TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 9, Pages 3655–3672
S 0002-9947(05)03758-X
Article electronically published on January 21, 2005

ASYMPTOTIC BEHAVIOUR OF ARITHMETICALLY COHEN-MACAULAY BLOW-UPS

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ABSTRACT. This paper addresses problems on arithmetic Macaulayfications of projective schemes. We give a surprising complete answer to a question poised by Cutkosky and Herzog. Let Y be the blow-up of a projective scheme $X = \operatorname{Proj} R$ along the ideal sheaf of $I \subset R$. It is known that there are embeddings $Y \cong \operatorname{Proj} k[(I^e)_c]$ for $c \geq d(I)e + 1$, where $d(I)$ denotes the maximal generating degree of I, and that there exists a Cohen-Macaulay ring of the form $k[(I^e)_c]$ (which gives an arithmetic Macaulayfication of X) if and only if $H^0(Y, \mathcal{O}_Y) = k$, $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, ..., \dim Y - 1$, and Y is equidimensional and Cohen-Macaulay. We show that under these conditions, there are well-determined invariants ε and e_0 such that $k[(I^e)_c]$ is Cohen-Macaulay for all $c > d(I)e + \varepsilon$ and $e > e_0$, and that these bounds are the best possible. We also investigate the existence of a Cohen-Macaulay Rees algebra of the form $R[(I^e)_c t]$. If R has negative a^*-invariant, we prove that such a Cohen-Macaulay Rees algebra exists if and only if $\pi_* \mathcal{O}_Y = \mathcal{O}_X$, $R^i \pi_* \mathcal{O}_Y = 0$ for $i > 0$, and Y is equidimensional and Cohen-Macaulay. Moreover, these conditions imply the Cohen-Macaulayness of $R[(I^e)_c t]$ for all $c > d(I)e + \varepsilon$ and $e > e_0$.

INTRODUCTION

Let X be a projective scheme over a field k. An arithmetic Macaulayfication of X is a proper birational morphism $\pi : Y \to X$ such that Y has an arithmetically Cohen-Macaulay embedding, i.e. there exists a Cohen-Macaulay standard graded k-algebra A such that $Y \cong \operatorname{Proj} A$. Inspired by the problem of resolution of singularities, it was asked when X has an arithmetic Macaulayfication. The local version of this problem (arithmetic Macaulayfication of local rings) has been extensively studied in the literature and recently solved by Kawasaki [22]. An important aspect of the global problem is to determine, given a proper birational morphism $Y \to X$, if Y has an arithmetically Cohen-Macaulay embedding, and if it does, which embeddings of Y are arithmetically Cohen-Macaulay.

Let R be a standard graded k-algebra and let $I \subset R$ be a homogeneous ideal such that $X = \operatorname{Proj} R$ and Y is the blow-up of X along the ideal sheaf of I. It was observed by Cutkosky and Herzog [9] that $Y \cong \operatorname{Proj} k[(I^e)_c]$ for $c \geq d(I)e + 1$,

Received by the editors January 10, 2004.
2000 *Mathematics Subject Classification.* Primary 14M05, 13H10, 13A30, 14E25.
Key words and phrases. Blow-up, Rees algebra, Cohen-Macaulay, projective embedding.
The second author was partially supported by the National Basic Research Program of Vietnam.

where $(I^e)_c$ denotes the vector space of forms of degree c of the ideal power I^e and $d(I)$ is the maximal degree of the elements of a homogeneous basis of I. In other words, Y can be embedded into a projective space by the complete linear system $|cE_0 - eE|$, where E denotes the exceptional divisor and E_0 is the pull-back of a general hyperplane in X. By [26] we know that there exists a Cohen-Macaulay ring $k[(I^e)_c]$ for $c \geq d(I)e + 1$ if and only if Y satisfies the following conditions:

- Y is equidimensional and Cohen-Macaulay,
- $H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, ..., \dim Y - 1$.

In the first part of this paper, we study the problem of which values of c and e is $k[(I^e)_c]$ a Cohen-Macaulay ring. This problem originated from a beautiful result of Geramita, Gimigliano and Pitteloud [13] which shows that if I is the defining ideal of a set of fat points in a projective space over a field of characteristic zero, then $k[I_c]$ is a Cohen-Macaulay ring for all $c \geq \operatorname{reg}(I)$, where $\operatorname{reg}(I)$ is the Castelnuovo-Mumford regularity of I. This result initiated the study on the Cohen-Macaulayness of algebras of the form $k[(I^e)_c]$ first in [7] and then in [9, 25, 26, 16]. In particular, Cutkosky and Herzog [9] showed that if I is a locally complete intersection ideal, then there exists a constant δ such that $k[(I^e)_c]$ is Cohen-Macaulay for $c \geq \delta e$. They raised the question of when there is a linear bound on c and e ensuring that $k[(I^e)_c]$ is a Cohen-Macaulay ring.

Our results will give a complete answer to this question. We show that if the above two conditions are satisfied, then there exist well-determined invariants ε and e_0 such that $k[(I^e)_c]$ is a Cohen-Macaulay ring for all $c > d(I)e + \varepsilon$ and $e > e_0$ (Theorem 2.2). The invariant e_0 is a projective version of the a^*-invariant, which is the largest non-vanishing degree of the graded local cohomology modules [29, 32]. The invariant ε comes from the asymptotic linearity of the Castelnuovo-Mumford regularity of powers of ideals ([31, 10, 23, 34]). We will see that the bounds $c > d(I)e + \varepsilon$ and $e > e_0$ are the best possible (Theorem 2.3 and Example 2.5). The existence of linear bounds on c and e is not hard to prove. The novelty we claim here is that an explicit description for the best possible bounds is obtained. Moreover, if the Rees algebra $R[It]$ is locally Cohen-Macaulay on X, then $e_0 = 0$ and we can replace the second condition by the weaker condition that $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, ..., \dim X - 1$ (Theorem 2.4). These results strengthen and unify all previously-known results on the Cohen-Macaulayness of $k[(I^e)_c]$ which were obtained by different methods.

In the second part of this paper, we investigate the more difficult question of when Y is an arithmetically Cohen-Macaulay blow-up of X; that is, when there exists a standard graded k-algebra R and an ideal $J \subset R$, such that $X = \operatorname{Proj} R$, Y is the blow-up of X along the ideal sheaf of J, and $R[Jt]$ is a Cohen-Macaulay ring. Given R and I, we will concentrate on ideals $J \subseteq I$ which are generated by the elements of $(I^e)_c$. It is obvious that I^e and J define the same ideal sheaf for $c \geq d(I)e$. Rees algebras of the form $R[I_ct]$ $(e = 1)$ have been studied first for the defining ideal of a set of points in [15] and then for locally complete intersection ideals in [8], where it was shown that there exists a constant λ such that $R[I_ct]$ is a Cohen-Macaulay ring for $c \geq \lambda$. This leads to the problem of whether there is a constant δ such that the Rees algebra $R[(I^e)_ct]$ is a Cohen-Macaulay ring for $c \geq \delta e$.

If $a^*(R) < 0$ (e.g. if R is a polynomial ring) we solve this problem by showing that there exists a Cohen-Macaulay ring $R[(I^e)_ct]$ with $c \geq d(I)e$ if and only if the following conditions are satisfied:

- Y is equidimensional and Cohen-Macaulay,
- $\pi_*\mathcal{O}_Y = \mathcal{O}_X$, $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$.

In particular, these conditions imply that $R[(I^e)_c t]$ is a Cohen-Macaulay ring for all $c > d(I)e + \varepsilon$ and $e > e_0$ (Theorem 3.4). From this it follows that there exists a Cohen-Macaulay algebra of the form $R[I_c t]$ with $c \geq d(I)$ if and only if $R[It]$ is locally Cohen-Macaulay on X and that $e_0 = 0$ in this case (Corollary 3.8). We would like to point out that this phenomenon does not hold in general. In fact, there exist examples with $a^*(R) \geq 0$ such that $R[(I^e)_{d(I)e}t]$ is a Cohen-Macaulay ring, whereas $R[(I^e)_c t]$ is not a Cohen-Macaulay ring for any $c > d(I)e$ (Example 3.5). Using the above result we obtain several new classes of Cohen-Macaulay Rees algebras. Furthermore, we show that if $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, then Y is an arithmetically Cohen-Macaulay blow-up of X if and only if Y is locally arithmetical Cohen-Macaulay on X (Theorem 3.12).

