

TANGENT ALGEBRAIC SUBVARIETIES OF VECTOR FIELDS

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ABSTRACT. An algebraic commutative group G is associated to any vector field D on a complete algebraic variety X . The group G acts on X and its orbits are the minimal subvarieties of X which are tangent to D . This group is computed in the case of a vector field on \mathbb{P}_n .

INTRODUCTION

Let X be an algebraic variety over an algebraically closed field k of characteristic 0 and let D be a vector field on X . A closed subvariety Y of X , defined by a sheaf of ideals I of \mathcal{O}_X , is said to be a tangent subvariety of D if $D(I) \subseteq I$. This condition implies that D induces a derivation of the sheaf $\mathcal{O}_Y = \mathcal{O}_X/I$, so that it defines a vector field on Y . The aim of this paper is to study the structure of the family of tangent subvarieties of a vector field D . At first sight it seems reasonable to conjecture that minimal tangent subvarieties of D form an algebraic family. Precisely:

Question: Does there exist a dense open set U in X and a surjective morphism of k -schemes $\pi : U \rightarrow Z$ whose fibres are just the minimal tangent subvarieties of D in U ?

Unfortunately, the answer is negative in general (we shall give a counterexample in Section 5). However, it is affirmative in an interesting case, that of global vector fields on a complete algebraic variety. For these fields we prove the following result:

Theorem. *Let D be a vector field on an integral projective variety X . There exists a connected commutative algebraic subgroup $G \subseteq \mathbf{Aut} X$ and a G -invariant dense open set U in X such that:*

- a) *Minimal tangent subvarieties of D on U are just orbits of G in U .*
- b) *The quotient variety $\pi : U \rightarrow Z = U/G$ exists. Hence, the fibres of π are the minimal tangent subvarieties of D in U .*
- c) *The sheaf \mathcal{O}_Z coincides with the sheaf of first integrals of D ; that is to say, for any open set V in Z , we have*

$$\Gamma(V, \mathcal{O}_Z) = \{f \in \Gamma(\pi^{-1}V, \mathcal{O}_X) : Df = 0\}.$$

This theorem may be extended to a distribution on X generated by global vector fields (see 3.6).

The completeness of X is used to assure the existence of the algebraic group $\mathbf{Aut} X$. In such a case, the Lie algebra of all vector fields to X is canonically isomorphic to the Lie algebra of the group $\mathbf{Aut} X$. Hence, the vector field D

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corresponds to some element \tilde{D} in the Lie algebra of $\mathbf{Aut} X$, that is to say, D is a fundamental vector field with respect to the action of $\mathbf{Aut} X$ on X . In fact, the former theorem holds when X is not complete, if D is assumed to be a fundamental field with respect to the action on X of some algebraic group.

The above mentioned commutative group G is defined to be the minimal tangent subvariety of \tilde{D} through the identity of $\mathbf{Aut} X$.

We determine the group G in the case of vector fields on the complex projective space \mathbb{P}_n . It is well known that any vector field on \mathbb{P}_n may be expressed as $D = \pi_*(\sum \lambda_{ij} z_j \frac{\partial}{\partial z_i})$, where $\pi: \mathbb{C}^{n+1} \rightarrow \mathbb{P}_n$ is the natural projection. The group G is determined by two integers s, δ associated to D as follows. Let us consider the field \mathbb{C} as an infinite dimensional affine space over \mathbb{Q} ; then s is the dimension of the minimal affine \mathbb{Q} -subspace of \mathbb{C} containing all the eigenvalues of the matrix (λ_{ij}) . The integer δ is 0 when the matrix (λ_{ij}) is diagonalizable and 1 otherwise. We prove the following result:

Theorem. *The group G associated to a vector field $D = \pi_*(\sum \lambda_{ij} z_j \frac{\partial}{\partial z_i})$ on the complex projective space \mathbb{P}_n is*

$$G = (\mathbb{G}_m)^s \times (\mathbb{G}_a)^\delta.$$

(\mathbb{G}_m and \mathbb{G}_a are the multiplicative and additive lines respectively.)

Corollary. *The dimension of the minimal tangent subvarieties of D is $s + \delta$ (generically). The field of meromorphic functions on \mathbb{P}_n which are first integrals of D has transcendence degree equal to $n - s - \delta$.*

1. PRELIMINARIES

From now on, k will be an algebraically closed field of characteristic 0 and X will be a k -scheme.

1.1. Differential algebras. Let us recall some elementary facts about differential algebras. A **differential k -algebra** is a k -algebra A endowed with a k -linear derivation $D : A \rightarrow A$. An ideal I of A is said to be a **differential ideal** if $D(I) \subseteq I$.

Proposition 1.1. *The nilradical of any differential algebra A is a differential ideal.*

Proof. If $a^n = 0$, applying D^n we obtain $n!(Da)^n + ba = 0$, so that Da is nilpotent. \square

Proposition 1.2. *Any minimal prime ideal of a differential noetherian k -algebra is a differential ideal.*

Proof. Replacing A by $A/\text{rad} A$, we may assume that $\text{rad} A = 0$. In such case, $0 = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$ where $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are the minimal prime ideals of A . Let $0 \neq b \in \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_r$. If $a \in \mathfrak{p}_1$, then $ab = 0$ because $ab \in \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r = 0$. Hence $(Da)b = -aDb$, so that both terms are 0, since $aDb \in \mathfrak{p}_1$ and $bDa \in \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_r$. Finally, from $(Da)b = 0$ it follows that $Da \in \mathfrak{p}_1$ because $b \notin \mathfrak{p}_1$. Therefore $D\mathfrak{p}_1 \subseteq \mathfrak{p}_1$. \square

1.2. Functor of points. The **functor of points** of X is defined to be the following contravariant functor on the category of k -schemes

$$X^\bullet(T) := \text{Hom}_k(T, X) .$$

For any two k -schemes X, Y we have (Yoneda's lemma)

$$\text{Hom}_k(X, Y) = \text{Hom}_{\text{funct.}}(X^\bullet, Y^\bullet) .$$

In particular, in order to define a morphism of k -schemes $X \rightarrow Y$ it is enough to construct a morphism of functors $X^\bullet \rightarrow Y^\bullet$. We shall use this fact later on.

Elements of $X^\bullet(T)$, that is to say, morphisms of k -schemes $x : T \rightarrow X$, are said to be **T -valued points** (or **T -points**) of X . Moreover, we use the following convention: Given a T -point $x : T \rightarrow X$ and a morphism of k -schemes $T' \rightarrow T$, the composition $T' \rightarrow T \xrightarrow{x} X$ will be denoted also by x , that is to say, we do not change the notation of a point after a basis change.

1.3. Infinitesimal automorphisms. Let $k[\varepsilon] = k[x]/(x^2)$ and let $i : X \hookrightarrow X_{k[\varepsilon]} = X \times_k k[\varepsilon]$ be the closed immersion defined by $\varepsilon = 0$.

