

DEGENERATION OF LINEAR SYSTEMS THROUGH FAT POINTS ON $K3$ SURFACES

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ABSTRACT. In this paper we introduce a technique to degenerate $K3$ surfaces and linear systems through fat points in general position on $K3$ surfaces. Using this degeneration we show that on generic $K3$ surfaces it is enough to prove that linear systems with one fat point are non-special in order to obtain the non-speciality of homogeneous linear systems through $n = 4^u 9^w$ fat points in general position. Moreover, we use this degeneration to obtain a result for homogeneous linear systems through $n = 4^u 9^w$ fat points in general position on a general quartic surface in \mathbb{P}^3 .

1. INTRODUCTION

In this paper we assume the ground field is the field of the complex numbers.

Let S be a smooth projective generic $K3$ surface (i.e. $\text{Pic}(S) \cong \mathbb{Z}$) and let H be the generator of $\text{Pic}(S)$.

Consider n points in general position on S , to each one of them associate a natural number m_i called the *multiplicity* of the point and let n_j be the number of points with multiplicity m_j .

For a linear system of curves in $|dH|$ with n_j general base points of multiplicity m_j for $j = 1, \dots, k$, define its *virtual dimension* v as $\dim |dH| - \sum n_i m_i (m_i + 1)/2$ and its *expected dimension* by $e = \max\{v, -1\}$. If the dimension of the linear system is l , then $v \leq e \leq l$.

Observe that it is possible to have $e < l$, since the conditions imposed by the points may be dependent. In this case we say that the system is *special*.

Linear systems through general fat points on rational surfaces have been studied by many authors (see e.g. [AH00, BZ03, CM98, CM01, Eva99]), but, as far as we know, on $K3$ surfaces, no results on the non-speciality of such systems are known.

In Section 3 we develop a technique to degenerate a $K3$ surface and linear systems through fat points on $K3$ surfaces, this degeneration is based on a degeneration of the projective plane which was developed by C. Ciliberto and R. Miranda and used by A. Buckley and M. Zompatori in [BZ03].

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In the subsequent section, we use this degeneration to prove that, a homogeneous linear system of curves in $|dH|$ with n general base points of multiplicity m is non-special if all linear systems of curves in $|dH|$ with one multiple base point are non-special.

Finally, in Section 5, we prove conjecture [DL03, Conjecture 2.3 (i)] for homogeneous linear systems of curves in $|dH|$ with $4^u 9^w$ fat points of multiplicity m , if either $v \geq 0$ or $v \leq -1$ and $u > 0$ or $2d \not\equiv 1 \pmod 3$.

2. PRELIMINARIES

Let S be a smooth projective generic $K3$ surface and let H be the generator of $\text{Pic}(S)$. Then H is ample, $H^2 = 2g - 2 \geq 2$ and $h^0(H) = g + 1$; moreover, H is very ample if $g \geq 3$ and if $g = 2$, H defines a double covering of \mathbb{P}^2 branched at an irreducible sextic (see [May72, Proposition 3]).

Consider Q_1, \dots, Q_n points in general position on S , for each one of these points fix a multiplicity m_1, \dots, m_n . Define $\mathcal{L} = \mathcal{L}^\gamma(d, m_1, \dots, m_n)$, with $\gamma = H^2 = 2g - 2$, as the linear system of curves in $|dH|$ with multiplicity at least m_i at Q_i for all i . By abuse of notation, let $\mathcal{L}^\gamma(d, m_1, \dots, m_n)$ also denote the associated sheaf on S . Let v denote the virtual dimension of \mathcal{L} . Then, using $\dim |dH| = \frac{\gamma d^2}{2} + 1$, we obtain that

$$(2.a) \quad v = \frac{\gamma d^2}{2} + 1 - \sum_{i=1}^n \frac{m_i(m_i + 1)}{2}.$$

Let S' be the blowing-up of S along the points Q_1, \dots, Q_n , let $\pi : S' \rightarrow S$ be the projection map, and let E_i be the exceptional divisor corresponding to Q_i . The linear system \mathcal{L} then corresponds to the system associated to the line bundle $\pi^*(\mathcal{O}_S(dH)) \otimes \mathcal{O}_{S'}(-m_1 E_1 - \dots - m_n E_n)$. By abuse of notation, we will denote this linear system and its associated line bundle on the blowing-up also by $\mathcal{L}^\gamma(d, m_1, \dots, m_n)$.

Similarly, by $\mathcal{L}^\gamma(d, m_1^{n_1}, \dots, m_y^{n_y})$, we denote the linear system of curves in $|dH|$ with n_i points (in general position) of multiplicity at least m_i (for all $i = 1, \dots, y$) as well as the associated sheaf on S and the corresponding linear system and its associated line bundle on the blowing-up of S along those $n_1 + \dots + n_y$ general points.

Let $Z = \sum_{i=1}^n m_i Q_i$ be the 0-dimensional scheme of S defined by the multiple points and consider the exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_S(dH) \otimes \mathcal{I}_Z \longrightarrow \mathcal{O}_S(dH) \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where \mathcal{I}_Z is the ideal sheaf of Z . Taking cohomology we obtain

$$(2.b) \quad v = h^0(\mathcal{O}_S(dH) \otimes \mathcal{I}(Z)) - h^1(\mathcal{O}_S(dH) \otimes \mathcal{I}(Z)) - 1,$$

because $h^1(\mathcal{O}_S(dH)) = 0$ (see e.g. [May72]).

Let e denote the expected dimension of the linear system \mathcal{L} , i.e. $e = \max\{-1, v\}$, and let l denote the dimension of \mathcal{L} . Then obviously we have that $v \leq e \leq l$.