Our approach is based on the facts that the Rees algebra $S = R[It]$ has a natural bi-gradation and that $k[(I^e)_c]$ can be viewed as a diagonal subalgebra of S [7]. As a consequence, the Cohen-Macaulayness of $k[(I^e)_c]$ can be characterized by means of the sheaf cohomology $H^i(Y, \mathcal{O}_Y(m, n))$. Using Leray spectral sequence and Serre-Grothendieck correspondence, we may pass from this sheaf cohomology to the local cohomology of I^n and of ω_n, where $\omega_S = \bigoplus_{n\in\mathbb{Z}} \omega_n$ denotes the graded canonical module of S. It was shown recently that there are linear bounds for the vanishing of the local cohomology of I^n and ω_n ([31, 10, 23, 34]). It turns out that these linear bounds yield a linear bound on c and e such that $k[(I^e)_c]$ is a Cohen-Macaulay ring. The Cohen-Macaulayness of the Rees algebra $R[(I^e)_c t]$ can be studied similarly by using a recent result of Hyry [19] which characterizes the Cohen-Macaulayness of a standard bi-graded algebra by means of sheaf cohomology.

The paper is organized as follows. In Section 1, we introduce the notion of a projective a^*-invariant which governs how sheaf cohomology behaves through blowing up morphisms. The material in this section is interesting on its own right. In Section 2, we study the Cohen-Macaulayness of rings of the form $k[(I^e)_c]$ which correspond to projective embeddings of Y. The last section of the paper deals with the problem of when Y is an arithmetically Cohen-Macaulay blow-up of X.

For unexplained notation and facts we refer the reader to the books [4, 5, 17].

1. a^*-invariants

Let R be an arbitrary commutative noetherian ring. Let $S = \bigoplus_{n\geq 0} S_n$ be a finitely generated graded algebra over R. We shall always use $S_+ = \bigoplus_{n>0} S_n$ to denote the ideal generated by the homogeneous elements of positive degrees of S. Given any finitely generated graded S-module F, the local cohomology module $H^i_{S_+}(F)$ is also a graded S-module. It is well known that $H^i_{S_+}(F)_n = 0$ for $n \gg 0$, $i \geq 0$. Put

$$a_i(F) = \begin{cases} -\infty & \text{if} \quad H^i_{S_+}(F) = 0, \\ \max\{n \mid H^i_{S_+}(F)_n \neq 0\} & \text{if} \quad H^i_{S_+}(F) \neq 0. \end{cases}$$

Note that $a(F) := a_{\dim F}(F)$ is called the a-invariant of F if S is a standard graded algebra over a field. The a^*-*invariant* of F is defined to be

$$a^*(F) := \max\{a_i(F) \mid i \geq 0\}.$$

This invariant was introduced in [32] and [29] in order to control the vanishing of graded local cohomology modules with different supports. It is closely related to the Castelnuovo-Mumford regularity via the equality

$$\operatorname{reg}(F) = \max\{a_i(F) + i|\ i \geq 0\}.$$

Here we are interested in the case when R is a standard graded algebra over a field k and $S = R[It]$ is the Rees algebra of a homogeneous ideal $I \subset R$ with $\operatorname{ht} I \geq 1$. This Rees algebra has a natural grading with $S_n = I^n t^n$. Let $\omega_S = \bigoplus_{n\in\mathbb{Z}} \omega_n$ denote the canonical graded module of S.

Lemma 1.1. *Let $S = R[It]$ be as above. If S is a Cohen-Macaulay ring, then $a^*(S) = -1$ and $a^*(\omega_S) = 0$.*

Proof. It is well known that $\dim S = \dim R + 1$. Since $S/S_+ = R$, we have $\operatorname{ht} S_+ = \dim S - \dim R = 1$. This implies $\operatorname{grade} S_+ = 1$. Hence $a^*(S) \geq -1$ by [32, Corollary 2.3]. On the other hand, the Cohen-Macaulayness of S implies $H^i_M(S) = 0$ for $i < \dim S$, where M denotes the maximal graded ideal of S. By [33, Corollary 3.2] we always have $H^{\dim S}_M(S)_n = 0$ for $n \geq 0$. Hence $H^i_M(S)_n = 0$ for all $n \geq 0$ and $i \geq 0$. By [19, Lemma 2.3] (or [32, Corollary 2.8]), this implies $H^i_{S_+}(S)_n = 0$ for all $n \geq 0$ and $i \geq 0$. Therefore, $a^*(S) = -1$.

Since ω_S is a Cohen-Macaulay module with $\operatorname{Hom}_S(\omega_S, \omega_S) \cong S$ [2, Proposition 2], we also have $H^i_M(\omega_S) = 0$ for $i < \dim S$ and, by local duality,

$$H^{\dim S}_M(\omega_S)_n \cong \operatorname{Hom}_S(\omega_S, \omega_S)_{-n} \cong S_{-n}.$$

Since $S_0 = R \neq 0$ and $S_{-n} = 0$ for $n > 0$, we can conclude that $a^*_X(\omega_S) = 0$. □

Let $X = \operatorname{Proj} R$. For each $\mathfrak{p} \in X$, the homogeneous localization $F_{(\mathfrak{p})}$ is a finitely generated graded module over $S_{(\mathfrak{p})}$. Hence, we can define the *projective a^*-invariant*

$$a^*_X(F) := \max\{a^*(F_{(\mathfrak{p})})|\ \mathfrak{p} \in X\}.$$

Note that $H^i_{S_{(\mathfrak{p})+}}(F_{(\mathfrak{p})}) = H^i_{S_+}(F)_{(\mathfrak{p})}$ (cf. [29, Remark 2.2]). Then we always have $a^*_X(F) \leq a^*(F)$. Hence $a^*_X(F)$ is a finite number. Since $a^*_X(F)$ is determined by the local structure of F on X, it can easily be estimated in certain situations. As a demonstration, we show how to estimate $a^*_X(F)$ in the following case which will play an important role in our further investigation.

We say that S is *locally Cohen-Macaulay on X* if $S_{(\mathfrak{p})}$ is a Cohen-Macaulay ring for every $\mathfrak{p} \in X$. This condition holds if, for instance, X is locally Cohen-Macaulay and $\mathcal{I}$ is locally a complete intersection.

Proposition 1.2. *Let $X = \operatorname{Proj} R$ and $S = R[It]$ be as above. Then $a^*_X(S) \geq -1$ and $a^*_X(\omega_S) \geq 0$. Equalities hold if S is locally Cohen-Macaulay on X.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal in X. Then $R_{(\mathfrak{p})}$ is an artinian ring. Since $\mathfrak{p} \not\supseteq I$, we have $I_{(\mathfrak{p})} = R_{(\mathfrak{p})}$. Hence $S_{(\mathfrak{p})} = R_{(\mathfrak{p})}[t]$ is a Cohen-Macaulay ring. By Lemma 1.1, this implies $a^*(S_{(\mathfrak{p})}) = -1$ and $a^*(\omega_{S_{(\mathfrak{p})}}) = 0$. Hence $a^*_X(S) \geq -1$ and $a^*_X(\omega_S) \geq 0$. This proves the first statement. The second statement is an immediate consequence of Lemma 1.1. □

Beside the natural $\mathbb{N}$-graded structure given by the degrees of t, the Rees algebra $S = R[It]$ also has a natural bi-gradation with

$$S_{(m,n)} = (I^n)_m t^n$$

for $(m,n) \in \mathbb{N}^2$. Let Y be the blow-up of X along the ideal sheaf of I. Then $Y = \operatorname{Proj} S$ with respect to this bi-gradation. If $F = \bigoplus_{(m,n)\in\mathbb{Z}^2} F_{(m,n)}$ is a finitely generated bi-graded S-module, then F is also an $\mathbb{Z}$-graded S-module with $F_n = \bigoplus_{m\in\mathbb{Z}} F_{(m,n)}$. Let $\widetilde{F}$ denote the sheaf associated to F on Y. We write $\tilde{F}(n)$ and $\tilde{F}(m,n)$ to denote the twisted $\mathcal{O}_Y$-modules with respect to the $\mathbb{N}$-gradation and the $\mathbb{N}^2$-gradation of S. Moreover, we denote by $\widetilde{F_n}$ the sheafification of F_n on X.

It turns out that $a_X^*(F)$ is a measure for when we can pass from the sheaf cohomology of $\widetilde{F}(m,n)$ on Y to that of $\widetilde{F_n}(m)$ on X.

