

## ASYMPTOTIC BEHAVIOUR OF ARITHMETICALLY COHEN-MACAULAY BLOW-UPS

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**ABSTRACT.** This paper addresses problems on arithmetic Macaulayfications of projective schemes. We give a surprising complete answer to a question posed by Cutkosky and Herzog. Let  $Y$  be the blow-up of a projective scheme  $X = \text{Proj } R$  along the ideal sheaf of  $I \subset R$ . It is known that there are embeddings  $Y \cong \text{Proj } k[(I^e)_c]$  for  $c \geq d(I)e + 1$ , where  $d(I)$  denotes the maximal generating degree of  $I$ , and that there exists a Cohen-Macaulay ring of the form  $k[(I^e)_c]$  (which gives an arithmetic Macaulayfication of  $X$ ) if and only if  $H^0(Y, \mathcal{O}_Y) = k$ ,  $H^i(Y, \mathcal{O}_Y) = 0$  for  $i = 1, \dots, \dim Y - 1$ , and  $Y$  is equidimensional and Cohen-Macaulay. We show that under these conditions, there are well-determined invariants  $\varepsilon$  and  $e_0$  such that  $k[(I^e)_c]$  is Cohen-Macaulay for all  $c > d(I)e + \varepsilon$  and  $e > e_0$ , and that these bounds are the best possible. We also investigate the existence of a Cohen-Macaulay Rees algebra of the form  $R[(I^e)_c t]$ . If  $R$  has negative  $a^*$ -invariant, we prove that such a Cohen-Macaulay Rees algebra exists if and only if  $\pi_* \mathcal{O}_Y = \mathcal{O}_X$ ,  $R^i \pi_* \mathcal{O}_Y = 0$  for  $i > 0$ , and  $Y$  is equidimensional and Cohen-Macaulay. Moreover, these conditions imply the Cohen-Macaulayness of  $R[(I^e)_c t]$  for all  $c > d(I)e + \varepsilon$  and  $e > e_0$ .

### INTRODUCTION

Let  $X$  be a projective scheme over a field  $k$ . An arithmetic Macaulayfication of  $X$  is a proper birational morphism  $\pi : Y \rightarrow X$  such that  $Y$  has an arithmetically Cohen-Macaulay embedding, i.e. there exists a Cohen-Macaulay standard graded  $k$ -algebra  $A$  such that  $Y \cong \text{Proj } A$ . Inspired by the problem of resolution of singularities, it was asked when  $X$  has an arithmetic Macaulayfication. The local version of this problem (arithmetic Macaulayfication of local rings) has been extensively studied in the literature and recently solved by Kawasaki [22]. An important aspect of the global problem is to determine, given a proper birational morphism  $Y \rightarrow X$ , if  $Y$  has an arithmetically Cohen-Macaulay embedding, and if it does, which embeddings of  $Y$  are arithmetically Cohen-Macaulay.

Let  $R$  be a standard graded  $k$ -algebra and let  $I \subset R$  be a homogeneous ideal such that  $X = \text{Proj } R$  and  $Y$  is the blow-up of  $X$  along the ideal sheaf of  $I$ . It was observed by Cutkosky and Herzog [9] that  $Y \cong \text{Proj } k[(I^e)_c]$  for  $c \geq d(I)e + 1$ ,

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where  $(I^e)_c$  denotes the vector space of forms of degree  $c$  of the ideal power  $I^e$  and  $d(I)$  is the maximal degree of the elements of a homogeneous basis of  $I$ . In other words,  $Y$  can be embedded into a projective space by the complete linear system  $|cE_0 - eE|$ , where  $E$  denotes the exceptional divisor and  $E_0$  is the pull-back of a general hyperplane in  $X$ . By [26] we know that there exists a Cohen-Macaulay ring  $k[(I^e)_c]$  for  $c \geq d(I)e + 1$  if and only if  $Y$  satisfies the following conditions:

- $Y$  is equidimensional and Cohen-Macaulay,
- $H^0(Y, \mathcal{O}_Y) = k$  and  $H^i(Y, \mathcal{O}_Y) = 0$  for  $i = 1, \dots, \dim Y - 1$ .

In the first part of this paper, we study the problem of which values of  $c$  and  $e$  is  $k[(I^e)_c]$  a Cohen-Macaulay ring. This problem originated from a beautiful result of Geramita, Gimigliano and Pitteloud [13] which shows that if  $I$  is the defining ideal of a set of fat points in a projective space over a field of characteristic zero, then  $k[I_c]$  is a Cohen-Macaulay ring for all  $c \geq \text{reg}(I)$ , where  $\text{reg}(I)$  is the Castelnuovo-Mumford regularity of  $I$ . This result initiated the study on the Cohen-Macaulayness of algebras of the form  $k[(I^e)_c]$  first in [7] and then in [9, 25, 26, 16]. In particular, Cutkosky and Herzog [9] showed that if  $I$  is a locally complete intersection ideal, then there exists a constant  $\delta$  such that  $k[(I^e)_c]$  is Cohen-Macaulay for  $c \geq \delta e$ . They raised the question of when there is a linear bound on  $c$  and  $e$  ensuring that  $k[(I^e)_c]$  is a Cohen-Macaulay ring.

Our results will give a complete answer to this question. We show that if the above two conditions are satisfied, then there exist well-determined invariants  $\varepsilon$  and  $e_0$  such that  $k[(I^e)_c]$  is a Cohen-Macaulay ring for all  $c > d(I)e + \varepsilon$  and  $e > e_0$  (Theorem 2.2). The invariant  $e_0$  is a projective version of the  $a^*$ -invariant, which is the largest non-vanishing degree of the graded local cohomology modules [29, 32]. The invariant  $\varepsilon$  comes from the asymptotic linearity of the Castelnuovo-Mumford regularity of powers of ideals ([31, 10, 23, 34]). We will see that the bounds  $c > d(I)e + \varepsilon$  and  $e > e_0$  are the best possible (Theorem 2.3 and Example 2.5). The existence of linear bounds on  $c$  and  $e$  is not hard to prove. The novelty we claim here is that an explicit description for the best possible bounds is obtained. Moreover, if the Rees algebra  $R[It]$  is locally Cohen-Macaulay on  $X$ , then  $e_0 = 0$  and we can replace the second condition by the weaker condition that  $H^0(X, \mathcal{O}_X) = k$  and  $H^i(X, \mathcal{O}_X) = 0$  for  $i = 1, \dots, \dim X - 1$  (Theorem 2.4). These results strengthen and unify all previously-known results on the Cohen-Macaulayness of  $k[(I^e)_c]$  which were obtained by different methods.

In the second part of this paper, we investigate the more difficult question of when  $Y$  is an arithmetically Cohen-Macaulay blow-up of  $X$ ; that is, when there exists a standard graded  $k$ -algebra  $R$  and an ideal  $J \subset R$ , such that  $X = \text{Proj } R$ ,  $Y$  is the blow-up of  $X$  along the ideal sheaf of  $J$ , and  $R[Jt]$  is a Cohen-Macaulay ring. Given  $R$  and  $I$ , we will concentrate on ideals  $J \subseteq I$  which are generated by the elements of  $(I^e)_c$ . It is obvious that  $I^e$  and  $J$  define the same ideal sheaf for  $c \geq d(I)e$ . Rees algebras of the form  $R[I_c t]$  ( $e = 1$ ) have been studied first for the defining ideal of a set of points in [15] and then for locally complete intersection ideals in [8], where it was shown that there exists a constant  $\lambda$  such that  $R[I_c t]$  is a Cohen-Macaulay ring for  $c \geq \lambda$ . This leads to the problem of whether there is a constant  $\delta$  such that the Rees algebra  $R[(I^e)_c t]$  is a Cohen-Macaulay ring for  $c \geq \delta e$ .

If  $a^*(R) < 0$  (e.g. if  $R$  is a polynomial ring) we solve this problem by showing that there exists a Cohen-Macaulay ring  $R[(I^e)_c t]$  with  $c \geq d(I)e$  if and only if the

following conditions are satisfied:

- $Y$  is equidimensional and Cohen-Macaulay,
- $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ ,  $R^i\pi_*\mathcal{O}_Y = 0$  for  $i > 0$ .

In particular, these conditions imply that  $R[(I^e)_c t]$  is a Cohen-Macaulay ring for all  $c > d(I)e + \varepsilon$  and  $e > e_0$  (Theorem 3.4). From this it follows that there exists a Cohen-Macaulay algebra of the form  $R[I_c t]$  with  $c \geq d(I)$  if and only if  $R[It]$  is locally Cohen-Macaulay on  $X$  and that  $e_0 = 0$  in this case (Corollary 3.8). We would like to point out that this phenomenon does not hold in general. In fact, there exist examples with  $a^*(R) \geq 0$  such that  $R[(I^e)_{d(I)e} t]$  is a Cohen-Macaulay ring, whereas  $R[(I^e)_c t]$  is not a Cohen-Macaulay ring for any  $c > d(I)e$  (Example 3.5). Using the above result we obtain several new classes of Cohen-Macaulay Rees algebras. Furthermore, we show that if  $H^0(X, \mathcal{O}_X) = k$  and  $H^i(X, \mathcal{O}_X) = 0$  for  $i > 0$ , then  $Y$  is an arithmetically Cohen-Macaulay blow-up of  $X$  if and only if  $Y$  is locally arithmetical Cohen-Macaulay on  $X$  (Theorem 3.12).

Our approach is based on the facts that the Rees algebra  $S = R[It]$  has a natural bi-gradation and that  $k[(I^e)_c]$  can be viewed as a diagonal subalgebra of  $S$  [7]. As a consequence, the Cohen-Macaulayness of  $k[(I^e)_c]$  can be characterized by means of the sheaf cohomology  $H^i(Y, \mathcal{O}_Y(m, n))$ . Using Leray spectral sequence and Serre-Grothendieck correspondence, we may pass from this sheaf cohomology to the local cohomology of  $I^n$  and of  $\omega_n$ , where  $\omega_S = \bigoplus_{n \in \mathbb{Z}} \omega_n$  denotes the graded canonical module of  $S$ . It was shown recently that there are linear bounds for the vanishing of the local cohomology of  $I^n$  and  $\omega_n$  ([31, 10, 23, 34]). It turns out that these linear bounds yield a linear bound on  $c$  and  $e$  such that  $k[(I^e)_c]$  is a Cohen-Macaulay ring. The Cohen-Macaulayness of the Rees algebra  $R[(I^e)_c t]$  can be studied similarly by using a recent result of Hyry [19] which characterizes the Cohen-Macaulayness of a standard bi-graded algebra by means of sheaf cohomology.