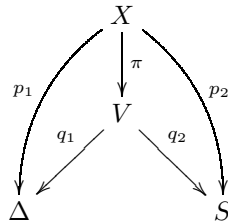
If $e < l$ (resp. $e = l$), we say the linear system \mathcal{L} is special (resp. non-special). Note that equation (2.b) shows that a non-empty linear system \mathcal{L} is non-special if and only if $h^1(\mathcal{L}) = 0$.

Analogously, on \mathbb{P}^2 , consider n general points Q_1, \dots, Q_n . Let $\mathcal{L}(d, m_1, \dots, m_n)$ denote the linear system of plane curves of degree d , having multiplicity at least m_i at Q_i for all $i = 1, \dots, n$. And let $\mathcal{L}(d, m_1^{n_1}, \dots, m_y^{n_y})$ denote the linear system

of plane curves of degree d having n_i points (in general position) of multiplicity at least m_i (for all $i = 1, \dots, y$). By abuse of notation, we also denote the associated sheafs on \mathbb{P}^2 and the corresponding linear systems and their associated line bundles on the blowings-up of \mathbb{P}^2 by $\mathcal{L}(d, m_1, \dots, m_n)$ and $\mathcal{L}(d, m_1^{n_1}, \dots, m_y^{n_y})$.

3. DEGENERATION OF LINEAR SYSTEMS ON $K3$ SURFACES

3.A. The degeneration of a $K3$ surface. Let S be a $K3$ surface and Δ a complex disc around the origin. Consider the product $V = S \times \Delta$ and its two projections $q_1 : V \rightarrow \Delta$ and $q_2 : V \rightarrow S$. Let V_t denote $S \times \{t\}$. Consider b general points in V_0 and blow up V along those b points. We then get a new threefold X and the maps $\pi : X \rightarrow V$, $p_1 = q_1 \circ \pi : X \rightarrow \Delta$ and $p_2 = q_2 \circ \pi : X \rightarrow S$; i.e. we obtain the following commutative diagram:



Let X_t be the fiber of p_1 over $t \in \Delta$. If $t \neq 0$, then $X_t \cong V_t$ is our $K3$ surface S . But, X_0 is the union of the proper transform \tilde{S} of V_0 and the b exceptional divisors \mathbf{P}_i . Obviously each \mathbf{P}_i is isomorphic to \mathbb{P}^2 and \tilde{S} is the blowing-up of our $K3$ surface S at the b general points with projection map $\mathfrak{b} : \tilde{S} \rightarrow S$.

Every \mathbf{P}_i intersects \tilde{S} transversally along a curve R_i , which is a line in \mathbf{P}_i and an exceptional divisor on \tilde{S} . When we want to indicate that we consider R_i in \tilde{S} , resp. \mathbf{P}_i , we denote it by E_i , resp. L_i .

Note that the map p_1 gives a flat family of surfaces over Δ , so X_0 can be seen as a degeneration of S .

3.B. The degeneration of a linear system on a $K3$ surface S . Let \mathcal{L} be a line bundle on S , and, for $k \in \mathbb{Z}$, define the line bundle $\mathcal{O}_X(\mathcal{L}, k)$ on X by

$$\mathcal{O}_X(\mathcal{L}, k) = p_2^*(\mathcal{L}) \otimes \mathcal{O}_X(k\tilde{S}).$$

The restriction of $\mathcal{O}_X(\mathcal{L}, k)$ to X_t , for $t \neq 0$, is then isomorphic to \mathcal{L} .

The restriction of $\mathcal{O}_X(\mathcal{L}, k)$ to X_0 , which we denote by $\mathcal{X}(\mathcal{L}, k)$, is a flat limit of the line bundle \mathcal{L} on the general fiber X_t , so $\mathcal{X}(\mathcal{L}, k)$ can be seen as a degeneration of the line bundle \mathcal{L} .

On any \mathbf{P}_i , we have that $\mathcal{X}(\mathcal{L}, k)|_{\mathbf{P}_i} = \mathcal{O}_X(\mathcal{L}, k)|_{\mathbf{P}_i} \cong \mathcal{O}_{\mathbb{P}^2}(k)$, since \tilde{S} intersects \mathbf{P}_i along a line.

On \tilde{S} we obtain that

$$\mathcal{X}(\mathcal{L}, k)|_{\tilde{S}} = \mathcal{O}_X(\mathcal{L}, k)|_{\tilde{S}} = p_2^*(\mathcal{L})|_{\tilde{S}} \otimes \mathcal{O}_X(k\tilde{S})|_{\tilde{S}}.$$

But, since $\tilde{S} \sim X_t - \sum_{i=1}^l \mathbf{P}_i$ as divisors on X and \mathbf{P}_i intersects \tilde{S} along R_i , this means that

$$\mathcal{X}(\mathcal{L}, k)|_{\tilde{S}} \cong \mathfrak{b}^*(\mathcal{L}) \otimes \mathcal{O}_{\tilde{S}}(-\sum_{i=1}^b kE_i).$$

Let Q_1, \dots, Q_n be general points on S , consider a zero-dimensional subscheme $Z = m_1Q_1 + \dots + m_nQ_n$ and let \mathcal{M} be the sheaf $\mathcal{L} \otimes \mathcal{I}_Z$, where \mathcal{I}_Z denotes the ideal sheaf that defines Z . Choose positive integers a_1, \dots, a_b such that $a_1 + \dots + a_b \leq n$. Now, for all $i \in \{1, \dots, b\}$, consider a_i general points on \mathbf{P}_i ; and take $n - \sum_{i=1}^b a_i$ general points on \tilde{S} . Denote those n points (on the \mathbf{P}_i and \tilde{S}) by Q'_i (any order will do), and let Z' be the zero-dimensional subscheme of X_0 given by $m_1Q'_1 + \dots + m_nQ'_n$.