An **infinitesimal automorphism** of X is defined to be an automorphism of $k[\varepsilon]$ -schemes $\tau_\varepsilon : X_{k[\varepsilon]} \rightarrow X_{k[\varepsilon]}$ satisfying the infinitesimal condition

$$(\tau_\varepsilon)|_{\varepsilon=0} = \text{id}_X ,$$

that is to say, $\tau_\varepsilon \circ i = i$.

Any vector field D on X determines an infinitesimal automorphism τ_ε of X given by the following morphism of $k[\varepsilon]$ -algebras:

$$\tau_\varepsilon : \mathcal{O}_X[\varepsilon] \longrightarrow \mathcal{O}_X[\varepsilon], \quad a \mapsto \tau_\varepsilon(a) = a + \varepsilon Da$$

and, conversely, any infinitesimal automorphism is clearly defined by a unique vector field.

1.4. Tangent subschemes. Let D be a vector field on X . A closed subscheme Y of X , defined by a sheaf of ideals I of \mathcal{O}_X , is said to be a **tangent subscheme** of D if $D(I) \subseteq I$. This condition implies that D induces a derivation of the sheaf $\mathcal{O}_Y = \mathcal{O}_X/I$, so that it defines a vector field on Y .

A closed subscheme Y of X is a tangent subscheme of D if and only if its functor of points Y^\bullet is stable under the corresponding infinitesimal automorphism τ_ε , that is to say, for any k -scheme T and any point $x \in X^\bullet(T)$, we have

$$x \in Y^\bullet(T) \Rightarrow \tau_\varepsilon(x) \in Y_{k[\varepsilon]}^\bullet(T_{k[\varepsilon]}) = Y^\bullet(T_{k[\varepsilon]})$$

where $\tau_\varepsilon(x)$ stands for the composition $T_{k[\varepsilon]} \xrightarrow{x} X_{k[\varepsilon]} \xrightarrow{\tau_\varepsilon} X_{k[\varepsilon]}$.

Given a closed point $x \in X$, we denote by Y_x the minimal tangent subscheme of D passing through x . Remark that $Y_{x'} \subseteq Y_x$ whenever $x' \in Y_x$.

From Propositions 1.1 and 1.2, it follows that Y_x is *reduced and irreducible*.

1.5. Zeros of a vector field. Let D be a vector field on X and let us consider the natural morphism $\Omega_X^1 \xrightarrow{D} \mathcal{O}_X$, $da \mapsto Da$. The image of this morphism is a certain sheaf of ideals I of \mathcal{O}_X . The **subscheme of zeros** of D is the closed subscheme Z_D of X defined by the sheaf of ideals I .

The subscheme of zeros Z_D may be defined in terms of the infinitesimal automorphism τ_ε corresponding to the vector field D . In fact, the functor of τ_ε -invariant points of X is representable by the subscheme of zeros Z_D as follows.

Lemma 1.3. *We have*

$$Z_D^\bullet(T) = \{x \in X^\bullet(T) : \tau_\varepsilon(x) = x\}$$

where the equality $\tau_\varepsilon(x) = x$ states the coincidence of the composition morphism $T_{k[\varepsilon]} \xrightarrow{x} X_{k[\varepsilon]} \xrightarrow{\tau_\varepsilon} X_{k[\varepsilon]}$ with the morphism $T_{k[\varepsilon]} \xrightarrow{x} X_{k[\varepsilon]}$.

Proof. A morphism $x : T \rightarrow X$ factors through Z_D if and only if the composition $\Omega_X^1 \xrightarrow{D} \mathcal{O}_X \xrightarrow{x} \mathcal{O}_T$ vanishes. Now, this condition is equivalent to the coincidence of the two following morphisms:

$$\begin{array}{ccc} \mathcal{O}_X[\varepsilon] & \xrightarrow{\tau_\varepsilon} \mathcal{O}_X[\varepsilon] \xrightarrow{x} \mathcal{O}_T[\varepsilon] & a \mapsto a + \varepsilon Da \mapsto x(a) + \varepsilon x(Da), \\ \mathcal{O}_X[\varepsilon] & \xrightarrow{x} \mathcal{O}_T[\varepsilon] & a \mapsto x(a). \end{array}$$

□

1.6. Differential morphisms. Let (A, D) and (A', D') be two differential k -algebras. A morphism of k -algebras $\varphi : A \rightarrow A'$ is said to be differential if it commutes with the given derivations: $\varphi(Da) = D'(\varphi(a))$.

Now let X and X' be two k -schemes and let D and D' be vector fields on X and X' respectively. A morphism of k -schemes $\varphi : X' \rightarrow X$ is said to be **differential** when the morphism of k -algebras $\varphi : \mathcal{O}_X(U) \rightarrow \mathcal{O}_{X'}(V)$ is differential, for any pair of open sets $U \subseteq X$ and $V \subseteq \varphi^{-1}(U)$.

Let τ_ε and τ'_ε be the infinitesimal automorphisms corresponding to D and D' respectively. A morphism of k -schemes $\varphi : X' \rightarrow X$ is differential if and only if the morphism $\varphi : X'_{k[\varepsilon]} \rightarrow X_{k[\varepsilon]}$ satisfies $\varphi \circ \tau'_\varepsilon = \tau_\varepsilon \circ \varphi$.

The following statement is trivial.

Proposition 1.4. *Let $\varphi : (X', D') \rightarrow (X, D)$ be a differential morphism of k -schemes. If Y is a tangent subscheme of (X, D) , then $\varphi^{-1}(Y) = Y \times_X X'$ is a tangent subscheme of (X', D') .*

2. ALGEBRAIC GROUP ASSOCIATED TO A VECTOR FIELD

In this section X will be a proper k -scheme.

Definition 2.1. Let us consider the functor of automorphisms of X , defined on the category of k -schemes,

$$F(T) := \text{Aut}_T(X \times_k T).$$

This functor is representable (see [9]) by an algebraic group **Aut** X , named **scheme of automorphisms** of X ([4]), that is to say,

$$(\mathbf{Aut} X)^\bullet(T) = \text{Aut}_T(X \times_k T).$$

Now let D be a vector field on X and let τ_ε be the corresponding infinitesimal automorphism. This automorphism induces an infinitesimal automorphism $\tilde{\tau}_\varepsilon$ of **Aut** X . We construct this automorphism $\tilde{\tau}_\varepsilon : (\mathbf{Aut} X)_{k[\varepsilon]} \rightarrow (\mathbf{Aut} X)_{k[\varepsilon]}$ by means of the functor of points: For any $k[\varepsilon]$ -scheme T we define

$$\tilde{\tau}_\varepsilon : \text{Aut}_T(X_{k[\varepsilon]} \times_{k[\varepsilon]} T) \rightarrow \text{Aut}_T(X_{k[\varepsilon]} \times_{k[\varepsilon]} T), \quad g \mapsto \tau_\varepsilon \circ g.$$

Note that $\tilde{\tau}_\varepsilon$ satisfies the infinitesimal condition $(\tilde{\tau}_\varepsilon)|_{\varepsilon=0} = \text{id}_{\mathbf{Aut} X}$.

