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# HARMONIC CALCULUS ON FRACTALS—A MEASURE GEOMETRIC APPROACH II

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ABSTRACT. Riesz potentials of fractal measures $\mu$ in metric spaces and their inverses are introduced. They define self–adjoint operators in the Hilbert space $L_2(\mu)$ and the former are shown to be compact.

In the Euclidean case the corresponding spectral asymptotics are derived with Besov space methods. The inverses of the Riesz potentials are fractal pseudodifferential operators. For the order two operator the spectral dimension coincides with the Hausdorff dimension of the underlying fractal.

## INTRODUCTION

In part I of this paper [7] the Laplace operator $\Delta_\mu$ with respect to an atomless finite Borel measure $\mu$ on an interval $(a,b)$ is introduced by

$$\Delta_\mu = \left(\frac{d}{d\mu}\right)^2 .$$

Then $g \in L_2(\mu)$ solves $\Delta_\mu g = f$ for $f \in L_2(\mu)$ with boundary conditions $g(a) = g_a$ and $g(b) = g_b$ if and only if

$$g(x) = g_b \frac{\mu\big((a,x)\big)}{\mu\big((a,b)\big)} + g_a \frac{\mu\big((x,b)\big)}{\mu\big((a,b)\big)} - \int_a^b G_\mu(x,y)\, f(y)\, \mu(dy), \tag{1}$$

where

$$G_\mu(x,y) := \begin{cases} \mu\big((a,x)\big)\ \mu\big((y,b)\big) & \text{if } x \le y, \\ \mu\big((a,y)\big)\ \mu\big((x,b)\big) & \text{if } x \ge y \end{cases}$$

is the corresponding Green's function for the Dirichlet boundary problem. In [24] it is indicated that the properties of $\Delta_\mu$ and an associated Dirichlet form may be derived from those of the Euclidean case, where $\mu = \lambda$ for the Lebesgue measure $\lambda$ on $(a,b)$, by a suitable scaling of the space according to the measure $\mu$. In particular, Weyl's spectral exponent $\mu_k \asymp k^2$ of the eigenvalues $\mu_k$ of $-\Delta_\mu$ obtained in part I for Cantor–type measures $\mu$ remains valid for arbitrary $\mu$.

Freiberg [6] extended the approach of part I to the operators

$$\Delta_{\mu\nu} := \frac{d}{d\mu}\,\frac{d}{d\nu} \tag{2}$$

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for a second measure $\nu$ of the above type. Here the analogue of (1) reads

$$g(x) = g_b \, \frac{\nu\big((a,x)\big)}{\nu\big((a,b)\big)} + g_a \, \frac{\nu\big((a,x)\big)}{\nu\big((a,b)\big)} - \int_a^b G_\nu(x,y) \; f(y) \; \mu(dy) \tag{3}$$

with Green's function

$$G_\nu(x,y) := \begin{cases} \nu\big((a,x)\big) \; \nu\big((y,b)\big) & \text{if } x \le y, \\ \nu\big((a,y)\big) \; \nu\big((x,b)\big) & \text{if } x \ge y. \end{cases}$$

The special case of Cantor measures $\mu$ and $\nu = \lambda$ was considered in Fujita [8] in connection with quasidiffusions on the real line. Strichartz [21], §2, demonstrates how Fujita's operator fits into the model of Kigami [12], who constructed the Laplacian $-\Delta_{\mu\lambda}$ by means of limits in finite difference schemes on the prefractals in the self–similar model.

In Kigami's higher–dimensional approach and the well–known equivalent stochastic constructions the spectral dimension of the Laplace operator and the Hausdorff dimension of the fractal support of $\mu$ differ. (References to part of the numerous literature on this topic may be found in part I.) By physical reasons it makes sense to ask for variants of the Laplace operator in a suitable metric where this difference disappears (Berry conjecture; see the discussion in Kigami and Lapidus [13]). In the case of the line it turns out that for these purposes the Lebesgue differentiation $\frac{d}{d\nu} = \frac{d}{d\lambda} = \frac{d}{dx}$ in (2) is to be replaced by a certain fractional differentiation $\big(\frac{d}{dx}\big)^\alpha$ for suitable $\alpha$.