Then we obtain that $\mathcal{X}(\mathcal{L}, k) \otimes \mathcal{I}_{Z'}$ is a degeneration of \mathcal{M} .

3.C. Homogeneous linear systems on generic K3 surfaces. Let S be a generic K3 surface and consider a homogeneous linear system $\mathcal{L} = \mathcal{L}^\gamma(d, m^n)$ on S . By abuse of notation, \mathcal{L} also denotes the corresponding sheaf.

Choose positive integers b and a such that $ab \leq n$, let X and $\mathcal{X}(\mathcal{O}_S(dH), k)$ be as constructed before. For all $1 \leq i \leq b$ and $1 \leq j \leq a$, let $Q'_{i,j}$ be a general point on \mathbf{P}_i and for $1 \leq j \leq n - ab$ let Q'_j be a general point on \tilde{S} . Now consider the zero-dimensional subscheme

$$Z' = \sum_{i=1}^{n-ab} mQ'_i + \sum_{\substack{i=1, \dots, b \\ j=1, \dots, a}} mQ'_{i,j}$$

On X_0 , resp. \mathbf{P}_i and \tilde{S} , define the sheaf $\mathcal{L}_0 := \mathcal{X}(\mathcal{O}_S(dH), k) \otimes \mathcal{I}_{Z'}$, resp. $\mathcal{L}_i := \mathcal{L}_0|_{\mathbf{P}_i}$ and $\mathcal{L}_{\tilde{S}} := \mathcal{L}_0|_{\tilde{S}}$. By abuse of notation, we also denote the corresponding linear systems on resp. X_0, \mathbf{P}_i and \tilde{S} , by resp. $\mathcal{L}_0, \mathcal{L}_i$ and $\mathcal{L}_{\tilde{S}}$. Then we see that a divisor in the linear system \mathcal{L}_0 on X_0 consists of a divisor $D_{\tilde{S}} \in \mathcal{L}_{\tilde{S}}$ and divisors $D_i \in \mathcal{L}_i$ such that $D_{\tilde{S}}|_{R_i} = D_i|_{R_i}$ for all i .

By $\mathcal{R}_{\tilde{S}}$, resp. \mathcal{R}_i , we denote the linear system on $\bigcup_{i=1}^b E_i$, resp. L_i , induced by $\mathcal{L}_{\tilde{S}}$, resp. \mathcal{L}_i .

Define the sheafs $\hat{\mathcal{L}}_{\tilde{S}} = \mathcal{L}_{\tilde{S}} \otimes \mathcal{O}_{\tilde{S}}(-\sum_{i=1}^b E_i)$ and $\hat{\mathcal{L}}_i = \mathcal{L}_i \otimes \mathcal{O}_{\mathbf{P}_i}(-L_i)$, and, again by abuse of notation, let $\hat{\mathcal{L}}_{\tilde{S}}$, resp. $\hat{\mathcal{L}}_i$, also denote the corresponding linear system on \tilde{S} , resp. \mathbf{P}_i .

Notation 3.1.

$$\begin{aligned} l &= \dim \mathcal{L}, \quad l_0 = \dim \mathcal{L}_0, \\ l_{\tilde{S}} &= \dim \mathcal{L}_{\tilde{S}}, \quad l_{\mathbf{P}} = \dim \mathcal{L}_i, \\ r_{\tilde{S}} &= \dim \mathcal{R}_{\tilde{S}}, \quad r_{\mathbf{P}} = \dim \mathcal{R}_i, \\ \hat{l}_{\tilde{S}} &= \dim \hat{\mathcal{L}}_{\tilde{S}}, \quad \hat{l}_{\mathbf{P}} = \dim \hat{\mathcal{L}}_i. \end{aligned}$$

Obviously we have the following:

$$(3.c) \quad l_{\tilde{S}} = r_{\tilde{S}} + \hat{l}_{\tilde{S}} + 1,$$

$$(3.d) \quad l_{\mathbf{P}} = r_{\mathbf{P}} + \hat{l}_{\mathbf{P}} + 1.$$

Let $\mathcal{R}_{\bigcup L_i}$ denote the linear system on $\bigcup_{i=1}^b R_i$ which consists of \mathcal{R}_j on R_j for all $j = 1, \dots, b$, and denote $r_{\bigcup L_i} = \dim \mathcal{R}_{\bigcup L_i}$. Then we have the following equality:

$$(3.e) \quad l_0 = \dim(\mathcal{R}_{\tilde{S}} \cap \mathcal{R}_{\bigcup L_i}) + b(\hat{l}_{\mathbf{P}} + 1) + \hat{l}_{\tilde{S}} + 1.$$

But, proceeding as in [BZ03, §2], we obtain the transversality of $\mathcal{R}_{\tilde{S}}$ and $\mathcal{R}_{\cup L_i}$; i.e.

$$\begin{aligned} \dim(\mathcal{R}_{\tilde{S}} \cap \mathcal{R}_{\cup L_i}) &= \max\{-1, r_{\tilde{S}} + r_{\cup L_i} - bk\} \\ (3.f) \qquad \qquad \qquad &= \max\{-1, r_{\tilde{S}} + br_{\mathbf{P}} - bk\}. \end{aligned}$$

For the virtual dimensions of the systems we introduce the following.

Notation 3.2.