The infinitesimal automorphism $\tilde{\tau}_\varepsilon$ corresponds to a certain tangent field \tilde{D} on $\mathbf{Aut} X$. Note that, by definition, $\tilde{\tau}_\varepsilon$ commutes with right translations in $\mathbf{Aut} X$, so that the field \tilde{D} is invariant under right translations.

Therefore, any vector field D on a proper k -scheme X has a canonically associated vector field \tilde{D} on the scheme of automorphisms $\mathbf{Aut} X$, which is invariant under right translations.

Moreover, it may be proved that the map $D \mapsto \tilde{D}$ defines an isomorphism of the Lie algebra of vector fields on X onto the Lie algebra of the group $\mathbf{Aut} X$ (we shall not use this fact).

Definition 2.2. Let D be a vector field on a proper k -scheme X and let \tilde{D} be the corresponding vector field on the scheme of automorphisms $\mathbf{Aut} X$, invariant under right translations. The **associated group** of D is defined to be the minimal tangent subscheme G of \tilde{D} passing through the identity of $\mathbf{Aut} X$. We shall prove that G is a commutative algebraic subgroup of $\mathbf{Aut} X$.

Proposition 2.3. G is a connected algebraic closed subgroup of $\mathbf{Aut} X$.

Proof. By definition G is a closed subscheme of $\mathbf{Aut} X$. Moreover, G is integral (hence connected) by Propositions 1.1 and 1.2. Finally, we have to show that the product $G \times G \rightarrow \mathbf{Aut} X$ factors through G and that the inverse morphism $\mathbf{Aut} X \rightarrow \mathbf{Aut} X$, $g \mapsto g^{-1}$ takes G into itself. Since G is reduced, it is enough to prove both statements in the case of closed points; that is to say, it is enough to show that $G^\bullet(k)$ is a subgroup of $(\mathbf{Aut} X)^\bullet(k)$.

Let $g \in G^\bullet(k)$. Since G is the minimal tangent subvariety of \tilde{D} through the identity and \tilde{D} is invariant under right translations, it follows that $G \cdot g$ is the minimal tangent subvariety of \tilde{D} passing through g . Hence $G \cdot g \subseteq G$, because G also passes through g . This inclusion is not strict, since otherwise, multiplying at right by g^n , we would obtain an infinite decreasing sequence of closed sets $G \cdot g^{n+1} \subset G \cdot g^n$, so contradicting the noetherian character of $\mathbf{Aut} X$. Hence $G \cdot g = G$ and then $G^\bullet(k) \cdot g = G^\bullet(k)$ for any $g \in G^\bullet(k)$. It follows readily that $G^\bullet(k)$ is a group. \square

Proposition 2.4. G is commutative.

Proof. Let us consider the infinitesimal automorphism

$$\sigma : (\mathbf{Aut} X)_{k[\varepsilon]} \longrightarrow (\mathbf{Aut} X)_{k[\varepsilon]}$$

defined, in terms of the functor of points, by the formula $\sigma(g) = \tau_\varepsilon \circ g \circ \tau_\varepsilon^{-1}$. This infinitesimal automorphism corresponds to a certain vector field on $\mathbf{Aut} X$. Let H be the corresponding closed subscheme of zeros. By Lemma 1.3, the functor of points of H is

$$H^\bullet(T) = \{g \in (\mathbf{Aut} X)^\bullet(T) : \sigma(g) = g\} = \{g \in (\mathbf{Aut} X)^\bullet(T) : \tau_\varepsilon \circ g = g \circ \tau_\varepsilon\}.$$

Remark that $H^\bullet(T)$ is a subgroup of $(\mathbf{Aut} X)^\bullet(T)$, hence H is a closed algebraic subgroup of $\mathbf{Aut} X$. Let us consider the center C of H , which is a commutative closed subgroup. The functor of points of the center is

$$C^\bullet(T) = \{c \in H^\bullet(T) : c \circ h = h \circ c \text{ for any } h \in H^\bullet(T') \text{ and any } T' \rightarrow T\}$$

(we refer to [2], II, §1 3.7 for the existence of the center). Let us prove that C is a tangent subscheme of \tilde{D} . According to 1.4, we have to show that the functor C^\bullet is stable by the automorphism $\tilde{\tau}_\varepsilon$: If $c \in C^\bullet(T)$, let us prove that

$\tilde{\tau}_\varepsilon(c) = \tau_\varepsilon \circ c \in C^\bullet(T_{k[\varepsilon]})$; in fact, for any morphism $T' \rightarrow T_{k[\varepsilon]}$ and any point $h \in H^\bullet(T')$ we have

$$(\tau_\varepsilon \circ c) \circ h = \tau_\varepsilon \circ (c \circ h) = \tau_\varepsilon \circ (h \circ c) = h \circ (\tau_\varepsilon \circ c) .$$

Finally, G is contained in C , because C is a tangent subscheme of \tilde{D} passing through the identity and G is minimal. Since C is commutative, so is G . \square

Note 2.5. A more general result than Proposition 2.4 was proved by Chevalley [1] in the realm of Lie algebras. In fact, if L is a Lie subalgebra of the Lie algebra of an algebraic group and L_{alg} stands for the minimal algebraic Lie subalgebra (in the sense that it is the Lie algebra of an algebraic subgroup) containing L , then [1], Chap. II, Th. 13, states that $[L, L] = [L_{\text{alg}}, L_{\text{alg}}]$; hence L_{alg} is abelian whenever L is. In particular, if $L = \langle D \rangle$, then L_{alg} is an abelian Lie subalgebra.

Let us consider the structure of the associated algebraic group G . According to the fundamental structure theorem of algebraic groups (see [12] or [13]), G has an affine connected normal subgroup N such that the quotient $A = G/N$ is an abelian variety. Moreover, any connected commutative affine group is a direct product of multiplicative and additive lines: $N = \mathbb{G}_m^r \times \mathbb{G}_a^\delta$. In conclusion, G is an extension of an abelian variety by a direct product of multiplicative and additive lines:

$$0 \longrightarrow \mathbb{G}_m^r \times \mathbb{G}_a^\delta \longrightarrow G \longrightarrow A \longrightarrow 0 .$$

When this extension is trivial (G being the associated group), one may easily prove that $\delta \leq 1$.

3. TANGENT SUBSCHEMES

Let us recall the notation of the former section: X is a proper scheme over an algebraically closed field k of characteristic zero, D is a vector field on X and G is the associated group. Recall that G is a subgroup of the group $\mathbf{Aut} X$, so that we have an obvious action $\mu : G \times X \rightarrow X$.