Proposition 1.3. *Let F be a finitely generated bi-graded S-module. For $n > a_X^*(F)$ we have*
(i) $\pi_*(\widetilde{F}(n)) = \widetilde{F_n}$ *and* $R^i\pi_*(\widetilde{F}(n)) = 0$ *for* $i > 0$,
(ii) $H^i(Y, \widetilde{F}(m,n)) \cong H^i(X, \widetilde{F_n}(m))$ *for all* $m \in \mathbb{Z}$ *and* $i \geq 0$.

Proof. Since (i) is a local statement, we only need to show that it holds locally. Let $\mathfrak{p}$ be a closed point of X, and consider the restriction $\pi_\mathfrak{p}$ of π over an affine open neighborhood $\operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}}$ of $\mathfrak{p}$

$$\pi_\mathfrak{p} : Y_\mathfrak{p} = Y \times_X \operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}} \to \operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}}.$$

We have $\tilde{F}\big|_{Y_\mathfrak{p}} = \widetilde{F_{(\mathfrak{p})}}$, where $\widetilde{F_{(\mathfrak{p})}}$ is the sheaf associated to $F_{(\mathfrak{p})}$ on $Y_\mathfrak{p}$. Thus,

$$R^i\pi_*(\tilde{F}(n))\Big|_{\operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}}} = R^i{\pi_\mathfrak{p}}_*(\widetilde{F_{(\mathfrak{p})}}(n)) = H^i(Y_\mathfrak{p}, \widetilde{F_{(\mathfrak{p})}}(n))\tilde{}.$$

On the other hand, we know by the Serre-Grothendieck correspondence that there are the exact sequence

$$0 \to H^0_{S_{(\mathfrak{p})+}}(F_{(\mathfrak{p})})_n \to (F_{(\mathfrak{p})})_n \to H^0(Y_\mathfrak{p}, \widetilde{F_{(\mathfrak{p})}}(n)) \to H^1_{S_{(\mathfrak{p})+}}(F_{(\mathfrak{p})})_n \to 0$$

and the isomorphisms $H^i(Y_\mathfrak{p}, \widetilde{F_{(\mathfrak{p})}}(n)) \cong H^{i+1}_{S_{(\mathfrak{p})+}}(F_{(\mathfrak{p})})_n$ for $i > 0$. By the definition of $a_X^*(F)$, we know that $H^i_{S_{(\mathfrak{p})+}}(F_{(\mathfrak{p})})_n$ for $n > a_X^*(F)$, $i > 0$. Thus,

$$R^i\pi_*(\tilde{F}(n))\Big|_{\operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}}} = H^i(Y_\mathfrak{p}, \widetilde{F_{(\mathfrak{p})}}(n))\tilde{} = \begin{cases} \widetilde{(F_n)_{(\mathfrak{p})}} & \text{if} \quad i = 0, \\ 0 & \text{if} \quad i > 0, \end{cases}$$

for $n > a_X^*(F)$.

To show (ii) we first observe that $\widetilde{F}(m,n) = \widetilde{F}(n) \otimes \pi^*\mathcal{O}_X(m)$. By the projection formula, we have

$$R^i\pi_*(\widetilde{F}(m,n)) = R^i\pi_*(\widetilde{F}(n)) \otimes \mathcal{O}_X(m) = \begin{cases} \widetilde{F_n}(m) & \text{if} \quad i = 0, \\ 0 & \text{if} \quad i > 0, \end{cases}$$

Hence the conclusion follows from the Leray spectral sequence

$$H^i(X, R^j\pi_*(\tilde{F}(m,n))) \Rightarrow H^{i+j}(Y, \tilde{F}(m,n)).$$

□

Let Y be the blow-up of a projective scheme X along an ideal sheaf $\mathcal{I}$. We say that Y is *locally arithmetic Cohen-Macaulay on* X if there exist R and I such that $X = \operatorname{Proj} R$, $\mathcal{I} = \widetilde{I}$ and $S = R[It]$ is locally Cohen-Macaulay on X.

Corollary 1.4. *Assume that Y is locally arithmetic Cohen-Macaulay on X. Then*
(i) $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ *and* $R^i\pi_*\mathcal{O}_Y = 0$ *for* $i > 0$,
(ii) $H^i(Y, \mathcal{O}_Y(m,0)) \cong H^i(X, \mathcal{O}_X(m))$ *for all* $m \in \mathbb{Z}$, $i \geq 0$.

Proof. With the above notations we have $a^*_X(S) = -1$ by Proposition 1.2. Hence the conclusion follows from Proposition 1.3 by taking $F = S$ and $n = 0$. $\square$

For each n, the graded R-module F_n has an a^*-invariant $a^*(F_n)$, which controls the vanishing of $H^i(X, \widetilde{F_n}(m))$ by the Grothendieck-Serre correspondence. On the other hand, since F is a finitely generated graded module over $S = R[It]$, there exists a number n_0 such that $F_n = I^{n-n_0}F_{n_0}$ for $n \geq n_0$. It was recently discovered that for any finitely generated graded R-module E, the Castelnuovo-Mumford regularity $\operatorname{reg}(I^nE)$ is bounded by a linear function on n with slope $d(I)$ [34, Theorem 2.2] (see also [10, 23] for the case R is a polynomial ring). By definition, we always have

$$a^*(I^nE) \leq \max\{a_i(I^nE) + i|\ i \geq 0\} = \operatorname{reg}(I^nE).$$

Therefore, $a^*(F_n)$ is bounded above by a linear function of the form $d(I)n + \varepsilon$ for $n \geq 1$.

We will denote by $\varepsilon(I)$ the smallest non-negative number such that

$$a^*(I^n) \leq d(I)n + \varepsilon(I)$$

for all $n \geq 1$. Since $\omega_S = \bigoplus_{n\in\mathbb{Z}} \omega_n$ is a finitely generated bi-graded S-module, there is a similar bound for $a^*(\omega_n)$. Note that the R-graded module ω_n is also called an *adjoint-type module* of I because of its relationship to the adjoint ideals [20]. We will denote by $\varepsilon^*(I)$ the smallest non-negative number such that

$$a_i(\omega_n) \leq d(I)n + \varepsilon^*(I)$$

for $i \geq 2$ and $n \geq 1$.

The meaning of these invariants will become more apparent in the next sections. Here we content ourselves with the following observations.

Lemma 1.5. *With the above notations we have*

(i) $H^0(X, \widetilde{S_n}(m)) = S_{(m,n)}$ *and* $H^i(X, \widetilde{S_n}(m)) = 0$ *for* $i > 0$ *and* $m > d(I)n + \varepsilon(I)$,

(ii) $H^i(X, \widetilde{\omega_n}(m)) = 0$ *for* $i > 0$ *and* $m > d(I)n + \varepsilon^*(I)$.

Proof. Since $S_n \cong I^n$, we have $H^i_{R_+}(S_n)_m = 0$ for $i \geq 0$, $m > d(I)n + \varepsilon(I)$ and $n \geq 1$. Hence the first statement follows from the Serre-Grothendieck correspondence, which gives the exact sequence

$$0 \to H^0_{R_+}(S_n)_m \to S_{(m,n)} \to H^0(X, \widetilde{S_n}(m)) \to H^1_{R_+}(S_n)_m \to 0$$

and the isomorphisms

$$H^i(X, \widetilde{S_n}(m)) \cong H^{i+1}_{R_+}(S_n)_m$$

for $i > 0$. The second statement can be similarly proved. $\square$

2. Arithmetically Cohen-Macaulay embeddings of blow-ups

Let X be a projective scheme over a field k. Let $Y \to X$ be the blowing up of X along an ideal sheaf $\mathcal{I}$. We say that Y has an *arithmetically Cohen-Macaulay embedding* if there exists a Cohen-Macaulay standard graded k-algebra A such that $Y \cong \operatorname{Proj} A$.

Let R be a finitely generated standard graded k-algebra and let $I \subset R$ be a homogeneous ideal such that $X = \operatorname{Proj} R$ and $\mathcal{I}$ is the ideal sheaf associated to I. Let $S = R[It]$ be the Rees algebra of R with respect to I. It is well known that $Y \cong \operatorname{Proj} k[(I^e)_c]$ for $c \geq d(I)e+1$ and $e \geq 1$, where $k[(I^e)_c]$ is the algebra generated by all forms of degree c of the ideal power I^e and $d(I)$ denotes the largest degree of a minimal set of homogeneous generators of I (cf. [9, Lemma 1.1]). There is the following simple criterion for the existence of a Cohen-Macaulay algebra $k[(I^e)_c]$ (which is at the same time a criterion for the existence of an arithmetically Cohen-Macaulay embedding).