The paper is organized as follows. In Section 1, we introduce the notion of a projective  $a^*$ -invariant which governs how sheaf cohomology behaves through blowing up morphisms. The material in this section is interesting on its own right. In Section 2, we study the Cohen-Macaulayness of rings of the form  $k[(I^e)_c]$  which correspond to projective embeddings of  $Y$ . The last section of the paper deals with the problem of when  $Y$  is an arithmetically Cohen-Macaulay blow-up of  $X$ .

For unexplained notation and facts we refer the reader to the books [4, 5, 17].

### 1. $a^*$ -INVARIANTS

Let  $R$  be an arbitrary commutative noetherian ring. Let  $S = \bigoplus_{n \geq 0} S_n$  be a finitely generated graded algebra over  $R$ . We shall always use  $S_+ = \bigoplus_{n > 0} S_n$  to denote the ideal generated by the homogeneous elements of positive degrees of  $S$ . Given any finitely generated graded  $S$ -module  $F$ , the local cohomology module  $H_{S_+}^i(F)$  is also a graded  $S$ -module. It is well known that  $H_{S_+}^i(F)_n = 0$  for  $n \gg 0$ ,  $i \geq 0$ . Put

$$a_i(F) = \begin{cases} -\infty & \text{if } H_{S_+}^i(F) = 0, \\ \max\{n \mid H_{S_+}^i(F)_n \neq 0\} & \text{if } H_{S_+}^i(F) \neq 0. \end{cases}$$

Note that  $a(F) := a_{\dim F}(F)$  is called the  $a$ -invariant of  $F$  if  $S$  is a standard graded algebra over a field. The  $a^*$ -invariant of  $F$  is defined to be

$$a^*(F) := \max\{a_i(F) \mid i \geq 0\}.$$

This invariant was introduced in [32] and [29] in order to control the vanishing of graded local cohomology modules with different supports. It is closely related to the Castelnuovo-Mumford regularity via the equality

$$\text{reg}(F) = \max\{a_i(F) + i \mid i \geq 0\}.$$

Here we are interested in the case when  $R$  is a standard graded algebra over a field  $k$  and  $S = R[It]$  is the Rees algebra of a homogeneous ideal  $I \subset R$  with  $\text{ht } I \geq 1$ . This Rees algebra has a natural grading with  $S_n = I^n t^n$ . Let  $\omega_S = \bigoplus_{n \in \mathbb{Z}} \omega_n$  denote the canonical graded module of  $S$ .

**Lemma 1.1.** *Let  $S = R[It]$  be as above. If  $S$  is a Cohen-Macaulay ring, then  $a^*(S) = -1$  and  $a^*(\omega_S) = 0$ .*

*Proof.* It is well known that  $\dim S = \dim R + 1$ . Since  $S/S_+ = R$ , we have  $\text{ht } S_+ = \dim S - \dim R = 1$ . This implies  $\text{grade } S_+ = 1$ . Hence  $a^*(S) \geq -1$  by [32, Corollary 2.3]. On the other hand, the Cohen-Macaulayness of  $S$  implies  $H_M^i(S) = 0$  for  $i < \dim S$ , where  $M$  denotes the maximal graded ideal of  $S$ . By [33, Corollary 3.2] we always have  $H_M^{\dim S}(S)_n = 0$  for  $n \geq 0$ . Hence  $H_M^i(S)_n = 0$  for all  $n \geq 0$  and  $i \geq 0$ . By [19, Lemma 2.3] (or [32, Corollary 2.8]), this implies  $H_{S_+}^i(S)_n = 0$  for all  $n \geq 0$  and  $i \geq 0$ . Therefore,  $a^*(S) = -1$ .

Since  $\omega_S$  is a Cohen-Macaulay module with  $\text{Hom}_S(\omega_S, \omega_S) \cong S$  [2, Proposition 2], we also have  $H_M^i(\omega_S) = 0$  for  $i < \dim S$  and, by local duality,

$$H_M^{\dim S}(\omega_S)_n \cong \text{Hom}_S(\omega_S, \omega_S)_{-n} \cong S_{-n}.$$

Since  $S_0 = R \neq 0$  and  $S_{-n} = 0$  for  $n > 0$ , we can conclude that  $a_X^*(\omega_S) = 0$ . □

Let  $X = \text{Proj } R$ . For each  $\mathfrak{p} \in X$ , the homogeneous localization  $F_{(\mathfrak{p})}$  is a finitely generated graded module over  $S_{(\mathfrak{p})}$ . Hence, we can define the *projective  $a^*$ -invariant*

$$a_X^*(F) := \max\{a^*(F_{(\mathfrak{p})}) \mid \mathfrak{p} \in X\}.$$

Note that  $H_{S_{(\mathfrak{p})}^+}^i(F_{(\mathfrak{p})}) = H_{S_+}^i(F)_{(\mathfrak{p})}$  (cf. [29, Remark 2.2]). Then we always have  $a_X^*(F) \leq a^*(F)$ . Hence  $a_X^*(F)$  is a finite number. Since  $a_X^*(F)$  is determined by the local structure of  $F$  on  $X$ , it can easily be estimated in certain situations. As a demonstration, we show how to estimate  $a_X^*(F)$  in the following case which will play an important role in our further investigation.

We say that  $S$  is *locally Cohen-Macaulay on  $X$*  if  $S_{(\mathfrak{p})}$  is a Cohen-Macaulay ring for every  $\mathfrak{p} \in X$ . This condition holds if, for instance,  $X$  is locally Cohen-Macaulay and  $\mathcal{I}$  is locally a complete intersection.

**Proposition 1.2.** *Let  $X = \text{Proj } R$  and  $S = R[It]$  be as above. Then  $a_X^*(S) \geq -1$  and  $a_X^*(\omega_S) \geq 0$ . Equalities hold if  $S$  is locally Cohen-Macaulay on  $X$ .*

*Proof.* Let  $\mathfrak{p}$  be a minimal prime ideal in  $X$ . Then  $R_{(\mathfrak{p})}$  is an artinian ring. Since  $\mathfrak{p} \not\supseteq I$ , we have  $I_{(\mathfrak{p})} = R_{(\mathfrak{p})}$ . Hence  $S_{(\mathfrak{p})} = R_{(\mathfrak{p})}[t]$  is a Cohen-Macaulay ring. By Lemma 1.1, this implies  $a^*(S_{(\mathfrak{p})}) = -1$  and  $a^*(\omega_{S_{(\mathfrak{p})}}) = 0$ . Hence  $a_X^*(S) \geq -1$  and  $a_X^*(\omega_S) \geq 0$ . This proves the first statement. The second statement is an immediate consequence of Lemma 1.1. □

Beside the natural  $\mathbb{N}$ -graded structure given by the degrees of  $t$ , the Rees algebra  $S = R[It]$  also has a natural bi-graduation with

$$S_{(m,n)} = (I^n)_m t^n$$

for  $(m, n) \in \mathbb{N}^2$ . Let  $Y$  be the blow-up of  $X$  along the ideal sheaf of  $I$ . Then  $Y = \text{Proj } S$  with respect to this bi-graduation. If  $F = \bigoplus_{(m,n) \in \mathbb{Z}^2} F_{(m,n)}$  is a finitely generated bi-graded  $S$ -module, then  $F$  is also an  $\mathbb{Z}$ -graded  $S$ -module with  $F_n = \bigoplus_{m \in \mathbb{Z}} F_{(m,n)}$ . Let  $\tilde{F}$  denote the sheaf associated to  $F$  on  $Y$ . We write  $\tilde{F}(n)$  and  $\tilde{F}(m, n)$  to denote the twisted  $\mathcal{O}_Y$ -modules with respect to the  $\mathbb{N}$ -graduation and the  $\mathbb{N}^2$ -graduation of  $S$ . Moreover, we denote by  $\widetilde{F}_n$  the sheafification of  $F_n$  on  $X$ .

It turns out that  $a_X^*(F)$  is a measure for when we can pass from the sheaf cohomology of  $\tilde{F}(m, n)$  on  $Y$  to that of  $\widetilde{F}_n(m)$  on  $X$ .

**Proposition 1.3.** *Let  $F$  be a finitely generated bi-graded  $S$ -module. For  $n > a_X^*(F)$  we have*

- (i)  $\pi_*(\tilde{F}(n)) = \widetilde{F}_n$  and  $R^i \pi_*(\tilde{F}(n)) = 0$  for  $i > 0$ ,
- (ii)  $H^i(Y, \tilde{F}(m, n)) \cong H^i(X, \widetilde{F}_n(m))$  for all  $m \in \mathbb{Z}$  and  $i \geq 0$ .

*Proof.* Since (i) is a local statement, we only need to show that it holds locally. Let  $\mathfrak{p}$  be a closed point of  $X$ , and consider the restriction  $\pi_{\mathfrak{p}}$  of  $\pi$  over an affine open neighborhood  $\text{Spec } \mathcal{O}_{X,\mathfrak{p}}$  of  $\mathfrak{p}$

$$\pi_{\mathfrak{p}} : Y_{\mathfrak{p}} = Y \times_X \text{Spec } \mathcal{O}_{X,\mathfrak{p}} \rightarrow \text{Spec } \mathcal{O}_{X,\mathfrak{p}}.$$

We have  $\tilde{F}|_{Y_{\mathfrak{p}}} = \widetilde{F}_{(\mathfrak{p})}$ , where  $\widetilde{F}_{(\mathfrak{p})}$  is the sheaf associated to  $F_{(\mathfrak{p})}$  on  $Y_{\mathfrak{p}}$ . Thus,

$$R^i \pi_{\mathfrak{p}*}(\tilde{F}(n)) \Big|_{\text{Spec } \mathcal{O}_{X,\mathfrak{p}}} = R^i \pi_{\mathfrak{p}*}(\widetilde{F}_{(\mathfrak{p})}(n)) = H^i(Y_{\mathfrak{p}}, \widetilde{F}_{(\mathfrak{p})}(n))^{\sim}.$$

On the other hand, we know by the Serre-Grothendieck correspondence that there are the exact sequence

$$0 \rightarrow H_{S_{(\mathfrak{p})+}^0}^0(F_{(\mathfrak{p})})_n \rightarrow (F_{(\mathfrak{p})})_n \rightarrow H^0(Y_{\mathfrak{p}}, \widetilde{F}_{(\mathfrak{p})}(n)) \rightarrow H_{S_{(\mathfrak{p})+}^1}^1(F_{(\mathfrak{p})})_n \rightarrow 0$$

and the isomorphisms  $H^i(Y_{\mathfrak{p}}, \widetilde{F}_{(\mathfrak{p})}(n)) \cong H_{S_{(\mathfrak{p})+}^{i+1}}^{i+1}(F_{(\mathfrak{p})})_n$  for  $i > 0$ . By the definition of  $a_X^*(F)$ , we know that  $H_{S_{(\mathfrak{p})+}^i}^i(F_{(\mathfrak{p})})_n$  for  $n > a_X^*(F)$ ,  $i > 0$ . Thus,

$$R^i \pi_{\mathfrak{p}*}(\tilde{F}(n)) \Big|_{\text{Spec } \mathcal{O}_{X,\mathfrak{p}}} = H^i(Y_{\mathfrak{p}}, \widetilde{F}_{(\mathfrak{p})}(n))^{\sim} = \begin{cases} (\widetilde{F}_n)_{(\mathfrak{p})} & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

for  $n > a_X^*(F)$ .