Let \tilde{D} be the right-invariant vector field on $\mathbf{Aut} X$ induced by D , and let $\tilde{\tau}_\varepsilon$ be the corresponding infinitesimal automorphism: $\tilde{\tau}_\varepsilon(g) = \tau_\varepsilon \circ g$. Let us consider \tilde{D} as a vector field on G (it may be done because G is a tangent subscheme of \tilde{D}). Then we may consider on $G \times X$ the vector field $(\tilde{D}, 0)$, the corresponding infinitesimal automorphism being $(\tilde{\tau}_\varepsilon, Id)$.

Lemma 3.1. *The natural action $\mu : G \times X \rightarrow X$ is a differential morphism. As well, for any closed point $x \in X$ the morphism $\mu_x : G = G \times \{x\} \subset G \times X \xrightarrow{\mu} X$ is also differential.*

Proof. We have to show that the action μ commutes with the respective infinitesimal automorphisms: $\mu \circ (\tilde{\tau}_\varepsilon, Id) = \tau_\varepsilon \circ \mu$. We prove it by means of the functor of points: Let $(g, x) \in G^\bullet(T) \times X^\bullet(T) = (G \times X)^\bullet(T)$, where T is a $k[\varepsilon]$ -scheme; we have

$$(\mu \circ (\tilde{\tau}_\varepsilon, Id))(g, x) = \mu(\tau_\varepsilon \cdot g, x) = (\tau_\varepsilon \cdot g)(x) = \tau_\varepsilon(g(x)) = \tau_\varepsilon(\mu(g, x)) = (\tau_\varepsilon \circ \mu)(g, x) .$$

The second statement may be proved in a similar way. \square

Theorem 3.2. *Let D be a vector field on a proper k -scheme X and let G be the associated group. For any closed point $x \in X$, the minimal tangent subscheme Y_x passing through x coincides with the closure of the orbit of x , that is,*

$$Y_x = \overline{G \cdot x}.$$

Proof. Let us consider the differential morphism $\mu_x : G \rightarrow X$, $g \mapsto g \cdot x$. By Proposition 1.4, $\mu_x^{-1}(Y_x)$ is a tangent subscheme of \overline{D} , which contains the identity of G . Since G is minimal, we conclude that $G \subseteq \mu_x^{-1}(Y_x)$, hence $\mu_x(G) = G \cdot x \subseteq Y_x$ and it follows that $\overline{G \cdot x} \subseteq Y_x$.

In order to show the reverse inclusion $Y_x \subseteq \overline{G \cdot x}$, it is enough to show that $\overline{G \cdot x}$ is a tangent subscheme of D , because Y_x is minimal. Let g_0 be the generic point of G and let $x_0 = \mu_x(g_0)$. It is clear that x_0 is the generic point of $\overline{G \cdot x} = \overline{\mu_x(G)}$. Let \mathfrak{p}_{x_0} be the maximal ideal of \mathcal{O}_{X,x_0} . The exact sequence

$$0 \longrightarrow \mathfrak{p}_{x_0} \longrightarrow \mathcal{O}_{X,x_0} \xrightarrow{\mu_x} \mathcal{O}_{G,g_0},$$

where the last morphism is differential, shows that \mathfrak{p}_{x_0} is a differential ideal of \mathcal{O}_{X,x_0} . Let \mathfrak{p} be the sheaf of ideals of the closed subscheme $\overline{G \cdot x}$. Since x_0 is the generic point of this subscheme, for any open set U in X we have

$$\mathfrak{p}(U) = \mathcal{O}_X(U) \cap \mathfrak{p}_{x_0}$$

(rigorously, $\mathfrak{p} = h^{-1}(\mathfrak{p}_{x_0})$ where $h : \mathcal{O}_X(U) \rightarrow \mathcal{O}_{X,x_0}$ is the natural morphism). It readily follows that $\mathfrak{p}(U)$ is a differential ideal of $\mathcal{O}_X(U)$. \square

Proposition 3.3. *Let $G \times X \rightarrow X$ be an action of a connected algebraic group G over an integral quasi-projective k -scheme X . There is a G -invariant dense open set U in X such that the geometric quotient $\pi : U \rightarrow U/G$ exists.*

We shall prove this result in the Appendix. Putting Theorem 3.2 and Proposition 3.3 together we obtain the following result.

Theorem 3.4. *Let D be a vector field on an integral projective k -scheme X and let G be the associated group. There exists a dense open set U in X such that:*

a) U is G -invariant and orbits of closed points in U are just minimal tangent subvarieties of the vector field D on U .

b) The geometric quotient $\pi : U \rightarrow Z = U/G$ exists. The sheaf \mathcal{O}_Z coincides with the sheaf of first integrals of D , that is, for any open set V in Z we have

$$\Gamma(V, \mathcal{O}_Z) = \{f \in \Gamma(\pi^{-1}V, \mathcal{O}_X) : Df = 0\}.$$

Proof. a) Let U be the open set whose existence states Proposition 3.3. The orbit of any closed point in U is a closed subset of U , since it is the fibre of $\pi : U \rightarrow U/G$ over a closed point of U/G . Applying Theorem 3.2 to the field D on U , we conclude that such orbits are just the minimal tangent subvarieties.

b) Since $\pi : U \rightarrow Z = U/G$ is a geometric quotient, the sheaf \mathcal{O}_Z coincides with the sheaf of G -invariant functions,

$$\Gamma(V, \mathcal{O}_Z) = \Gamma(\pi^{-1}V, \mathcal{O}_X)^G,$$

so that we have to show that a function $f \in \Gamma(\pi^{-1}V, \mathcal{O}_X)$ is G -invariant if and only if $Df = 0$.

Given a closed point $x \in \pi^{-1}V$, let us consider the minimal tangent subvariety $Y_x = G \cdot x$ of D passing through x . If f is G -invariant, then $f|_{Y_x}$ is constant, so that $(Df)(x) = (D(f|_{Y_x}))(x) = 0$.

Conversely, if $Df=0$, then we consider the ideal $I = (f - \lambda)$, where $\lambda := f(x)$. It is a differential ideal, hence it defines a tangent closed subscheme T of D . Since $x \in T$ and Y_x is minimal, we have $Y_x = G \cdot x \subseteq T$, hence $f = \lambda$ on $Y_x = G \cdot x$. Therefore f is constant on the orbits and we conclude that f is G -invariant. \square

Remark 3.5. The above theorem holds when X is not complete if D is assumed to be a fundamental vector field with respect to the action $\mu: \mathcal{G} \times X \rightarrow X$ of some algebraic group \mathcal{G} , i.e, $D = \mu_*(\tilde{D}, 0)$ for some right-invariant vector field \tilde{D} on \mathcal{G} . Recalling Definition 2.2, the **associated group** G to the vector field D is defined to be the minimal tangent subvariety of \tilde{D} passing through the identity of \mathcal{G} . The results of Sections 2 and 3 remain valid, so that the associated group G is a commutative connected algebraic subgroup of \mathcal{G} , and the minimal tangent subvarieties of D are the closure of the orbits of G on X .