Lemma 2.1 ([26, Corollary 3.5])**.** *There exists a Cohen-Macaulay ring $k[(I^e)_c]$ for $c \geq d(I)e+1$ if and only if the following conditions are satisfied:*
(i) *Y is equidimensional and Cohen-Macaulay,*
(ii) *$H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, ..., \dim Y - 1$.*

The proof of [26] used a deep result on the relationship between the local cohomology modules of a bi-graded algebra and its diagonal subalgebras [7]. However, the above lemma simply follows from the basic fact that (i) and (ii) are equivalent to the existence of an arithmetically Cohen-Macaulay Veronese embedding of Y (cf. [8, Lemma 1.1]). In fact, the Veronese subalgebras of $k[I_c]$ are exactly the algebras of the form $k[(I^e)_{ce}]$ for $c \geq d(I)+1$, $e \geq 1$. We notice that the statements of [26, Corollary 3.5] and [8, Lemma 1.1] missed the equidimensional condition.

In this section we will determine for which values of c and e is $k[(I^e)_c]$ a Cohen-Macaulay ring. First, we show that there are well-determined invariants ε and e_0 such that $k[(I^e)_c]$ is a Cohen-Macaulay ring for all $c > d(I)e + \varepsilon$ and $e > e_0$.

Theorem 2.2. *Let R be a standard graded k-algebra and let $I \subset R$ be a homogeneous ideal with $\operatorname{ht} I \geq 1$. Let Y be the blow-up of $X = \operatorname{Proj} R$ along the ideal sheaf of I and $S = R[It]$. Assume that*
(i) *Y is equidimensional and Cohen-Macaulay,*
(ii) *$H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, ..., \dim Y - 1$.*
Then $k[(I^e)_c]$ is a Cohen-Macaulay ring for $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e > \max\{a_X^*(S), a_X^*(\omega_S)\}$.

Note first that we always have $\max\{a_X^*(S), a_X^*(\omega_S)\} \geq 0$ by Proposition 1.2 and $\max\{\varepsilon(I), \varepsilon^*(I)\} \geq 0$ by the definition of $\varepsilon(I)$ and $\varepsilon^*(I)$.

Proof. Let $A = k[(I^e)_c]$. Since $c \geq de+1$, we have $Y \cong \operatorname{Proj} A$ [9, Lemma 1.1]. On the other hand, the Rees algebra $S = R[It]$ has a natural bi-gradation with $S_{(m,n)} = (I^n)_m t^n$ and $Y = \operatorname{Proj} S$. Moreover, we may view A as a diagonal subalgebra of S; that is, $A = \bigoplus_{n \in \mathbb{N}} S_{(cn,en)}$ [7, Lemma 1.2]. From this it follows that $A(n)\tilde{} = \mathcal{O}_Y(cn, en)$. Therefore, the Serre-Grothendieck correspondence yields the exact sequence

$$0 \longrightarrow H^0_{A_+}(A) \longrightarrow A \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^0(Y, \mathcal{O}_Y(cn, en)) \longrightarrow H^1_{A_+}(A) \longrightarrow 0$$

and the isomorphisms

$$\bigoplus_{n \in \mathbb{Z}} H^i(Y, \mathcal{O}_Y(cn, en)) \cong H^{i+1}_{A_+}(A)$$

for $i \geq 1$. It is well known that A is a Cohen-Macaulay ring if and only if $H^i_{A_+}(A) = 0$ for $i \neq \dim A$. Therefore, A is a Cohen-Macaulay ring if we can show

$$H^0(Y, \mathcal{O}_Y(cn, en)) = A_n = \begin{cases} 0 & \text{for } n < 0, \\ k & \text{for } n = 0, \\ (I^{en})_{cn} & \text{for } n > 0, \end{cases}$$
$$H^i(Y, \mathcal{O}_Y(cn, en)) = 0 \ (i = 1, ..., \dim Y - 1).$$

For $n = 0$, this follows from the assumption $H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, ..., \dim Y - 1$.

For $n > 0$ we have $cn > d(I)en + \varepsilon(I)n \geq d(I)en + \varepsilon(I)$ and $en > a^*_X(S)n \geq a^*_X(S)$. Hence, using Proposition 1.3 and Lemma 1.5 we get

$$H^0(Y, \mathcal{O}_Y(cn, en)) = H^0(X, \widetilde{I^{en}}(cn)) = (I^{en})_{cn},$$
$$H^i(Y, \mathcal{O}_Y(cn, en)) = H^i(X, \widetilde{I^{en}}(cn)) = 0, \ i = 1, ..., \dim Y - 1.$$

For $n < 0$ we have

$$H^i(Y, \mathcal{O}_Y(cn, en)) = H^{\dim Y - i}(Y, \omega_Y(-cn, -en))$$

for $i \geq 0$. Serre duality can be applied here because Y is equidimensional and Cohen-Macaulay. Since $-cn > -d(I)en - \varepsilon^*(I)n \geq -d(I)en + \varepsilon^*(I)$ and $-en > -a^*_X(\omega_S)n \geq a^*_X(\omega_S)$, using Proposition 1.3 and Lemma 1.5 we get

$$H^{\dim Y - i}(Y, \omega_Y(-cn, -en)) = H^{\dim Y - i}(X, \widetilde{(\omega_S)_{-en}}(-cn)) = 0$$

for $i < \dim Y$. So we get $H^i(Y, \mathcal{O}_Y(cn, en)) = 0$ for all $n < 0$ and $i = 0, ..., \dim Y - 1$. The proof of Theorem 2.2 is now complete. □

The following theorem shows that the bound $e > \max\{a^*_X(S), a^*_X(\omega_S)\}$ of Theorem 2.2 is the best possible.

Theorem 2.3. *Let the notations and assumptions be as in Theorem* 2.2. *Let*

$$e_0 = \max\{a^*_X(S), a^*_X(\omega_S)\}.$$

Then $k[(I^{e_0})_c]$ is not a Cohen-Macaulay ring for $c \gg 0$ if $e_0 \geq 1$.

Proof. Let $A = k[(I^{e_0})_c]$ for $c \gg 0$. As we have seen in the proof of Theorem 2.2, A is not Cohen-Macaulay if $H^0(Y, \mathcal{O}_Y(c, e_0)) \neq (I^{e_0})_c$ or $H^i(Y, \mathcal{O}_Y(c, e_0)) \neq 0$ or $H^i(Y, \mathcal{O}_Y(-c, -e_0)) \neq 0$ for some $i = 1, ..., \dim Y - 1$.

We shall first consider the case $e_0 = a^*_X(S)$. Let q be the smallest integer such that $e_0 = \max\{a_q(S_{(\mathfrak{p})})|\ \mathfrak{p} \in X\}$. Then

$$H^i_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0} = 0, \ i < q, \text{ for all } \mathfrak{p} \in X,$$
$$H^q_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0} \neq 0 \text{ for some } \mathfrak{p} \in X.$$

It is a classical result that there exists $\dim R_{(\mathfrak{p})}$ elements in $I_{(\mathfrak{p})}$ which generates an ideal with the same radical as $I_{(\mathfrak{p})}$. The same also holds for the ideal $S_{(\mathfrak{p})+} = I_{(\mathfrak{p})}t$. From this it follows that $H^{\dim R_{(\mathfrak{p})}+1}_{S_{(\mathfrak{p})+}}(E) = 0$ for any $R_{(\mathfrak{p})}$-module E (cf. [4, Corollary 3.3.3]). Hence

$$q \leq \max\{\dim R_{(\mathfrak{p})}|\ \mathfrak{p} \in X\} = \dim Y.$$

Let $Y_{\mathfrak{p}} = Y \times_X \operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}}$. The Serre-Grothendieck correspondence yields the exact sequence

$$0 \to H^0_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0} \to (S_{(\mathfrak{p})})_{e_0} \to H^0(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) \to H^1_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0} \to 0,$$

and isomorphisms $H^i(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) \cong H^{i+1}_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0}$, $i \geq 1$.

