CREMER FIXED POINTS AND SMALL CYCLES

LIA PETRACOVICI

ABSTRACT. Let $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and let (p_n/q_n) denote the sequence of convergents to the regular continued fraction of α. Let f be a function holomorphic at the origin, with a power series of the form $f(z) = \lambda z + \sum_{n=2}^{\infty} a_n z^n$. We assume that for infinitely many n we simultaneously have (i) $\log \log q_{n+1} \geq 3 \log q_n$, (ii) the coefficients a_{1+q_n} stay outside two small disks, and (iii) the series $f(z)$ is lacunary, with $a_j = 0$ for $2 + q_n \leq j \leq q_n^{1+q_n} - 1$. We then prove that $f(z)$ has infinitely many periodic orbits in every neighborhood of the origin.

INTRODUCTION

Consider a map $f(z) = \lambda z + O(z^2)$ holomorphic on a neighborhood of the origin, with a fixed point of multiplier λ at 0. *Linearizing f* means finding a conformal map h such that $h(0) = 0$ and $(h^{-1} \circ f \circ h)(z) = \lambda z$ near 0. For $|\lambda| \neq 0, 1$, this is always possible (Koenigs, see [3]). For $\lambda = 0$ this is only possible for $f \equiv 0$, and for λ a root of unity, f is linearizable if and only if some iterate of f is the identity map. The remaining case is $\lambda = e^{2\pi i \alpha}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and then we say that 0 is an *irrational fixed point*. Such a point is either a *Siegel point* or a *Cremer point*, according as a local linearization is possible or not (see [1], [3]).

Let (p_n/q_n) denote the sequence of convergents to the continued fraction of α. In 1972 Brjuno (see [3]) showed that if α satisfies the *Brjuno condition*, i.e. if

$$(B) \qquad \sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty,$$

then any holomorphic germ with a fixed point of multiplier λ at 0 is linearizable.

In 1987 Yoccoz [6] proved that Brjuno's condition is optimal. More precisely, he showed that if

$$(\neg B) \qquad \sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} = \infty,$$

then the quadratic map $f(z) = \lambda z + z^2$ is not linearizable at the origin. Moreover, this fixed point has the *"small cycles property"*: Every neighborhood of the origin contains infinitely many periodic orbits (see [3]).

It is natural to ask whether every Cremer fixed point has the small cycles property. Let us assume that $\sum_{n=1}^{\infty}(\log q_{n+1})/q_n = \infty$, so that a Cremer fixed point can exist.

A negative answer was given by Pérez-Marco [4] in 1990. He showed that if α satisfies the arithmetic condition

$$(\neg B') \qquad \sum_{n=1}^{\infty} \frac{\log\log q_{n+1}}{q_n} = \infty,$$

then there exists $f(z) = \lambda z + O(z^2)$, holomorphic and injective in the unit disk, such that every orbit of f contained in the unit disk has the origin as an accumulation point. Such an f has a Cremer fixed point at the origin and no small cycles.

The answer is positive when the above series converges (Pérez-Marco, [4]). More precisely, if

$$(B') \qquad \sum_{n=1}^{\infty} \frac{\log\log q_{n+1}}{q_n} < \infty,$$

then every non-linearizable holomorphic germ $f(z) = \lambda z + O(z^2)$ has small cycles at the origin [4].

In the case of polynomial germs, more can be said about the question of periodic orbits accumulating at a non-linearizable fixed point. When α satisfies condition (B'), the answer is given by Pérez-Marco's theorem above.

When α satisfies condition $(\neg B')$ in such a way that the terms of the series are unbounded, i.e., when both $(\neg B')$ and

$$(C_\infty) \qquad \sup_{n \geq 1} \frac{\log\log q_{n+1}}{q_n} = \infty$$

hold, then any polynomial germ is non-linearizable and has a sequence of periodic orbits accumulating at the origin (Pérez-Marco, [4]).

Thus, when studying the existence of small cycles near a Cremer fixed point of a polynomial germ, the only case left is

$$(\neg B') \qquad \sum_{n=1}^{\infty} \frac{\log\log q_{n+1}}{q_n} = \infty$$

and

$$(\neg C_\infty) \qquad \sup_{n \geq 1} \frac{\log\log q_{n+1}}{q_n} < \infty.$$

The main result of this paper is the following (see Theorem 2.3 for a precise statement). We consider $f(z) = \lambda z + a_2 z^2 + \ldots + \ldots + a_n z^n + \ldots$ analytic on a disk centered at the origin. We assume that for infinitely many n we simultaneously have (i) $\log\log q_{n+1} \geq 3 \log q_n$, (ii) the coefficients a_{1+q_n} stay outside two small disks, and (iii) the series $f(z)$ is lacunary, with $a_j = 0$ for $2 + q_n \leq j \leq q_n^{1+q_n} - 1$. We then prove that $f(z)$ has infinitely many periodic orbits in every neighborhood of the origin.