In the present paper we start to treat the case of $\mathbb{R}^n$ with $n > 1$ instead of the line. Some measure geometric motivations for our approach may be found in the survey paper [25]. Part of the theory will be developed in section 1 for general metric spaces. For the Euclidean case and Ahlfors $d$–regular finite Borel measures $\mu$ with compact support $\Gamma$, where $0 < d < n$, we interpret the fractal $d$–set $\Gamma$ as the boundary of the closure of its open complement. As in differential geometry of manifolds we postulate that this boundary has no boundary. Therefore we will deal with boundary free problems, and the father of our 'Laplacian' is the Euclidean boundary free variant $\Delta$ with Green's functions $G(x,y) = \text{const}\,|x-y|^{-(n-2)}$ for $-\Delta$. It is well known that the fractional powers $(-\Delta)^{-\sigma/2} =: I^\sigma$, $\sigma > 0$, possess the Green's functions $G^\sigma(x,y) = \text{const}\,|x-y|^{-(n-\sigma)}$, i.e.,

$$I^\sigma \, f(x) := \text{const} \int |x-y|^{-(n-\sigma)} \, f(y) \, dy \, .$$

These are the Riesz potentials of order $s$ which have been studied, e.g. in Adams and Hedberg [1], Rubin [18] and Samko, Kilbas and Marichev [19] (see also the references therein). By Fourier transformation methods $I^\sigma$ may be extended to $\sigma \in \mathbb{R}$ and distributions $f$.

If we consider a certain trace in the sense of Jonsson and Wallin [11], Jonsson [10] and Triebel [22] of a modification $\widetilde{I}^\sigma$ of $I^\sigma$ on the fractal $d$–set $\Gamma$, we arrive at the Riesz potential of order $s$ with respect to the measure $\mu$:

$$I^s_\mu \, f(x) := \text{const} \int |x-y|^{-(d-s)} \, f(y) \, \mu(dy), \tag{4}$$

$f \in L_2(\mu)$, where $\sigma = s + n - d$ and $n - d$ is the fractal defect (Theorem 3.1). The modification $\widetilde{I}^\sigma$ is chosen such that its kernel coincides with the Riesz kernel on $\Gamma$ and with a Bessel type kernel at infinity. By means of the trace property we

infer that the image $I^s_\mu\big(L_2(\mu)\big)$ lies in the fractal Besov space $B^s_{2,\infty}(\Gamma)$. The Fourier analytic background from the measure theoretic side is closely related to the papers of Strichartz [20] and Lau [14].

Moreover, $I^s_\mu$ is invertible and positive (Theorem 3.2). Thus, considering $I^s_\mu : L_2(\mu) \to L^s_2(\mu)$ as an isometry, the space $L^s_2(\mu)$ of Riesz potentials of order $s$ becomes a Hilbert space. The inverses $D^s_\mu := (I^s_\mu)^{-1}$ may be interpreted as nonlocal fractal pseudodifferential operators.

Using a method of Triebel [22] developed for other operators, we obtain the spectral asymptotics $k^{-s/d}$ for the Riesz potentials $I^s_\mu$ (Theorem 3.3). In particular, for the eigenvalues $\mu_k$ of $D^2_\mu$ the Berry conjecture $\mu_k \asymp k^{2/d}$ holds true.

Note that (4) is a boundary free higher dimensional variant of (3) where $\nu\big((x,y)\big) := y - x$ is to be replaced by $(y-x)^\alpha$ for certain $\alpha$.