If $q \leq 1$, then $H^0(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) \neq (S_{(\mathfrak{p})})_{e_0} = I^{e_0}_{(\mathfrak{p})}$ for some $\mathfrak{p} \in X$. From this it follows, as in the proof of Proposition 1.3, that $\pi_*(\mathcal{O}_Y(e_0)) \neq \widetilde{I^{e_0}}$. But $\pi_*(\mathcal{O}_Y(e_0))(c)$ and $\widetilde{I^{e_0}}(c)$ are generated by global sections for $c \gg 0$. Therefore, by the projection formula we have

$$H^0(X, \pi_*(\mathcal{O}_Y(c, e_0))) = H^0(X, \pi_*(\mathcal{O}_Y(e_0))(c)) \neq H^0(X, \widetilde{I^{e_0}}(c)) \ = (I^{e_0})_c$$

for $c \gg 0$. Moreover,

$$H^0(Y, \mathcal{O}_Y(c, e_0)) = H^0(X, \pi_*(\mathcal{O}_Y(c, e_0))).$$

Hence $H^0(Y, \mathcal{O}_Y(c, e_0)) \neq (I^{e_0})_c$.

If $q \geq 2$, then the Serre-Grothendieck sequence implies $H^i(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) = 0$ for all $\mathfrak{p} \in X$, $0 < i < q-1$, and $H^{q-1}(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) \neq 0$ for some $\mathfrak{p} \in X$. From this it follows, as in the proof of Proposition 1.3, that

$$\begin{aligned} &R^i\pi_*(\mathcal{O}_Y(e_0)) = 0 \text{ for } 0 < i < q-1, \\ &R^{q-1}\pi_*(\mathcal{O}_Y(e_0)) \neq 0. \end{aligned}$$

By the projection formula, we have

$$\begin{aligned} &R^i\pi_*(\mathcal{O}_Y(c, e_0)) = R^i\pi_*(\mathcal{O}_Y(e_0)) \otimes \mathcal{O}_X(c) = 0 \text{ for } 0 < i < q-1, \\ &R^{q-1}\pi_*(\mathcal{O}_Y(c, e_0)) = R^{q-1}\pi_*(\mathcal{O}_Y(e_0)) \otimes \mathcal{O}_X(c) \neq 0. \end{aligned}$$

Since $\pi_*(\mathcal{O}_Y(c, e_0)) = \pi_*(\mathcal{O}_Y(e_0))(c)$, we also have $H^{q-1}(X, \pi_*(\mathcal{O}_Y(c, e_0))) = 0$ for $c \gg 0$. Therefore, using Leray spectral sequence

$$H^i(X, R^j\pi_*(\mathcal{O}_Y(m, e_0))) \Rightarrow H^{i+j}(Y, \mathcal{O}_Y(m, e_0))$$

we can deduce that

$$H^{q-1}(Y, \mathcal{O}_Y(c, e_0)) = H^0(X, R^{q-1}\pi_*(\mathcal{O}_Y(c, e_0)))$$

for $c \gg 0$. But $R^{q-1}\pi_*(\mathcal{O}_Y(c, e_0))$ is generated by global sections for $c \gg 0$. So we get $H^{q-1}(Y, \mathcal{O}_Y(c, e_0)) \neq 0$.

Let us now consider the case $e_0 = a^*_X(\omega_S)$. Let q be the smallest integer such that $e_0 = \max\{a_q((\omega_S)_{(\mathfrak{p})}) |\ \mathfrak{p} \in X\}$. For $\mathfrak{p} \in X$ we have $(\omega_S)_{(\mathfrak{p})} = \bigoplus_{n>0} H^0(Y_{\mathfrak{p}}, \omega_{Y_{\mathfrak{p}}}(n))$ (see [20, 2.5.2(1) and 2.6.2]). From this it follows that $[H^i_{S_{(\mathfrak{p})+}}((\omega_S)_{(\mathfrak{p})})]_n = 0$ for $n > 0$, $i = 0, 1$. Since $e_0 > 0$, this implies $q > 1$. Similarly as in the first case, we can also show that $q \leq \dim Y$ and that $H^{q-1}(Y, \omega_Y(c, e_0)) \neq 0$ for $c \gg 0$. By Serre duality we get

$$H^{\dim Y - q + 1}(Y, \mathcal{O}_Y(-c, -e_0)) = H^{q-1}(Y, \omega_Y(c, e_0)) \neq 0$$

for $c \gg 0$. This completes the proof of Theorem 2.3. $\square$

We shall see later in Example 2.5 that the bound $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ of Theorem 2.2 is sharp.

Now we want to study the problem when there exists a Cohen-Macaulay ring of the form $k[(I^e)_c]$ for $e \geq 1$.

Theorem 2.4. *Let R be an equidimensional standard graded k-algebra and let I be a homogeneous ideal of R with $\operatorname{ht} I \geq 1$. Let $X = \operatorname{Proj} R$ and $S = R[It]$. Assume that S is locally Cohen-Macaulay on X. Then, there exists a Cohen-Macaulay ring $k[(I^e)_c]$ with $c \geq d(I)e + 1$ if and only if $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, ..., \dim X - 1$. In particular, this condition implies that $k[(I^e)_c]$ is a Cohen-Macaulay ring for $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.*

Proof. Let Y be the blow-up of X along the ideal sheaf of I. The assumption implies that Y is equidimensional and Cohen-Macaulay. Since S is locally Cohen-Macaulay over X, Y is locally arithmetic Cohen-Macaulay over X. Applying Corollary 1.4, we have $H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and $H^i(Y, \mathcal{O}_Y) = H^i(X, \mathcal{O}_X)$ for $i > 0$. Therefore, the first statement follows from Lemma 2.1. Moreover, we have $\max\{a_X^*(S), a_X^*(\omega_S)\} = 0$ by Proposition 1.2. Hence the second statement follows from Theorem 2.2. □

Note that the condition $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, ..., \dim X - 1$ is satisfied if R is a Cohen-Macaulay ring.

The following example shows that the bound $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ is sharp.

Example 2.5. Let $R = k[x_0, x_1, x_2]$ and $I = (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4)$. It is easy to see that $S = R[It]$ is locally Cohen-Macaulay on $X = \operatorname{Proj} R$. We have $I^n = (x_1, x_2)^{4n}$ for all $n \geq 2$. We have

$$a^*(I^n) = \begin{cases} 4 & \text{if} \quad n = 1, \\ 4n - 1 & \text{if} \quad n \geq 2. \end{cases}$$

From this it follows that $\varepsilon(I) = 0$. To compute $\varepsilon^*(I)$ we approximate I by the ideal $J = (x_1, x_2)^4$. Put $S^* = R[Jt]$. Then we have the exact sequence

$$0 \to R[It] \to R[Jt] \to k \to 0.$$

From this it follows that $\omega_S = \omega_{S^*}$. Note that S^* is a Veronese subring of the ring $T = R[(x_1, x_2)t]$ and that T is a Gorenstein ring with $\omega_T = T(-2)$. Then $\omega_{S^*} = \bigoplus_{n \geq 1}(x_1, x_2)^{4n-2}$. We have

$$a^*(\omega_n) = a^*((x_1, x_2)^{4n-2}) = 4n - 3$$

for $n \geq 1$. Hence $\varepsilon^*(I) = 0$. By Theorem 2.4, these facts imply that $k[(I^e)_c]$ is Cohen-Macaulay for $c > 4e$ and $e \geq 1$ (which can be also verified directly). On the other hand, for $c = 4$ and $e = 1$, the ring $k[I_4] = k[x_1^4, x_1^3x_2, x_1x_2^3, x_2^4]$ is not Cohen-Macaulay.

There have been various criteria for the Cohen-Macaulayness of Rees algebras (cf. [33, 18, 27, 30, 1, 21, 28]), so that one can construct various classes of ideals I for which S is locally Cohen-Macaulay on X. We list here only the most interesting applications of Theorem 2.4.