Thus, for irrational numbers α satisfying both conditions $(\neg B')$ and $(\neg C_\infty)$, the holomorphic germs with the imposed conditions on the Taylor coefficients have Cremer fixed points with the small cycles property. So, one can have small cycles either with or without condition (C_∞). We note that we can easily find irrational numbers α that satisfy condition (i), as well as both conditions $(\neg B')$ and $(\neg C_\infty)$.

We describe here the idea of the proof. The first two assumptions (number-theoretic and coefficients outside disks) imply that the partial sums up to the $q_n + 1$ power of z have periodic orbits which lie entirely in small disks whose size goes to zero as $n \to \infty$. The last condition (lacunarity) guarantees that we can apply Rouché's theorem to get periodic cycles of f itself inside any neighborhood of the origin.

This work is part of the author's Ph.D. thesis under the supervision of Aimo Hinkkanen. The author is grateful to Alexandru Zaharescu for helpful comments.

1. PRELIMINARIES

Definition 1.1 ([5])**.** Let $k \in \mathbb{N}$. A *composition of* k *into* m *parts* is an ordered partition $k = n_1 + n_2 + \ldots + n_m$, where the n_i, for $1 \le i \le m$, are positive integers.

Lemma 1.2 ([5])**.** *Let $p_m(k)$ denote the number of compositions of the number k into exactly m parts. Then*

$$(1.1) \qquad p_m(k) = \binom{k-1}{m-1} \ \text{for } 1 \le m \le k.$$

Lemma 1.3. *Let $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots + a_m z^m + \cdots$, and let*

$$f^{\circ k}(z) = (\underbrace{f \circ f \circ \ldots \circ f}_{k})(z) = \lambda^k z + a_{k,2} z^2 + \ldots + a_{k,m} z^m + \ldots.$$

Assuming that λ is not a root of unity, we have

$$(1.2) \qquad a_{k,m} = \frac{a_m \lambda^{k-1}(\lambda^{k(m-1)} - 1)}{\lambda^{m-1} - 1} + P_{k,m}(\lambda, a_2, \ldots, a_{m-1}),$$

where $P_{k,m}$ is a polynomial in $m-1$ variables $(\lambda, a_2, \ldots, a_{m-1})$ with integer coefficients, with $P_{k,m}(0, 0, \ldots, 0) = 0$.

Proof. The proof can be done by induction on the number k of iterations. $\qquad \square$

Lemma 1.4. *For any integers $k \ge 1$, $m \ge 1$, any sequence of complex numbers $(a_n)_{n \ge 1}$, and any complex number $\gamma \ne 0$ we have*

$$(1.3) \qquad P_{k,m}(a_1, \gamma a_2, \gamma^2 a_3, \ldots, \gamma^{m-2} a_{m-1}) = \gamma^{m-1} P_{k,m}(a_1, a_2, \ldots, a_{m-1}).$$

Proof. Let $f(z) = \lambda z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = f(\gamma z)/\gamma$. For any $k \ge 1$ we then have $g^{\circ k}(z) = f^{\circ k}(\gamma z)/\gamma$. The coefficient of z^m in each side of this equality is given by Lemma 2. We identify the two coefficients to get the conclusion. $\qquad \square$

2. RESULTS

In what follows we will work with numbers $\lambda = e^{2\pi i \alpha}$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let p/q be one of the convergents to the continued fraction of α, so that λ is close to $e^{2\pi i p/q}$.

Choose for now $k = q$ and $m = q + 1$ in Lemma 1.3. We write the q-th iterate of f as $f^{\circ q}(z) = \lambda^q + a_{q,2} z^2 + \ldots + a_{q,n} z^n + \ldots$. By equation (1.2) in Lemma 1.3 we have

$$\frac{\lambda}{q} a_{q,1+q} = \frac{\lambda}{q} P_{q,1+q}(\lambda, a_2, \ldots, a_q) + a_{1+q} \frac{\lambda^q}{q} \cdot \frac{\lambda^{q^2} - 1}{\lambda^q - 1}.$$

This can be rewritten as

$$\frac{\lambda}{q} a_{q,1+q} = \left(\frac{\lambda}{q} P_{q,1+q}(\lambda, a_2, \ldots, a_q) + a_{1+q} \right) + a_{1+q} \frac{\left(\lambda^q + \lambda^{2q} + \ldots + \lambda^{q^2} - q \right)}{q}.$$

We have

$$\lambda^q + \lambda^{2q} + \ldots + \lambda^{q^2} - q = \sum_{j=1}^{q}(\lambda^{jq} - 1) = (\lambda^q - 1)\sum_{j=1}^{q}\sum_{l=0}^{j-1}\lambda^{lq},$$

and therefore we get

$$\left|\lambda^q + \lambda^{2q} + \ldots + \lambda^{q^2} - q\right| \le |\lambda^q - 1|\sum_{j=1}^{q}j = \frac{q(q+1)}{2}|\lambda^q - 1|.$$