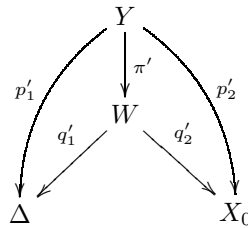
$$\begin{aligned} v &= \text{vdim } \mathcal{L}, \\ v_{\mathbf{P}} &= \text{vdim } \mathcal{L}_i, \quad v_{\tilde{S}} = \text{vdim } \mathcal{L}_{\tilde{S}}, \\ \hat{v}_{\tilde{S}} &= \text{vdim } \hat{\mathcal{L}}_{\tilde{S}}, \quad \hat{v}_{\mathbf{P}} = \text{vdim } \hat{\mathcal{L}}_i. \end{aligned}$$

Since $\mathcal{L} = \mathcal{L}^\gamma(d, m^n)$, $\mathcal{L}_{\tilde{S}} \cong \mathcal{L}^\gamma(d, k^b, m^{n-ab})$, $\hat{\mathcal{L}}_{\tilde{S}} \cong \mathcal{L}^\gamma(d, (k+1)^b, m^{n-ab})$, $\mathcal{L}_i \cong \mathcal{L}(k, m^a)$ and $\hat{\mathcal{L}}_i \cong \mathcal{L}(k-1, m^a)$, a simple calculation shows that

$$\begin{aligned} v &= v_{\tilde{S}} + b\hat{v}_{\mathbf{P}} + b = v_{\tilde{S}} + b(v_{\mathbf{P}} - k) \\ (3.g) \qquad \qquad \qquad &= \hat{v}_{\tilde{S}} + bv_{\mathbf{P}} + b = \hat{v}_{\tilde{S}} + b(\hat{v}_{\mathbf{P}} + k + 2). \end{aligned}$$

3.D. Remark. Let S be a $K3$ surface, \mathcal{L} a line bundle on S and let X and $\mathcal{X}(\mathcal{L}, k)$ be as before.

Now do the construction of §3.A and §3.B using X_0 instead of S and $\mathcal{X}(\mathcal{L}, k)$ (or $\mathcal{X}(\mathcal{L}, k) \otimes \mathcal{I}_{Z'}$) instead of \mathcal{L} (or $\mathcal{L} \otimes \mathcal{I}_Z$); i.e. consider $W = X_0 \times \Delta$, blow up W along b' general points on $\tilde{S} \times \{0\} \subset W_0 = X_0 \times \{0\}$ and obtain the following commutative diagram:



So we obtain a degeneration of X_0 , and a degeneration of $\mathcal{X}(\mathcal{L}, k)$ (or $\mathcal{X}(\mathcal{L}, k) \otimes \mathcal{I}_{Z'}$).

We call this a *double degeneration* of S and \mathcal{L} (or $\mathcal{L} \otimes \mathcal{I}_Z$), and, continuing in the same way, we can obtain an η -uple degeneration of S and \mathcal{L} (or $\mathcal{L} \otimes \mathcal{I}_Z$), for any $\eta \geq 2$.

4. APPLYING THE DEGENERATION TO HOMOGENEOUS LINEAR SYSTEMS ON GENERIC $K3$ SURFACES

From now on we assume that we work on a generic $K3$ surface S , with $H^2 = \gamma$, where H is the generator of $\text{Pic } S$.

The main result of this section is the following.

Theorem 4.1. *If $\mathcal{L}^\gamma(d, \mu)$ is non-special for all μ , then $\mathcal{L} = \mathcal{L}^\gamma(d, m^n)$ with $n = 4^u 9^w$ is non-special for all positive integers m, u and w .*

To make the proof of this theorem more transparent we first fix some notation and state a few auxiliary results.

Let $c \in \{4, 9\}$ such that $c|n$ and consider the degeneration (X_0, \mathcal{L}_0) of (S, \mathcal{L}) obtained by using the construction explained in §3 with $a = c$, $b = n/c$ and $k \in \mathbb{N}$. Note that, since \mathcal{L}_0 is a degeneration of \mathcal{L} , $l_0 \geq l \geq e \geq v$.

On \mathbb{P}^2 , let $\mathcal{L}(\delta, \mu^\nu)$ denote the linear system of plane curves of degree δ having multiplicity μ at ν points (in general position).

The following is a well-known result, and can easily be checked using, for instance, the results of [Har85].

Lemma 4.2. *On \mathbb{P}^2 the linear system $\mathcal{L}(\delta, \mu^c)$ with $c \in \{4, 9\}$ is non-special for all δ and μ . \square*

Remark 4.3. Lemma 4.2 implies, in particular, that \mathcal{L}_i and $\widehat{\mathcal{L}}_i$ are non-special linear systems on \mathbf{P}_i .

Claim 4.4. *If $v \geq -1$, $v_{\widehat{\mathcal{L}}_i} \geq -1$, $v_{\mathbf{P}} \geq -1$ and $\mathcal{L}_{\widehat{\mathcal{L}}_i}$ and $\widehat{\mathcal{L}}_{\widehat{\mathcal{L}}_i}$ are non-special systems; then $\dim(\mathcal{R}_{\widehat{\mathcal{L}}_i} \cap \mathcal{R}_{\cup L_i}) = r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk$.*

Proof. Note that the conditions of the theorem immediately imply that $l_{\widehat{\mathcal{L}}_i} = v_{\widehat{\mathcal{L}}_i}$ and $l_{\mathbf{P}} = v_{\mathbf{P}}$.

Since $\dim(\mathcal{R}_{\widehat{\mathcal{L}}_i} \cap \mathcal{R}_{\cup L_i}) = \max\{-1, r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk\}$, it suffices to show that $r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk \geq -1$.

