TANGENT ALGEBRAIC SUBVARIETIES OF VECTOR FIELDS

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ABSTRACT. An algebraic commutative group \( G \) is associated to any vector field \( D \) on a complete algebraic variety \( X \). The group \( G \) acts on \( X \) and its orbits are the minimal subvarieties of \( X \) which are tangent to \( D \). This group is computed in the case of a vector field on \( \mathbb{P}^n \).

INTRODUCTION

Let \( X \) be an algebraic variety over an algebraically closed field \( k \) of characteristic 0 and let \( D \) be a vector field on \( X \). A closed subvariety \( Y \) of \( X \), defined by a sheaf of ideals \( I \) of \( \mathcal{O}_X \), is said to be a tangent subvariety of \( D \) if \( D(I) \subseteq I \). This condition implies that \( D \) induces a derivation of the sheaf \( \mathcal{O}_Y = \mathcal{O}_X/I \), so that it defines a vector field on \( Y \). The aim of this paper is to study the structure of the family of tangent subvarieties of a vector field \( D \). At first sight it seems reasonable to conjecture that minimal tangent subvarieties of \( D \) form an algebraic family. Precisely:

Question: Does there exist a dense open set \( U \) in \( X \) and a surjective morphism of \( k \)-schemes \( \pi : U \to Z \) whose fibres are just the minimal tangent subvarieties of \( D \) in \( U \)?

Unfortunately, the answer is negative in general (we shall give a counterexample in Section 5). However, it is affirmative in an interesting case, that of global vector fields on a complete algebraic variety. For these fields we prove the following result:

**Theorem.** Let \( D \) be a vector field on an integral projective variety \( X \). There exists a connected commutative algebraic subgroup \( G \subseteq \text{Aut} X \) and a \( G \)-invariant dense open set \( U \) in \( X \) such that:

a) Minimal tangent subvarieties of \( D \) on \( U \) are just orbits of \( G \) in \( U \).

b) The quotient variety \( \pi : U \to Z = U/G \) exists. Hence, the fibres of \( \pi \) are the minimal tangent subvarieties of \( D \) in \( U \).

c) The sheaf \( \mathcal{O}_Z \) coincides with the sheaf of first integrals of \( D \); that is to say, for any open set \( V \) in \( Z \), we have

\[
\Gamma(V, \mathcal{O}_Z) = \{ f \in \Gamma(\pi^{-1}V, \mathcal{O}_X) : Df = 0 \}.
\]

This theorem may be extended to a distribution on \( X \) generated by global vector fields (see 3.6).

The completeness of \( X \) is used to assure the existence of the algebraic group \( \text{Aut} X \). In such a case, the Lie algebra of all vector fields to \( X \) is canonically isomorphic to the Lie algebra of the group \( \text{Aut} X \). Hence, the vector field \( D \)
corresponds to some element $\tilde{D}$ in the Lie algebra of $\text{Aut} \, X$, that is to say, $D$ is a fundamental vector field with respect to the action of $\text{Aut} \, X$ on $X$. In fact, the former theorem holds when $X$ is not complete, if $D$ is assumed to be a fundamental field with respect to the action on $X$ of some algebraic group.

The above mentioned commutative group $G$ is defined to be the minimal tangent subvariety of $\tilde{D}$ through the identity of $\text{Aut} \, X$.

We determine the group $G$ in the case of vector fields on the complex projective space $\mathbb{P}_n$. It is well known that any vector field on $\mathbb{P}_n$ may be expressed as $D = \pi^*(\sum \lambda_{ij} z_j \frac{\partial}{\partial z_i})$, where $\pi: \mathbb{C}^{n+1} \to \mathbb{P}_n$ is the natural projection. The group $G$ is determined by two integers $s, \delta$ associated to $D$ as follows. Let us consider the field $\mathbb{C}$ as an infinite dimensional affine space over $\mathbb{Q}$; then $s$ is the dimension of the minimal affine $\mathbb{Q}$–subspace of $\mathbb{C}$ containing all the eigenvalues of the matrix $(\lambda_{ij})$. The integer $\delta$ is 0 when the matrix $(\lambda_{ij})$ is diagonalizable and 1 otherwise. We prove the following result:

**Theorem.** The group $G$ associated to a vector field $D = \pi^*(\sum \lambda_{ij} z_j \frac{\partial}{\partial z_i})$ on the complex projective space $\mathbb{P}_n$ is

$$G = (\mathbb{G}_m)^s \times (\mathbb{G}_a)^\delta.$$ 

($\mathbb{G}_m$ and $\mathbb{G}_a$ are the multiplicative and additive lines respectively.)

**Corollary.** The dimension of the minimal tangent subvarieties of $D$ is $s+\delta$ (generically). The field of meromorphic functions on $\mathbb{P}_n$ which are first integrals of $D$ has transcendence degree equal to $n - s - \delta$.

1. Preliminaries

From now on, $k$ will be an algebraically closed field of characteristic 0 and $X$ will be a $k$–scheme.

1.1. **Differential algebras.** Let us recall some elementary facts about differential algebras. A differential $k$–algebra is a $k$–algebra $A$ endowed with a $k$–linear derivation $D : A \to A$. An ideal $I$ of $A$ is said to be a differential ideal if $D(I) \subseteq I$.

**Proposition 1.1.** The nilradical of any differential algebra $A$ is a differential ideal.

**Proof.** If $a^n = 0$, applying $D^n$ we obtain $n!(Da)^n + ba = 0$, so that $Da$ is nilpotent. □

**Proposition 1.2.** Any minimal prime ideal of a differential noetherian $k$–algebra is a differential ideal.

**Proof.** Replacing $A$ by $A/\text{rad} \, A$, we may assume that $\text{rad} \, A = 0$. In such case, $0 = p_1 \cap \cdots \cap p_r$, where $p_1, \ldots, p_r$ are the minimal prime ideals of $A$. Let $0 \neq b \in p_2 \cap \cdots \cap p_r$. If $a \in p_1$, then $ab = 0$ because $ab \in p_1 \cap \cdots \cap p_r = 0$. Hence $(Da)b = -aDb$, so that both terms are 0, since $aDb \in p_1$ and $bDa \in p_2 \cap \cdots \cap p_r$. Finally, from $(Da)b = 0$ it follows that $Da \in p_1$ because $b \notin p_1$. Therefore $Dp_1 \subseteq p_1$. □
1.2. **Functor of points.** The functor of points of $X$ is defined to be the following contravariant functor on the category of $k$–schemes

$$X^\bullet(T) := \text{Hom}_k(T, X).$$

For any two $k$–schemes $X, Y$ we have (Yoneda’s lemma)

$$\text{Hom}_k(X, Y) = \text{Hom}_{\text{funct.}}(X^\bullet, Y^\bullet).$$

In particular, in order to define a morphism of $k$–schemes $X \to Y$ it is enough to construct a morphism of functors $X^\bullet \to Y^\bullet$. We shall use this fact later on.

Elements of $X^\bullet(T)$, that is to say, morphisms of $k$–schemes $x : T \to X$, are said to be $T$–valued points (or $T$–points) of $X$. Moreover, we use the following convention: Given a $T$-point $x : T \to X$ and a morphism of $k$–schemes $T' \to T$, the composition $T' \to T \xrightarrow{x} X$ will be denoted also by $x$, that is to say, we do not change the notation of a point after a basis change.