Corollary 2.6. *Let R be a Cohen-Macaulay standard graded k-algebra. Let $I \subset R$ be a homogeneous ideal with $\operatorname{ht} I \geq 1$ which is a locally complete intersection. Then $k[(I^e)_c]$ is a Cohen-Macaulay ring for all $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.*

Proof. Let $X = \operatorname{Proj} R$. The assumption on I means that $I_\mathfrak{p}$ is a complete intersection ideal in $R_\mathfrak{p}$ for $\mathfrak{p} \in X$. Therefore, $R_{(\mathfrak{p})}[I_{(\mathfrak{p})}t]$ is Cohen-Macaulay for all $\mathfrak{p} \in X$. Hence, $S = R[It]$ is locally Cohen-Macaulay on X. The result follows from Theorem 2.4. □

Corollary 2.7. *Let R be a polynomial ring over a field k of characteristic zero and $I \subset R$ a non-singular homogeneous ideal with* $\operatorname{ht} I \geq 1$. *Then, $k[(I^e)_c]$ is a Cohen-Macaulay ring for $c > d(I)e + \varepsilon(I)$ and $e \geq 1$.*

Proof. The assumption implies that I is locally a complete intersection. Hence $S = R[It]$ is locally Cohen-Macaulay on $X = \operatorname{Proj} R$. Let $Y = \operatorname{Proj} S$. Then Y is a projective non-singular scheme. Let m, n be positive integers with $m \geq d(I)n + 1$. Then $\mathcal{O}_Y(m,n)$ is a very ample invertible sheaf on Y because $Y \cong \operatorname{Proj} k[(I^n)_m]$ [9, Lemma 1.1]. Let ω_S be the canonical module of S and $\omega_Y = \widetilde{\omega_S}$. Then $H^i(Y, \omega_Y(m,n)) = 0$ for $i \geq 1$ by Kodaira's vanishing theorem. On the other hand, we have

$$H^i(Y, \omega_Y(m,n)) = H^i(X, \widetilde{(\omega_S)_n}(m))$$

by Proposition 1.3. Therefore, $H^i(X, \widetilde{(\omega_S)_n}(m)) = 0$ for $i \geq 1$. Using the Serre-Grothendieck correspondence we can deduce that $H^i_{R_+}((\omega_S)_n)_m = 0$ for $i \geq 2$. Hence $\varepsilon^*(I) = 0$. Now, the conclusion follows from Corollary 2.6. $\square$

Remark 2.8. A similar result to Theorem 2.4 was already given by Cutkosky and Herzog [9, Theorem 4.1] when R is Cohen-Macaulay. Their result shows the existence of a constant δ such that $k[(I^e)_c]$ is Cohen-Macaulay for $c \geq \delta e$, $e > 0$, under some assumptions on the associated graded ring $\bigoplus_{n \geq 0} I^n/I^{n+1}$. It is not hard to see that these assumptions imply $\max\{a^*_X(S), a^*_X(\omega_S)\} \leq 0$ (see [9, Lemma 2.1 and Lemma 2.2]). Hence their result is also a consequence of Theorem 2.2. Similar statements to the above two corollaries were also given in [9] but without any information on the slope δ.

It is not easy to compute $\varepsilon(I)$ explicitly, even when I is a non-singular ideal in a polynomial ring. By a famous result of Bertram, Ein and Lazarsfeld [3] we know that if I is the ideal of a smooth complex variety cut out scheme-theoretically by hypersurfaces of degree $d_1 \geq \cdots \geq d_m$, then

$$a_i(I^n) \leq d_1 n + d_2 + \cdots + d_m - \operatorname{ht} I$$

for $i \geq 2$ and $n \geq 1$. However, we do not know any bound for $a_1(I^n)$ in terms of $d_1, ..., d_m$. It would be of interest to find such a bound. In general, if we happen to know the minimal free resolution of S over a bi-graded polynomial ring, then we can estimate $\varepsilon(I)$ in terms of the shifts of syzygy modules of the resolution [10].

In the case when I is the defining ideal of a scheme of fat points, we know an explicit bound for $a^*(I^n)$, namely $a^*(I^n) \leq \operatorname{reg}(I)n$ for all $n \geq 1$ [6, 13]. As a consequence, we immediately obtain the following result of Geramita, Gimigliano and Pitteloud.

Corollary 2.9 ([13, Theorem 2.4])**.** *Let R be a polynomial ring over a field k of characteristic zero, and let $I \subset R$ be the defining ideal of a scheme of fat points in* $\operatorname{Proj} R$. *Then, $k[(I^e)_c]$ is a Cohen-Macaulay ring for $c > \operatorname{reg}(I)e$ and $e \geq 1$.*

Proof. By definition, the ideal I has the form $I = \bigcap_{i=1}^s \mathfrak{p}_i^{m_i}$, where $\mathfrak{p}_i$ is the defining prime ideal of a closed point in $X = \operatorname{Proj} R$ and $m_i \in \mathbb{N}$. Then $R_{(\mathfrak{p})}[I_{(\mathfrak{p})}t]$ is Cohen-Macaulay for all $\mathfrak{p} \in X$. In fact, we may assume that $\mathfrak{p} = \mathfrak{p}_i$ for some i. Then $\mathfrak{p}$ is a complete intersection and $R_{(\mathfrak{p})}[I_{(\mathfrak{p})}t] = R_{(\mathfrak{p})}[\mathfrak{p}^{m_i}_{(\mathfrak{p})}t]$ is a Veronese subalgebra of $R_{(\mathfrak{p})}[\mathfrak{p}_{(\mathfrak{p})}t]$. Since $R_{(\mathfrak{p})}[\mathfrak{p}_{(\mathfrak{p})}t]$ is a Cohen-Macaulay ring, so is $R_{(\mathfrak{p})}[I_{(\mathfrak{p})}t]$. Thus,

$S = R[It]$ is locally Cohen-Macaulay on X. This argument also shows that $Y = \operatorname{Proj} S$ is smooth. Using the Kodaira vanishing theorem we can show, as in the proof of Corollary 2.7, that $\varepsilon^*(I) = 0$. The conclusion now follows from the proof of Theorem 2.4 when we replace the slope $d(I)$ by $\operatorname{reg}(I) \geq d(I)$ and $\varepsilon(I)$ by 0 because of the bound $a^*(I^n) \leq \operatorname{reg}(I)n$. □

It was asked in [7] whether there exists a Cohen-Macaulay ring $k[(I^e)_c]$ for $c \gg e \gg 0$ if R is a polynomial ring and $R[It]$ is Cohen-Macaulay. This question has been positively settled in [25, Theorem 4.5]. We can make this result more precise as follows.

Corollary 2.10. *Let R be a Cohen-Macaulay standard graded k-algebra. Let $I \subset R$ be a homogeneous ideal with $\operatorname{ht} I \geq 1$ such that $R[It]$ is Cohen-Macaulay. Then $k[(I^e)_c]$ is a Cohen-Macaulay ring for all $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.*

3. Arithmetically Cohen-Macaulay blow-ups

Let X be a projective scheme over a field k. Let $\pi : Y \to X$ be the blowing up of X along an ideal sheaf $\mathcal{I}$. We say that Y is an *arithmetically Cohen-Macaulay blow-up* of X if there is a standard graded k-algebra R and a homogeneous ideal $J \subset R$ with $\operatorname{ht} J \geq 1$ such that $X = \operatorname{Proj} R$, $\mathcal{I} = \widetilde{J}$, and $R[Jt]$ is a Cohen-Macaulay ring. The aim of this section is to characterize arithmetically Cohen-Macaulay blow-ups.

Let R be a finitely generated standard graded k-algebra, and let I be a homogeneous ideal of R with $\operatorname{ht} I \geq 1$, such that $X = \operatorname{Proj} R$ and $\mathcal{I} = \widetilde{I}$. Let $d(I)$ denote the maximal degree of the elements of a homogeneous basis of I. For any ideal J generated by $(I^e)_c$ with $c \geq d(I)e$ we have $J_n = (I^e)_n$ for all $n \geq c$ so that $\mathcal{I}^e = \widetilde{J}$. Hence $Y = \operatorname{Proj} R[Jt]$. The Rees algebra $R[(I^e)_c t] = R[Jt]$ is called a *truncated Rees algebra* of I^e [15, 8]. We may strengthen the problem on the characterization of arithmetically Cohen-Macaulay blow-ups by asking the question: When does there exist a Cohen-Macaulay truncated Rees algebra $R[(I^e)_c t]$? To solve this problem we shall need the following result of Hyry.

Let T be a standard bi-graded algebra over a field k, that is, T is generated over k by the elements of degree $(1,0)$ and $(0,1)$. Let M denote the maximal graded ideal of T and define

$$a^1(T) := \max\{m \mid \text{there is } n \text{ such that } H_M^{\dim T}(T)_{(m,n)} \neq 0\},$$
$$a^2(T) := \max\{n \mid \text{there is } m \text{ such that } H_M^{\dim T}(T)_{(m,n)} \neq 0\}.$$

Theorem 3.1 ([19, Theorem 2.5])**.** *Let T be a standard bi-graded k-algebra with $a^1(T)$, $a^2(T) < 0$. Let $Y = \operatorname{Proj} T$. Then T is Cohen-Macaulay if and only if the following conditions are satisfied:*

$$H^0(Y, T(m,n)\tilde{\ }) \cong T_{(m,n)} \text{ for } m, n \geq 0,$$
$$H^i(Y, T(m,n)\tilde{\ }) = 0 \text{ for } m, n \geq 0,\ i > 0,$$
$$H^i(Y, T(m,n)\tilde{\ }) = 0 \text{ for } m, n < 0,\ i < \dim T - 2.$$

Let $J \subset R$ be an arbitrary ideal generated by forms of degree c and put $T = R[Jt]$. Then T can be equipped with another bi-gradation given by

$$T_{(m,n)} = (J^n)_{m+cn} t^n$$

for $(m,n) \in \mathbb{N}^2$. With this bi-gradation, T is a standard bi-graded k-algebra. Comparing with the natural bi-gradation of T considered in the preceding sections, we see that both bi-gradations share the same bi-homogeneous elements and the same relevant bi-graded ideals. Therefore, $\operatorname{Proj} T$ with respect to these bi-gradations are isomorphic.