Because of (3.c) and (3.d), we have that

$$r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk = l_{\widehat{\mathcal{L}}_i} - \widehat{l}_{\widehat{\mathcal{L}}_i} - 1 + b(l_{\mathbf{P}} - \widehat{l}_{\mathbf{P}} - 1) - bk.$$

If $\widehat{v}_{\widehat{\mathcal{L}}_i} \geq -1$ and $\widehat{v}_{\mathbf{P}} \geq -1$, then $\widehat{l}_{\widehat{\mathcal{L}}_i} = \widehat{v}_{\widehat{\mathcal{L}}_i}$ and $\widehat{l}_{\mathbf{P}} = \widehat{v}_{\mathbf{P}}$. So

$$\begin{aligned} r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk &= v_{\widehat{\mathcal{L}}_i} - \widehat{v}_{\widehat{\mathcal{L}}_i} - 1 + b(v_{\mathbf{P}} - \widehat{v}_{\mathbf{P}} - 1) - bk \\ &= b(k + 1) - 1 \geq -1. \end{aligned}$$

If $\widehat{v}_{\widehat{\mathcal{L}}_i} \leq -2$ and $\widehat{v}_{\mathbf{P}} \geq -1$, then $\widehat{l}_{\widehat{\mathcal{L}}_i} = -1$ and $\widehat{l}_{\mathbf{P}} = \widehat{v}_{\mathbf{P}}$. So

$$\begin{aligned} r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk &= v_{\widehat{\mathcal{L}}_i} + b(v_{\mathbf{P}} - \widehat{v}_{\mathbf{P}} - 1) - bk \\ &= v_{\widehat{\mathcal{L}}_i} \geq -1. \end{aligned}$$

If $\widehat{v}_{\widehat{\mathcal{L}}_i} \geq -1$ and $\widehat{v}_{\mathbf{P}} \leq -2$, then $\widehat{l}_{\widehat{\mathcal{L}}_i} = \widehat{v}_{\widehat{\mathcal{L}}_i}$ and $\widehat{l}_{\mathbf{P}} = -1$. So

$$\begin{aligned} r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk &= v_{\widehat{\mathcal{L}}_i} - \widehat{v}_{\widehat{\mathcal{L}}_i} - 1 + bv_{\mathbf{P}} - bk \\ &= b(1 + v_{\mathbf{P}}) - 1 \geq -1. \end{aligned}$$

If $\widehat{v}_{\widehat{\mathcal{L}}_i} \leq -2$ and $\widehat{v}_{\mathbf{P}} \leq -2$, then $\widehat{l}_{\widehat{\mathcal{L}}_i} = -1$ and $\widehat{l}_{\mathbf{P}} = -1$. So

$$\begin{aligned} r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk &= v_{\widehat{\mathcal{L}}_i} + b(v_{\mathbf{P}} - k) \\ &\stackrel{(3.g)}{=} v \geq -1. \end{aligned}$$

\square

Lemma 4.5. *If $v \geq -1$, $v_{\widehat{\mathcal{L}}_i} \geq -1$, $v_{\mathbf{P}} \geq -1$ and $\mathcal{L}_{\widehat{\mathcal{L}}_i}$ and $\widehat{\mathcal{L}}_{\widehat{\mathcal{L}}_i}$ are non-special systems; then \mathcal{L} is non-special.*

Proof. Because \mathcal{L}_0 is a degeneration of \mathcal{L} , we know that $v \leq l \leq l_0$; so it suffices to prove that $l_0 = v$.

Using (3.e), (3.f) and Claim 4.4 we obtain that

$$l_0 = r_{\widehat{\mathcal{L}}_i} + br_{\mathbf{P}} - bk + b(\widehat{l}_{\mathbf{P}} + 1) + \widehat{l}_{\widehat{\mathcal{L}}_i} + 1.$$

So, using (3.c) and (3.d), we see that

$$l_0 = l_{\widehat{\mathcal{L}}_i} + b(l_{\mathbf{P}} - k) = v_{\widehat{\mathcal{L}}_i} + b(v_{\mathbf{P}} - k) \stackrel{(3.g)}{=} v.$$

\square

Lemma 4.6. *Let $\mathcal{L} = \mathcal{L}^\gamma(d, m^n)$ with $n = 4^u 9^w$, $d, m, u, w \in \mathbb{N}$, $u + w > 0$ and $v \geq -1$. Take c, b, X_0 and \mathcal{L}_0 as before. Then $\exists k \in \mathbb{N}$ such that $v_{\tilde{\mathcal{S}}} \geq -1$ and $v_{\mathbf{P}} \geq -1$.*

Proof. Because $v_{\tilde{\mathcal{S}}} = \frac{\gamma}{2}d^2 + 1 - b\frac{k(k+1)}{2}$ and $v_{\mathbf{P}} = \frac{k(k+3)}{2} - c\frac{m(m+1)}{2}$, the inequalities $v_{\tilde{\mathcal{S}}} \geq -1$ and $v_{\mathbf{P}} \geq -1$ are equivalent to

$$(4.h) \quad k^2 + k - \alpha \leq 0, \quad \text{with } \alpha = \frac{1}{b}(\gamma d^2 + 4),$$

$$(4.i) \quad k^2 + 3k - \beta \geq 0, \quad \text{with } \beta = cm(m + 1) - 2.$$

But this is the same as

$$(4.j) \quad k \in \left[\frac{-1 - \sqrt{1 + 4\alpha}}{2}, \frac{-1 + \sqrt{1 + 4\alpha}}{2} \right],$$

$$(4.k) \quad k \in \left] -\infty, \frac{-3 - \sqrt{9 + 4\beta}}{2} \right] \cup \left[\frac{-3 + \sqrt{9 + 4\beta}}{2}, \infty \right[.$$

So proving the statement is equivalent to proving that there exists a positive integer k such that both (4.j) and (4.k) are satisfied, i.e. it is enough to show that

$$\frac{-3 + \sqrt{9 + 4\beta}}{2} + 1 \leq \frac{-1 + \sqrt{1 + 4\alpha}}{2}.$$

A simple calculation shows that the previous inequality is equivalent to $\alpha \geq \beta + 2$, which is in turn equivalent to $v \geq -1$. \square

Lemma 4.7. *If $v \leq -1$, $\hat{v}_{\tilde{\mathcal{S}}} \leq -1$, $\hat{v}_{\mathbf{P}} \leq -1$ and $\mathcal{L}_{\tilde{\mathcal{S}}}$ and $\hat{\mathcal{L}}_{\tilde{\mathcal{S}}}$ are non-special systems, then \mathcal{L} is non-special and thus empty.*

Proof. Since $v_{\tilde{\mathcal{S}}} = \hat{v}_{\tilde{\mathcal{S}}} + b(k + 1)$ and $v_{\mathbf{P}} = \hat{v}_{\mathbf{P}} + k + 1$, we see that $v_{\tilde{\mathcal{S}}} \leq b(k + 1) - 1$ and $v_{\mathbf{P}} \leq k$.