1.3. **Infinitesimal automorphisms.** Let $k[\varepsilon] = k[x]/(x^2)$ and let $i : X \hookrightarrow X_{k[\varepsilon]} = X \times_k k[\varepsilon]$ be the closed immersion defined by $\varepsilon = 0$.

An infinitesimal automorphism of $X$ is defined to be an automorphism of $k[\varepsilon]$–schemes $\tau_\varepsilon : X_{k[\varepsilon]} \to X_{k[\varepsilon]}$ satisfying the infinitesimal condition

$$(\tau_\varepsilon)_\varepsilon = 0 = \text{id}_X,$$

that is to say, $\tau_\varepsilon \circ i = i$.

Any vector field $D$ on $X$ determines an infinitesimal automorphism $\tau_\varepsilon$ of $X$ given by the following morphism of $k[\varepsilon]$–algebras:

$$\tau_\varepsilon : \mathcal{O}_X[\varepsilon] \longrightarrow \mathcal{O}_X[\varepsilon], \quad a \mapsto \tau_\varepsilon(a) = a + \varepsilon Da$$

and, conversely, any infinitesimal automorphism is clearly defined by a unique vector field.

1.4. **Tangent subschemes.** Let $D$ be a vector field on $X$. A closed subscheme $Y$ of $X$, defined by a sheaf of ideals $I$ of $\mathcal{O}_X$, is said to be a tangent subscheme of $D$ if $D(I) \subseteq I$. This condition implies that $D$ induces a derivation of the sheaf $\mathcal{O}_Y = \mathcal{O}_X/I$, so that it defines a vector field on $Y$.

A closed subscheme $Y$ of $X$ is a tangent subscheme of $D$ if and only if its functor of points $Y^\bullet$ is stable under the corresponding infinitesimal automorphism $\tau_\varepsilon$, that is to say, for any $k$–scheme $T$ and any point $x \in X^\bullet(T)$, we have

$$x \in Y^\bullet(T) \Rightarrow \tau_\varepsilon(x) \in Y^\bullet_{k[\varepsilon]}(T_{k[\varepsilon]}) = Y^\bullet(T_{k[\varepsilon]})$$

where $\tau_\varepsilon(x)$ stands for the composition $T_{k[\varepsilon]} \longrightarrow X_{k[\varepsilon]} \longrightarrow X_{k[\varepsilon]}$.

Given a closed point $x \in X$, we denote by $Y_x$ the minimal tangent subscheme of $D$ passing through $x$. Remark that $Y_{x'} \subseteq Y_x$ whenever $x' \in Y_x$.

From Propositions [1.1] and [1.2] it follows that $Y_x$ is reduced and irreducible.

1.5. **Zeros of a vector field.** Let $D$ be a vector field on $X$ and let us consider the natural sheaf of ideals $I$ of $\mathcal{O}_X$. The subscheme of zeros of $D$ is the closed subscheme $Z_D$ of $X$ defined by the sheaf of ideals $I$.

The subscheme of zeros $Z_D$ may be defined in terms of the infinitesimal automorphism $\tau_\varepsilon$ corresponding to the vector field $D$. In fact, the functor of $\tau_\varepsilon$–invariant points of $X$ is representable by the subscheme of zeros $Z_D$ as follows.
Lemma 1.3. We have
\[ Z^n_D(T) = \{ x \in X^\bullet(T) : \tau_\varepsilon(x) = x \} \]
where the equality \( \tau_\varepsilon(x) = x \) states the coincidence of the composition morphism \( T_{k[\varepsilon]} \xrightarrow{\varepsilon} X_{k[\varepsilon]} \xrightarrow{\tau_\varepsilon} X_{k[\varepsilon]} \) with the morphism \( T_{k[\varepsilon]} \xrightarrow{\varepsilon} X_{k[\varepsilon]} \).

Proof. A morphism \( x : T \rightarrow X \) factors through \( Z_D \) if and only if the composition \( \Omega_X \xrightarrow{D} \mathcal{O}_X \xrightarrow{x} \mathcal{O}_T \) vanishes. Now, this condition is equivalent to the coincidence of the two following morphisms:
\[
\mathcal{O}_X[\varepsilon] \xrightarrow{\tau_\varepsilon} \mathcal{O}_X[\varepsilon] \xrightarrow{x} \mathcal{O}_T[\varepsilon] \quad a \mapsto a + \varepsilon Da \mapsto x(a) + \varepsilon x(Da),
\]
\[
\mathcal{O}_X[\varepsilon] \xrightarrow{x} \mathcal{O}_T[\varepsilon] \quad a \mapsto x(a) .
\]

\[ \square \]

1.6. Differential morphisms. Let \((A, D)\) and \((A', D')\) be two differential \(k\)-algebras. A morphism of \(k\)-algebras \( \varphi : A \rightarrow A' \) is said to be differential if it commutes with the given derivations: \( \varphi(Da) = D'(\varphi(a)) \).

Now let \( X \) and \( X' \) be two \( k \)-schemes and let \( D \) and \( D' \) be vector fields on \( X \) and \( X' \) respectively. A morphism of \( k \)-schemes \( \varphi : X' \rightarrow X \) is said to be differential when the morphism of \( k \)-algebras \( \varphi : \mathcal{O}_X(U) \rightarrow \mathcal{O}_{X'}(V) \) is differential, for any pair of open sets \( U \subseteq X \) and \( V \subseteq \varphi^{-1}(U) \).

Let \( \tau_\varepsilon \) and \( \tau'_\varepsilon \) be the infinitesimal automorphisms corresponding to \( D \) and \( D' \) respectively. A morphism of \( k \)-schemes \( \varphi : X' \rightarrow X \) is differential if and only if the morphism \( \varphi : X'_{k[\varepsilon]} \rightarrow X_{k[\varepsilon]} \) satisfies \( \varphi \circ \tau'_\varepsilon = \tau_\varepsilon \circ \varphi \).

The following statement is trivial.

Proposition 1.4. Let \( \varphi : (X', D') \rightarrow (X, D) \) be a differential morphism of \( k \)-schemes. If \( Y \) is a tangent subscheme of \((X, D)\), then \( \varphi^{-1}(Y) = Y \times_X X' \) is a tangent subscheme of \((X', D')\).

2. Algebraic group associated to a vector field

In this section \( X \) will be a proper \( k \)-scheme.