Hence the following inequality holds:

$$(2.1) \qquad |a_{q,1+q}| \ge q\left|\frac{\lambda}{q}P_{q,1+q}(\lambda, a_2, \ldots, a_q) + a_{1+q}\right| - |a_{1+q}|\frac{q(q+1)}{2}|\lambda^q - 1|.$$

Thus, the coefficient $a_{q,q+1}$ is bounded away from zero, if a_{1+q} is suitably chosen. For example, if we choose $a_{1+q} \ne 0$ inside the unit disk, and outside the disk of radius $\sqrt{|\lambda^q - 1|}$ centered at $-\lambda P_{q,1+q}(\lambda, a_2, \cdots, a_q)/q$, we find that

$$(2.2) \qquad |a_{q,1+q}| \ge q\sqrt{|\lambda^q - 1|} - \frac{q(q+1)}{2}|\lambda^q - 1|.$$

This observation will be used to prove the first theorem.

Theorem 2.1. *Let $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and let p/q be one of the convergents to the continued fraction of α. Let $h(z) = \lambda z + a_2 z^2 + \cdots + a_{1+q}z^{1+q}$, $a_{1+q} \ne 0$, and let C be an upper bound for the absolute values of the coefficients of h. We assume that the following conditions are satisfied:*

- *(i) a_{1+q} is chosen in the unit disk, but outside the disk of radius $\sqrt{|\lambda^q - 1|}$ centered at $-\lambda P_{q,1+q}(\lambda, a_2, \cdots, a_q)/q$; and*
- *(ii) $|\lambda^q - 1| \le \frac{1}{(4Ceq)^{2q^2}}$.*

Then $h(z)$ has an entire periodic orbit, distinct from the origin, inside the disk of radius $\frac{1}{(2C)^q}$ centered at the origin.

Proof. Write $N = (q+1)^q - 1$. Let $z_1, z_2, \cdots, z_N$ denote the non-zero periodic points of period q of $h(z)$. They are the roots of the equation

$$\frac{h^{\circ q}(z) - z}{z} = (\lambda^q - 1) + a_{q,2}z + \cdots + a_{q,1+q}z^q + \cdots + a_{q,N+1}z^N = 0,$$

and so their reciprocals $\frac{1}{z_1}, \frac{1}{z_2}, \cdots, \frac{1}{z_N}$ are the roots of the equation

$$(\lambda^q - 1)z^N + a_{q,2}z^{N-1} + \cdots + a_{q,1+q}z^{N-q} + \cdots + a_{q,N+1} = 0.$$

Now we have

$$(2.3) \qquad \sum_{1 \le j_1 < j_2 < \cdots < j_q < N}\frac{1}{z_{j_1}z_{j_2}\cdots z_{j_q}} = \frac{(-1)^q a_{q,1+q}}{\lambda^q - 1}.$$

Using the remark (2.2) above, we note that we have

$$(2.4) \qquad |a_{q,1+q}| \ge \sqrt{|\lambda^q - 1|}, \text{ if } |\lambda^q - 1| \le \frac{4(q-1)^2}{q^2(q+1)^2}.$$

This holds by assumption (ii), and so we get that

$$(2.5) \qquad \sum_{1 \le j_1 < j_2 < \cdots < j_q \le N}\frac{1}{|z_{j_1}z_{j_2}\cdots z_{j_q}|} \ge \frac{1}{\sqrt{|\lambda^q - 1|}}.$$

Since the number $\binom{N}{q}$ of q-tuples $(z_{j_1}, z_{j_2}, \cdots, z_{j_q})$ is less than N^q, there exists such a q-tuple for which

$$(2.6) \qquad \frac{1}{|z_{j_1} z_{j_2} \cdots z_{j_q}|} \geq \frac{1}{N^q \sqrt{|\lambda^q - 1|}}.$$

This implies that one of these roots, say z_{j_1} satisfies

$$(2.7) \qquad |z_{j_1}| \leq N|\lambda^q - 1|^{\frac{1}{2q}} < (q+1)^q |\lambda^q - 1|^{\frac{1}{2q}}.$$

If the right-hand side of (2.7) is small, then clearly not only z_{j_1}, but the entire orbit of z_{j_1} will be close to 0. To see this, note that we have $|h(z)| \leq 2|z|$, if we have $\sum_{j=2}^{q+1} |a_j||z|^j \leq |z|$, which is true, e.g., if $|z| \leq 1/\sum_{j=2}^{q+1} |a_j|$ and $|z| \leq 1$. Let $R = \min\{1, 1/\sum_{j=2}^{q+1} |a_j|\}$.