To show (ii) we first observe that  $\tilde{F}(m, n) = \tilde{F}(n) \otimes \pi^* \mathcal{O}_X(m)$ . By the projection formula, we have

$$R^i \pi_{\mathfrak{p}*}(\tilde{F}(m, n)) = R^i \pi_{\mathfrak{p}*}(\tilde{F}(n)) \otimes \mathcal{O}_X(m) = \begin{cases} \widetilde{F}_n(m) & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

Hence the conclusion follows from the Leray spectral sequence

$$H^i(X, R^j \pi_{\mathfrak{p}*}(\tilde{F}(m, n))) \Rightarrow H^{i+j}(Y, \tilde{F}(m, n)).$$

□

Let  $Y$  be the blow-up of a projective scheme  $X$  along an ideal sheaf  $\mathcal{I}$ . We say that  $Y$  is *locally arithmetic Cohen-Macaulay on  $X$*  if there exist  $R$  and  $I$  such that  $X = \text{Proj } R$ ,  $\mathcal{I} = \tilde{I}$  and  $S = R[It]$  is locally Cohen-Macaulay on  $X$ .

**Corollary 1.4.** *Assume that  $Y$  is locally arithmetic Cohen-Macaulay on  $X$ . Then*

- (i)  $\pi_* \mathcal{O}_Y = \mathcal{O}_X$  and  $R^i \pi_* \mathcal{O}_Y = 0$  for  $i > 0$ ,
- (ii)  $H^i(Y, \mathcal{O}_Y(m, 0)) \cong H^i(X, \mathcal{O}_X(m))$  for all  $m \in \mathbb{Z}$ ,  $i \geq 0$ .

*Proof.* With the above notations we have  $a_X^*(S) = -1$  by Proposition 1.2. Hence the conclusion follows from Proposition 1.3 by taking  $F = S$  and  $n = 0$ .  $\square$

For each  $n$ , the graded  $R$ -module  $F_n$  has an  $a^*$ -invariant  $a^*(F_n)$ , which controls the vanishing of  $H^i(X, \widetilde{F}_n(m))$  by the Grothendieck-Serre correspondence. On the other hand, since  $F$  is a finitely generated graded module over  $S = R[It]$ , there exists a number  $n_0$  such that  $F_n = I^{n-n_0}F_{n_0}$  for  $n \geq n_0$ . It was recently discovered that for any finitely generated graded  $R$ -module  $E$ , the Castelnuovo-Mumford regularity  $\text{reg}(I^n E)$  is bounded by a linear function on  $n$  with slope  $d(I)$  [34, Theorem 2.2] (see also [10, 23] for the case  $R$  is a polynomial ring). By definition, we always have

$$a^*(I^n E) \leq \max\{a_i(I^n E) + i \mid i \geq 0\} = \text{reg}(I^n E).$$

Therefore,  $a^*(F_n)$  is bounded above by a linear function of the form  $d(I)n + \varepsilon$  for  $n \geq 1$ .

We will denote by  $\varepsilon(I)$  the smallest non-negative number such that

$$a^*(I^n) \leq d(I)n + \varepsilon(I)$$

for all  $n \geq 1$ . Since  $\omega_S = \bigoplus_{n \in \mathbb{Z}} \omega_n$  is a finitely generated bi-graded  $S$ -module, there is a similar bound for  $a^*(\omega_n)$ . Note that the  $R$ -graded module  $\omega_n$  is also called an *adjoint-type module* of  $I$  because of its relationship to the adjoint ideals [20]. We will denote by  $\varepsilon^*(I)$  the smallest non-negative number such that

$$a_i(\omega_n) \leq d(I)n + \varepsilon^*(I)$$

for  $i \geq 2$  and  $n \geq 1$ .

The meaning of these invariants will become more apparent in the next sections. Here we content ourselves with the following observations.

**Lemma 1.5.** *With the above notations we have*

- (i)  $H^0(X, \widetilde{S}_n(m)) = S_{(m,n)}$  and  $H^i(X, \widetilde{S}_n(m)) = 0$  for  $i > 0$  and  $m > d(I)n + \varepsilon(I)$ ,
- (ii)  $H^i(X, \widetilde{\omega}_n(m)) = 0$  for  $i > 0$  and  $m > d(I)n + \varepsilon^*(I)$ .

*Proof.* Since  $S_n \cong I^n$ , we have  $H_{R_+}^i(S_n)_m = 0$  for  $i \geq 0$ ,  $m > d(I)n + \varepsilon(I)$  and  $n \geq 1$ . Hence the first statement follows from the Serre-Grothendieck correspondence, which gives the exact sequence

$$0 \rightarrow H_{R_+}^0(S_n)_m \rightarrow S_{(m,n)} \rightarrow H^0(X, \widetilde{S}_n(m)) \rightarrow H_{R_+}^1(S_n)_m \rightarrow 0$$

and the isomorphisms

$$H^i(X, \widetilde{S}_n(m)) \cong H_{R_+}^{i+1}(S_n)_m$$

for  $i > 0$ . The second statement can be similarly proved.  $\square$

## 2. ARITHMETICALLY COHEN-MACAULAY EMBEDDINGS OF BLOW-UPS

Let  $X$  be a projective scheme over a field  $k$ . Let  $Y \rightarrow X$  be the blowing up of  $X$  along an ideal sheaf  $\mathcal{I}$ . We say that  $Y$  has an *arithmetically Cohen-Macaulay embedding* if there exists a Cohen-Macaulay standard graded  $k$ -algebra  $A$  such that  $Y \cong \text{Proj } A$ .

Let  $R$  be a finitely generated standard graded  $k$ -algebra and let  $I \subset R$  be a homogeneous ideal such that  $X = \text{Proj } R$  and  $\mathcal{I}$  is the ideal sheaf associated to  $I$ . Let  $S = R[It]$  be the Rees algebra of  $R$  with respect to  $I$ . It is well known

that  $Y \cong \text{Proj } k[(I^e)_c]$  for  $c \geq d(I)e + 1$  and  $e \geq 1$ , where  $k[(I^e)_c]$  is the algebra generated by all forms of degree  $c$  of the ideal power  $I^e$  and  $d(I)$  denotes the largest degree of a minimal set of homogeneous generators of  $I$  (cf. [9, Lemma 1.1]). There is the following simple criterion for the existence of a Cohen-Macaulay algebra  $k[(I^e)_c]$  (which is at the same time a criterion for the existence of an arithmetically Cohen-Macaulay embedding).

**Lemma 2.1** ([26, Corollary 3.5]). *There exists a Cohen-Macaulay ring  $k[(I^e)_c]$  for  $c \geq d(I)e + 1$  if and only if the following conditions are satisfied:*

- (i)  $Y$  is equidimensional and Cohen-Macaulay,
- (ii)  $H^0(Y, \mathcal{O}_Y) = k$  and  $H^i(Y, \mathcal{O}_Y) = 0$  for  $i = 1, \dots, \dim Y - 1$ .

The proof of [26] used a deep result on the relationship between the local cohomology modules of a bi-graded algebra and its diagonal subalgebras [7]. However, the above lemma simply follows from the basic fact that (i) and (ii) are equivalent to the existence of an arithmetically Cohen-Macaulay Veronese embedding of  $Y$  (cf. [8, Lemma 1.1]). In fact, the Veronese subalgebras of  $k[I_c]$  are exactly the algebras of the form  $k[(I^e)_{ce}]$  for  $c \geq d(I) + 1$ ,  $e \geq 1$ . We notice that the statements of [26, Corollary 3.5] and [8, Lemma 1.1] missed the equidimensional condition.

In this section we will determine for which values of  $c$  and  $e$  is  $k[(I^e)_c]$  a Cohen-Macaulay ring. First, we show that there are well-determined invariants  $\varepsilon$  and  $e_0$  such that  $k[(I^e)_c]$  is a Cohen-Macaulay ring for all  $c > d(I)e + \varepsilon$  and  $e > e_0$ .

**Theorem 2.2.** *Let  $R$  be a standard graded  $k$ -algebra and let  $I \subset R$  be a homogeneous ideal with  $\text{ht } I \geq 1$ . Let  $Y$  be the blow-up of  $X = \text{Proj } R$  along the ideal sheaf of  $I$  and  $S = R[I t]$ . Assume that*

- (i)  $Y$  is equidimensional and Cohen-Macaulay,
- (ii)  $H^0(Y, \mathcal{O}_Y) = k$  and  $H^i(Y, \mathcal{O}_Y) = 0$  for  $i = 1, \dots, \dim Y - 1$ .

*Then  $k[(I^e)_c]$  is a Cohen-Macaulay ring for  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  and  $e > \max\{a_X^*(S), a_X^*(\omega_S)\}$ .*

Note first that we always have  $\max\{a_X^*(S), a_X^*(\omega_S)\} \geq 0$  by Proposition 1.2 and  $\max\{\varepsilon(I), \varepsilon^*(I)\} \geq 0$  by the definition of  $\varepsilon(I)$  and  $\varepsilon^*(I)$ .