Remark 3.6. Theorem 3.4 may be generalized to a distribution generated by global vector fields. Let X be an integral projective k -scheme and let L be a vector subspace of the space of all global vector fields on X .

A closed subscheme of X is said to be a **tangent subscheme of L** if it is a tangent subscheme of any $D \in L$. Recall that each vector field D on X corresponds with a right-invariant vector field \tilde{D} on $\mathbf{Aut} X$. Let \tilde{L} be the space of all right-invariant vector fields \tilde{D} on $\mathbf{Aut} X$ such that $D \in L$.

The results of Sections 2 and 3 (and their proofs) may be extended for the distribution L :

1. *The minimal tangent subscheme of \tilde{L} , passing through the identity of $\mathbf{Aut} X$, is a connected closed algebraic subgroup G of $\mathbf{Aut} X$.*
2. *If L is an abelian Lie algebra, then G is a commutative group.*
3. *The minimal tangent subscheme Y_x of L passing through a closed point $x \in X$ is the closure of the orbit, $Y_x = \overline{G \cdot x}$.*
4. *There exists a dense G -invariant open subset U in X such that:*
 - a) *Orbits of closed points in U are just minimal tangent subvarieties of L on U ;*
 - b) *The geometric quotient $\pi : U \rightarrow Z = U/G$ exists and the sheaf \mathcal{O}_Z coincides with the sheaf of first integrals of L , that is, for any open set V in Z we have*

$$\Gamma(V, \mathcal{O}_Z) = \{f \in \Gamma(\pi^{-1}V, \mathcal{O}_X) : Df = 0 \text{ for any } D \in L\}.$$

4. VECTOR FIELDS ON \mathbb{P}_n

The aim of this section is to calculate the associated algebraic group of any vector field on the complex projective space \mathbb{P}_n .

Lemma 4.1. *Let us consider the differential algebras*

$$\mathbb{C}[z_1, \dots, z_r], D = \sum \mu_i z_i \frac{\partial}{\partial z_i},$$

$$\mathbb{C}[z_0, z_1, \dots, z_r], D = \frac{\partial}{\partial z_0} + \sum_{i>0} \mu_i z_i \frac{\partial}{\partial z_i}$$

where $\mu_1, \dots, \mu_r \in \mathbb{C}$. If μ_1, \dots, μ_r are linearly independent over \mathbb{Q} , then any non-null differential ideal of these algebras contains some monomial $z_1^{a_1} \cdots z_r^{a_r}$.

Proof. The same argument holds in both algebras. It is easy to check that monomials $\lambda z_1^{a_1} \cdots z_r^{a_r}$ are the only eigenvectors of D . Let E_n be the vector subspace of all polynomials of degree $\leq n$ and note that $D(E_n) \subseteq E_n$ in both cases. Let I be a non-null differential ideal. It is clear that $I \cap E_n \neq 0$ when $n \gg 0$. Since the dimension of $I \cap E_n$ is finite, the linear map $D : I \cap E_n \rightarrow I \cap E_n$ has some eigenvector, hence I contains some monomial. \square

Corollary 4.2. *If μ_1, \dots, μ_r are linearly independent over \mathbb{Q} , then the differential algebras*

$$\begin{aligned} \mathbb{C}[z_1, \dots, z_r, (z_1 \cdots z_r)^{-1}], \quad D = \sum \mu_i z_i \frac{\partial}{\partial z_i}, \\ \mathbb{C}[z_0, z_1, \dots, z_r, (z_1 \cdots z_r)^{-1}], \quad D = \frac{\partial}{\partial z_0} + \sum_{i>0} \mu_i z_i \frac{\partial}{\partial z_i} \end{aligned}$$

have no non-trivial differential ideal.

Proposition 4.3. *If $\mu_1, \dots, \mu_r \in \mathbb{C}$ are linearly independent over \mathbb{Q} , then the following differential algebras of holomorphic functions on \mathbb{C} ,*

$$\mathbb{C}[e^{\mu_1 t}, \dots, e^{\mu_r t}, e^{-\sum \mu_i t}], \quad \mathbb{C}[t, e^{\mu_1 t}, \dots, e^{\mu_r t}, e^{-\sum \mu_i t}]$$

(endowed with the derivation $\frac{\partial}{\partial t}$) have no non-trivial differential ideal.

Proof. Let us consider the obvious differential epimorphisms

$$\begin{aligned} \mathbb{C}[z_1, \dots, z_r, \frac{1}{z_1 \cdots z_r}] &\longrightarrow \mathbb{C}[e^{\mu_1 t}, \dots, e^{\mu_r t}, e^{-\sum \mu_i t}], \\ \mathbb{C}[z_0, z_1, \dots, z_r, \frac{1}{z_1 \cdots z_r}] &\longrightarrow \mathbb{C}[t, e^{\mu_1 t}, \dots, e^{\mu_r t}, e^{-\sum \mu_i t}], \end{aligned}$$

where the algebras on the left are endowed with the derivations considered in Corollary 4.2. These epimorphisms are isomorphisms because both have null kernel by Corollary 4.2. \square

Remark 4.4. By the last isomorphism in the former proof, the holomorphic functions $t, e^{\mu_1 t}, \dots, e^{\mu_r t}$ are algebraically independent whenever μ_1, \dots, μ_r are linearly independent over \mathbb{Q} .

4.5. Let $M = (\lambda_{ij})$ be an $n \times n$ matrix with complex coefficients. Let us consider the **linear vector field**

$$D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$$

on \mathbb{C}^n . The corresponding infinitesimal automorphism τ_ε of \mathbb{C}^n is a $\mathbb{C}[\varepsilon]$ -linear transformation with matrix $Id + \varepsilon M \equiv e^{\varepsilon M}$. This automorphism induces an infinitesimal automorphism $\tilde{\tau}_\varepsilon$ of the full linear group Gl_n ,

$$(Gl_n)_{\mathbb{C}[\varepsilon]} \xrightarrow{\tilde{\tau}_\varepsilon} (Gl_n)_{\mathbb{C}[\varepsilon]} \quad g \mapsto \tau_\varepsilon \circ g,$$

whose corresponding vector field \tilde{D} on Gl_n is obviously right-invariant. Note that D is a fundamental vector field with respect to the natural action $\mu: Gl_n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$, since we have $\mu_*(\tilde{D}, 0) = D$ because $\mu \circ (\tilde{\tau}_\varepsilon, Id) = \tau_\varepsilon \circ \mu$.

According to Remark 3.5, the **associated group** G of the linear vector field D is defined to be the minimal tangent subvariety of \tilde{D} passing through the identity of Gl_n . We know that G is a commutative connected algebraic subgroup of Gl_n ,

and that the minimal tangent subvarieties of D are the closure of the orbits of G on \mathbb{C}^n .