Lemma 3.2. *Let $T = R[Jt]$ be as above. Then*
(i) $a^1(T) \leq \max\{a^*(J^n) - nc|\ n \geq 0\}$,
(ii) $a^2(T) < 0$.

Proof. To prove (i) we will show more, namely, that $H^i_M(T)_{(m,n)} = 0$ for $m > \max\{a^*(J^n) - nc|\ n \geq 0\}$ and $i \geq 0$. Let T_1 denote the ideal of T generated by the homogeneous elements of degree $(1, 0)$. Then, by [19, Lemma 2.3], we only need to show that $H^i_{T_1}(T)_{(m,n)} = 0$ for $m > \max\{a^*(J^n) - nc|\ n \geq 0\}$ and $i \geq 0$. Since T_1 is generated by R_+, we always have

$$H^i_{T_1}(T)_{(m,n)} = \begin{cases} 0 & \text{for} \quad n < 0, \\ H^i_{R_+}(J^n)_{m+nc} & \text{for} \quad n \geq 0. \end{cases}$$

But $H^i_{R_+}(J^n)_{m+nc} = 0$ for $m + nc > a^*(J^n)$, $n \geq 0$. Therefore, $H^i_{T_1}(T)_{(m,n)} = 0$ for $m > \max\{a^*(J^n) - nc|\ n \geq 0\}$, as required.

To prove (ii) we first observe that

$$a^2(T) = \max\{n|\ H^{\dim T}_M(T)_n \neq 0\},$$

where the $\mathbb{Z}$-gradation comes from the natural grading $T_n = J^n t^n$, $n \geq 0$. Therefore, the conclusion $a^2(T) < 0$ follows from [33, Corollary 3.2]. □

Corollary 3.3. *Let R be a standard graded k-algebra with $a^*(R) < 0$ and let $I \subset R$ be a homogeneous ideal with $\operatorname{ht} I \geq 1$. Let $T = R[(I^e)_c t]$ for some fixed integers $c > d(I)e + \varepsilon(I)$ and $e \geq 1$. Then $a^1(T) < 0$ and $a^2(T) < 0$.*

Proof. Let J be the ideal of R generated by $(I^e)_c$. By Lemma 3.2 we only need to prove that $a^*(J^n) < nc$ for $n \geq 0$. For $n = 0$, this follows from the assumption $a^*(R) < 0$. For $n \geq 1$, we will approximate $a^*(J)$ by $a^*(I^{en})$. Since J^n is generated by elements of degree cn and since $cn > d(I)en \geq d(I^{en})$, we have $(I^{en}/J^n)_m = 0$ for $m \geq cn$. From this it follows that $H^0(I^{en}/J^n) = I^{en}/J^n$ and $H^i(I^{en}/J^n) = 0$ for $i > 0$. Therefore, from the exact sequence

$$0 \longrightarrow J^n \longrightarrow I^{en} \longrightarrow I^{en}/J^n \longrightarrow 0$$

we can deduce that $H^i(J^n)_m = H^i(I^{en})_m$ for $m \geq cn$ and $i \geq 0$. This implies

$$a^*(J^n) \leq \max\{cn - 1, a^*(I^{en})\}.$$

By the definition of $\varepsilon(I)$ we have $a^*(I^{en}) \leq d(I)en + \varepsilon(I) \leq cn - 1$. Therefore, $a^*(J^n) \leq cn - 1$ for $n \geq 1$. □

We are now ready to give a necessary and sufficient condition for the existence of a Cohen-Macaulay truncated Rees algebra.

Theorem 3.4. *Let R be a standard graded k-algebra with $a^*(R) < 0$ and let $I \subset R$ be a homogeneous ideal with $\operatorname{ht} I \geq 1$. Let $X = \operatorname{Proj} R$, $S = R[It]$ and $Y = \operatorname{Proj} S$. Then there exists a Cohen-Macaulay ring $R[(I^e)_c t]$ with $c \geq d(I)e$ if and only if the following conditions are satisfied:*
(i) *Y is equidimesional and Cohen-Macaulay,*
(ii) *$\pi_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^i \pi_* \mathcal{O}_Y = 0$ for $i > 0$.*
Moreover, these conditions imply that $R[(I^e)_c t]$ is a Cohen-Macaulay ring for $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e > \max\{a^*_X(S), a^*_X(\omega_S)\}$.

Proof. Let J be the ideal of R generated by $(I^e)_c$ and $T = R[Jt]$ for a fixed pair of positive integers c, e with $c \geq d(I)e$. Then $Y \cong \operatorname{Proj} T$. If T is a Cohen-Macaulay ring, then (i) is obviously satisfied and Y is locally arithmetic Cohen-Macaulay over X. (ii) follows from Corollary 1.4.

To prove the converse we equip T with the aforementioned bi-gradation. Set $e_0 = \max\{a_X^*(S), a_X^*(\omega_S)\}$. We will use Theorem 3.1 to prove that T is Cohen-Macaulay for $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e > e_0$. By Corollary 3.3 we have $a^1(T) < 0$ and $a^2(T) < 0$. From the bi-gradation of T we see that

$$T(m,n)\tilde{} = \mathcal{O}_Y(m + cn, en),$$

where $\mathcal{O}_Y(m + cn, n)$ denotes the twisted $\mathcal{O}_Y$-module with respect to the natural bi-gradation of S. If $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$, then we can show as in the proof of Proposition 1.3 that

$$H^i(Y, \mathcal{O}_Y(m, 0)) = H^i(X, \mathcal{O}_X(m))$$

for $i \geq 0$. Since $a^*(R) < 0$, we have $H^i_{R_+}(R)_m = 0$ for all $m \geq 0$ and $i \geq 0$. Using the Serre-Grothendieck correspondence between sheaf cohomology of X and local cohomology of R we can deduce that $H^0(X, \mathcal{O}_X(m)) = R_m$ and $H^i(X, \mathcal{O}_X(m)) = 0$ for $i > 0$. Therefore,

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(m, 0)) &= R_m = T_{(m,0)}, \\ H^i(Y, \mathcal{O}_Y(m, 0)) &= 0, \ i > 0. \end{aligned}$$

For $m \geq 0$ and $n > 0$ we have $m + cn > d(I)en + \varepsilon(I)$. Therefore, using Proposition 1.3 and Lemma 1.5 we get

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(m + cn, en)) &= T_{(m,n)}, \\ H^i(Y, \mathcal{O}_Y(m + cn, en)) &= 0, \ i > 0, \end{aligned}$$

for $e > e_0$. For $m, n < 0$ we can show, similarly as above, that

$$H^i(Y, \omega_Y(-m - cn, -en)) = 0$$

for $i > 0$ and $e > e_0$. If Y is equidimensional and Cohen-Macaulay, we can apply Serre duality and obtain

$$H^i(Y, \mathcal{O}_Y(m + cn, en)) = 0, \ i < \dim Y.$$

Passing from $\mathcal{O}_Y(m + cn, en)$ to $T(m,n)\tilde{}$ we get

$$\begin{aligned} &H^0(Y, T(m,n)\tilde{}) \cong T_{(m,n)} \text{ for } m, n \geq 0, \\ &H^i(Y, T(m,n)\tilde{}) = 0 \text{ for } m, n \geq 0, \ i > 0, \\ &H^i(Y, T(m,n)\tilde{}) = 0 \text{ for } m, n < 0, \ i < \dim T - 2. \end{aligned}$$

By Theorem 3.1, these conditions imply that T is a Cohen-Macaulay ring. The proof of Theorem 3.4 is now complete. □

The following example shows that the condition $a^*(R) < 0$ is not necessary for the existence of a Cohen-Macaulay truncated Rees algebra. It also shows that in general, the existence of a Cohen-Macaulay truncated Rees algebra does not imply the existence of a linear bound on c ensuring the Cohen-Macaulayness of $R[(I^e)_c t]$.