If $v_{\tilde{\mathcal{S}}} \leq -1$ and $v_{\mathbf{P}} \leq -1$, then $r_{\tilde{\mathcal{S}}} = r_{\mathbf{P}} = -1$. If $v_{\tilde{\mathcal{S}}} \leq -1$ and $v_{\mathbf{P}} \geq -1$, then $r_{\tilde{\mathcal{S}}} = -1$ and $r_{\mathbf{P}} = v_{\mathbf{P}} \leq k$. If $v_{\tilde{\mathcal{S}}} \geq -1$ and $v_{\mathbf{P}} \leq -1$, then $r_{\tilde{\mathcal{S}}} = v_{\tilde{\mathcal{S}}} \leq b(k + 1) - 1$ and $r_{\mathbf{P}} = -1$. If $v_{\tilde{\mathcal{S}}} \geq -1$ and $v_{\mathbf{P}} \geq -1$, then $r_{\tilde{\mathcal{S}}} = v_{\tilde{\mathcal{S}}}$ and $r_{\mathbf{P}} = v_{\mathbf{P}}$. So in any case we obtain that $r_{\tilde{\mathcal{S}}} + b r_{\mathbf{P}} - b k \leq -1$, i.e. $\dim(\mathcal{R}_{\tilde{\mathcal{S}}} \cap \mathcal{R}_{\cup L_i}) = -1$.

Using (3.e) and $\hat{l}_{\tilde{\mathcal{S}}} = \hat{l}_{\mathbf{P}} = -1$, we see that $l_0 = \hat{l}_{\tilde{\mathcal{S}}} + k(\hat{l}_{\mathbf{P}} + 1) = -1$. \square

Lemma 4.8. *Let $\mathcal{L} = \mathcal{L}^\gamma(d, m^n)$ with $n = 4^u 9^w$, $d, m, u, w \in \mathbb{N}$, $u + w > 0$ and $v \leq -1$. Take c, b, X_0 and \mathcal{L}_0 as before. Then $\exists k \in \mathbb{N}$ such that $\hat{v}_{\tilde{\mathcal{S}}} \leq -1$ and $\hat{v}_{\mathbf{P}} \leq -1$.*

Proof. Because $\hat{v}_{\tilde{\mathcal{S}}} = \frac{\gamma}{2}d^2 + 1 - b\frac{(k+1)(k+2)}{2}$ and $\hat{v}_{\mathbf{P}} = \frac{(k-1)(k+2)}{2} - c\frac{m(m+1)}{2}$, the inequalities $\hat{v}_{\tilde{\mathcal{S}}} \leq -1$ and $\hat{v}_{\mathbf{P}} \leq -1$ are equivalent to

$$k^2 + 3k - \alpha \geq 0, \quad \text{with } \alpha = \frac{1}{b}(\gamma d^2 + 4) - 2,$$

$$k^2 + k - \beta \leq 0, \quad \text{with } \beta = cm(m + 1).$$

Proceeding as in the proof of Lemma 4.6, we obtain that it is sufficient to prove that

$$\frac{-1 + \sqrt{1 + 4\beta}}{2} \geq \frac{-3 + \sqrt{9 + 4\alpha}}{2} + 1,$$

which is equivalent to $\beta \geq \alpha + 2$. This last inequality is equivalent to $v \geq -1$. \square

Proof of Theorem 4.1. Using induction, the result follows immediately from Lemmas 4.5 and 4.6 if $v \geq -1$, and from Lemmas 4.7 and 4.8 if $v \leq -1$. \square

4.A. Using a higher order degeneration. Using the η -uple degeneration explained in Remark 3.D, we can, proceeding as above, prove that the non-speciality of $\mathcal{L}^\gamma(d, \mu_1, \dots, \mu_\eta)$ for all μ_1, \dots, μ_η implies the non-speciality of

$$\mathcal{L} = \mathcal{L}^\gamma(d, m_1^{n_1}, \dots, m_\eta^{n_\eta})$$

with $n_i = 4^{u_i} 9^{w_i}$ for all $m_1, \dots, m_\eta, u_1, \dots, u_\eta, w_1, \dots, w_\eta$.

As we will see in Proposition 5.3, proving the non-speciality of systems $\mathcal{L}^\gamma(d, \mu)$ is already rather complex. In fact, in Proposition 5.3 we only prove when such a system is non-special for general generic K3 surfaces with $\gamma = 4$.

4.B. Remark. As proved in [CM01, Theorem 6.1], the Segre conjecture on planar linear systems through fat points implies that the only special homogeneous linear systems $\mathcal{L}(k, m^n)$ are the ones with $n \in \{2, 3, 5, 6, 7, 8\}$. So if the Segre conjecture is true, we can do the degeneration using, not only $a = 4$ or 9 , but $a \in \{4\} \cup \mathbb{Z}_{\geq 9}$. In this way we can then prove Theorem 4.1, for any n which can be written as a product of powers of numbers in $\{4\} \cup \mathbb{Z}_{\geq 9}$, and thus, in particular, for any $n \geq 9$. Note that the idea of using the results on \mathbb{P}^2 to reduce the problem of linear systems through fat points to a problem involving a single fat point has previously come up in the closely related context of Seshadri constants (see [Roe03, Theorem 3 and Corollary 8])

5. THE NON-SPECIALITY OF LINEAR SYSTEMS $\mathcal{L}^\gamma(d, m^n)$ WITH $n = 4^u 9^w$ ON GENERAL GENERIC K3 SURFACES WITH $\gamma = 4$

The main result of this section is the following.