*Proof.* Let  $A = k[(I^e)_c]$ . Since  $c \geq de + 1$ , we have  $Y \cong \text{Proj } A$  [9, Lemma 1.1]. On the other hand, the Rees algebra  $S = R[I t]$  has a natural bi-gradation with  $S_{(m,n)} = (I^m)_n t^n$  and  $Y = \text{Proj } S$ . Moreover, we may view  $A$  as a diagonal subalgebra of  $S$ ; that is,  $A = \bigoplus_{n \in \mathbb{N}} S_{(cn, en)}$  [7, Lemma 1.2]. From this it follows that  $A(n)^\sim = \mathcal{O}_Y(cn, en)$ . Therefore, the Serre-Grothendieck correspondence yields the exact sequence

$$0 \longrightarrow H_{A_+}^0(A) \longrightarrow A \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^0(Y, \mathcal{O}_Y(cn, en)) \longrightarrow H_{A_+}^1(A) \longrightarrow 0$$

and the isomorphisms

$$\bigoplus_{n \in \mathbb{Z}} H^i(Y, \mathcal{O}_Y(cn, en)) \cong H_{A_+}^{i+1}(A)$$

for  $i \geq 1$ . It is well known that  $A$  is a Cohen-Macaulay ring if and only if  $H_{A_+}^i(A) = 0$  for  $i \neq \dim A$ . Therefore,  $A$  is a Cohen-Macaulay ring if we can show

$$H^0(Y, \mathcal{O}_Y(cn, en)) = A_n = \begin{cases} 0 & \text{for } n < 0, \\ k & \text{for } n = 0, \\ (I^{en})_{cn} & \text{for } n > 0, \end{cases}$$

$$H^i(Y, \mathcal{O}_Y(cn, en)) = 0 \quad (i = 1, \dots, \dim Y - 1).$$

For  $n = 0$ , this follows from the assumption  $H^0(Y, \mathcal{O}_Y) = k$  and  $H^i(Y, \mathcal{O}_Y) = 0$  for  $i = 1, \dots, \dim Y - 1$ .

For  $n > 0$  we have  $cn > d(I)en + \varepsilon(I)n \geq d(I)en + \varepsilon(I)$  and  $en > a_X^*(S)n \geq a_X^*(S)$ . Hence, using Proposition 1.3 and Lemma 1.5 we get

$$H^0(Y, \mathcal{O}_Y(cn, en)) = H^0(X, \widetilde{I^{en}}(cn)) = (I^{en})_{cn},$$

$$H^i(Y, \mathcal{O}_Y(cn, en)) = H^i(X, \widetilde{I^{en}}(cn)) = 0, \quad i = 1, \dots, \dim Y - 1.$$

For  $n < 0$  we have

$$H^i(Y, \mathcal{O}_Y(cn, en)) = H^{\dim Y - i}(Y, \omega_Y(-cn, -en))$$

for  $i \geq 0$ . Serre duality can be applied here because  $Y$  is equidimensional and Cohen-Macaulay. Since  $-cn > -d(I)en - \varepsilon^*(I)n \geq -d(I)en + \varepsilon^*(I)$  and  $-en > -a_X^*(\omega_S)n \geq a_X^*(\omega_S)$ , using Proposition 1.3 and Lemma 1.5 we get

$$H^{\dim Y - i}(Y, \omega_Y(-cn, -en)) = H^{\dim Y - i}(X, \widetilde{(\omega_S)_{-en}}(-cn)) = 0$$

for  $i < \dim Y$ . So we get  $H^i(Y, \mathcal{O}_Y(cn, en)) = 0$  for all  $n < 0$  and  $i = 0, \dots, \dim Y - 1$ . The proof of Theorem 2.2 is now complete.  $\square$

The following theorem shows that the bound  $e > \max\{a_X^*(S), a_X^*(\omega_S)\}$  of Theorem 2.2 is the best possible.

**Theorem 2.3.** *Let the notations and assumptions be as in Theorem 2.2. Let*

$$e_0 = \max\{a_X^*(S), a_X^*(\omega_S)\}.$$

*Then  $k[(I^{e_0})_c]$  is not a Cohen-Macaulay ring for  $c \gg 0$  if  $e_0 \geq 1$ .*

*Proof.* Let  $A = k[(I^{e_0})_c]$  for  $c \gg 0$ . As we have seen in the proof of Theorem 2.2,  $A$  is not Cohen-Macaulay if  $H^0(Y, \mathcal{O}_Y(c, e_0)) \neq (I^{e_0})_c$  or  $H^i(Y, \mathcal{O}_Y(c, e_0)) \neq 0$  or  $H^i(Y, \mathcal{O}_Y(-c, -e_0)) \neq 0$  for some  $i = 1, \dots, \dim Y - 1$ .

We shall first consider the case  $e_0 = a_X^*(S)$ . Let  $q$  be the smallest integer such that  $e_0 = \max\{a_q(S_{(\mathfrak{p})}) \mid \mathfrak{p} \in X\}$ . Then

$$H_{S_{(\mathfrak{p})+}^i}^i(S_{(\mathfrak{p})})_{e_0} = 0, \quad i < q, \quad \text{for all } \mathfrak{p} \in X,$$

$$H_{S_{(\mathfrak{p})+}^q}^q(S_{(\mathfrak{p})})_{e_0} \neq 0 \quad \text{for some } \mathfrak{p} \in X.$$

It is a classical result that there exists  $\dim R_{(\mathfrak{p})}$  elements in  $I_{(\mathfrak{p})}$  which generates an ideal with the same radical as  $I_{(\mathfrak{p})}$ . The same also holds for the ideal  $S_{(\mathfrak{p})+} = I_{(\mathfrak{p})}t$ . From this it follows that  $H_{S_{(\mathfrak{p})+}^{\dim R_{(\mathfrak{p})}+1}}^{\dim R_{(\mathfrak{p})}+1}(E) = 0$  for any  $R_{(\mathfrak{p})}$ -module  $E$  (cf. [4, Corollary 3.3.3]). Hence

$$q \leq \max\{\dim R_{(\mathfrak{p})} \mid \mathfrak{p} \in X\} = \dim Y.$$

Let  $Y_{\mathfrak{p}} = Y \times_X \text{Spec } \mathcal{O}_{X,\mathfrak{p}}$ . The Serre-Grothendieck correspondence yields the exact sequence

$$0 \rightarrow H^0_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0} \rightarrow (S_{(\mathfrak{p})})_{e_0} \rightarrow H^0(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) \rightarrow H^1_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0} \rightarrow 0,$$

and isomorphisms  $H^i(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) \cong H^{i+1}_{S_{(\mathfrak{p})+}}(S_{(\mathfrak{p})})_{e_0}$ ,  $i \geq 1$ .

If  $q \leq 1$ , then  $H^0(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) \neq (S_{(\mathfrak{p})})_{e_0} = I_{(\mathfrak{p})}^{e_0}$  for some  $\mathfrak{p} \in X$ . From this it follows, as in the proof of Proposition 1.3, that  $\pi_*(\mathcal{O}_Y(e_0)) \neq \widetilde{I}^{e_0}$ . But  $\pi_*(\mathcal{O}_Y(e_0))(c)$  and  $\widetilde{I}^{e_0}(c)$  are generated by global sections for  $c \gg 0$ . Therefore, by the projection formula we have

$$H^0(X, \pi_*(\mathcal{O}_Y(c, e_0))) = H^0(X, \pi_*(\mathcal{O}_Y(e_0))(c)) \neq H^0(X, \widetilde{I}^{e_0}(c)) = (I^{e_0})_c$$

for  $c \gg 0$ . Moreover,

$$H^0(Y, \mathcal{O}_Y(c, e_0)) = H^0(X, \pi_*(\mathcal{O}_Y(c, e_0))).$$

Hence  $H^0(Y, \mathcal{O}_Y(c, e_0)) \neq (I^{e_0})_c$ .

If  $q \geq 2$ , then the Serre-Grothendieck sequence implies  $H^i(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) = 0$  for all  $\mathfrak{p} \in X$ ,  $0 < i < q - 1$ , and  $H^{q-1}(Y_{\mathfrak{p}}, \widetilde{S_{(\mathfrak{p})}}(e_0)) \neq 0$  for some  $\mathfrak{p} \in X$ . From this it follows, as in the proof of Proposition 1.3, that

$$\begin{aligned} R^i \pi_*(\mathcal{O}_Y(e_0)) &= 0 \text{ for } 0 < i < q - 1, \\ R^{q-1} \pi_*(\mathcal{O}_Y(e_0)) &\neq 0. \end{aligned}$$

By the projection formula, we have

$$\begin{aligned} R^i \pi_*(\mathcal{O}_Y(c, e_0)) &= R^i \pi_*(\mathcal{O}_Y(e_0)) \otimes \mathcal{O}_X(c) = 0 \text{ for } 0 < i < q - 1, \\ R^{q-1} \pi_*(\mathcal{O}_Y(c, e_0)) &= R^{q-1} \pi_*(\mathcal{O}_Y(e_0)) \otimes \mathcal{O}_X(c) \neq 0. \end{aligned}$$

Since  $\pi_*(\mathcal{O}_Y(c, e_0)) = \pi_*(\mathcal{O}_Y(e_0))(c)$ , we also have  $H^{q-1}(X, \pi_*(\mathcal{O}_Y(c, e_0))) = 0$  for  $c \gg 0$ . Therefore, using Leray spectral sequence

$$H^i(X, R^j \pi_*(\mathcal{O}_Y(m, e_0))) \Rightarrow H^{i+j}(Y, \mathcal{O}_Y(m, e_0))$$

we can deduce that

$$H^{q-1}(Y, \mathcal{O}_Y(c, e_0)) = H^0(X, R^{q-1} \pi_*(\mathcal{O}_Y(c, e_0)))$$

for  $c \gg 0$ . But  $R^{q-1} \pi_*(\mathcal{O}_Y(c, e_0))$  is generated by global sections for  $c \gg 0$ . So we get  $H^{q-1}(Y, \mathcal{O}_Y(c, e_0)) \neq 0$ .

