R}^n}^{F_n}(J, g) = \Lambda(F_n(g)) = \Lambda(F_n(id)) = 1. \]

Let us denote \( i_{F_n,J}(F_n(g), \bar{\alpha}) = i_n(g, \bar{\alpha}) \) for \( \bar{\alpha} \in \text{Fix}(F_n(g)) \). Then
\[ 1 = I_{\mathbb{R}^n}^{F_n}(J, g) = \sum_{\bar{\alpha} \subset \{-1, 0, 1\}} i_n(g, \bar{\alpha}). \]

Using the commutativity and the homotopy invariance properties of the fixed point index as in the proof of Proposition 7, it is not difficult to see that
\[ i_n(g, 1) = i_n(g, -1) = i_n(g, \{-1, 1\}) = 1, \]
\[ i_n(g, \{-1, 0\}) = i_n(g, \{0, 1\}) = i_{n-1}(g, 0), \]
and
\[ i_n(g, \{-1, 0, 1\}) = i_{n-2}(g, 0). \]

Then, for \textit{n} > 2,
\[ 1 = I_{\mathbb{R}^n}^{F_n}(J, g) = i_n(g, 0) + 2i_{n-1}(g, 0) + i_{n-2}(g, 0) + 3. \]

Since \( I_{\mathbb{R}^n}^{F_n}(\{0\}, 2Id) = i_n(g, 0) \), by an induction argument on the last formula we finish the proof. \( \square \)

\textbf{Proof of Theorem 2.} Since \{0\} is an isolated invariant set, the eigenvalues \( \{\lambda_1, \ldots, \lambda_m\} \) of \( f \) have modulus different from 1. The first equality of the theorem, which reduces the study of the fixed point index in \( \mathbb{R}^m \) to the cases of \( \mathbb{R} \) and \( \mathbb{R}^2 \), is easy to prove by using the techniques employed in the proof of Proposition 6.

On the other hand, the computation of \( I_{\mathbb{R}^n}^{F_n}(\{0\}, D(2)) \) follows from studying the dynamical system \( g_1 : J \to J \) defined as \( g_1(x) = x^{1/3} \) with \( J = [-1, 1] \). In fact,
\[ 1 = \chi(F_n(J)) = I_{\mathbb{R}^n}^{F_n}(J, g_1) = \sum_{\bar{\alpha} \subset \{-1, 0, 1\}} i_n(g_1, \bar{\alpha}), \]
and we compute \( I_{\mathbb{R}^n}^{F_n}(\{0\}, D(2)) = i_n(g_1, 0) \) from the above equality. In an analogous way we have \( I_{\mathbb{R}^n}^{F_n}(\{0\}, D(-2)) \) with \( g_2(x) = -x^{1/3} \).

It only remains to compute \( I_{\mathbb{R}^n}^{F_n}(\{0\}, D(2,-2)) \). Let us consider the dynamical systems \( F = s \circ f : S^2 \to S^2 \) and \( G = s \circ g : S^2 \to S^2 \), where \( s : S^2 \to S^2 \) is a symmetry with respect to the plane \( \{z = 0\} \) and \( f, g : S^2 \to S^2 \) are the dynamical systems shown in Figure 6.

For the dynamical system given by \( F \), the fixed point \( p \) is of type \( D(2,-2) \) and \( q \) is an attractor. The fixed points \( a \) of \( G \) are of type \( D(2,-1/2) \) and \( D(-2,1/2) \) respectively. The pairs \( \{c_1, c_2\} \) and \( \{d_1, d_2\} \) are attracting periodic orbits of period 2.
We have $F \cong G$. Therefore, if $n \geq 2$,
\[
I_{S^2}^F(S^2, G) = I_{S^2}^F(S^2, F) = i_n(F, p) + i_n(F, q) + i_n(F, \{p, q\}) \\
= i_n(F, p) + 1 + i_{n-1}(F, p).
\]

Let us prove the equality $I_{S^2}^F(S^2, G) = 1$. By the additivity property of the fixed point index,
\[
I_{S^2}^F(S^2, G) = \sum_{\bar{\alpha} \subset \{a, b, \{c_1, c_2\}, \{d_1, d_2\}\}} i_n(G, \bar{\alpha}).
\]

We have that $i_n(G, a) = I_{R^2}^F(\{0\}, D(2))$ and $i_n(G, b) = I_{R^2}^F(\{0\}, D(-2))$. The only difficulty is to compute $i_n(G, \{a, b\})$.

Let $J_1 = J_2 = [-1,1]$. We denote by $X = J_1 \cup J_2$ the disjoint union of the intervals. Let us consider the map $h : X \to X$, defined as $h(x) = x^{1/3}$ if $x \in J_1$ and $h(x) = -x^{1/3}$ if $x \in J_2$.

Since $\chi(F_1(X)) = I_{\bar{X}}(X, h) = 2$ and $\chi(F_n(X)) = I_{\bar{X}}(X, h) = 3$ if $n > 1$, we can prove that
\[
i_n(h, \{0, 0\}) = \begin{cases} 
-1 & \text{if } n = 4k + 2, \\
1 & \text{if } n = 4k + 3, \\
0 & \text{otherwise},
\end{cases}
\]
for $k \in \mathbb{N}$.

Since $i_n(h, \{0, 0\}) = i_n(G, \{a, b\})$, the equality $I_{S^2}^F(S^2, G) = 1$ follows from (3).

Then $i_n(F, p) + i_{n-1}(F, p) = 0$. Since $i_1(F, p) = -1$, we obtain the value of $I_{R^2}^F(\{0\}, D(2, -2)) = i_n(F, p)$, and the proof is finished. \[\square\]

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**References**


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