Definition 2.1. Let us consider the functor of automorphisms of \( X \), defined on the category of \( k \)-schemes,
\[ F(T) := \text{Aut}_T(X \times_k T) . \]
This functor is representable (see [1]) by an algebraic group \textbf{Aut} \( X \), named scheme of automorphisms of \( X \) ([1]), that is to say,
\[ (\text{Aut} \ X)^\bullet(T) = \text{Aut}_T(X \times_k T) . \]

Now let \( D \) be a vector field on \( X \) and let \( \tau_\varepsilon \) be the corresponding infinitesimal automorphism. This automorphism induces an infinitesimal automorphism \( \tilde{\tau}_\varepsilon \) of \textbf{Aut} \( X \). We construct this automorphism \( \tilde{\tau}_\varepsilon : (\text{Aut} \ X)_{k[\varepsilon]} \rightarrow (\text{Aut} \ X)_{k[\varepsilon]} \) by means of the functor of points: For any \( k[\varepsilon] \)-scheme \( T \) we define
\[ \tilde{\tau}_\varepsilon : \text{Aut}_T(X_{k[\varepsilon]} \times_{k[\varepsilon]} T) \rightarrow \text{Aut}_T(X_{k[\varepsilon]} \times_{k[\varepsilon]} T), \quad g \mapsto \tau_\varepsilon \circ g . \]
Note that \( \tilde{\tau}_\varepsilon \) satisfies the infinitesimal condition \( (\tilde{\tau}_\varepsilon)_{|_{\varepsilon=0}} = \text{id}_{\text{Aut} \ X} \).
The infinitesimal automorphism \( \tilde{\tau}_c \) corresponds to a certain tangent field \( \tilde{D} \) on \( \text{Aut} \, X \). Note that, by definition, \( \tilde{\tau}_c \) commutes with right translations in \( \text{Aut} \, X \), so that the field \( \tilde{D} \) is invariant under right translations.

Therefore, any vector field \( D \) on a proper \( k \)-scheme \( X \) has a canonically associated vector field \( \tilde{D} \) on the scheme of automorphisms \( \text{Aut} \, X \), which is invariant under right translations.

Moreover, it may be proved that the map \( D \mapsto \tilde{D} \) defines an isomorphism of the Lie algebra of vector fields on \( X \) onto the Lie algebra of the group \( \text{Aut} \, X \) (we shall not use this fact).

**Definition 2.2.** Let \( D \) be a vector field on a proper \( k \)-scheme \( X \) and let \( \tilde{D} \) be the corresponding vector field on the scheme of automorphisms \( \text{Aut} \, X \), invariant under right translations. The **associated group** of \( D \) is defined to be the minimal tangent subscheme \( G \) of \( \tilde{D} \) passing through the identity of \( \text{Aut} \, X \). We shall prove that \( G \) is a commutative algebraic subgroup of \( \text{Aut} \, X \).

**Proposition 2.3.** \( G \) is a connected algebraic closed subgroup of \( \text{Aut} \, X \).

**Proof.** By definition \( G \) is a closed subscheme of \( \text{Aut} \, X \). Moreover, \( G \) is integral (hence connected) by Propositions 1.1 and 1.2. Finally, we have to show that the product \( G \times G \rightarrow \text{Aut} \, X \) factors through \( G \) and that the inverse morphism \( \text{Aut} \, X \rightarrow \text{Aut} \, X \), \( g \mapsto g^{-1} \) takes \( G \) into itself. Since \( G \) is reduced, it is enough to prove both statements in the case of closed points; that is to say, it is enough to show that \( G^*(k) \) is a subgroup of \( (\text{Aut} \, X)^*(k) \).

Let \( g \in G^*(k) \). Since \( G \) is the minimal tangent subvariety of \( \tilde{D} \) through the identity and \( \tilde{D} \) is invariant under right translations, it follows that \( G \cdot g \) is the minimal tangent subvariety of \( \tilde{D} \) passing through \( g \). Hence \( G \cdot g \subseteq G \), because \( G \) also passes through \( g \). This inclusion is not strict, since otherwise, multiplying at right by \( g^n \), we would obtain an infinite decreasing sequence of closed sets \( G \cdot g^{n+1} \subset G \cdot g^n \), so contradicting the noetherian character of \( \text{Aut} \, X \). Hence \( G \cdot g = G \) and then \( G^*(k) \cdot g = G^*(k) \) for any \( g \in G^*(k) \). It follows readily that \( G^*(k) \) is a group. \( \square \)

**Proposition 2.4.** \( G \) is commutative.

**Proof.** Let us consider the infinitesimal automorphism

\[ \sigma : (\text{Aut} \, X)_{k[e]} \rightarrow (\text{Aut} \, X)_{k[e]} \]

defined, in terms of the functor of points, by the formula \( \sigma(g) = \tau_c \circ g \circ \tau_c^{-1} \). This infinitesimal automorphism corresponds to a certain vector field on \( \text{Aut} \, X \). Let \( H \) be the corresponding closed subscheme of zeros. By Lemma 1.3 the functor of points of \( H \) is

\[ H^*(T) = \{ g \in (\text{Aut} \, X)^*(T) : \sigma(g) = g \} = \{ g \in (\text{Aut} \, X)^*(T) : \tau_c \circ g = g \circ \tau_c \} . \]

Remark that \( H^*(T) \) is a subgroup of \( (\text{Aut} \, X)^*(T) \), hence \( H \) is a closed algebraic subgroup of \( \text{Aut} \, X \). Let us consider the center \( C \) of \( H \), which is a commutative closed subgroup. The functor of points of the center is

\[ C^*(T) = \{ c \in H^*(T) : c \circ h = h \circ c \ \text{for any} \ h \in H^*(T') \ \text{and any} \ T' \rightarrow T \} \]

(we refer to [2], II, §1 3.7 for the existence of the center). Let us prove that \( C \) is a tangent subscheme of \( \tilde{D} \). According to 1.3 we have to show that the functor \( C^* \) is stable by the automorphism \( \tilde{\tau}_c \): If \( c \in C^*(T) \), let us prove that
\[ \tau_\varepsilon(c) = \tau_\varepsilon \circ c \in C^*(T_{k[x]}) \]; in fact, for any morphism \( T' \to T_{k[x]} \) and any point \( h \in H^\bullet(T') \) we have
\[
(\tau_\varepsilon \circ c) \circ h = \tau_\varepsilon \circ (c \circ h) = \tau_\varepsilon \circ (h \circ c) = h \circ (\tau_\varepsilon \circ c).
\]

Finally, \( G \) is contained in \( C \), because \( C \) is a tangent subscheme of \( \hat{D} \) passing through the identity and \( G \) is minimal. Since \( C \) is commutative, so is \( G \). \( \square \)

**Note 2.5.** A more general result than Proposition 2.4 was proved by Chevalley \[1\] in the realm of Lie algebras. In fact, if \( L \) is a Lie subalgebra of the Lie algebra of an algebraic group and \( L_{\text{alg}} \) stands for the minimal algebraic Lie subalgebra (in the sense that it is the Lie algebra of an algebraic subgroup) containing \( L \), then \[1\], Chap. II, Th. 13, states that \( [L, L] = [L_{\text{alg}}, L_{\text{alg}}] \); hence \( L_{\text{alg}} \) is abelian whenever \( L \) is. In particular, if \( L = \langle D \rangle \), then \( L_{\text{alg}} \) is an abelian Lie subalgebra.

Let us consider the structure of the associated algebraic group \( G \). According to the fundamental structure theorem of algebraic groups (see \[12\] or \[13\]), \( G \) has an affine connected normal subgroup \( N \) such that the quotient \( A = G/N \) is an abelian variety. Moreover, any connected commutative affine group is a direct product of multiplicative and additive lines: \( N = G_r^r \times G_a^\delta \). In conclusion, \( G \) is an extension of an abelian variety by a direct product of multiplicative and additive lines:
\[
0 \to G_r^r \times G_a^\delta \to G \to A \to 0.
\]

When this extension is trivial \((G \text{ being the associated group})\), one may easily prove that \( \delta \leq 1 \).