Let us now consider the case  $e_0 = a_X^*(\omega_S)$ . Let  $q$  be the smallest integer such that  $e_0 = \max\{a_q((\omega_S)_{(\mathfrak{p})}) \mid \mathfrak{p} \in X\}$ . For  $\mathfrak{p} \in X$  we have  $(\omega_S)_{(\mathfrak{p})} = \bigoplus_{n>0} H^0(Y_{\mathfrak{p}}, \omega_{Y_{\mathfrak{p}}}(n))$  (see [20, 2.5.2(1) and 2.6.2]). From this it follows that  $[H^i_{S_{(\mathfrak{p})+}}((\omega_S)_{(\mathfrak{p})})]_n = 0$  for  $n > 0$ ,  $i = 0, 1$ . Since  $e_0 > 0$ , this implies  $q > 1$ . Similarly as in the first case, we can also show that  $q \leq \dim Y$  and that  $H^{q-1}(Y, \omega_Y(c, e_0)) \neq 0$  for  $c \gg 0$ . By Serre duality we get

$$H^{\dim Y - q + 1}(Y, \mathcal{O}_Y(-c, -e_0)) = H^{q-1}(Y, \omega_Y(c, e_0)) \neq 0$$

for  $c \gg 0$ . This completes the proof of Theorem 2.3. □

We shall see later in Example 2.5 that the bound  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  of Theorem 2.2 is sharp.

Now we want to study the problem when there exists a Cohen-Macaulay ring of the form  $k[(I^e)_c]$  for  $e \geq 1$ .

**Theorem 2.4.** *Let  $R$  be an equidimensional standard graded  $k$ -algebra and let  $I$  be a homogeneous ideal of  $R$  with  $\text{ht } I \geq 1$ . Let  $X = \text{Proj } R$  and  $S = R[It]$ . Assume that  $S$  is locally Cohen-Macaulay on  $X$ . Then, there exists a Cohen-Macaulay ring  $k[(I^e)_c]$  with  $c \geq d(I)e + 1$  if and only if  $H^0(X, \mathcal{O}_X) = k$  and  $H^i(X, \mathcal{O}_X) = 0$  for  $i = 1, \dots, \dim X - 1$ . In particular, this condition implies that  $k[(I^e)_c]$  is a Cohen-Macaulay ring for  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  and  $e \geq 1$ .*

*Proof.* Let  $Y$  be the blow-up of  $X$  along the ideal sheaf of  $I$ . The assumption implies that  $Y$  is equidimensional and Cohen-Macaulay. Since  $S$  is locally Cohen-Macaulay over  $X$ ,  $Y$  is locally arithmetic Cohen-Macaulay over  $X$ . Applying Corollary 1.4, we have  $H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$  and  $H^i(Y, \mathcal{O}_Y) = H^i(X, \mathcal{O}_X)$  for  $i > 0$ . Therefore, the first statement follows from Lemma 2.1. Moreover, we have  $\max\{a_X^*(S), a_X^*(\omega_S)\} = 0$  by Proposition 1.2. Hence the second statement follows from Theorem 2.2.  $\square$

Note that the condition  $H^0(X, \mathcal{O}_X) = k$  and  $H^i(X, \mathcal{O}_X) = 0$  for  $i = 1, \dots, \dim X - 1$  is satisfied if  $R$  is a Cohen-Macaulay ring.

The following example shows that the bound  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  is sharp.

**Example 2.5.** Let  $R = k[x_0, x_1, x_2]$  and  $I = (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4)$ . It is easy to see that  $S = R[It]$  is locally Cohen-Macaulay on  $X = \text{Proj } R$ . We have  $I^n = (x_1, x_2)^{4n}$  for all  $n \geq 2$ . We have

$$a^*(I^n) = \begin{cases} 4 & \text{if } n = 1, \\ 4n - 1 & \text{if } n \geq 2. \end{cases}$$

From this it follows that  $\varepsilon(I) = 0$ . To compute  $\varepsilon^*(I)$  we approximate  $I$  by the ideal  $J = (x_1, x_2)^4$ . Put  $S^* = R[Jt]$ . Then we have the exact sequence

$$0 \rightarrow R[It] \rightarrow R[Jt] \rightarrow k \rightarrow 0.$$

From this it follows that  $\omega_S = \omega_{S^*}$ . Note that  $S^*$  is a Veronese subring of the ring  $T = R[(x_1, x_2)t]$  and that  $T$  is a Gorenstein ring with  $\omega_T = T(-2)$ . Then  $\omega_{S^*} = \bigoplus_{n \geq 1} (x_1, x_2)^{4n-2}$ . We have

$$a^*(\omega_n) = a^*((x_1, x_2)^{4n-2}) = 4n - 3$$

for  $n \geq 1$ . Hence  $\varepsilon^*(I) = 0$ . By Theorem 2.4, these facts imply that  $k[(I^e)_c]$  is Cohen-Macaulay for  $c > 4e$  and  $e \geq 1$  (which can be also verified directly). On the other hand, for  $c = 4$  and  $e = 1$ , the ring  $k[I_4] = k[x_1^4, x_1^3x_2, x_1x_2^3, x_2^4]$  is not Cohen-Macaulay.

There have been various criteria for the Cohen-Macaulayness of Rees algebras (cf. [33, 18, 27, 30, 1, 21, 28]), so that one can construct various classes of ideals  $I$  for which  $S$  is locally Cohen-Macaulay on  $X$ . We list here only the most interesting applications of Theorem 2.4.

**Corollary 2.6.** *Let  $R$  be a Cohen-Macaulay standard graded  $k$ -algebra. Let  $I \subset R$  be a homogeneous ideal with  $\text{ht } I \geq 1$  which is a locally complete intersection. Then  $k[(I^e)_c]$  is a Cohen-Macaulay ring for all  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  and  $e \geq 1$ .*

*Proof.* Let  $X = \text{Proj } R$ . The assumption on  $I$  means that  $I_{\mathfrak{p}}$  is a complete intersection ideal in  $R_{\mathfrak{p}}$  for  $\mathfrak{p} \in X$ . Therefore,  $R_{(\mathfrak{p})}[I_{(\mathfrak{p})}t]$  is Cohen-Macaulay for all  $\mathfrak{p} \in X$ . Hence,  $S = R[It]$  is locally Cohen-Macaulay on  $X$ . The result follows from Theorem 2.4.  $\square$

**Corollary 2.7.** *Let  $R$  be a polynomial ring over a field  $k$  of characteristic zero and  $I \subset R$  a non-singular homogeneous ideal with  $\text{ht } I \geq 1$ . Then,  $k[(I^e)_c]$  is a Cohen-Macaulay ring for  $c > d(I)e + \varepsilon(I)$  and  $e \geq 1$ .*

*Proof.* The assumption implies that  $I$  is locally a complete intersection. Hence  $S = R[It]$  is locally Cohen-Macaulay on  $X = \text{Proj } R$ . Let  $Y = \text{Proj } S$ . Then  $Y$  is a projective non-singular scheme. Let  $m, n$  be positive integers with  $m \geq d(I)n + 1$ . Then  $\mathcal{O}_Y(m, n)$  is a very ample invertible sheaf on  $Y$  because  $Y \cong \text{Proj } k[(I^n)_m]$  [9, Lemma 1.1]. Let  $\omega_S$  be the canonical module of  $S$  and  $\omega_Y = \widetilde{\omega}_S$ . Then  $H^i(Y, \omega_Y(m, n)) = 0$  for  $i \geq 1$  by Kodaira’s vanishing theorem. On the other hand, we have

$$H^i(Y, \omega_Y(m, n)) = H^i(X, \widetilde{(\omega_S)_n(m)})$$

by Proposition 1.3. Therefore,  $H^i(X, \widetilde{(\omega_S)_n(m)}) = 0$  for  $i \geq 1$ . Using the Serre-Grothendieck correspondence we can deduce that  $H^i_{R_+}((\omega_S)_n)_m = 0$  for  $i \geq 2$ . Hence  $\varepsilon^*(I) = 0$ . Now, the conclusion follows from Corollary 2.6.  $\square$

*Remark 2.8.* A similar result to Theorem 2.4 was already given by Cutkosky and Herzog [9, Theorem 4.1] when  $R$  is Cohen-Macaulay. Their result shows the existence of a constant  $\delta$  such that  $k[(I^e)_c]$  is Cohen-Macaulay for  $c \geq \delta e$ ,  $e > 0$ , under some assumptions on the associated graded ring  $\bigoplus_{n \geq 0} I^n/I^{n+1}$ . It is not hard to see that these assumptions imply  $\max\{a^*_X(S), a^*_X(\omega_S)\} \leq 0$  (see [9, Lemma 2.1 and Lemma 2.2]). Hence their result is also a consequence of Theorem 2.2. Similar statements to the above two corollaries were also given in [9] but without any information on the slope  $\delta$ .

It is not easy to compute  $\varepsilon(I)$  explicitly, even when  $I$  is a non-singular ideal in a polynomial ring. By a famous result of Bertram, Ein and Lazarsfeld [3] we know that if  $I$  is the ideal of a smooth complex variety cut out scheme-theoretically by hypersurfaces of degree  $d_1 \geq \dots \geq d_m$ , then

$$a_i(I^n) \leq d_1 n + d_2 + \dots + d_m - \text{ht } I$$

for  $i \geq 2$  and  $n \geq 1$ . However, we do not know any bound for  $a_1(I^n)$  in terms of  $d_1, \dots, d_m$ . It would be of interest to find such a bound. In general, if we happen to know the minimal free resolution of  $S$  over a bi-graded polynomial ring, then we can estimate  $\varepsilon(I)$  in terms of the shifts of syzygy modules of the resolution [10].

In the case when  $I$  is the defining ideal of a scheme of fat points, we know an explicit bound for  $a^*(I^n)$ , namely  $a^*(I^n) \leq \text{reg}(I)n$  for all  $n \geq 1$  [6, 13]. As a consequence, we immediately obtain the following result of Geramita, Gimigliano and Pitteloud.