If z_{j_1} satisfies the inequality

$$(2.8) \qquad |z_{j_1}| \leq 2^{-(q-2)} R,$$

then we find that any element $h^{\circ r}(z_{j_1})$, $0 \leq r \leq q-1$, of the orbit of z_{j_1} satisfies

$$(2.9) \qquad |h^{\circ r}(z_{j_1})| \leq 2^r |z_{j_1}| \leq 2^q (q+1)^q |\lambda^q - 1|^{\frac{1}{2q}}.$$

We still need to show that z_{j_1} satisfies (2.8). Since we have (2.7), it suffices to check that

$$(2.10) \qquad (q+1)|\lambda^q - 1|^{\frac{1}{2q}} \leq 2^{-(q-2)} R,$$

or

$$|\lambda^q - 1| \leq \frac{R^{2q}}{(q+1)^{2q^2} 2^{2q(q-2)}} ,$$

which is true by (ii) if

$$\frac{1}{4Ceq} \leq \frac{R^{1/q}}{(q+1)2^{1-2/q}} .$$

This can be rewritten as

$$(2.11) \qquad C \geq \frac{q+1}{2eq4^{1/q} R^{1/q}} .$$

It is easy to check that this holds if C is large enough. Hence z_{j_1} satisfies the inequality (2.8).

Since by the assumption (ii) we have $|\lambda^q - 1| \leq 1/(4Ceq)^{2q^2}$, it follows from (2.9) that, for any r with $0 \leq r \leq q-1$ we have

$$(2.12) \qquad |h^{\circ r}(z_{j_1})| \leq 2^q (q+1)^q \frac{1}{(4Ceq)^q} \leq \frac{1}{(2C)^q}.$$

We conclude that $h(z)$ has an entire periodic orbit in the disk of radius $1/(2C)^q$ centered at the origin, and so the proof of Theorem 2.1 is complete. $\qquad \square$

Remark 2.2. Now let p/q be one of the convergents p_n/q_n to the continued fraction of α. We know that in this case, the following inequalities hold (see [3]):

$$(2.13) \qquad \frac{2}{q_{n+1}} \leq |\lambda^{q_n} - 1| \leq \frac{2\pi}{q_{n+1}},$$

and so, the above condition (ii) in Theorem 2.1 is satisfied if we have, for instance,

$$(2.14) \qquad \log q_{n+1} \geq 3q_n{}^2 \log(4Ceq_n),$$

or if we have

$$(2.15) \qquad \log q_{n+1} \geq q_n{}^3,$$

which is the same as condition (ii) in Theorem 2.1, namely

$$(2.16) \qquad \log \log q_{n+1} \geq 3 \log q_n.$$

Theorem 2.3. *Let $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and let $(p_n/q_n)_{n \geq 1}$ denote the convergents to the continued fraction of α. Let $f(z) = \lambda z + a_2 z^2 + \ldots + a_n z^n + \ldots$ be analytic on a disk centered at the origin.*

Assume that for infinitely many numbers n we simultaneously have:

(I) *$\log\log q_{n+1} \geq 3 \log q_n$;*

(II) *a_{1+q_n} lies outside the disk of radius $\sqrt{|\lambda^{q_n} - 1|}$ centered at*

$$-\lambda P_{q_n, 1+q_n}(\lambda, a_2, \ldots, a_{q_n})/q_n,$$

and $|a_{1+q_n}| > (q_n)^{-q_n{}^2}$; and

(III) *$a_j = 0$, for $2 + q_n \leq j \leq q_n{}^{1+q_n} - 1$.*

Then $f(z)$ has infinitely many periodic orbits in any neighborhood of the origin.

Proof. <u>CASE 1.</u> We first assume that the coefficients $(a_n)_{n \geq 2}$ are bounded by a positive constant C. Without loss of generality, we can assume that $C \geq 1$. Fix q_n large with properties (I), (II) and (III) above. Let $M_n = q_n^{(1+q_n)-1}$.

The assumptions (I) and (II), together with Theorem 2.1, imply that the mapping $h(z) = \lambda z + a_2 z^2 + \ldots + a_{1+q_n} z^{1+q_n}$ has a periodic orbit in the disk $\{z : |z| \leq \frac{1}{(2C)^{q_n}}\}$.

We will show that if q_n is large enough, then $f(z)$ will have a periodic orbit in a disk slightly larger, for example, in the disk $\{z : |z| \leq \frac{3}{(2C)^{q_n}}\}$.

From the proof of Theorem 2.1 (see equations (2.7) and (2.12)), we know that $\frac{h^{\circ q_n}(z) - z}{z}$ has a root z_{j_1} inside the disk $\{z : |z| \leq \frac{1}{(4C)^{q_n}}\}$.