### 3. Tangent Subschemes

Let us recall the notation of the former section: \( X \) is a proper scheme over an algebraically closed field \( k \) of characteristic zero, \( D \) is a vector field on \( X \) and \( G \) is the associated group. Recall that \( G \) is a subgroup of the group \( \text{Aut} X \), so that we have an obvious action \( \mu : G \times X \to X \).

Let \( \hat{D} \) be the right-invariant vector field on \( \text{Aut} X \) induced by \( D \), and let \( \tilde{\tau}_\varepsilon \) be the corresponding infinitesimal automorphism: \( \tilde{\tau}_\varepsilon(g) = \tau_\varepsilon \circ g \). Let us consider \( \hat{D} \) as a vector field on \( G \) (it may be done because \( G \) is a tangent subscheme of \( \hat{D} \)). Then we may consider on \( G \times X \) the vector field \((\hat{D}, 0)\), the corresponding infinitesimal automorphism being \((\tilde{\tau}_\varepsilon, \text{Id})\).

**Lemma 3.1.** The natural action \( \mu : G \times X \to X \) is a differential morphism. As well, for any closed point \( x \in X \) the morphism \( \mu_x : G = G \times \{x\} \subset G \times X \xrightarrow{\mu} X \) is also differential.

**Proof.** We have to show that the action \( \mu \) commutes with the respective infinitesimal automorphisms: \( \mu \circ (\tilde{\tau}_\varepsilon, \text{Id}) = \tau_\varepsilon \circ \mu \). We prove it by means of the functor of points: Let \((g, x) \in G^\bullet(T) \times X^\bullet(T) = (G \times X)^\bullet(T)\), where \( T \) is a \( k[x] \)-scheme; we have
\[
(\mu \circ (\tilde{\tau}_\varepsilon, \text{Id}))(g, x) = \mu(\tau_\varepsilon \cdot g, x) = (\tau_\varepsilon \cdot g)(x) = \tau_\varepsilon(g(x)) = (\tau_\varepsilon \circ \mu)(g, x).
\]

The second statement may be proved in a similar way. \( \square \)
Theorem 3.2. Let $D$ be a vector field on a proper $k$–scheme $X$ and let $G$ be the associated group. For any closed point $x \in X$, the minimal tangent subscheme $Y_x$ passing through $x$ coincides with the closure of the orbit of $x$, that is,

$$Y_x = \overline{G \cdot x}.$$

Proof. Let us consider the differential morphism $\mu_x : G \to X, g \mapsto g \cdot x$. By Proposition 1.4, $\mu_x^{-1}(Y_x)$ is a tangent subscheme of $D$, which contains the identity of $G$. Since $G$ is minimal, we conclude that $G \subseteq \mu_x^{-1}(Y_x)$, hence $\mu_x(G) = G \cdot x \subseteq Y_x$ and it follows that $G \cdot x \subseteq Y_x$.

In order to show the reverse inclusion $Y_x \subseteq \overline{G \cdot x}$, it is enough to show that $G \cdot x$ is a tangent subscheme of $D$, because $Y_x$ is minimal. Let $g_0$ be the generic point of $G$ and let $x_0 = \mu_x(g_0)$. It is clear that $x_0$ is the generic point of $\overline{G \cdot x} = \mu_x(G)$. Let $p_{x_0}$ be the maximal ideal of $\mathcal{O}_{X,x_0}$. The exact sequence

$$0 \to p_{x_0} \to \mathcal{O}_{X,x_0} \overset{\mu_{x_0}}{\to} \mathcal{O}_{G,g_0}$$

where the last morphism is differential, shows that $p_{x_0}$ is a differential ideal of $\mathcal{O}_{X,x_0}$. Let $p$ be the sheaf of ideals of the closed subscheme $G \cdot x$. Since $x_0$ is the generic point of this subscheme, for any open set $U$ in $X$ we have

$$p(U) = \mathcal{O}_X(U) \cap p_{x_0}$$

(rigorously, $p = h^{-1}(p_{x_0})$ where $h : \mathcal{O}_X(U) \to \mathcal{O}_{X,x_0}$ is the natural morphism). It readily follows that $p(U)$ is a differential ideal of $\mathcal{O}_X(U)$. \hfill \Box

Proposition 3.3. Let $G \times X \to X$ be an action of a connected algebraic group $G$ over an integral quasi–projective $k$–scheme $X$. There is a $G$–invariant dense open set $U$ in $X$ such that the geometric quotient $\pi : U \to U/G$ exists.

We shall prove this result in the Appendix. Putting Theorem 3.2 and Proposition 3.3 together we obtain the following result.

Theorem 3.4. Let $D$ be a vector field on an integral projective $k$–scheme $X$ and let $G$ be the associated group. There exists a dense open set $U$ in $X$ such that:

a) $U$ is $G$–invariant and orbits of closed points in $U$ are just minimal tangent subvarieties of the vector field $D$ on $U$.

b) The geometric quotient $\pi : U \to Z = U/G$ exists. The sheaf $\mathcal{O}_Z$ coincides with the sheaf of first integrals of $D$, that is, for any open set $V$ in $Z$ we have

$$\Gamma(V,\mathcal{O}_Z) = \{ f \in \Gamma(\pi^{-1}V,\mathcal{O}_X) : Df = 0 \}.$$

Proof. a) Let $U$ be the open set whose existence states Proposition 3.3. The orbit of any closed point in $U$ is a closed subset of $U$, since it is the fibre of $\pi : U \to U/G$ over a closed point of $U/G$. Applying Theorem 3.2 to the field $D$ on $U$, we conclude that such orbits are just the minimal tangent subvarieties.

b) Since $\pi : U \to Z = U/G$ is a geometric quotient, the sheaf $\mathcal{O}_Z$ coincides with the sheaf of $G$–invariant functions,

$$\Gamma(V,\mathcal{O}_Z) = \Gamma(\pi^{-1}V,\mathcal{O}_X)^G,$$

so that we have to show that a function $f \in \Gamma(\pi^{-1}V,\mathcal{O}_X)$ is $G$–invariant if and only if $Df = 0$.

Given a closed point $x \in \pi^{-1}V$, let us consider the minimal tangent subvariety $Y_x = G \cdot x$ of $D$ passing through $x$. If $f$ is $G$–invariant, then $f|_{Y_x}$ is constant, so that $(Df)(x) = (D(f|_{Y_x}))(x) = 0$. 