**Corollary 2.9** ([13, Theorem 2.4]). *Let  $R$  be a polynomial ring over a field  $k$  of characteristic zero, and let  $I \subset R$  be the defining ideal of a scheme of fat points in  $\text{Proj } R$ . Then,  $k[(I^e)_c]$  is a Cohen-Macaulay ring for  $c > \text{reg}(I)e$  and  $e \geq 1$ .*

*Proof.* By definition, the ideal  $I$  has the form  $I = \bigcap_{i=1}^s \mathfrak{p}_i^{m_i}$ , where  $\mathfrak{p}_i$  is the defining prime ideal of a closed point in  $X = \text{Proj } R$  and  $m_i \in \mathbb{N}$ . Then  $R_{(\mathfrak{p})}[I_{(\mathfrak{p})}t]$  is Cohen-Macaulay for all  $\mathfrak{p} \in X$ . In fact, we may assume that  $\mathfrak{p} = \mathfrak{p}_i$  for some  $i$ . Then  $\mathfrak{p}$  is a complete intersection and  $R_{(\mathfrak{p})}[I_{(\mathfrak{p})}t] = R_{(\mathfrak{p})}[\mathfrak{p}_{(\mathfrak{p})}^{m_i}t]$  is a Veronese subalgebra of  $R_{(\mathfrak{p})}[\mathfrak{p}_{(\mathfrak{p})}t]$ . Since  $R_{(\mathfrak{p})}[\mathfrak{p}_{(\mathfrak{p})}t]$  is a Cohen-Macaulay ring, so is  $R_{(\mathfrak{p})}[I_{(\mathfrak{p})}t]$ . Thus,

$S = R[It]$  is locally Cohen-Macaulay on  $X$ . This argument also shows that  $Y = \text{Proj } S$  is smooth. Using the Kodaira vanishing theorem we can show, as in the proof of Corollary 2.7, that  $\varepsilon^*(I) = 0$ . The conclusion now follows from the proof of Theorem 2.4 when we replace the slope  $d(I)$  by  $\text{reg}(I) \geq d(I)$  and  $\varepsilon(I)$  by 0 because of the bound  $a^*(I^n) \leq \text{reg}(I)n$ .  $\square$

It was asked in [7] whether there exists a Cohen-Macaulay ring  $k[(I^e)_c]$  for  $c \gg e \gg 0$  if  $R$  is a polynomial ring and  $R[It]$  is Cohen-Macaulay. This question has been positively settled in [25, Theorem 4.5]. We can make this result more precise as follows.

**Corollary 2.10.** *Let  $R$  be a Cohen-Macaulay standard graded  $k$ -algebra. Let  $I \subset R$  be a homogeneous ideal with  $\text{ht } I \geq 1$  such that  $R[It]$  is Cohen-Macaulay. Then  $k[(I^e)_c]$  is a Cohen-Macaulay ring for all  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  and  $e \geq 1$ .*

### 3. ARITHMETICALLY COHEN-MACAULAY BLOW-UPS

Let  $X$  be a projective scheme over a field  $k$ . Let  $\pi : Y \rightarrow X$  be the blowing up of  $X$  along an ideal sheaf  $\mathcal{I}$ . We say that  $Y$  is an *arithmetically Cohen-Macaulay blow-up* of  $X$  if there is a standard graded  $k$ -algebra  $R$  and a homogeneous ideal  $J \subset R$  with  $\text{ht } J \geq 1$  such that  $X = \text{Proj } R$ ,  $\mathcal{I} = \tilde{J}$ , and  $R[Jt]$  is a Cohen-Macaulay ring. The aim of this section is to characterize arithmetically Cohen-Macaulay blow-ups.

Let  $R$  be a finitely generated standard graded  $k$ -algebra, and let  $I$  be a homogeneous ideal of  $R$  with  $\text{ht } I \geq 1$ , such that  $X = \text{Proj } R$  and  $\mathcal{I} = \tilde{I}$ . Let  $d(I)$  denote the maximal degree of the elements of a homogeneous basis of  $I$ . For any ideal  $J$  generated by  $(I^e)_c$  with  $c \geq d(I)e$  we have  $J_n = (I^e)_n$  for all  $n \geq c$  so that  $\tilde{\mathcal{I}}^e = \tilde{J}$ . Hence  $Y = \text{Proj } R[Jt]$ . The Rees algebra  $R[(I^e)_c t] = R[Jt]$  is called a *truncated Rees algebra* of  $I^e$  [15, 8]. We may strengthen the problem on the characterization of arithmetically Cohen-Macaulay blow-ups by asking the question: When does there exist a Cohen-Macaulay truncated Rees algebra  $R[(I^e)_c t]$ ? To solve this problem we shall need the following result of Hyry.

Let  $T$  be a standard bi-graded algebra over a field  $k$ , that is,  $T$  is generated over  $k$  by the elements of degree  $(1, 0)$  and  $(0, 1)$ . Let  $M$  denote the maximal graded ideal of  $T$  and define

$$a^1(T) := \max\{m \mid \text{there is } n \text{ such that } H_M^{\dim T}(T)_{(m,n)} \neq 0\},$$

$$a^2(T) := \max\{n \mid \text{there is } m \text{ such that } H_M^{\dim T}(T)_{(m,n)} \neq 0\}.$$

**Theorem 3.1** ([19, Theorem 2.5]). *Let  $T$  be a standard bi-graded  $k$ -algebra with  $a^1(T), a^2(T) < 0$ . Let  $Y = \text{Proj } T$ . Then  $T$  is Cohen-Macaulay if and only if the following conditions are satisfied:*

$$H^0(Y, T(m, n)^\sim) \cong T_{(m,n)} \text{ for } m, n \geq 0,$$

$$H^i(Y, T(m, n)^\sim) = 0 \text{ for } m, n \geq 0, \quad i > 0,$$

$$H^i(Y, T(m, n)^\sim) = 0 \text{ for } m, n < 0, \quad i < \dim T - 2.$$

Let  $J \subset R$  be an arbitrary ideal generated by forms of degree  $c$  and put  $T = R[Jt]$ . Then  $T$  can be equipped with another bi-graduation given by

$$T_{(m,n)} = (J^n)_{m+cn} t^n$$

for  $(m, n) \in \mathbb{N}^2$ . With this bi-graduation,  $T$  is a standard bi-graded  $k$ -algebra. Comparing with the natural bi-graduation of  $T$  considered in the preceding sections, we

see that both bi-gradations share the same bi-homogeneous elements and the same relevant bi-graded ideals. Therefore,  $\text{Proj } T$  with respect to these bi-gradations are isomorphic.

**Lemma 3.2.** *Let  $T = R[Jt]$  be as above. Then*

- (i)  $a^1(T) \leq \max\{a^*(J^n) - nc \mid n \geq 0\}$ ,
- (ii)  $a^2(T) < 0$ .

*Proof.* To prove (i) we will show more, namely, that  $H_M^i(T)_{(m,n)} = 0$  for  $m > \max\{a^*(J^n) - nc \mid n \geq 0\}$  and  $i \geq 0$ . Let  $T_1$  denote the ideal of  $T$  generated by the homogeneous elements of degree  $(1, 0)$ . Then, by [19, Lemma 2.3], we only need to show that  $H_{T_1}^i(T)_{(m,n)} = 0$  for  $m > \max\{a^*(J^n) - nc \mid n \geq 0\}$  and  $i \geq 0$ . Since  $T_1$  is generated by  $R_+$ , we always have

$$H_{T_1}^i(T)_{(m,n)} = \begin{cases} 0 & \text{for } n < 0, \\ H_{R_+}^i(J^n)_{m+nc} & \text{for } n \geq 0. \end{cases}$$

But  $H_{R_+}^i(J^n)_{m+nc} = 0$  for  $m + nc > a^*(J^n)$ ,  $n \geq 0$ . Therefore,  $H_{T_1}^i(T)_{(m,n)} = 0$  for  $m > \max\{a^*(J^n) - nc \mid n \geq 0\}$ , as required.

To prove (ii) we first observe that

$$a^2(T) = \max\{n \mid H_M^{\dim T}(T)_n \neq 0\},$$

where the  $\mathbb{Z}$ -gradation comes from the natural grading  $T_n = J^n t^n$ ,  $n \geq 0$ . Therefore, the conclusion  $a^2(T) < 0$  follows from [33, Corollary 3.2]. □

**Corollary 3.3.** *Let  $R$  be a standard graded  $k$ -algebra with  $a^*(R) < 0$  and let  $I \subset R$  be a homogeneous ideal with  $\text{ht } I \geq 1$ . Let  $T = R[(I^e)_c t]$  for some fixed integers  $c > d(I)e + \varepsilon(I)$  and  $e \geq 1$ . Then  $a^1(T) < 0$  and  $a^2(T) < 0$ .*

*Proof.* Let  $J$  be the ideal of  $R$  generated by  $(I^e)_c$ . By Lemma 3.2 we only need to prove that  $a^*(J^n) < nc$  for  $n \geq 0$ . For  $n = 0$ , this follows from the assumption  $a^*(R) < 0$ . For  $n \geq 1$ , we will approximate  $a^*(J)$  by  $a^*(I^{en})$ . Since  $J^n$  is generated by elements of degree  $cn$  and since  $cn > d(I)en \geq d(I^{en})$ , we have  $(I^{en}/J^n)_m = 0$  for  $m \geq cn$ . From this it follows that  $H^0(I^{en}/J^n) = I^{en}/J^n$  and  $H^i(I^{en}/J^n) = 0$  for  $i > 0$ . Therefore, from the exact sequence

$$0 \longrightarrow J^n \longrightarrow I^{en} \longrightarrow I^{en}/J^n \longrightarrow 0$$

we can deduce that  $H^i(J^n)_m = H^i(I^{en})_m$  for  $m \geq cn$  and  $i \geq 0$ . This implies

$$a^*(J^n) \leq \max\{cn - 1, a^*(I^{en})\}.$$

By the definition of  $\varepsilon(I)$  we have  $a^*(I^{en}) \leq d(I)en + \varepsilon(I) \leq cn - 1$ . Therefore,  $a^*(J^n) \leq cn - 1$  for  $n \geq 1$ . □

We are now ready to give a necessary and sufficient condition for the existence of a Cohen-Macaulay truncated Rees algebra.

**Theorem 3.4.** *Let  $R$  be a standard graded  $k$ -algebra with  $a^*(R) < 0$  and let  $I \subset R$  be a homogeneous ideal with  $\text{ht } I \geq 1$ . Let  $X = \text{Proj } R$ ,  $S = R[It]$  and  $Y = \text{Proj } S$ . Then there exists a Cohen-Macaulay ring  $R[(I^e)_c t]$  with  $c \geq d(I)e$  if and only if the following conditions are satisfied:*

- (i)  $Y$  is equidimensional and Cohen-Macaulay,
- (ii)  $\pi_* \mathcal{O}_Y = \mathcal{O}_X$  and  $R^i \pi_* \mathcal{O}_Y = 0$  for  $i > 0$ .