In distinction to approaches with boundaries (see, e.g., Metz [16], Kigami and Lapidus [13] and Denker and Sato [2]) the unique harmonic function with respect to the Laplace operator is identically zero. This simplifies the study of related (nonlinear) pseudodifferential equations. (See Falconer [4] and Falconer and Hu [5] for the Laplace operators on the variational fractals of Mosco [17].) According to the above definitions, in our case such fractal differential equations may be transformed into traditional integral equations, and existence of a solution can be proved by compactness arguments. This, as well as the existence of associated Dirichlet forms and Markov processes in the sense of Fukushima, Oshima and Takeda [9], will be shown elsewhere. In order to obtain Laplace type operators of this kind, one has to change the metric in the Euclidean case.

In sections 1 and 2 we investigate the more general case of $s$–potentials,

$$\mathcal{P}^s_\mu f(x) := \int \varrho(x,y)^{-s}\, f(y)\, \mu(dy), \;\; f \in L_2(\mu), \tag{5}$$

for Ahlfors (upper) $d$–regular finite measures $\mu$ in metric spaces $(X,\varrho)$, where $0 < s < d$. Riesz potentials and 'pseudofifferential' operators are derived notions. In particular, we show that $\mathcal{P}^s_\mu$ is a self–adjoint compact operator in the Hilbert space $L_2(\mu)$.

Recall that the $s$–potential functions $U^s_\mu(x) := \int \varrho(x,y)^{-s}\, d\mu(y)$ play an important role in fractal dimension theory. Furthermore, straightforward calculation shows that the Riesz potential function of order $\alpha$ considered by Maly and Mosco [15] in the spirit of Lagrangian theory for $d$–regular measures $\mu$ is equivalent to our $U^{d-\alpha}_\mu$. Thus, we obtain close relationships to former approaches to problems of analysis on fractals.

## 1. $s$–potentials of Borel measures in metric spaces

Throughout this paper let $(X,\varrho)$ be a metric space, let $d > 0$, and let $\mu$ be a finite Borel measure on $X$. For simplicity we frequently assume that $\mu(X) = 1$.

**Definition.** $\mu$ is said to be *Ahlfors upper $d$–regular* if there is a constant $c_1 > 0$ such that

$$\mu\big(B(x,r)\big) \le c_1\, r^d \tag{1.1}$$

for any $r > 0$ and $x \in \operatorname{supp}\mu$, where $B(x,r)$ denotes the closed ball with centre $x$ and radius $r$. If

$$c_2\, r^d \le \mu\big(B(x,r)\big) \le c_1\, r^d \tag{1.2}$$

for some constants $c_1$, $c_2 > 0$, any $0 < r < r_0$ (for some $r_0$) and $x \in \operatorname{supp} \mu$, then $\mu$ is called *Ahlsfors d–regular* (briefly a *d–measure*).

The upper density comparison theorem implies that any $d$–measure $\mu$ is equivalent to Hausdorff measure $\mathcal{H}^d$ restricted to the support of $\mu$. (This is well known from the Euclidean case and proved by Edgar [3], 1.5.14, for metric spaces.)

**Proposition 1.1.** *If $\mu$ is Ahlfors upper d–regular and $0 \le s < d$, then we have*

$$\int \varrho(x,y)^{-s}\ \mu(dy) \le 2 + c_1\ s(d-s)^{-1} =: c(s)$$

*for all $x \in \operatorname{supp} \mu$.*

The proof is well known from the Euclidean case. For completeness we will repeat it here:

For $x$ as above consider $m_x(r) := \mu(B(x,r))$ as the distribution function of a normed measure on $(0,+\infty)$ and decompose

$$\begin{aligned}
&\int \varrho(x,y)^{-s}\ \mu(dy) = \int_0^\infty r^{-s}\ dm_x(r) \\
&= \int_0^1 r^{-s}\ dm_x(r) + \int_1^\infty r^{-s}\ dm_x(r) \le \int_0^1 r^{-s}\ dm_x(r) + 1 \\
&= r^{-s}\ m_x(r)\ \Big|_0^1 + s\int_0^1 r^{-s-1}\ m_x(r)\ dr + 1 \\
&\text{(by Fubini)} \\
&\le 1 - \lim_{r\searrow 0} r^{-s}\ m_x(r) + s\int_0^1 r^{-s-1}\ c_1\ r^d\ dr + 1 \\
&\text{(by (1.1))} \\
&= 2 - 0 + c_1\ s(d-s)^{-1}\ r^{d-s}\ \Big|_0^1 = 2 + c_1\ s(d-s)^{-1}.
\end{aligned}$$

□

*Remark.* In capacity theory and fractal geometry

$$U_\mu^s(x) = \int \varrho(x,y)^{-s}\ \mu(dy)$$

is called an *s–potential function* and its mean value

$$E_\mu^s := \int U_\mu^s(x)\ \mu(dx) = \iint \varrho(x,y)^{-s}\ \mu(dy)\ \mu(dx)$$

an *s–energy* of the measure $\mu$.

Thus, we have proved that the $s$–potential functions of $\mu$ are *bounded*. Moreover, we infer the following *uniform integrability* of the kernel $\varrho(x,\cdot)^{-s}$:

**Proposition 1.2.** *Under the conditions of Proposition* 1.1 *we have for any Borel set $A \subset X$*

$$\int_A \varrho(x,y)^{-s}\ d\mu(y) \le c(sp)^{1/p}\ \mu(A)^{1/q}$$

*at ($\mu$–almost) all $x \in \operatorname{supp} \mu$, if $p,q \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $sp < d$.*

*Proof.* The Hölder inequality implies

$$\int_A \varrho(x,y)^{-s}\ \mu(dy) \le \left(\int \varrho(x,y)^{-sp}\ \mu(dy)\right)^{1/p}\ \mu(A)^{1/q}.$$

Then Proposition 1.1 yields the assertion. □

Finally, the $s$–potentials are *continuous*:

**Proposition 1.3.** *If $\mu$ is upper $d$–regular and $0 < s < d$, then we have*

$$\lim_{\varrho(x,z)\to 0} \int \left|\ \varrho(x,y)^{-s} - \varrho(z,y)^{-s}\right|\ \mu(dy) = 0$$

*uniformly in $(x,z)$.*

*Proof.* For fixed $\delta > 0$ we obtain

$$\begin{aligned}
&\int \left|\varrho(x,y)^{-s} - \varrho(z,y)^{-s}\right| \mu(dy)\\
&= \int_{\{\varrho(x,y)>\delta\}\cap\{\varrho(z,y)>\delta\}} \frac{\left|\varrho(z,y)^s - \varrho(x,y)^s\right|}{\varrho(x,y)^s\,\varrho(z,y)^s}\ \mu(dy)\\
&+ \int_{\{\varrho(x,y)\le\delta\}\cup\{\varrho(z,y)\le\delta\}} \left|\ \varrho(x,y)^{-s} - \varrho(z,y)^{-s}\right|\ \mu(dy)\,.
\end{aligned}$$

The first summand tends to 0 as $\varrho(x,z)\to 0$ uniformly in $x,z$. The second summand does not exceed

$$\begin{aligned}
&\int_{\{\varrho(x,y)\le\delta\}} \left(\varrho(x,y)^{-s} + \varrho(z,y)^{-s}\right)\ \mu(dy)\\
&+ \int_{\{\varrho(z,y)\le\delta\}} \left(\varrho(x,y)^{-s} + \varrho(z,y)^{-s}\right)\ \mu(dy)\\
&\le\ \text{const}\left(\mu\left\{y : \varrho(x,y)\le\delta\right\}^{1/q} + \mu\left\{y : \varrho(z,y)\le\delta\right\}^{1/q}\right)\\
&\le\ \text{const}\ \ \delta^{d/q}
\end{aligned}$$

for any $x,z$ by Proposition 1.2 and the upper $d$–regularity of $\mu$. Letting $\delta\to 0$ we conclude the assertion. □