Theorem 5.1. *Let S be a general generic K3 surface with $\gamma = 4$ and let $\mathcal{L} = \mathcal{L}^4(d, m^n)$, with $n = 4^u 9^w$, $u, w \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$.*

(1) *If $v(\mathcal{L}) \geq 0$, then the linear system \mathcal{L} is non-special.*

(2) *If $v(\mathcal{L}) \leq -1$ and either $u > 0$ or $2d \not\equiv 1 \pmod{3}$, then \mathcal{L} is non-special unless $n = 1$, $m = 2d$ and $d \geq 2$. In the latter case $\dim(\mathcal{L}^4(d, 2d)) = 0 > \text{edim}(\mathcal{L}^4(d, 2d)) = -1$ and $\mathcal{L}^4(d, 2d) = dC$, with C the unique element of $\mathcal{L}^4(1, 2)$.*

Remark 5.2. Note that the previous theorem implies that [DL03, Conjecture 2.3 (i)] is true for homogeneous linear systems with $n = 4^u 9^w$ fat points on S in the following cases:

- (i) $v \geq 0$, or
- (ii) $v \leq -1$ and $u > 0$, or
- (iii) $v \leq -1$ and $2d \not\equiv 1 \pmod{3}$.

We will prove Theorem 5.1 by using the degeneration as introduced in §4 and the following.

Proposition 5.3. *Let S be a general generic K3 surface with $\gamma = 4$. Then $\mathcal{L} = \mathcal{L}^4(d, \mu)$, with $d, \mu \in \mathbb{Z}_{>0}$, is non-special unless $\mu = 2d$ and $d \geq 2$. In the latter case $\dim(\mathcal{L}^4(d, 2d)) = 0 > \text{vdim}(\mathcal{L}^4(d, 2d)) = -1$ and $\mathcal{L}^4(d, 2d) = dC$, with C the unique element of $\mathcal{L}^4(1, 2)$.*

Proof. The map $\phi : S \rightarrow \mathbb{P}^3$ corresponding to $|H|$ (with H the generator of $\text{Pic } S$) is an embedding, so we may look at S as a quartic surface in \mathbb{P}^3 . The unique element C of $\mathcal{L}^4(1, 2)$ is then the divisor on $S \subset \mathbb{P}^3$ defined by the tangent plane to S at

this general point P . So C is an irreducible plane curve of degree 4 having a node at P (and no other singularities).

Blow up S along the point P , and recall that, by abuse of notation, $\mathcal{L}^4(d, \mu)$ also denotes the complete linear system and the line bundle on this blowing-up corresponding to $\mathcal{L}^4(d, \mu)$ on S . Let \tilde{C} denote the strict transform of C on this blowing-up; then $g(\tilde{C}) = 2$.

Assume $\mu = 2d$ ($d \geq 2$) and consider the following sequence:

$$0 \longrightarrow \mathcal{L}^4(d-1, 2d-2) \longrightarrow \mathcal{L}^4(d, 2d) \longrightarrow \mathcal{L}^4(d, 2d) \otimes \mathcal{O}_{\tilde{C}} \longrightarrow 0.$$

Because $\deg(\mathcal{L}^4(d, 2d) \otimes \mathcal{O}_{\tilde{C}}) = 0 = 2g(\tilde{C}) - 4$, we know that $h^0(\mathcal{L}^4(d, 2d) \otimes \mathcal{O}_{\tilde{C}}) = 0$ unless $\mathcal{L}^4(d, 2d) \otimes \mathcal{O}_{\tilde{C}} = \mathcal{O}_{\tilde{C}}$. But $\mathcal{L}^4(d, 2d) \otimes \mathcal{O}_{\tilde{C}} = |dg_4^2 - 2d(P_1 + P_2)|$, where P_1 and P_2 are the intersection points of \tilde{C} with the exceptional curve. So this would mean that $|dg_4^2 - 2d(P_1 + P_2)| = \mathcal{O}_{\tilde{C}}$. But, since P is a general point on the general quartic S , we know that $g_4^2 = |K_{\tilde{C}} + P_1 + P_2|$; so we would obtain that $|dK_{\tilde{C}} - d(P_1 + P_2)| = \mathcal{O}_{\tilde{C}}$, or thus that $|dK_{\tilde{C}}| = |d(P_1 + P_2)|$, which is not true (because P is a general element of S). So $h^0(\mathcal{L}^4(d, 2d)) = h^0(\mathcal{L}^4(d-1, 2(d-1)))$ for all $d \geq 2$. Using this a number of times we thus obtain that $h^0(\mathcal{L}^4(d, 2d)) = h^0(\mathcal{L}^4(1, 2)) = 1$, so dC is the only divisor in $\mathcal{L}^4(d, 2d)$.

It now follows immediately that $\dim \mathcal{L} = -1$ if $\mu \geq 2d + 1$, since the only divisor in $\mathcal{L}^4(d, 2d)$ has multiplicity exactly $2d$ in P .