*Moreover, these conditions imply that  $R[(I^e)_c t]$  is a Cohen-Macaulay ring for  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  and  $e > \max\{a_X^*(S), a_X^*(\omega_S)\}$ .*

*Proof.* Let  $J$  be the ideal of  $R$  generated by  $(I^e)_c$  and  $T = R[Jt]$  for a fixed pair of positive integers  $c, e$  with  $c \geq d(I)e$ . Then  $Y \cong \text{Proj } T$ . If  $T$  is a Cohen-Macaulay ring, then (i) is obviously satisfied and  $Y$  is locally arithmetic Cohen-Macaulay over  $X$ . (ii) follows from Corollary 1.4.

To prove the converse we equip  $T$  with the aforementioned bi-graduation. Set  $e_0 = \max\{a_X^*(S), a_X^*(\omega_S)\}$ . We will use Theorem 3.1 to prove that  $T$  is Cohen-Macaulay for  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  and  $e > e_0$ . By Corollary 3.3 we have  $a^1(T) < 0$  and  $a^2(T) < 0$ . From the bi-graduation of  $T$  we see that

$$T(m, n)^\sim = \mathcal{O}_Y(m + cn, en),$$

where  $\mathcal{O}_Y(m + cn, n)$  denotes the twisted  $\mathcal{O}_Y$ -module with respect to the natural bi-graduation of  $S$ . If  $\pi_*\mathcal{O}_Y = \mathcal{O}_X$  and  $R^i\pi_*\mathcal{O}_Y = 0$  for  $i > 0$ , then we can show as in the proof of Proposition 1.3 that

$$H^i(Y, \mathcal{O}_Y(m, 0)) = H^i(X, \mathcal{O}_X(m))$$

for  $i \geq 0$ . Since  $a^*(R) < 0$ , we have  $H_{R_+}^i(R)_m = 0$  for all  $m \geq 0$  and  $i \geq 0$ . Using the Serre-Grothendieck correspondence between sheaf cohomology of  $X$  and local cohomology of  $R$  we can deduce that  $H^0(X, \mathcal{O}_X(m)) = R_m$  and  $H^i(X, \mathcal{O}_X(m)) = 0$  for  $i > 0$ . Therefore,

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(m, 0)) &= R_m = T_{(m,0)}, \\ H^i(Y, \mathcal{O}_Y(m, 0)) &= 0, \quad i > 0. \end{aligned}$$

For  $m \geq 0$  and  $n > 0$  we have  $m + cn > d(I)en + \varepsilon(I)$ . Therefore, using Proposition 1.3 and Lemma 1.5 we get

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(m + cn, en)) &= T_{(m,n)}, \\ H^i(Y, \mathcal{O}_Y(m + cn, en)) &= 0, \quad i > 0, \end{aligned}$$

for  $e > e_0$ . For  $m, n < 0$  we can show, similarly as above, that

$$H^i(Y, \omega_Y(-m - cn, -en)) = 0$$

for  $i > 0$  and  $e > e_0$ . If  $Y$  is equidimensional and Cohen-Macaulay, we can apply Serre duality and obtain

$$H^i(Y, \mathcal{O}_Y(m + cn, en)) = 0, \quad i < \dim Y.$$

Passing from  $\mathcal{O}_Y(m + cn, en)$  to  $T(m, n)^\sim$  we get

$$\begin{aligned} H^0(Y, T(m, n)^\sim) &\cong T_{(m,n)} \text{ for } m, n \geq 0, \\ H^i(Y, T(m, n)^\sim) &= 0 \text{ for } m, n \geq 0, \quad i > 0, \\ H^i(Y, T(m, n)^\sim) &= 0 \text{ for } m, n < 0, \quad i < \dim T - 2. \end{aligned}$$

By Theorem 3.1, these conditions imply that  $T$  is a Cohen-Macaulay ring. The proof of Theorem 3.4 is now complete.  $\square$

The following example shows that the condition  $a^*(R) < 0$  is not necessary for the existence of a Cohen-Macaulay truncated Rees algebra. It also shows that in general, the existence of a Cohen-Macaulay truncated Rees algebra does not imply the existence of a linear bound on  $c$  ensuring the Cohen-Macaulayness of  $R[(I^e)_c t]$ .

**Example 3.5.** Take  $R = k[x, y, z]/(xy^2 - z^3)$ , the coordinate ring of a cuspidal plane curve, and  $I = (x) \subseteq R$ , a homogeneous ideal with  $\text{ht } I = 1$ . Then  $R$  is a two-dimensional Cohen-Macaulay ring with  $a^*(R) = 0$ . It is obvious that  $R[(I^e)_c t] = R[It]$  is a Cohen-Macaulay ring for  $e \geq 1$ . For  $c > e$  we have  $R[(I^e)_c t] \cong R[(x, y, z)^{c-e} t]$ . It is easy to check that the reduction number of the ideal  $(x, y, z)^{c-e} t$  is greater than 1. By [14], this implies that  $R[(x, y, z)^{c-e} t]$  is not Cohen-Macaulay for any  $c > e$ .

Now we will show that the bound  $e > e_0$  in Theorem 3.4 is once again best possible.

**Theorem 3.6.** *Let the notations and assumptions be as in Theorem 3.4. Let*

$$e_0 = \max\{a_X^*(S), a_X^*(\omega_S)\}.$$

*Then  $R[(I^{e_0})_c t]$  is not a Cohen-Macaulay ring for  $c \geq d(I)e_0$  if  $e_0 \geq 1$ .*

*Proof.* Let  $T = R[(I^{e_0})_c t]$  for some  $c \geq d(I)e_0$ , and suppose  $e_0 \geq 1$ . Note that  $(I^{e_0})_c$  and  $I^{e_0}$  defines the same ideal sheaf in  $\mathcal{O}_X$ . Consider the natural  $\mathbb{N}$ -grading of  $T$  and  $S$  given by the degree of  $t$ . For any  $\mathfrak{p} \in X$ , the ring  $T_{(\mathfrak{p})}$  is isomorphic to the  $e_0$ -th Veronese subring of  $S_{(\mathfrak{p})}$ . Hence

$$\begin{aligned} H_{T_{(\mathfrak{p})+}}^i(T_{(\mathfrak{p})})_1 &= H_{S_{(\mathfrak{p})+}}^i(S_{(\mathfrak{p})})_{e_0}, \\ H_{T_{(\mathfrak{p})+}}^i((\omega_T)_{(\mathfrak{p})})_1 &= H_{S_{(\mathfrak{p})+}}^i((\omega_S)_{(\mathfrak{p})})_{e_0}, \end{aligned}$$

for  $i \geq 0$ . By the definition of  $e_0$  there exists  $\mathfrak{p} \in X$  and  $i \geq 0$  such that either  $H_{S_{(\mathfrak{p})+}}^i(S_{(\mathfrak{p})})_{e_0} \neq 0$  or  $H_{S_{(\mathfrak{p})+}}^i((\omega_S)_{(\mathfrak{p})})_{e_0} \neq 0$ . Therefore,  $\max\{a^*(T), a^*(\omega_T)\} \geq 1$ . By Lemma 1.1, this implies that  $T$  is not a Cohen-Macaulay ring.  $\square$

From Theorem 3.4 we can derive the following sufficient condition for the existence of a truncated Cohen-Macaulay Rees algebra.

**Theorem 3.7.** *Let  $R$  be an equidimensional standard graded  $k$ -algebra with  $a^*(R) < 0$  and let  $I \subset R$  be a homogeneous ideal with  $\text{ht } I \geq 1$ . Let  $X = \text{Proj } R$  and  $S = R[It]$ . Assume that  $S$  is locally Cohen-Macaulay on  $X$ . Then  $R[(I^e)_c t]$  is a Cohen-Macaulay ring for  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  and  $e \geq 1$ .*

*Proof.* It is obvious that the assumptions imply that  $Y$  is equidimensional and Cohen-Macaulay. The condition  $\pi_* \mathcal{O}_Y = \mathcal{O}_X$  and  $R^i \pi_* \mathcal{O}_Y = 0$  for  $i > 0$  follows from Corollary 1.4. Hence the conclusion follows from Theorem 3.4.  $\square$

The above condition is also a necessary condition for the existence of a truncated Cohen-Macaulay Rees algebra of the form  $R[I_c t]$  ( $e = 1$ ).

**Corollary 3.8.** *Let  $R$  be a standard graded  $k$ -algebra with  $a^*(R) < 0$  and let  $I \subset R$  be a homogeneous ideal with  $\text{ht } I \geq 1$ . Let  $X = \text{Proj } R$  and  $S = R[It]$ . Then there exists a Cohen-Macaulay ring  $R[I_c t]$  with  $c \geq d(I)$  if and only if  $S$  is locally Cohen-Macaulay on  $X$ .*

*Proof.* By Theorem 3.7 we only need to show that if  $R[I_c t]$  is a Cohen-Macaulay ring for some  $c \geq d(I)$ , then  $S$  is locally Cohen-Macaulay on  $X$ . But this is obvious because  $(I_c)$  and  $I$  define the same ideal sheaf and  $R[I_c t]$  is locally Cohen-Macaulay on  $X$ .  $\square$

Using Theorem 3.7 we obtain several classes of Cohen-Macaulay Rees algebras.

**Corollary 3.9** (cf. [8, Corollary 2.2.1(2)] for the case  $e = 1$ ). *Let  $R$  be a Cohen-Macaulay standard graded  $k$ -algebra with  $a(R) < 0$ . Let  $I \subset R$  be a homogeneous ideal with  $\text{ht } I \geq 1$  which is locally a complete intersection. Then  $R[(I^e)_c t]$  is a Cohen-Macaulay ring for all  $c > d(I)e + \max\{\varepsilon(I), \varepsilon^*(I)\}$  and  $e \geq 1$ .*

*Proof.* As in the proof of Corollary 2.6,  $S = R[It]$  is locally Cohen-Macaulay over  $X = \text{Proj } R$ . Since the assumption on  $R$  implies  $a^*(R) < 0$ , the conclusion follows from Theorem 3.7.  $\square$

**Corollary 3.10.** *Let  $R$  be a polynomial ring over a field  $k$  of characteristic zero and let  $I \subset R$  be a non-singular homogeneous ideal. Then  $R[(I^e)_c t]$  is a Cohen-Macaulay ring for all  $c > d(I)e + \varepsilon(I)$  and  $e \geq 1$ .*

*Proof.* We have seen in the proof of Corollary 2.7 that  $\varepsilon^*(I) = 0$ . Hence the assertion follows from Corollary 3.9.  $\square$

**Corollary 3.11** (cf. [15, Theorem 2.4] for the case  $e = 1$ ). *Let  $R$  be a polynomial ring over a field  $k$  of characteristic zero and let  $I \subset R$  be the defining ideal of a scheme of fat points in  $\text{Proj } R$ . Then  $R[(I^e)_c t]$  is a Cohen-Macaulay ring for  $c > \text{reg}(I)e$ .*

*Proof.* The proof follows from Theorem 3.7 with the same lines of arguments as in the proof of Corollary 2.9.  $\square$

Now we will use Theorem 3.7 to find a criterion for arithmetically Cohen-Macaulay blow-ups. Recall that the blow-up  $Y$  of a projective scheme  $X$  along an ideal sheaf  $\mathcal{I}$  is said to be *locally arithmetic Cohen-Macaulay on  $X$*  if there exist a standard graded algebra  $R$  over a field and a homogeneous ideal  $I \subset R$  such that  $X = \text{Proj } R$ ,  $\mathcal{I} = \tilde{I}$  and  $S = R[It]$  is locally Cohen-Macaulay on  $X$ .