Conversely, if $Df=0$, then we consider the ideal $I = (f - \lambda)$, where $\lambda := f(x)$. It is a differential ideal, hence it defines a tangent closed subscheme $T$ of $D$. Since $x \in T$ and $Y_x$ is minimal, we have $Y_x = G \cdot x \subseteq T$, hence $f = \lambda$ on $Y_x = G \cdot x$. Therefore $f$ is constant on the orbits and we conclude that $f$ is $G$-invariant. □

Remark 3.5. The above theorem holds when $X$ is not complete if $D$ is assumed to be a fundamental vector field with respect to the action $\mu: G \times X \to X$ of some algebraic group $G$, i.e., $D = \mu_*(\bar{D},0)$ for some right-invariant vector field $\bar{D}$ on $G$. Recalling Definition 2.2, the associated group $G$ to the vector field $D$ is defined to be the minimal tangent subvariety of $\bar{D}$ passing through the identity of $G$. The results of Sections 2 and 3 remain valid, so that the associated group $G$ is a commutative connected algebraic subgroup of $G$, and the minimal tangent subvarieties of $D$ are the closure of the orbits of $G$ on $X$.

Remark 3.6. Theorem 3.4 may be generalized to a distribution generated by global vector fields. Let $X$ be an integral projective $k$-scheme and let $L$ be a vector subspace of the space of all global vector fields on $X$.

A closed subscheme of $X$ is said to be a tangent subscheme of $L$ if it is a tangent subscheme of any $D \in L$. Recall that each vector field $D$ on $X$ corresponds with a right-invariant vector field $\bar{D}$ on $\text{Aut} \, X$. Let $\bar{L}$ be the space of all right-invariant vector fields $\bar{D}$ on $\text{Aut} \, X$ such that $D \in L$.

The results of Sections 2 and 3 (and their proofs) may be extended for the distribution $L$:

1. The minimal tangent subscheme of $\bar{L}$, passing through the identity of $\text{Aut} \, X$, is a connected algebraic subgroup $G$ of $\text{Aut} \, X$.
2. If $L$ is an abelian Lie algebra, then $G$ is a commutative group.
3. The minimal tangent subscheme $Y_x$ of $L$ passing through a closed point $x \in X$ is the closure of the orbit, $Y_x = G \cdot x$.
4. There exists a dense $G$-invariant open subset $U$ in $X$ such that:
   a) Orbits of closed points in $U$ are just minimal tangent subvarieties of $L$ on $U$;
   b) The geometric quotient $\pi: U \to Z = U/G$ exists and the sheaf $\mathcal{O}_Z$ coincides with the sheaf of first integrals of $L$, that is, for any open set $V$ in $Z$ we have
      \[ \Gamma(V, \mathcal{O}_Z) = \{ f \in \Gamma(\pi^{-1}V, \mathcal{O}_X): Df = 0 \text{ for any } D \in L \}. \]

4. Vector fields on $\mathbb{P}_n$

The aim of this section is to calculate the associated algebraic group of any vector field on the complex projective space $\mathbb{P}_n$.

Lemma 4.1. Let us consider the differential algebras

\[ \mathbb{C}[z_1, \ldots, z_r], \quad D = \sum \mu_i z_i \frac{\partial}{\partial z_i}, \]

\[ \mathbb{C}[z_0, z_1, \ldots, z_r], \quad D = \frac{\partial}{\partial z_0} + \sum_{i>0} \mu_i z_i \frac{\partial}{\partial z_i} \]

where $\mu_1, \ldots, \mu_r \in \mathbb{C}$. If $\mu_1, \ldots, \mu_r$ are linearly independent over $\mathbb{Q}$, then any non-null differential ideal of these algebras contains some monomial $z_1^{a_1} \cdots z_r^{a_r}$.
Proof. The same argument holds in both algebras. It is easy to check that monomials $\lambda z_1^{a_1} \cdots z_r^{a_r}$ are the only eigenvectors of $D$. Let $E_n$ be the vector subspace of all polynomials of degree $\leq n$ and note that $D(E_n) \subseteq E_n$ in both cases. Let $I$ be a non-null differential ideal. It is clear that $I \cap E_n \neq 0$ when $n \gg 0$. Since the dimension of $I \cap E_n$ is finite, the linear map $D : I \cap E_n \to I \cap E_n$ has some eigenvector, hence $I$ contains some monomial. □

Corollary 4.2. If $\mu_1, \ldots, \mu_r$ are linearly independent over $\mathbb{Q}$, then the differential algebras

$$
\mathbb{C}[z_1, \ldots, z_r, (z_1 \cdots z_r)^{-1}], \quad D = \sum \mu_i z_i \frac{\partial}{\partial z_i},
$$

$$
\mathbb{C}[z_0, z_1, \ldots, z_r, (z_1 \cdots z_r)^{-1}], \quad D = \frac{\partial}{\partial z_0} + \sum_{i>0} \mu_i z_i \frac{\partial}{\partial z_i}
$$

have no non-trivial differential ideal.

Proposition 4.3. If $\mu_1, \ldots, \mu_r \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the following differential algebras of holomorphic functions on $\mathbb{C}$,

$$
\mathbb{C}[e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_i t}], \quad \mathbb{C}[t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_i t}]
$$

(endowed with the derivation $\frac{\partial}{\partial t}$) have no non-trivial differential ideal.

Proof. Let us consider the obvious differential epimorphisms

$$
\mathbb{C}[z_1, \ldots, z_r, \frac{1}{z_1 \cdots z_r}] \to \mathbb{C}[e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_i t}],
$$

$$
\mathbb{C}[z_0, z_1, \ldots, z_r, \frac{1}{z_1 \cdots z_r}] \to \mathbb{C}[t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_i t}],
$$

where the algebras on the left are endowed with the derivations considered in Corollary 4.2. These epimorphisms are isomorphisms because both have null kernel by Corollary 4.2. □

Remark 4.4. By the last isomorphism in the former proof, the holomorphic functions $t, e^{\mu_1 t}, \ldots, e^{\mu_r t}$ are algebraically independent whenever $\mu_1, \ldots, \mu_r$ are linearly independent over $\mathbb{Q}$.

4.5. Let $M = (\lambda_{ij})$ be an $n \times n$ matrix with complex coefficients. Let us consider the linear vector field

$$
D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}
$$
on $\mathbb{C}^n$. The corresponding infinitesimal automorphism $\tau_\varepsilon$ of $\mathbb{C}^n$ is a $\mathbb{C}[\varepsilon]$-linear transformation with matrix $Id + \varepsilon M \equiv e^{\varepsilon M}$. This automorphism induces an infinitesimal automorphism $\tilde{\tau}_\varepsilon$ of the full linear group $Gl_n$,

$$(Gl_n)_{\mathbb{C}[\varepsilon]} \xrightarrow{\tilde{\tau}_\varepsilon} (Gl_n)_{\mathbb{C}[\varepsilon]} \quad g \mapsto \tau_\varepsilon \circ g,$$

whose corresponding vector field $\tilde{D}$ on $Gl_n$ is obviously right-invariant. Note that $D$ is a fundamental vector field with respect to the natural action $\mu : Gl_n \times \mathbb{C}^n \to \mathbb{C}^n$, since we have $\mu_\ast(D, 0) = D$ because $\mu \circ (\tilde{\tau}_\varepsilon, Id) = \tau_\varepsilon \circ \mu$.

According to Remark 8.6, the associated group $G$ of the linear vector field $D$ is defined to be the minimal tangent subvariety of $\tilde{D}$ passing through the identity of $Gl_n$. We know that $G$ is a commutative connected algebraic subgroup of $Gl_n$. 
and that the minimal tangent subvarieties of \( D \) are the closure of the orbits of \( G \) on \( \mathbb{C}^n \).