Now consider the case $\mu \leq 2d - 1$. If we can prove that \mathcal{L} is non-special for $\mu = 2d - 1$, then the non-speciality follows for all $\mu \leq 2d - 1$ (since $\text{vdim } \mathcal{L}^4(d, 2d - 1) = d + 1$). For $d = 1$, there is nothing to prove, since P is a general point of S . So we may assume that $d \geq 2$ and that the $\mathcal{L}^4(d', 2d' - 1)$ is non-special for $d' \leq d - 1$. Now consider the following sequence:

$$0 \longrightarrow \mathcal{L}^4(d-1, 2d-3) \longrightarrow \mathcal{L}^4(d, 2d-1) \longrightarrow \mathcal{L}^4(d, 2d-1) \otimes \mathcal{O}_{\tilde{C}} \longrightarrow 0.$$

Because

$$\deg(\mathcal{L}^4(d, 2d-1) \otimes \mathcal{O}_{\tilde{C}}) = 2 = 2g(\tilde{C}) - 2,$$

we know that $h^1(\mathcal{L}^4(d, 2d-1) \otimes \mathcal{O}_{\tilde{C}}) = 0$ unless $\mathcal{L}^4(d, 2d-1) \otimes \mathcal{O}_{\tilde{C}} = K_{\tilde{C}}$ (the canonical class on \tilde{C}). But $\mathcal{L}^4(d, 2d) \otimes \mathcal{O}_{\tilde{C}} = |dg_4^2 - (2d-1)(P_1 + P_2)|$ and $g_4^2 = |K_{\tilde{C}} + P_1 + P_2|$ (where P_1 and P_2 are, as before, the intersection points of \tilde{C} with the exceptional curve). Then this would mean that $|(d-1)K_{\tilde{C}}| = |(d-1)(P_1 + P_2)|$, which is not true (because P is a general element of S). So $h^1(\mathcal{L}^4(d, 2d-1) \otimes \mathcal{O}_{\tilde{C}}) = 0$ and, by hypotheses, $h^1(\mathcal{L}^4(d-1, 2d-3)) = 0$, thus also $h^1(\mathcal{L}^4(d, 2d-1)) = 0$. \square

Proof of Theorem 5.1. (1) In case $n = 1$ we are done because of Proposition 5.3. So we assume $n > 1$, or equivalently $u + w \geq 1$. Consider the degeneration (X_0, \mathcal{L}_0) of (S, \mathcal{L}) obtained by using the construction explained in §3 with $a = n$, $b = 1$ and $k = 2d - 1$. Then we have that $\mathcal{L}_{\tilde{S}} = \mathcal{L}^4(d; 2d - 1)$, $\hat{\mathcal{L}}_{\tilde{S}} = \mathcal{L}^4(d; 2d)$, $\mathcal{L}_1 = \mathcal{L}(2d - 1; m^n)$ and $\hat{\mathcal{L}}_1 = \mathcal{L}(2d - 2; m^n)$. So, because of Proposition 5.3, we know that $l_{\tilde{S}} = v_{\tilde{S}} (= d + 1)$ and $\hat{l}_{\tilde{S}} = 0 (> \hat{v}_{\tilde{S}})$. A simple calculation shows that $v_{\mathbf{P}} = v + d - 2 \geq v \geq 0$. Moreover, because of [BZ03, Theorem 4], we know that \mathcal{L}_1 and $\hat{\mathcal{L}}_1$ are non-special. Proceeding as in the proof of Claim 4.4, one can easily check (using $v \geq 0$) that $\dim(\mathcal{R}_{\tilde{S}} \cap \mathcal{R}_{\cup L_i}) = r_{\tilde{S}} + r_{\mathbf{P}} - 2d + 1$. So, using (3.e), we obtain that

$$l_0 = r_{\tilde{S}} + r_{\mathbf{P}} - 2d + 1 + \hat{l}_{\tilde{S}} + l_{\mathbf{P}} + 2 = l_{\tilde{S}} + l_{\mathbf{P}} - 2d + 1 = v_{\tilde{S}} + v_{\mathbf{P}} - k = v.$$

(2) Again, if $n = 1$ we are done because of Proposition 5.3. So we assume $n > 1$, or equivalently $u + w \geq 1$. As in the proof of Theorem 4.1, we can use Lemmas 4.7 and 4.8, up to the last step. For this last step, we have $c = n$ and $b = 1$, and we will take $k = 2d$. In this case, we can no longer use Lemma 4.7 since $\mathcal{L}_{\tilde{\mathcal{S}}}$ is special. If we can prove, however, that $\dim(\mathcal{R}_{\tilde{\mathcal{S}}} \cap \mathcal{R}_{\cup L_i}) = \hat{l}_{\tilde{\mathcal{S}}} = \hat{l}_{\mathbf{P}} = -1$, then we still obtain $l_0 = -1$, and thus also $l = -1$. Since $\dim(\mathcal{R}_{\tilde{\mathcal{S}}} \cap \mathcal{R}_{\cup L_i}) = \max\{-1, r_{\tilde{\mathcal{S}}} + (r_{\mathbf{P}} - 2d)\}$, $r_{\tilde{\mathcal{S}}} = 0$ and $r_{\mathbf{P}} = l_{\mathbf{P}}$ it is enough to prove that, for $k = 2d$, $v_{\mathbf{P}} \leq 2d - 1$ and $\hat{v}_{\mathbf{P}} \leq -1$. These inequalities are equivalent to respectively $v \leq -d$ and $v \leq 1 - d$, so it is sufficient to have $v \leq -d$.

In case $u > 0$, we can make sure that in the last step $c = 4$, and we obtain

$$v \leq -1 \iff m \geq d \iff v \leq 1 - 2d.$$

In case $u = 0$, we have $c = 9$ in the last step, and using the fact that $2d \not\equiv 1 \pmod{3}$, a simple calculation shows that

$$v \leq -1 \iff m \geq \frac{2d}{3} \iff v \leq 1 - 3d. \quad \square$$

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