Indeed, it may be proved that the associated group G of a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ coincides with the so-called differential Galois group of the linear differential system $z_i(t)' = \sum_j \lambda_{ij} z_j(t)$. See Note 4.7 below.

Let $\mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ be the algebra of holomorphic functions on \mathbb{C} generated by the coefficients of the matrix e^{Mt} and the function $\det(e^{-Mt})$. It is a differential algebra with the derivation $\frac{\partial}{\partial t}$.

Lemma 4.6. *The group G associated to a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ is*

$$G = \text{Spec } \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$$

where $M = (\lambda_{ij})$.

Proof. Let $Gl_n = \text{Spec } \mathbb{C}[x_{ij}, \det(x_{ij})^{-1}]$. The infinitesimal automorphism $\tilde{\tau}_\varepsilon$ is

$$\tilde{\tau}_\varepsilon((x_{ij})) = \tau_\varepsilon \circ (x_{ij}) = e^{M\varepsilon} \circ (x_{ij}) = (Id + \varepsilon M) \circ (x_{ij}) = (x_{ij}) + \varepsilon M \circ (x_{ij}),$$

so that the derivation \tilde{D} is (in matrix form)

$$(*) \quad (\tilde{D}x_{ij}) = M \circ (x_{ij}).$$

Let us consider the obvious epimorphism

$$\mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \longrightarrow \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$$

which transforms the coefficients of the universal matrix (x_{ij}) into the coefficients of the matrix e^{Mt} . Equality (*) states that it is a differential morphism when the first algebra is endowed with the derivation \tilde{D} and the second algebra is endowed with $\frac{\partial}{\partial t}$. Therefore $\text{Spec } \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ is a tangent subscheme of \tilde{D} . Clearly, this closed subscheme passes through the identity of Gl_n when $t = 0$. In order to show that it is the minimal tangent subscheme, it is enough to show that the differential algebra $\mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ has no non-trivial differential ideal.

After a linear coordinate change, we may assume that the matrix M is in Jordan form. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of M , then it is easy to check that

$$\mathbb{C}[e^{Mt}, \det(e^{-Mt})] = \mathbb{C}[\delta t, e^{\lambda_1 t}, \dots, e^{\lambda_n t}, e^{-\sum \lambda_i t}]$$

where $\delta = 0$ if M is diagonalizable and $\delta = 1$ otherwise.

Let μ_1, \dots, μ_r be a base of the \mathbb{Q} -vector space $\mathbb{Q}\lambda_1 + \dots + \mathbb{Q}\lambda_n \subset \mathbb{C}$, that is to say, $\mathbb{Q}\lambda_1 + \dots + \mathbb{Q}\lambda_n = \mathbb{Q}\mu_1 \oplus \dots \oplus \mathbb{Q}\mu_r$. It is easy to check that

$$\mathbb{C}[\delta t, e^{\lambda_1 t}, \dots, e^{\lambda_n t}, e^{-\sum \lambda_i t}] = \mathbb{C}[\delta t, e^{\mu_1 t}, \dots, e^{\mu_r t}, e^{-\sum \mu_j t}].$$

Now, by Proposition 4.3, we conclude that $\mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ has no non-trivial differential ideal. □

Note 4.7. Let us explain the relation of the associated group G with the Galois theory of differential equations.

We have shown in the former proof that $\mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ is a simple differential ring, i.e., it has no non-trivial differential ideal. According to ([10], Def. 1.15), this ring is the *Picard-Vessiot ring* of the differential equation $z' = Mz$, since e^M is a *fundamental matrix*. Let L be the field of fractions of the Picard-Vessiot ring,

which is named the *Picard–Vessiot field* of the equation $z' = Mz$ ([10], Def. 1.21). Now, by Lemma 4.6, L is the field of functions of the closed algebraic subgroup $(G, \tilde{D}) \subset (Gl_n, \tilde{D})$. The action of G on itself by right–translations induces an action of G on L by differential automorphisms. It is immediate to check that any G -invariant function is constant: $L^G = \mathbb{C}$. Then, by the Galois correspondence ([10], Prop. 1.34), we conclude that G is the group of all differential automorphisms of L , i.e., G is the *differential Galois group* of the equation $z' = Mz$.

The following theorem improves a statement of ([8], Prop. 3.27), about the differential Galois group of a linear differential system with constant coefficients.

Theorem 4.8. *The associated group of a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ on \mathbb{C}^n is*

$$G = \mathbb{G}_m^r \times \mathbb{G}_a^\delta$$

where r stands for the dimension of the \mathbb{Q} -vector space $\mathbb{Q}\lambda_1 + \dots + \mathbb{Q}\lambda_n \subset \mathbb{C}$ spanned by the eigenvalues of the matrix $M = (\lambda_{ij})$, and $\delta = 0$ when M is diagonalizable and $\delta = 1$ otherwise.

Proof. Again we put $Gl_n = \text{Spec } \mathbb{C}[x_{ij}, \det(x_{ij})^{-1}]$. The group law in Gl_n is determined by the coproduct

$$\mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \hookrightarrow \mathbb{C}[y_{ij}, \det(y_{ij})^{-1}] \otimes \mathbb{C}[z_{ij}, \det(z_{ij})^{-1}], \quad (x_{ij}) = (y_{ij}) \circ (z_{ij}).$$

The group law in $G = \text{Spec } \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ is defined by the induced coproduct in the quotient algebra $\mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \twoheadrightarrow \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$, that is,

$$\begin{aligned} \mathbb{C}[e^{Mt}, \det(e^{-Mt})] &\hookrightarrow \mathbb{C}[e^{Mu}, \det(e^{-Mu})] \otimes \mathbb{C}[e^{Mv}, \det(e^{-Mv})], \\ e^{Mt} &= e^{Mu} \circ e^{Mv} = e^{M(u+v)}. \end{aligned}$$

In other words, this coproduct takes each function $f(t)$ into $f(u + v)$.

As shown in the proof of Lemma 4.6, the coordinate ring of G has the form

$$\mathbb{C}[\delta t, e^{\mu_1 t}, \dots, e^{\mu_r t}, e^{-\sum \mu_j t}].$$

Recall that the functions $t, e^{\mu_1 t}, \dots, e^{\mu_r t}$ are algebraically independent (Remark 4.4). With the coproduct $f(t) \mapsto f(u + v)$, the Hopf algebra

$$\mathbb{C}[\delta t, e^{\mu_1 t}, \dots, e^{\mu_r t}, e^{-\sum \mu_j t}]$$

has an obvious decomposition as a tensor product of Hopf algebras:

$$\mathbb{C}[\delta t, e^{\mu_1 t}, \dots, e^{\mu_r t}, e^{-\sum \mu_j t}] = \mathbb{C}[\delta t] \otimes \mathbb{C}[e^{\mu_1 t}, e^{-\mu_1 t}] \otimes \dots \otimes \mathbb{C}[e^{\mu_r t}, e^{-\mu_r t}],$$

hence $G = \mathbb{G}_a^\delta \times \mathbb{G}_m \times \dots \times \mathbb{G}_m$. □

Remark 4.9. By Theorem 3.4a, there exists a G -invariant dense open set U in \mathbb{C}^n such that the orbits of G in U coincide with the minimal tangent subvarieties of D in U . The isotropy subgroup of any point of U is the identity subgroup. Let us give a (summarized) proof of this fact: Every (flat) family of subgroups of $G = \mathbb{G}_m^r \times \mathbb{G}_a^\delta$ is a constant family, hence all the points of U have the same isotropy subgroup (generically); since G acts faithfully on U , we conclude that such a subgroup is the identity. Therefore, the orbits of G in U have the same dimension as G , that is to say:

Minimal tangent subvarieties of a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ on \mathbb{C}^n have dimension $r + \delta$ (generically).