We now choose a radius r with $\frac{2}{(4C)^{q_n}} \leq r \leq \frac{3}{(4C)^{q_n}}$, for which no roots of the equation $h^{\circ q_n}(z) - z = 0$ lie on the circle $\{z : |z| = r\}$. Since the number of the non-zero roots of this equation is $N = (q_n + 1)^{q_n} - 1$, we can in fact choose an $r \in [\frac{2}{(4C)^{q_n}}, \frac{3}{(4C)^{q_n}}]$ such that the distance between the circle $\{z : |z| = r\}$ and any of the roots $z_1, z_2, \ldots, z_N$ is at least $1/(2N(4C)^{q_n})$.

Let

$$
\begin{aligned}
H(z) &= h^{\circ q_n}(z) - z, \\
(2.17) \qquad F(z) &= f^{\circ q_n}(z) - z.
\end{aligned}
$$

We want to show that

$$(2.18) \qquad |H(z) - F(z)| < |H(z)|, \text{ for } |z| = r.$$

Assuming that (2.18) holds, Rouché 's theorem implies that $H(z)$ and $F(z)$ have the same number of roots inside the circle $\{z : |z| = r\}$. Since $H(z)$ has at least one such root, namely z_{j_1}, and since $H(z)$ and $F(z)$ have a zero of the same order at the origin, it will follow that $F(z)$ has a root, call it w, inside this circle. We also need to show that w satisfies a condition similar to (2.8), for instance,

$$(2.19) \qquad |w| \leq 2^{-(q_n-2)} R,$$

where R has the property that $|f(\zeta)| \leq 2|\zeta|$ for all ζ with $|\zeta| \leq R$. A short calculation shows that we can take, for example, $R = 1/(1 + C)$.

We know that w is situated inside the circle of radius r centered at the origin, and therefore for (2.19) to hold it is sufficient to check that

$$(2.20) \qquad \frac{3}{(4C)^{q_n}} \leq 2^{-(q_n-2)} \frac{1}{1+C}.$$

This is true for any $C \geq 1$ and any $q_n \geq 1$.

It follows that $|f(w)| \leq 2|w|$. Inductively we find that, for any r with $0 \leq r \leq q_n - 1$, we have

$$(2.21) \qquad |f^{\circ r}(w)| \leq 2^r \frac{3}{(4C)^{q_n}}.$$

This means that the entire orbit $\{w, f(w), f^{\circ 2}(w), \ldots, f^{\circ(q_n-1)}(w)\}$ lies inside the circle of radius $3/(2C)^{q_n}$ centered at origin, and we are done.

Therefore, it remains for us to show that (2.18) holds for q_n large enough. In order to do this it is sufficient to provide a lower bound for $H(z)$, and an upper bound for the error $|H(z) - F(z)|$ on the circle $\{z : |z| = r\}$.

Fix z with $|z| = r$. The leading coefficient of $H(z) = h^{\circ q_n}(z) - z$ equals

$$A = (a_{1+q_n})^{1+(1+q_n)+(1+q_n)^2+\ldots+(1+q_n)^{q_n-1}} = (a_{1+q_n})^{N/q_n}.$$

Now

$$H(z) = Az \prod_{1 \leq j \leq N} (z - z_j),$$

and since each z_j is at least $1/(2N(4C)^{q_n})$ away from z, it follows that

$$(2.22) \qquad |H(z)| \geq \frac{|A|r}{(2N(4C)^{q_n})^N}, \quad \text{where } |A| = |a_{1+q_n}|^{N/q_n}.$$

We now check that the choice of $|a_{1+q_n}|$ in assumption (II) is such that we have

$$(2.23) \qquad |A| > \frac{1}{(2N(4C)^{q_n})^N}.$$

Indeed, this happens if

$$\frac{1}{q_n^{q_n^2}} > \frac{1}{(2N(4C)^{q_n})^{q_n}},$$

which is equivalent to

$$q_n^{q_n} < 2N(4C)^{q_n} = 2\left((1+q_n)^{q_n} - 1\right)(4C)^{q_n}.$$

This last inequality holds for large enough q_n. Indeed, since $C \geq 1$, for all large n we have

$$2q_n^{q_n}4^{q_n} \leq 2\left((1+q_n)^{q_n} - 1\right)(4C)^{q_n}.$$

Thus from (2.22) and from the choice of r we get a lower bound for $|H(z)|$; that is,

$$(2.24) \qquad |H(z)| \geq \frac{2}{(2N(4C)^{1+q_n})^{2N}}.$$

Next, we want to find an upper bound for $|H(z) - F(z)|$. Recall that

$$|H(z) - F(z)| = |f^{\circ q_n}(z) - h^{\circ q_n}(z)|,$$

and also

$$\begin{aligned} h(z) &= \lambda z + a_2 z^2 + \ldots + a_{1+q_n} z^{1+q_n}, \\ f(z) &= \lambda z + a_2 z^2 + \ldots + a_{1+q_n} z^{1+q_n} + a_{M_n} z^{M_n} + \ldots. \end{aligned}$$

It follows from equation (2.17) and Lemma 1.3 that $H(z) - F(z)$ is a power series in z with terms having degree at least M_n. In fact, $H(z) - F(z)$ is exactly the tail of $f^{\circ q_n}(z)$ starting at degree M_n. Thus, in order to estimate $|H(z) - F(z)|$ we first need to estimate the coefficients of $f^{\circ q_n}(z)$.