Example 3.5. Take $R = k[x,y,z]/(xy^2 - z^3)$, the coordinate ring of a cuspidal plane curve, and $I = (x) \subseteq R$, a homogeneous ideal with $\operatorname{ht} I = 1$. Then R is a two-dimensional Cohen-Macaulay ring with $a^*(R) = 0$. It is obvious that $R[(I^e)_e t] = R[It]$ is a Cohen-Macaulay ring for $e \geq 1$. For $c > e$ we have $R[(I^e)_c t] \cong R[(x,y,z)^{c-e}t]$. It is easy to check that the reduction number of the ideal $(x,y,z)^{c-e}$ is greater than 1. By [14], this implies that $R[(x,y,z)^{c-e}t]$ is not Cohen-Macaulay for any $c > e$.

Now we will show that the bound $e > e_0$ in Theorem 3.4 is once again best possible.

Theorem 3.6. *Let the notations and assumptions be as in Theorem* 3.4*. Let*

$$e_0 = \max\{a_X^*(S), a_X^*(\omega_S)\}.$$

Then $R[(I^{e_0})_c t]$ is not a Cohen-Macaulay ring for $c \geq d(I)e_0$ if $e_0 \geq 1$.

Proof. Let $T = R[(I^{e_0})_c t]$ for some $c \geq d(I)e_0$, and suppose $e_0 \geq 1$. Note that $(I^{e_0})_c$ and I^{e_0} defines the same ideal sheaf in $\mathcal{O}_X$. Consider the natural $\mathbb{N}$-grading of T and S given by the degree of t. For any $\mathfrak{p} \in X$, the ring $T_{(\mathfrak{p})}$ is isomorphic to the e_0-th Veronese subring of $S_{(\mathfrak{p})}$. Hence

$$H^i_{T_{(\mathfrak{p})+}}(T_{(\mathfrak{p})})_1 = H^i_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0},$$
$$H^i_{T_{(\mathfrak{p})+}}((\omega_T)_{(\mathfrak{p})})_1 = H^i_{S_{(\mathfrak{p})+}}((\omega_S)_{(\mathfrak{p})})_{e_0},$$

for $i \geq 0$. By the definition of e_0 there exists $\mathfrak{p} \in X$ and $i \geq 0$ such that either $H^i_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0} \neq 0$ or $H^i_{S_{(\mathfrak{p})+}}((\omega_S)_{(\mathfrak{p})})_{e_0} \neq 0$. Therefore, $\max\{a^*(T), a^*(\omega_T)\} \geq 1$. By Lemma 1.1, this implies that T is not a Cohen-Macaulay ring. □

From Theorem 3.4 we can derive the following sufficient condition for the existence of a truncated Cohen-Macaulay Rees algebra.

Theorem 3.7. *Let R be an equidimensional standard graded k-algebra with $a^*(R) < 0$ and let $I \subset R$ be a homogeneous ideal with $\operatorname{ht} I \geq 1$. Let $X = \operatorname{Proj} R$ and $S = R[It]$. Assume that S is locally Cohen-Macaulay on X. Then $R[(I^e)_c t]$ is a Cohen-Macaulay ring for $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.*

Proof. It is obvious that the assumptions imply that Y is equidimensional and Cohen-Macaulay. The condition $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$ follows from Corollary 1.4. Hence the conclusion follows from Theorem 3.4. □

The above condition is also a necessary condition for the existence of a truncated Cohen-Macaulay Rees algebra of the form $R[I_c t]$ $(e = 1)$.

Corollary 3.8. *Let R be a standard graded k-algebra with $a^*(R) < 0$ and let $I \subset R$ be a homogeneous ideal with $\operatorname{ht} I \geq 1$. Let $X = \operatorname{Proj} R$ and $S = R[It]$. Then there exists a Cohen-Macaulay ring $R[I_c t]$ with $c \geq d(I)$ if and only if S is locally Cohen-Macaulay on X.*

Proof. By Theorem 3.7 we only need to show that if $R[I_c t]$ is a Cohen-Macaulay ring for some $c \geq d(I)$, then S is locally Cohen-Macaulay on X. But this is obvious because (I_c) and I define the same ideal sheaf and $R[I_c t]$ is locally Cohen-Macaulay on X. □

Using Theorem 3.7 we obtain several classes of Cohen-Macaulay Rees algebras.

Corollary 3.9 (cf. [8, Corollary 2.2.1(2)] for the case $e = 1$)**.** *Let R be a Cohen-Macaulay standard graded k-algebra with $a(R) < 0$. Let $I \subset R$ be a homogeneous ideal with $\operatorname{ht} I \geq 1$ which is locally a complete intersection. Then $R[(I^e)_c t]$ is a Cohen-Macaulay ring for all $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$ and $e \geq 1$.*

Proof. As in the proof of Corollary 2.6, $S = R[It]$ is locally Cohen-Macaulay over $X = \operatorname{Proj} R$. Since the assumption on R implies $a^*(R) < 0$, the conclusion follows from Theorem 3.7. □

Corollary 3.10. *Let R be a polynomial ring over a field k of characteristic zero and let $I \subset R$ be a non-singular homogeneous ideal. Then $R[(I^e)_c t]$ is a Cohen-Macaulay ring for all $c > d(I)e + \varepsilon(I)$ and $e \geq 1$.*

Proof. We have seen in the proof of Corollary 2.7 that $\varepsilon^*(I) = 0$. Hence the assertion follows from Corollary 3.9. □

Corollary 3.11 (cf. [15, Theorem 2.4] for the case $e = 1$)**.** *Let R be a polynomial ring over a field k of characteristic zero and let $I \subset R$ be the defining ideal of a scheme of fat points in $\operatorname{Proj} R$. Then $R[(I^e)_c t]$ is a Cohen-Macaulay ring for $c > \operatorname{reg}(I)e$.*

Proof. The proof follows from Theorem 3.7 with the same lines of arguments as in the proof of Corollary 2.9. □

Now we will use Theorem 3.7 to find a criterion for arithmetically Cohen-Macaulay blow-ups. Recall that the blow-up Y of a projective scheme X along an ideal sheaf $\mathcal{I}$ is said to be *locally arithmetic Cohen-Macaulay on X* if there exist a standard graded algebra R over a field and a homogeneous ideal $I \subset R$ such that $X = \operatorname{Proj} R$, $\mathcal{I} = \widetilde{I}$ and $S = R[It]$ is locally Cohen-Macaulay on X.

Theorem 3.12. *Let X be a projective scheme over a field k such that $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Let $Y \longrightarrow X$ be a proper birational morphism. Then Y is an arithmetically Cohen-Macaulay blow-up if and only if Y is equidimensional and locally arithmetic Cohen-Macaulay on X.*

Proof. Suppose Y is an arithmetically Cohen-Macaulay blow-up of X. Let R be a standard graded algebra over k, and let I be a homogeneous ideal of R, such that $X = \operatorname{Proj} R$, Y is the blow-up of X along the ideal sheaf $\widetilde{I}$, and $S = R[It]$ is a Cohen-Macaulay ring. Then, $\mathcal{O}_{X,x}[\mathcal{I}_x t] = S_{(\mathfrak{p})}$ is obviously Cohen-Macaulay for all $\mathfrak{p} \in X$. Thus, Y is locally arithmetic Cohen-Macaulay on X.

Conversely, suppose Y is equidimensional and locally arithmetic Cohen-Macaulay on Y. Then there exist a standard graded k-algebra R and a homogeneous ideal $I \subset R$ such that $X = \operatorname{Proj} R$, Y is the blow-up of X along the ideal sheaf of I, and $R[It]$ is locally Cohen-Macaulay on X. The assumption on the sheaf cohomology of X implies that $H^i_{R_+}(R)_0 = 0$ for $i \geq 0$. Without restriction we may replace R by a suitable Veronese subalgebra and obtain $H^i_{R_+}(R)_n = 0$ for all $n \geq 0$ or, equivalently, $a^*(R) < 0$. Now we may apply Theorem 3.7 to find a Cohen-Macaulay Rees algebra $R[I_c t]$ with $c \gg 0$. Since the ideal (I_c) defines the same ideal sheaf $\widetilde{I}$, we can conclude that Y is an arithmetically blow-up of X. □

Note added in proof

After the manuscript was sent for publication, we were informed that Olga Lavila-Vidal obtained similar results in her thesis (ArXiv:math.AC/0407041) as our Theorem 2.4 and Corollary 2.6. She did not have an explicit description for the constant term as we did though.

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