By 3.4b, the field of rational functions on the quotient variety $Z = U/G$ coincides with the field of rational functions on \mathbb{C}^n which are first integrals of D . Since $\dim Z = \dim U - \dim G = n - r - \delta$, we conclude that:

The field of all rational functions on \mathbb{C}^n which are first integrals of a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ has transcendence degree $n - r - \delta$.

This field may be computed by methods of Linear Algebra [3].

4.10. Let \mathbb{P}_n be the n -dimensional projective space and let $\pi : \mathbb{C}^{n+1} \rightarrow \mathbb{P}_n$ be the natural projection. It is well known that any vector field D on \mathbb{P}_n is the projection by π of some linear vector field $D_0 = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ on \mathbb{C}^{n+1} . Moreover, such a linear field is unique up to the addition of a vector field proportional to $\sum z_i \frac{\partial}{\partial z_i}$. These facts follow readily from the standard exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_n} \rightarrow \text{Hom}_{\mathcal{O}_{\mathbb{P}_n}}(\mathcal{O}_{\mathbb{P}_n}(-1), \mathcal{O}_{\mathbb{P}_n}^{n+1}) \rightarrow \mathcal{D}_{\mathbb{P}_n} \rightarrow 0,$$

where $\mathcal{O}_{\mathbb{P}_n}(-1)$ is the sheaf of sections of the tautological line bundle on \mathbb{P}_n , $\mathcal{D}_{\mathbb{P}_n}$ is the sheaf of vector fields on \mathbb{P}_n and $\text{Hom}_{\mathcal{O}_{\mathbb{P}_n}}(\mathcal{O}_{\mathbb{P}_n}(-1), \mathcal{O}_{\mathbb{P}_n}^{n+1})$ is the sheaf of π -projectable vector fields.

Theorem 4.11. *Let $D = \pi_*(\sum \lambda_{ij} z_i \frac{\partial}{\partial z_i})$ be a vector field on \mathbb{P}_n . Its associated group is*

$$G = \mathbb{G}_m^s \times \mathbb{G}_a^\delta$$

where s stands for the dimension of the minimal \mathbb{Q} -affine subspace of \mathbb{C} containing all the eigenvalues of the matrix (λ_{ij}) , and $\delta = 0$ when such a matrix is diagonalizable and $\delta = 1$ otherwise.

Proof. Let $D_0 = \sum \lambda_{ij} z_i \frac{\partial}{\partial z_i}$. After the addition of a vector field proportional to $\sum z_i \frac{\partial}{\partial z_i}$, we may assume that the matrix $M = (\lambda_{ij})$ has the eigenvalue 0. In such case the minimal \mathbb{Q} -affine subspace containing the eigenvalues of (λ_{ij}) is a \mathbb{Q} -vector space. By Theorem 4.8, the group associated to D_0 is

$$G_0 = \mathbb{G}_m^s \times \mathbb{G}_a^\delta.$$

Let $0 \neq v \in \mathbb{C}^{n+1}$ be an eigenvector of M of eigenvalue 0. Note that v is a fixed point of the infinitesimal automorphism τ_ε^0 (corresponding to D_0) since $\tau_\varepsilon^0(v) = e^{\varepsilon M}(v) = (Id + \varepsilon M)(v) = v$.

Let H_v be the stabilizer of v , which is a closed algebraic subgroup of Gl_{n+1} . The functor of points of H_v is

$$H_v^\bullet(T) = \{g \in Gl_{n+1}^\bullet(T) : g(v) = v\}.$$

We have that H_v is a tangent subvariety of (Gl_{n+1}, \tilde{D}_0) , since

$$g \in H_v^\bullet(T) \Rightarrow \tilde{\tau}_\varepsilon^0(g) = \tau_\varepsilon^0 \circ g \in H_v^\bullet(T).$$

Since G_0 is minimal we obtain that $G_0 \subseteq H_v$.

Analogously, denoting by H_p the stabilizer of $p = \pi(v) \in \mathbb{P}_n$ with respect to the action of $PGL_{n+1} = \mathbf{Aut} \mathbb{P}_n$, we may prove that H_p is a tangent subvariety of (PGL_{n+1}, \tilde{D}) and then $G \subseteq H_p$.

Now, it is immediate that the natural epimorphism $Gl_{n+1} \rightarrow PGL_{n+1}$ induces a differential isomorphism $(H_v, \tilde{D}_0) \rightarrow (H_p, \tilde{D})$. Via this isomorphism, we conclude that $G_0 = G$. □

Remark 4.12. The same arguments used in Remark 4.9 show that:

Minimal tangent subvarieties of a vector field D on \mathbb{P}_n have dimension $s + \delta$ (generically).

The field of all rational functions on \mathbb{P}_n which are first integrals of a vector field has transcendence degree $n - s - \delta$. (The case $n = 2$ is well known; see [7], pp. 12-16.)

5. A COUNTEREXAMPLE

Without the hypothesis of X being complete, it does not follow the existence, for any vector field D on X , of a dense open set U and a projection $\varphi : U \rightarrow Z$ whose fibres are the minimal tangent subvarieties of D in U .

As a counterexample, let us consider the field

$$D = z_1 z_4 \frac{\partial}{\partial z_4} + z_2 z_5 \frac{\partial}{\partial z_5} + z_3 z_6 \frac{\partial}{\partial z_6}$$

on $X = \mathbb{C}^6$. It is clear that D is tangent to any 3-plane $z_1 = \lambda_1, z_2 = \lambda_2, z_3 = \lambda_3$. On these planes D is a linear vector field with associated group \mathbb{G}_m^r , where r is the dimension of the \mathbb{Q} -vector space $\mathbb{Q}\lambda_1 + \mathbb{Q}\lambda_2 + \mathbb{Q}\lambda_3$ (see Theorem 4.8). In each 3-plane, the minimal tangent subvarieties generically have dimension r (Remark 4.9). Now, points $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ with $r = 3$ form a dense set in \mathbb{C}^3 (and so do points with $r = 2$). Therefore, minimal tangent subvarieties of dimension 3, as well as those of dimension 2, form a dense subset of \mathbb{C}^6 . This fact prevents the existence of such projection $\varphi : U \rightarrow Z$, due to the semicontinuity of the dimension of the fibres of any algebraic morphism.