**Theorem 3.12.** *Let  $X$  be a projective scheme over a field  $k$  such that  $H^0(X, \mathcal{O}_X) = k$  and  $H^i(X, \mathcal{O}_X) = 0$  for  $i > 0$ . Let  $Y \rightarrow X$  be a proper birational morphism. Then  $Y$  is an arithmetically Cohen-Macaulay blow-up if and only if  $Y$  is equidimensional and locally arithmetic Cohen-Macaulay on  $X$ .*

*Proof.* Suppose  $Y$  is an arithmetically Cohen-Macaulay blow-up of  $X$ . Let  $R$  be a standard graded algebra over  $k$ , and let  $I$  be a homogeneous ideal of  $R$ , such that  $X = \text{Proj } R$ ,  $Y$  is the blow-up of  $X$  along the ideal sheaf  $\tilde{I}$ , and  $S = R[It]$  is a Cohen-Macaulay ring. Then,  $\mathcal{O}_{X,x}[\mathcal{I}_x t] = S_{(\mathfrak{p})}$  is obviously Cohen-Macaulay for all  $\mathfrak{p} \in X$ . Thus,  $Y$  is locally arithmetic Cohen-Macaulay on  $X$ .

Conversely, suppose  $Y$  is equidimensional and locally arithmetic Cohen-Macaulay on  $Y$ . Then there exist a standard graded  $k$ -algebra  $R$  and a homogeneous ideal  $I \subset R$  such that  $X = \text{Proj } R$ ,  $Y$  is the blow-up of  $X$  along the ideal sheaf of  $I$ , and  $R[It]$  is locally Cohen-Macaulay on  $X$ . The assumption on the sheaf cohomology of  $X$  implies that  $H_{R_+}^i(R)_0 = 0$  for  $i \geq 0$ . Without restriction we may replace  $R$  by a suitable Veronese subalgebra and obtain  $H_{R_+}^i(R)_n = 0$  for all  $n \geq 0$  or, equivalently,  $a^*(R) < 0$ . Now we may apply Theorem 3.7 to find a Cohen-Macaulay Rees algebra  $R[I_c t]$  with  $c \gg 0$ . Since the ideal  $(I_c)$  defines the same ideal sheaf  $\tilde{I}$ , we can conclude that  $Y$  is an arithmetically blow-up of  $X$ .  $\square$

## NOTE ADDED IN PROOF

After the manuscript was sent for publication, we were informed that Olga Lavila-Vidal obtained similar results in her thesis (ArXiv:math.AC/0407041) as our Theorem 2.4 and Corollary 2.6. She did not have an explicit description for the constant term as we did though.

## REFERENCES

- [1] I. M. Aberbach, C. Huneke and N.V. Trung. *Reduction numbers, Briançon-Skoda theorems, and depth of Rees rings*. *Compositio Math.* **97** (1995), 403-434. MR1353282 (96g:13002)
- [2] Y. Aoyama. *On the depth and the projective dimension of the canonical module*. *Japanese J. Math.* **6** (1980), 61-66. MR0615014 (82h:13007)
- [3] A. Bertram, L. Ein and R. Lazarsfeld. *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties*. *J. Amer. Math. Soc.* **4** (1991), no. 3, 587-602. MR1092845 (92g:14014)
- [4] M. Brodmann and R. Sharp. *Local cohomology*. Cambridge University Press, 1998. MR1613627 (99h:13020)
- [5] W. Bruns and J. Herzog. *Cohen-Macaulay rings*. Cambridge University Press, 1993. MR1251956 (95h:13020)
- [6] K. A. Chandler. *Regularity of the powers of an ideal*. *Commun. Algebra.* **25** (1997), 3773-3776. MR1481564 (98i:13040)
- [7] A. Conca, J. Herzog, N.V. Trung and G. Valla. *Diagonal subalgebras of bi-graded algebras and embeddings of blow-ups of projective spaces*. *American Journal of Math.* **119** (1997), 859-901. MR1465072 (99d:13001)
- [8] S.D. Cutkosky and H. Tàì Hà. *Arithmetic Macaulayfication of projective schemes*. *J. Pure Appl. Algebra*. To appear.
- [9] S.D. Cutkosky and J. Herzog. *Cohen-Macaulay coordinate rings of blowup schemes*. *Comment. Math. Helv.* **72** (1997), 605-617. MR1600158 (99d:13028)
- [10] S.D. Cutkosky, J. Herzog and N.V. Trung. *Asymptotic behaviour of the Castelnuovo-Mumford regularity*. *Compositio Math.* **118** (1999), 243-261. MR1711319 (2000f:13037)
- [11] A.V. Geramita and A. Gimigliano. *Generators for the defining ideal of certain rational surfaces*. *Duke Mathematical Journal.* **62** (1991), no. 1, 61-83. MR1104323 (92f:14031)
- [12] A.V. Geramita, A. Gimigliano and B. Harbourne. *Projectively normal but superabundant embeddings of rational surfaces in projective space*. *J. Algebra.* **169** (1994), no. 3, 791-804. MR1302116 (96f:14044)
- [13] A.V. Geramita, A. Gimigliano and Y. Pitteloud. *Graded Betti numbers of some embedded rational  $n$ -folds*. *Math. Ann.* **301** (1995), 363-380. MR1314592 (96f:13022)
- [14] S. Goto and Y. Shimoda, *On the Rees algebras of Cohen-Macaulay rings*. *Lect. Notes in Pure and Appl. Math.* 68, Marcel-Dekker, 1979, 201-231. MR0655805 (84a:13021)
- [15] H. Tàì Hà. *On the Rees algebra of certain codimension two perfect ideals*. *Manu. Math.* **107** (2002), 479-501. MR1906772 (2003d:13002)
- [16] H. Tàì Hà. *Projective embeddings of projective schemes blown up at subschemes*. *Math. Z.* **246** (2004), 111-124. MR2031448
- [17] R. Hartshorne. *Algebraic Geometry*. Graduate Text **52**. Springer-Verlag, 1977. MR0463157 (57:3116)
- [18] S. Huckaba and C. Huneke. *Rees algebras of ideals having small analytic deviation*. *Trans. Amer. Math. Soc.* **339** (1993), no. 1, 373-402. MR1123455 (93k:13008)
- [19] E. Hyry. *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*. *Trans. Amer. Math. Soc.* **351** (1999), no. 6, 2213-2232. MR1467469 (99i:13005)
- [20] E. Hyry and K. Smith. *On a Non-Vanishing Conjecture of Kawamata and the Core of an Ideal*. *Amer. J. Math.* **125** (2003), 1349-1410. MR2018664
- [21] B. Johnston and D. Katz. *Castelnuovo regularity and graded rings associated to an ideal*. *Proc. Amer. Math. Soc.* **123** (1995), 727-734. MR1231300 (95d:13005)
- [22] T. Kawasaki. *On arithmetic Macaulayfication of local rings*. *Trans. Amer. Math. Soc.* **354**, 123-149. MR1859029 (2002i:13001)
- [23] V. Kodiyalam. *Asymptotic behaviour of Castelnuovo-Mumford regularity*. *Proc. Amer. Math. Soc.* **128** (2000), 407-411. MR1621961 (2000c:13027)

- [24] D. Mumford. *Varieties defined by quadratic equations*. C.I.M.E. III. (1969), 29-100. MR0282975 (44:209)
- [25] O. Lavila-Vidal. *On the Cohen-Macaulay property of diagonal subalgebras of the Rees algebra*. Manu. Math. **95** (1998), 47-58. MR1492368 (99i:13006)
- [26] O. Lavila-Vidal. *On the existence of Cohen-Macaulay coordinate rings of blow-up schemes*. Preprint.
- [27] J. Lipman. *Cohen-Macaulayness in graded algebras*. Math. Res. Letters **1** (1994), 149-157. MR1266753 (95d:13006)
- [28] C. Polini and B. Ulrich. *Necessary and sufficient conditions for the Cohen-Macaulayness of blow-up algebras*. Compositio Math. **119** (1999), no. 2, 185-207. MR1723128 (2001f:13007)
- [29] R. Sharp. *Bass numbers in the graded case,  $a$ -invariant formula, and an analogue of Falting's annihilator theorem*. J. Algebra. **222** (1999), no. 1, 246-270. MR1728160 (2000j:13027)
- [30] A. Simis, B. Ulrich, and W. Vasconcelos. *Cohen-Macaulay Rees algebras and degrees of polynomial equations*. Math. Ann. **301** (1995), 421-444. MR1324518 (96a:13005)
- [31] I. Swanson. *Powers of ideals. Primary decompositions, Artin-Rees lemma and regularity*. Math. Ann. **307** (1997), 299-313. MR1428875 (97j:13005)
- [32] N.V. Trung. *The largest non-vanishing degree of graded local cohomology modules*. J. Algebra. **215** (1999), no. 2, 481-499. MR1686202 (2000f:13038)
- [33] N.V. Trung and S. Ikeda. *When is the Rees algebra Cohen-Macaulay?* Comm. Algebra. **17** (1989), no. 12, 2893-2922. MR1030601 (91a:13009)
- [34] N.V. Trung and H-J. Wang. *On the asymptotic linearity of Castelnuovo-Mumford regularity*. Preprint. arXiv:math.AC/0212161.

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