Let $a_1 = \lambda$, so that we can write $f(z) = \sum_{n \geq 1} a_n z^n$. We have

$$
\begin{aligned}
f^{\circ 2}(z) &= \sum_{m \geq 1} a_m \left(\sum_{n \geq 1} a_n z^n \right)^m \\
&= \sum_{k=1}^{\infty} z^k \left[\sum_{m \leq k} a_m \left(\sum_{n_1 + \cdots + n_m = k, n_i \geq 1} a_{n_1} a_{n_2} \cdots a_{n_m} \right) \right] = \sum_{k=1}^{\infty} z^k A_{k,2},
\end{aligned}
$$

where

$$
A_{k,2} = \sum_{m \leq k} a_m \left(\sum_{n_1 + \cdots + n_m = k, n_i \geq 1} a_{n_1} a_{n_2} \cdots a_{n_m} \right).
$$

By induction we find that

$$
(2.25) \qquad f^{\circ q}(z) = \sum_{k=1}^{\infty} z^k A_{k,q},
$$

where

$$
A_{k,q} = \sum_{m \leq k} A_{m,q-1} \left(\sum_{n_1 + \ldots + n_m = k, n_i \geq 1} a_{n_1} a_{n_2} \cdots a_{n_m} \right).
$$

Let $p_m(k)$ denote the number of compositions of k into m parts. Using Lemma 1.2, we can estimate the coefficient $A_{k,2}$ of z^k in $f^{\circ 2}(z)$ as follows:

$$
\begin{aligned}
|A_{k,2}| &\leq \sum_{1 \leq m \leq k} |a_m| \sum_{n_1 + \ldots + n_m = k, n_i \geq 1} |a_{n_1}| \cdots |a_{n_m}| \\
&\leq \sum_{1 \leq m \leq k} p_m(k) C^{m+1} = C^2 (1 + C)^{k-1}, \text{ for } k > 1.
\end{aligned}
$$

By induction we see that the coefficient $A_{k,q}$ of z^k in the iterate $f^{\circ q}(z)$ satisfies

$$
\begin{aligned}
|A_{k,q}| &\leq C^q (1 + C + C^2 + \ldots + C^{q-1})^{k-1} \\
&= C^q \left(\frac{C^q - 1}{C - 1} \right)^{k-1} \leq C^q (C^q)^{k-1} = C^{qk},
\end{aligned}
$$

the last inequality being satisfied for large enough q (we can always take $C > 2$). Thus we have

$$
(2.26) \qquad |A_{k,q}| \leq C^{qk}, \text{ for large } q.
$$

We can now estimate

$$
\begin{aligned}
|H(z) - F(z)| &\leq \sum_{k \geq M_n} |A_{k,q_n}| \cdot |z|^k \\
(2.27) \qquad &\overset{(2.26)}{\leq} \sum_{k \geq M_n} C^{q_n k} \cdot \left(\frac{3}{(4C)^{q_n}} \right)^k \leq 2 \left(\frac{3}{4^{q_n}} \right)^{M_n},
\end{aligned}
$$

when n is large enough. On the other hand, we have already found a lower bound (2.24) for $|H(z)|$:

$$|H(z)| \geq \frac{2}{(2N(4C)^{1+q_n})^{2N}}.$$

It is easy to see that

$$N = (1+q_n)^{q_n} - 1 < q_n{}^{q_n} \left(\frac{1+q_n}{q_n}\right)^{q_n} < eq_n{}^{q_n},$$

and so we get the following lower bound for $|H(z)|$:

$$(2.28) \qquad |H(z)| \geq \frac{2}{(2eq_n{}^{q_n}(4C)^{1+q_n})^{2eq_n{}^{q_n}}}.$$

In order for (2.18) to hold it is sufficient to have

$$(2.29) \qquad 2\left(\frac{3}{4q_n}\right)^{M_n} < \frac{2}{(2eq_n{}^{q_n}(4C)^{1+q_n})^{2eq_n{}^{q_n}}},$$

which is the same as

$$(2.30) \qquad \left(2eq_n{}^{q_n}(4C)^{1+q_n}\right)^{2eq_n{}^{q_n}} < \left(\frac{4q_n}{3}\right)^{M_n}.$$