Indeed, it may be proved that the associated group \( G \) of a linear vector field \( D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i} \) coincides with the so-called differential Galois group of the linear differential system \( z_i(t) = \sum_j \lambda_{ij} z_j(t) \). See Note 4.7 below.

Let \( \mathbb{C}[e^{Mt}, \det(e^{-Mt})] \) be the algebra of holomorphic functions on \( \mathbb{C} \) generated by the coefficients of the matrix \( e^{Mt} \) and the function \( \det(e^{-Mt}) \). It is a differential algebra with the derivation \( \frac{D}{M} \).

**Lemma 4.6.** The group \( G \) associated to a linear vector field \( D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i} \) is

\[
G = \text{Spec} \mathbb{C}[e^{Mt}, \det(e^{-Mt})]
\]

where \( M = (\lambda_{ij}) \).

**Proof.** Let \( GL_n = \text{Spec} \mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \). The infinitesimal automorphism \( \tilde{\tau}_x \) is

\[
(\tilde{\tau}_x((x_{ij}))) = \tau_x \circ (x_{ij}) = e^{M_t} \circ (x_{ij}) = (Id + \varepsilon M) \circ (x_{ij}) = (x_{ij}) + \varepsilon M \circ (x_{ij}),
\]

so that the derivation \( \tilde{D} \) is (in matrix form)

\[
(\tilde{D}x_{ij}) = M \circ (x_{ij}).
\]

Let us consider the obvious epimorphism

\[
\mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \longrightarrow \mathbb{C}[e^{Mt}, \det(e^{-Mt})]
\]

which transforms the coefficients of the universal matrix \( (x_{ij}) \) into the coefficients of the matrix \( e^{Mt} \). Equality (*) states that it is a differential morphism when the first algebra is endowed with the derivation \( \frac{D}{M} \) and the second algebra is endowed with \( \Delta \). Therefore \( \text{Spec} \mathbb{C}[e^{Mt}, \det(e^{-Mt})] \) is a tangent subscheme of \( \tilde{D} \). Clearly, this closed subscheme passes through the identity of \( GL_n \) when \( t = 0 \). In order to show that it is the minimal tangent subscheme, it is enough to show that the differential algebra \( \mathbb{C}[e^{Mt}, \det(e^{-Mt})] \) has no non-trivial differential ideal.

After a linear coordinate change, we may assume that the matrix \( M \) is in Jordan form. If \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( M \), then it is easy to check that

\[
\mathbb{C}[e^{Mt}, \det(e^{-Mt})] = \mathbb{C}[\delta t, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}, e^{-\Sigma \lambda_i t}]
\]

where \( \delta = 0 \) if \( M \) is diagonalizable and \( \delta = 1 \) otherwise.

Let \( \mu_1, \ldots, \mu_r \) be a base of the \( \mathbb{Q} \)-vector space \( \mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_n \subset \mathbb{C} \), that is to say, \( \mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_n = \mathbb{Q}\mu_1 + \cdots + \mathbb{Q}\mu_r \). It is easy to check that

\[
\mathbb{C}[\delta t, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}, e^{-\Sigma \lambda_i t}] = \mathbb{C}[\delta t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\Sigma \mu_i t}].
\]

Now, by Proposition 4.3, we conclude that \( \mathbb{C}[e^{Mt}, \det(e^{-Mt})] \) has no non-trivial differential ideal. \( \square \)

**Note 4.7.** Let us explain the relation of the associated group \( G \) with the Galois theory of differential equations.

We have shown in the former proof that \( \mathbb{C}[e^{Mt}, \det(e^{-Mt})] \) is a simple differential ring, i.e., it has no non-trivial differential ideal. According to [10, Def. 1.15], this ring is the Picard-Vessiot ring of the differential equation \( z' = Mz \), since \( e^{Mt} \) is a fundamental matrix. Let \( L \) be the field of fractions of the Picard-Vessiot ring,
which is named the Picard–Vessiot field of the equation \( z' = Mz \) \([10]\), Def. 1.21).

Now, by Lemma \([4.6]\) \( L \) is the field of functions of the closed algebraic subgroup \((G, \delta) \subset (Gl_n, \delta)\). The action of \( G \) on itself by right–translations induces an action of \( G \) on \( L \) by differential automorphisms. It is immediate to check that any \( G \)-invariant function is constant: \( L^G = \mathbb{C} \). Then, by the Galois correspondence \([10]\), Prop. 1.34), we conclude that \( G \) is the group of all differential automorphisms of \( L \), i.e., \( G \) is the differential Galois group of the equation \( z' = Mz \).

The following theorem improves a statement of \([8]\), Prop. 3.27), about the differential Galois group of a linear differential system with constant coefficients.

**Theorem 4.8.** The associated group of a linear vector field \( D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i} \) on \( \mathbb{C}^n \) is

\[
G = \mathbb{G}_m^r \times \mathbb{G}_a^\delta
\]

where \( r \) stands for the dimension of the \( \mathbb{Q} \)-vector space \( \mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_n \subset \mathbb{C} \) spanned by the eigenvalues of the matrix \( M = (\lambda_{ij}) \), and \( \delta = 0 \) when \( M \) is diagonalizable and \( \delta = 1 \) otherwise.

**Proof.** Again we put \( Gl_n = \text{Spec} \mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \). The group law in \( Gl_n \) is determined by the coproduct

\[
\mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \hookrightarrow \mathbb{C}[y_{ij}, \det(y_{ij})^{-1}] \otimes \mathbb{C}[z_{ij}, \det(z_{ij})^{-1}], \quad (x_{ij}) = (y_{ij}) \circ (z_{ij}) .
\]

The group law in \( G = \text{Spec} \mathbb{C}[e^{Mt}, \det(e^{-Mt})] \) is defined by the induced coproduct in the quotient algebra \( \mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \rightarrow \mathbb{C}[e^{Mt}, \det(e^{-Mt})] \), that is,

\[
\mathbb{C}[e^{Mt}, \det(e^{-Mt})] \rightarrow \mathbb{C}[e^{Mu}, \det(e^{-Mu})] \otimes \mathbb{C}[e^{Mv}, \det(e^{-Mv})],
\]

where \( e^{Mt} = e^{Mu} \circ e^{Mv} = e^{M(u+v)} \).

In other words, this coproduct takes each function \( f(t) \) into \( f(u+v) \).

As shown in the proof of Lemma \([4.6]\) the coordinate ring of \( G \) has the form

\[
\mathbb{C}[\delta t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_i t}] .
\]

Recall that the functions \( t, e^{\mu_1 t}, \ldots, e^{\mu_r t} \) are algebraically independent (Remark \([4.3]\)). With the coproduct \( f(t) \mapsto f(u+v) \), the Hopf algebra

\[
\mathbb{C}[\delta t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_i t}]
\]

has an obvious decomposition as a tensor product of Hopf algebras:

\[
\mathbb{C}[\delta t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_i t}] = \mathbb{C}[\delta t] \otimes \mathbb{C}[e^{\mu_1 t}, e^{-\mu_1 t}] \otimes \cdots \otimes \mathbb{C}[e^{\mu_r t}, e^{-\mu_r t}],
\]

hence \( G = \mathbb{G}_m^r \times \mathbb{G}_m \times \ldots \times \mathbb{G}_m \).