6. THE CASE OF POSITIVE CHARACTERISTIC

In this section, k denotes a field of characteristic $p > 0$. Let $\mathbb{G}_a = \text{Spec } k[t]$ be the additive line. The subscheme $G := \text{Spec } k[t]/(t^p)$ is a finite infinitesimal subgroup of \mathbb{G}_a .

Proposition 6.1. *Let X be a k -scheme. Each vector field D on X defines an action $\mu : G \times_k X \rightarrow X$,*

$$\mu^* : \mathcal{O}_X \rightarrow \mathcal{O}_X[t]/(t^p), \quad \mu^*(f) = \sum_{r=0}^{p-1} \frac{1}{r!} D^r(f) t^r .$$

Conversely, any action $\mu : G \times_k X \rightarrow X$ is defined by a unique vector field D on X .

The proof of Proposition 6.1 is a simple exercise.

Let D be a vector field on X and let $\mu : G \times_k X \rightarrow X$ be the corresponding action. If Y is a tangent subscheme of (X, D) , then the corresponding action $G \times_k Y \rightarrow Y$ is the restriction to Y of the action of G on X . Therefore, a closed subscheme Y of X is tangent to D if and only if it is G -invariant. We conclude that the minimal tangent subscheme passing through a closed point x is the orbit $G \cdot x =$ scheme-theoretic image of $G \times x \subseteq G \times X \xrightarrow{\mu} X$, that is to say, $G \cdot x$ is the closed subscheme of X defined by the ideal of all functions f such that $f(x) = Df(x) = \dots = D^{p-1}f(x) = 0$. Note that each orbit has a unique point, but it is not a reduced scheme in general.

APPENDIX A. QUOTIENTS BY ALGEBRAIC GROUPS

Theorem A.1 (Rosenlicht [11]). *Let $\mu: G \times_k X \rightarrow X$ be an action of an affine algebraic group G on an integral quasi-projective variety X . There exists a G -invariant dense open subset $U \subseteq X$ such that the geometric quotient $U \rightarrow U/G$ exists.*

The purpose of this Appendix is to extend Rosenlicht's result to the case of an algebraic group G non-necessarily affine.

Quotients by abelian varieties. Let $\mu: A \times_k X \rightarrow X$ be an action of an abelian variety A on an integral quasi-projective variety X .

Lemma A.2. *The action $\mu: A \times_k X \rightarrow X$, $(a, x) \mapsto a \cdot x$, and the morphism $\phi: A \times_k X \rightarrow X \times_k X$, $(a, x) \mapsto (a \cdot x, x)$, are projective morphisms.*

Proof. The isomorphism $\varphi: A \times X \rightarrow A \times X$, $(a, x) \mapsto (a, a \cdot x)$, makes commutative the triangle

$$\begin{array}{ccc} A \times X & \xrightarrow{\varphi} & A \times X \\ & \mu \searrow & \swarrow p_2 \\ & & X \end{array}$$

Since any abelian variety is projective, the map $p_2: A \times X \rightarrow X$ is a projective morphism and then the above commutative triangle implies that $\mu: A \times X \rightarrow X$ is also a projective morphism. Finally, $\phi = \mu \times p_2$ is a projective morphism because μ and p_2 are also. \square

Let R be the image of the map $\phi: A \times_k X \rightarrow X \times_k X$, $(a, x) \mapsto (a \cdot x, x)$, that is to say, R is the equivalence relation defined by the action of A over X . By the previous lemma, R is a closed subset of $X \times_k X$. We shall consider R as a closed subscheme of $X \times_k X$ with the reduced structure.

Lemma A.3. *The projection $p_1: R \rightarrow X$ is a projective morphism.*

Proof. Since $\phi: A \times X \rightarrow R$ is surjective and the composition morphism $\mu = p_1 \circ \phi: A \times X \rightarrow R \rightarrow X$ is proper, it is easy to check that the valuative criterion of properness ([6], II, Th. 4.7) holds for the morphism $p_1: R \rightarrow X$.

Moreover, $R \subseteq X \times X \xrightarrow{p_1} X$ is a quasi-projective morphism, hence we conclude that $p_1: R \rightarrow X$ is a projective morphism. \square

Given an A -stable open subset U of X , we write $R_U := p_1^{-1}(U)$, i.e., R_U is the equivalence relation defined by the action of A over U .

Lemma A.4. *There exists an A -invariant dense open subset U of X such that $p_1: R_U \rightarrow U$ is a flat morphism.*

Proof. By the semicontinuity character of the Hilbert polynomial of the fibres of a projective morphism $p_1: R \rightarrow X$, we have that the subset U of all points $x \in X$, whose fibre $p_1^{-1}(x)$ has the same Hilbert polynomial than the fibre of the generic point, is a dense open subset of X . This open subset U is invariant by the action of A over X , since the projection $p_1: R \rightarrow X$ is an A -equivariant morphism (the action of A over R is defined by the formula $a \cdot (x_1, x_2) = (a \cdot x_1, x_2)$).

Finally, since the fibres of $p_1: R_U \rightarrow U$ have the same Hilbert polynomial, we conclude (see [6], III, Th. 9.9) that $p_1: R_U \rightarrow U$ is a flat morphism. \square

Theorem A.5. *Let $\mu: A \times_k X \rightarrow X$ be an action of an abelian variety A on an integral quasi-projective variety X . There exists an A -invariant dense open subset U of X such that the geometric quotient $\pi: U \rightarrow U/A$ exists.*

Proof. By the previous lemmas, there exists an A -invariant dense open subset U of X such that $p_1: R_U \rightarrow U$ is a projective flat morphism. By a theorem of Grothendieck ([5], V, Th. 7.1), there exists the quotient map $\pi: U \rightarrow U/A$. \square

The general case.

Theorem A.6. *Let $\mu: G \times_k X \rightarrow X$ be an action of a connected smooth group G on an integral quasi-projective variety X . There exists a G -invariant dense open subset $U \subseteq X$ such that the geometric quotient $U \rightarrow U/G$ exists.*

Proof. By the structure theorem of algebraic groups [12], there exists a normal affine subgroup G_0 of G such that $A = G/G_0$ is an abelian variety. By Rosenlicht's result, there exists a G_0 -invariant dense open subset $U_0 \subseteq X$ such that the geometric quotient $\pi: U_0 \rightarrow U_0/G_0$ exists. Taking $G \cdot U_0$ instead of U_0 , we may assume that U_0 is G -invariant. By Theorem A.5, there exists an A -invariant dense open subset V in U_0/G_0 such that the geometric quotient $V \rightarrow V/A$ exists. Then $U := \pi^{-1}V$ is the desired open set, since $U/G = (U/G_0)/A = V/A$. \square

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