It is enough to check that

$$M_n(q_n - 1)\log 4 > (2eq_n{}^{q_n})\log\left(2e(4C)^{1+q_n}q_n{}^{q_n}\right).$$

We note that $2e(4C)^{1+q_n}q_n{}^{q_n} < q_n{}^{2q_n}$ for large n. Since $M_n = (q_n)^{1+q_n}$, it suffices to show that

$$(q_n)^{1+q_n}(q_n - 1)\log 4 \quad > \quad 2eq_n{}^{q_n}\log q_n{}^{2q_n},$$

or

$$(q_n - 1)\log 4 > 4e\log q_n.$$

We know that q_n increases at least exponentially with n (see [2]), and so this last inequality will easily be true for large enough n. Thus, for large n, the inequality (2.30) holds, and so does (2.28).

Recall that $M_n = q_n{}^{1+q_n}$ is the degree of the first term following $a_{q_n}z^{q_n}$ in the power series of $f(z)$ whose coefficient is allowed to be non-zero . We want to check that $M_n < q_{n+1}$ for large n, or equivalently, $\log M_n < \log q_{n+1}$. By assumption (I) we have $\log q_{n+1} \geq q_n{}^3$, and thus it is enough to check that $(1+q_n)\log q_n < q_n{}^3$, which holds if $2\log q_n < q_n{}^2$. This is clearly true for large enough n. Therefore we have

$$q_n < M_n = q_n{}^{1+q_n} < q_{n+1}, \text{ for all large } n.$$

We have proved that (2.18) holds, and by Rouché's theorem and the observations above, we conclude that $f(z)$ has an entire periodic orbit inside the disk of radius $3/(2C)^{q_n}$ centered at the origin. Hence, the proof is complete in the first case.

GENERAL CASE. The general case can be reduced to the previous case as follows. Let $\varepsilon > 0$ be such that f is analytic on $B(0,\varepsilon)$. Take $\gamma = \varepsilon/(1+\varepsilon)$ and consider

$$(2.31) \qquad g(z) = \frac{f(\gamma z)}{\gamma}.$$

Then z_1 is a periodic point of period q of $f(z)$ if and only if z_1/γ is a periodic point of period q of $g(z)$.

Note that $|z| < 1 + \varepsilon$ implies $|\gamma z| < \varepsilon$, and so it follows that $g(z)$ is defined and analytic on $B(0, 1 + \varepsilon)$. Write $g(z) = \sum_{n=0}^{\infty} b_n z^n, |z| < 1 + \varepsilon$. We would like the power series $g(z)$ to have bounded coefficients. Since $g(z)$ is analytic on the disk centered at 0 of radius $1 + \varepsilon > 1$, an easy application of Cauchy's integral formula shows that the coefficients b_n are bounded, say by a positive constant C. Without loss of generality, we can assume that $C \geq 1$.

Next, we want to apply Case 1 to $g(z)$. Fix q_n large enough so that conditions (I), (II), and (III) hold simultaneously.

It is easy to see that $b_1 = \lambda$ and $b_n = a_n \gamma^{n-1}, n > 1$. Together with (III), this implies that

$$(2.32) \qquad b_{2+q_n} = b_{3+q_n} = \cdots = b_{q_n^{1+q_n}-1} = 0.$$

By Lemma 1.3 and assumption (II) we get

$$\left| b_{1+q_n} + \frac{\lambda P_{q_n, 1+q_n}(\lambda, b_2, b_3, \ldots, b_{q_n})}{q_n} \right|$$

$$= \left| \gamma^{q_n} a_{1+q_n} + \frac{\lambda P_{q_n, 1+q_n}(\lambda, \gamma a_2, \gamma^2 a_3, \ldots, \gamma^{q_n-1} a_{q_n})}{q_n} \right|$$

$$(2.33) \qquad = \gamma^{q_n} \left| a_{1+q_n} + \frac{\lambda P_{q_n, 1+q_n}(\lambda, a_2, \ldots, a_{q_n})}{q_n} \right| > \gamma^{q_n} \sqrt{|\lambda^{q_n} - 1|}.$$

Let $p(z) = \lambda z + b_2 z^2 + \cdots + b_{1+q_n} z^{1+q_n}$. In order to be able to apply the proof from Case 1, we need to prove that $p(z)$ has an entire periodic orbit inside the disk $\{z : |z| \leq \frac{1}{(2C)^{q_n}}\}$.