\( \square \)

**Remark 4.9.** By Theorem \([4.3]\), there exists a \( G \)-invariant dense open set \( U \) in \( \mathbb{C}^n \) such that the orbits of \( G \) in \( U \) coincide with the minimal tangent subvarieties of \( D \) in \( U \). The isotropy subgroup of any point of \( U \) is the identity subgroup. Let us give a (summarized) proof of this fact: Every (flat) family of subgroups of \( G = \mathbb{G}_m^r \times \mathbb{G}_a^\delta \) is a constant family, hence all the points of \( U \) have the same isotropy subgroup (generically); since \( G \) acts faithfully on \( U \), we conclude that such a subgroup is the identity. Therefore, the orbits of \( G \) in \( U \) have the same dimension as \( G \), that is to say:

**Minimal tangent subvarieties of a linear vector field** \( D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i} \) on \( \mathbb{C}^n \) have dimension \( r + \delta \) (generically).
By [3.3], the field of rational functions on the quotient variety $Z = U/G$ coincides with the field of rational functions on $\mathbb{C}^n$ which are first integrals of $D$. Since $\dim Z = \dim U - \dim G = n - r - \delta$, we conclude that:

The field of all rational functions on $\mathbb{C}^n$ which are first integrals of a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$, has transcendence degree $n - r - \delta$.

This field may be computed by methods of Linear Algebra [3].

4.10. Let $\mathbb{P}_n$ be the $n$–dimensional projective space and let $\pi : \mathbb{C}^{n+1} \to \mathbb{P}_n$ be the natural projection. It is well known that any vector field $D$ on $\mathbb{P}_n$ is the projection by $\pi$ of some linear vector field $D_0 = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ on $\mathbb{C}^{n+1}$. Moreover, such a linear field is unique up to the addition of a vector field proportional to $\sum z_i \frac{\partial}{\partial z_i}$. These facts follow readily from the standard exact sequence

$$0 \to \mathcal{O}(\mathbb{P}_n) \to Hom_{\mathcal{O}(\mathbb{P}_n)}(\mathcal{O}(\mathbb{P}_n(-1)), \mathcal{O}(\mathbb{P}_n)^{n+1}) \to \mathcal{D}_{\mathbb{P}_n} \to 0,$$

where $\mathcal{O}(\mathbb{P}_n(-1))$ is the sheaf of sections of the tautological line bundle on $\mathbb{P}_n$, $\mathcal{D}_{\mathbb{P}_n}$ is the sheaf of vector fields on $\mathbb{P}_n$ and $Hom_{\mathcal{O}(\mathbb{P}_n)}(\mathcal{O}(\mathbb{P}_n(-1)), \mathcal{O}(\mathbb{P}_n)^{n+1})$ is the sheaf of $\pi$-projectable vector fields.

**Theorem 4.11.** Let $D = \pi_* (\sum \lambda_{ij} z_i \frac{\partial}{\partial z_i})$ be a vector field on $\mathbb{P}_n$. Its associated group is

$$G = \mathbb{G}_m^s \times \mathbb{G}_a^\delta$$

where $s$ stands for the dimension of the minimal $\mathbb{Q}$–affine subspace of $\mathbb{C}$ containing all the eigenvalues of the matrix $(\lambda_{ij})$, and $\delta = 0$ when such a matrix is diagonalizable and $\delta = 1$ otherwise.

**Proof.** Let $D_0 = \sum \lambda_{ij} z_i \frac{\partial}{\partial z_i}$. After the addition of a vector field proportional to $\sum z_i \frac{\partial}{\partial z_i}$, we may assume that the matrix $M = (\lambda_{ij})$ has the eigenvalue 0. In such case the minimal $\mathbb{Q}$–affine subspace containing the eigenvalues of $(\lambda_{ij})$ is a $\mathbb{Q}$–vector space. By Theorem [2,8] the group associated to $D_0$ is

$$G_0 = \mathbb{G}_m^s \times \mathbb{G}_a^\delta.$$ 

Let $0 \neq v \in \mathbb{C}^{n+1}$ be an eigenvector of $M$ of eigenvalue 0. Note that $v$ is a fixed point of the infinitesimal automorphism $\tau^0_\varepsilon$ (corresponding to $D_0$) since $\tau^0_\varepsilon(v) = e^{\varepsilon M}(v) = (\text{Id} + \varepsilon M)(v) = v$.

Let $H_v$ be the stabilizer of $v$, which is a closed algebraic subgroup of $GL_{n+1}$. The functor of points of $H_v$ is

$$H_v^*(T) = \{ g \in GL^\bullet_{n+1}(T) : g(v) = v \}.$$ 

We have that $H_v$ is a tangent subvariety of $(GL_{n+1}, \tilde{D}_0)$, since

$$g \in H_v^*(T) \Rightarrow \tau^0_\varepsilon(g) = \tau^0_\varepsilon \circ g \in H_v^*(T).$$

Since $G_0$ is minimal we obtain that $G_0 \subseteq H_v$.

Analogously, denoting by $H_p$ the stabilizer of $p = \pi(v) \in \mathbb{P}_n$ with respect to the action of $PGl_{n+1} = \text{Aut} \mathbb{P}_n$, we may prove that $H_p$ is a tangent subvariety of $(PGl_{n+1}, \tilde{D})$ and then $G \subseteq H_p$.

Now, it is immediate that the natural epimorphism $GL_{n+1} \to PGl_{n+1}$ induces a differential isomorphism $(H_v, \tilde{D}_0) \to (H_p, \tilde{D})$. Via this isomorphism, we conclude that $G_0 = G$. \qed
Remark 4.12. The same arguments used in Remark 4.9 show that:

Minimal tangent subvarieties of a vector field $D$ on $\mathbb{P}_n$ have dimension $s + \delta$ (generically).

The field of all rational functions on $\mathbb{P}_n$ which are first integrals of a vector field has transcendence degree $n - s - \delta$. (The case $n = 2$ is well known; see [7], pp. 12-16.)

5. A COUNTEREXAMPLE

Without the hypothesis of $X$ being complete, it does not follow the existence, for any vector field $D$ on $X$, of a dense open set $U$ and a projection $\varphi : U \to Z$ whose fibres are the minimal tangent subvarieties of $D$ in $U$.

As a counterexample, let us consider the field

$$D = z_1 z_4 \frac{\partial}{\partial z_4} + z_2 z_5 \frac{\partial}{\partial z_5} + z_3 z_6 \frac{\partial}{\partial z_6}$$

on $X = \mathbb{C}^6$. It is clear that $D$ is tangent to any 3–plane $z_1 = \lambda_1, z_2 = \lambda_2, z_3 = \lambda_3$. On these planes $D$ is a linear vector field with associated group $G_{\mathbb{C}^r}$, where $r$ is the dimension of the $\mathbb{Q}$–vector space $\mathbb{Q}\lambda_1 + \mathbb{Q}\lambda_2 + \mathbb{Q}\lambda_3$ (see Theorem 4.8). In each 3–plane, the minimal tangent subvarieties generically have dimension $r$ (Remark 4.9). Now, points $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ with $r = 3$ form a dense set in $\mathbb{C}^3$ (and so do points with $r = 2$). Therefore, minimal tangent subvarieties of dimension 3, as well as those of dimension 2, form a dense subset of $\mathbb{C}^6$. This fact prevents the existence of such projection $\varphi : U \to Z$, due to the semicontinuity of the dimension of the fibres of any algebraic morphism.

6. THE CASE OF POSITIVE CHARACTERISTIC

In this section, $k$ denotes a field of characteristic $p > 0$. Let $G_a = \text{Spec } k[t]$ be the additive line. The subscheme $G := \text{Spec } k[t]/(t^p)$ is a finite infinitesimal subgroup of $G_a$.

Proposition 6.1. Let $X$ be a $k$–scheme. Each vector field $D$ on $X$ defines an action $\mu : G \times_k X \to X$,

$$\mu^* : \mathcal{O}_X \to \mathcal{O}_X[t]/(t^p), \quad \mu^*(f) = \sum_{r=0}^{p-1} \frac{1}{r!} D^r(f) t^r.$$

Conversely, any action $\mu : G \times_k X \to X$ is defined by a unique vector field $D$ on $X$.

The proof of Proposition 6.1 is a simple exercise.

Let $D$ be a vector field on $X$ and let $\mu : G \times_k X \to X$ be the corresponding action. If $Y$ is a tangent subscheme of $(X, D)$, then the corresponding action $G \times_k Y \to Y$ is the restriction to $Y$ of the action of $G$ on $X$. Therefore, a closed subscheme $Y$ of $X$ is tangent to $D$ if and only if it is $G$–invariant. We conclude that the minimal tangent subscheme passing through a closed point $x$ is the orbit $G \cdot x = \text{scheme–theoretic image of } G \times x \subseteq G \times X \xrightarrow{\mu} X$, that is to say, $G \cdot x$ is the closed subscheme of $X$ defined by the ideal of all functions $f$ such that $f(x) = D f(x) = \cdots = D^{p-1} f(x) = 0$. Note that each orbit has a unique point, but it is not a reduced scheme in general.
APPENDIX A. QUOTIENTS BY ALGEBRAIC GROUPS

Theorem A.1 (Rosenlicht[11]). Let $\mu: G \times_k X \to X$ be an action of an affine algebraic group $G$ on an integral quasi–projective variety $X$. There exists a $G$–invariant dense open subset $U \subseteq X$ such that the geometric quotient $U \to U/G$ exists.

The purpose of this Appendix is to extend Rosenlicht’s result to the case of an algebraic group $G$ non-necessarily affine.

Quotients by abelian varieties. Let $\mu: A \times_k X \to X$ be an action of an abelian variety $A$ on an integral quasi–projective variety $X$.

Lemma A.2. The action $\mu: A \times_k X \to X$, $(a, x) \mapsto a \cdot x$, and the morphism $\phi: A \times_k X \to X \times_k X$, $(a, x) \mapsto (a \cdot x, x)$, are projective morphisms.

Proof. The isomorphism $\varphi: A \times X \to A \times X$, $(a, x) \mapsto (a, a \cdot x)$, makes commutative the triangle
\[ A \times X \xrightarrow{\varphi} A \times X \]
\[ \mu \downarrow \]
\[ \mu \quad \phi \quad p_2 \]

Since any abelian variety is projective, the map $p_2: A \times X \to X$ is a projective morphism and then the above commutative triangle implies that $\mu: A \times X \to X$ is also a projective morphism. Finally, $\phi = \mu \times p_2$ is a projective morphism because $\mu$ and $p_2$ are also. \qed

Let $R$ be the image of the map $\phi: A \times_k X \to X \times_k X$, $(a, x) \mapsto (a \cdot x, x)$, that is to say, $R$ is the equivalence relation defined by the action of $A$ over $X$. By the previous lemma, $R$ is a closed subset of $X \times_k X$. We shall consider $R$ as a closed subscheme of $X \times_k X$ with the reduced structure.

Lemma A.3. The projection $p_1: R \to X$ is a projective morphism.

Proof. Since $\phi: A \times X \to R$ is surjective and the composition morphism $\mu = p_1 \circ \phi: A \times X \to R \to X$ is proper, it is easy to check that the valuative criterion of properness (II, II, Th. 4.7) holds for the morphism $p_1: R \to X$.

Moreover, $R \subseteq X \times X \xrightarrow{p_1} X$ is a quasi-projective morphism, hence we conclude that $p_1: R \to X$ is a projective morphism. \qed

Given an $A$–stable open subset $U$ of $X$, we write $R_U := p_1^{-1}(U)$, i.e., $R_U$ is the equivalence relation defined by the action of $A$ over $U$.

Lemma A.4. There exists an $A$–invariant dense open subset $U$ of $X$ such that $p_1: R_U \to U$ is a flat morphism.

Proof. By the semicontinuity character of the Hilbert polynomial of the fibres of a projective morphism $p_1: R \to X$, we have that the subset $U$ of all points $x \in X$, whose fibre $p_1^{-1}(x)$ has the same Hilbert polynomial than the fibre of the generic point, is a dense open subset of $X$. This open subset $U$ is invariant by the action of $A$ over $X$, since the projection $p_1: R \to X$ is an $A$–equivariant morphism (the action of $A$ over $R$ is defined by the formula $a \cdot (x_1, x_2) = (a \cdot x_1, x_2)$).

Finally, since the fibres of $p_1: R_U \to U$ have the same Hilbert polynomial, we conclude (see [3], III, Th. 9.9) that $p_1: R_U \to U$ is a flat morphism. \qed
Theorem A.5. Let $\mu: A \times_k X \to X$ be an action of an abelian variety $A$ on an integral quasi-projective variety $X$. There exists an $A$–invariant dense open subset $U$ of $X$ such that the geometric quotient $\pi: U \to U/A$ exists.

Proof. By the previous lemmas, there exists an $A$–invariant dense open subset $U$ of $X$ such that $p_1: R_U \to U$ is a projective flat morphism. By a theorem of Grothendieck \cite[V, Th. 7.1]{5}, there exists the quotient map $\pi: U \to U/A$. $\square$

The general case.

Theorem A.6. Let $\mu: G \times_k X \to X$ be an action of a connected smooth group $G$ on an integral quasi–projective variety $X$. There exists a $G$–invariant dense open subset $U \subseteq X$ such that the geometric quotient $U \to U/G$ exists.

Proof. By the structure theorem of algebraic groups \cite{12}, there exists a normal affine subgroup $G_0$ of $G$ such that $A = G/G_0$ is an abelian variety. By Rosenlicht’s result, there exists a $G_0$–invariant dense open subset $U_0 \subseteq X$ such that the geometric quotient $\pi: U_0 \to U_0/G_0$ exists. Taking $G \cdot U_0$ instead of $U_0$, we may assume that $U_0$ is $G$–invariant. By Theorem A.5 there exists an $A$–invariant dense open subset $V$ in $U_0/G_0$ such that the geometric quotient $V \to V/A$ exists. Then $U := \pi^{-1}V$ is the desired open set, since $U/G = (U/G_0)/A = V/A$. $\square$

References


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