Following the proof of Theorem 2.1 (refer to equations (2.1), (2.3), and (2.6)), we deduce that $p(z)$ has a periodic point z_{j_1}, of period q_n, satisfying

$$(2.34) \qquad |z_{j_1}| \leq \gamma(1 + q_n)^{q_n} |\lambda^{q_n} - 1|^{\frac{1}{2q_n}}.$$

We want to show that not only z_{j_1}, but its entire orbit is close to 0. We need to check, as in the proof of Theorem 2.1, that z_{j_1} also satisfies

$$(2.35) \qquad |z_{j_1}| \leq 2^{-(q_n-2)} R,$$

where $R = \min\{1, 1/\sum_{j=2}^{1+q_n} |b_j|\}$. It suffices to check that

$$(2.36) \qquad \gamma(1 + q_n)^{q_n} |\lambda^{q_n} - 1|^{\frac{1}{2q_n}} \leq 2^{-(q_n-2)} R.$$

By (2.13) and assumption (I), we know that $|\lambda^{q_n} - 1| \leq 1/(4Ceq_n)^{2q_n^2}$. As in the proof of Theorem 2.1, we can check that

$$\frac{1}{4Ceq_n} \leq \frac{R^{1/q_n}}{(1 + q_n) 2^{1-2/q_n} \gamma^{1/q_n}}, \quad \text{for large } n.$$

Thus z_{j_1} satisfies (2.35), and we conclude that the entire orbit will satisfy

$$|p^{\circ r}(z_{j_1})| \leq \gamma 2^{q_n} (1 + q_n)^{q_n} |\lambda^{q_n} - 1|^{\frac{1}{2q_n}} \quad \text{for any } 0 \leq r \leq q_n - 1.$$

This whole orbit will be contained in the disk of radius $1/(2C)^{q_n}$ centered at the origin, if

$$(2.37) \qquad |\lambda^{q_n} - 1|^{\frac{1}{2q_n}} < \frac{1}{\gamma} \cdot \frac{1}{(4Cq_n)^{q_n}}.$$

Taking (2.13) and assumption (I) into account, it is easy to check that this holds for all large n. Therefore, $p(z)$ has an entire orbit in the disk of radius $1/(2C)^{q_n}$ centered at the origin.

Now let

$$P(z) = p^{\circ q_n}(z) - z,$$
$$G(z) = g^{\circ q_n}(z) - z.$$

Let $B = |b_{1+q_n}|^{N/q_n}$. Everything from Case 1 goes through if the analogue of (2.23) holds, i.e., if

$$(2.38) \qquad |B| > \frac{1}{(2N(4C)^{q_n})^{q_n}} \quad \text{for all large } n.$$

Since $b_{1+q_n} = a_{1+q_n}\gamma^{q_n}$ and $|a_{1+q_n}| > (q_n)^{-q_n^2}$, inequality (2.38) holds for all large n if

$$(2.39) \qquad \frac{1}{q_n{}^{q_n^2}} \cdot \gamma^{q_n} > \frac{1}{(2N(4C)^{q_n})^{q_n}},$$

where $N = (1+q_n)^{q_n} - 1$. Since we know that $C \geq 1$, it is sufficient for us to check that

$$(2.40) \qquad q_n{}^{q_n} < 2(1+q_n)^{q_n}\gamma 4^{q_n},$$

which, for any choice of $\gamma = \varepsilon/(1+\varepsilon) > 0$, clearly holds for all large n. Thus, by Case 1, the function $g(z)$ has infinitely many periodic orbits in any small neighborhood of 0.

Since $g(z) = f(\gamma z)/\gamma$, every periodic orbit $\{z, g(z), \ldots, g^{\circ(q-1)}(z)\}$ contained in a disk of radius s centered at 0 corresponds to a periodic orbit of f, namely $\{\gamma z, f(\gamma z), \ldots, f^{\circ(q-1)}(\gamma z)\}$, contained in a disk of radius γs centered at 0.

We conclude that $f(z)$ has infinitely many periodic orbits in any neighborhood of the origin, and the proof of Theorem 2.3 is complete. $\square$

References

1. L. Geyer, *Linearization of Structurally Stable Polynomials*, Progress in Holomorphic Dynamics, Pitman Research Notes in Mathematics Series **387**(1998), 27-30. MR 99m:58154
2. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 3rd edition, Oxford at the Clarendon Press, 1954. MR 16:673c
3. J. Milnor, *Dynamics in One Complex Variable*, Introductory Lectures, 2nd edition, Vieweg, 2000. MR 2002i:37057
4. R. Pérez-Marco, *Sur les dynamiques holomorphes non-linéarisables et une conjecture de V.I. Arnold,* Ann. Sci. École Norm. Sup.(4), **26**(1993), 565-644. MR 95a:58103
5. J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley&Sons Inc., 1958. MR 20:3077
6. J.-C. Yoccoz, *Theorème de Siegel, polynômes quadratiques et nombres de Brjuno*, Astérisque **231**(1995), 3-88. MR 96m:58214

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

E-mail address: `petracvc@math.uiuc.edu`

Current address: Department of Mathematics, Western Illinois University, 1 University Circle, Macomb, Illinois 61455

E-mail address: `L-Petracovici@wiu.edu`