Propositions 1.1–1.3 enable us to study properties of the *integral operator*

$$\mathcal{P}^s_\mu f(x) := \int \varrho(x,y)^{-s}\ f(y)\ \mu(dy) \tag{1.3}$$

on the Hilbert space $L_2(\mu)$ of square integrable, real functions $f$ on $X$ with scalar product $\langle f,g\rangle_\mu := \int fg\,d\mu$. It turns out that for $\mu$ and $s$ as before $\mathcal{P}^s_\mu$ is a bounded map from $L_2(\mu)$ into $L_2(\mu)$. Moreover, we have the following.

**Theorem 1.4.** *If $\mu$ is upper $d$–regular and $0 < s < d$, then $\mathcal{P}^s_\mu$ is determined, and the image set of the unit ball $B_\mu$ in $L_2(\mu)$ under the mapping $\mathcal{P}^s_\mu$ is in $L_2(\mu)$*

(i) *equibounded,*
(ii) *equiintegrable,*
(iii) *equicontinuous if $X = \mathbb{R}^n$.*

*Proof.* For $f \in B_\mu$ and any Borel set $A \subset X$ we get by the Cauchy–Schwarz inequality applied to the normed measure $U_\mu^s(x)^{-1}\ \varrho(x,y)^{-s}\ \mu(dy)$:

$$\begin{aligned}
&\int_A \left( \int f(y)\, \varrho(x,y)^{-s}\, \mu(dy) \right)^2 \mu(dx) \\
&\le \int_A \int f(y)^2\, \varrho(x,y)^{-s}\, \mu(dy)\, U_\mu^s(x)\, \mu(dx) \\
&= \int f(y)^2 \int_A \varrho(x,y)^{-s}\, U_\mu^s(x)\, \mu(dx)\, \mu(dy) \\
&\le \int f(y)^2\, \mu(dy) \sup_x U_\mu^s(x) \sup_y \int_A \varrho(x,y)^{-s}\, \mu(dx) \\
&\le c(s)\ c(sp)^{1/p}\ \mu(A)^{1/q}
\end{aligned}$$

in view of Propositions 1.1 and 1.2.

This proves (i) and (ii). For (iii) we similarly estimate

$$\begin{aligned}
&\int \left( \int f(y)\, |x-y|^{-s}\, \mu(dy) - \int f(y)\, |x-u-y|^{-s}\, \mu(dy) \right)^2 \mu(dx) \\
&= \int \left( \int f(y)\ \left( |x-y|^{-s} - |x-u-y|^{-s} \right) \mu(dy) \right)^2 \mu(dx) \\
&\le \iint f(y)^2\, k(x,u) \big| |x-y|^{-s} - |x-u-y|^{-s} \big|\, \mu(dy)\, \mu(dx) \\
&\le 2 \sup_x U_\mu^s(x) \sup_y \int \big| |x-y|^{-s} - |x-u-y|^{-s} \big|\, \mu(dx) \int f(y)^2\, \mu(dy),
\end{aligned}$$

where $k(x,u) = \int \big| |x-y|^{-s} - |x-u-y|^{-s} \big|\, \mu(dy) \le U_\mu^s(x) + U_\mu^s(x-u)$. According to Proposition 1.1, $U_\mu^s$ is bounded by $c(s)$. It remains to apply Proposition 1.3 as $u \to 0$. □

For higher order integrable functions the following uniform continuity property holds true.

**Proposition 1.5.** *If $\mu$ is upper $d$–regular, $0 < s < d$, and $f \in L_r(\mu)$ with $\frac{1}{r} + \frac{1}{t} = 1$, $st < d$, then the image function $\mathcal{P}_\mu^s f$ is everywhere determined and uniformly continuous on* $\operatorname{supp} \mu$.

*Proof.* The Cauchy–Schwarz inequality and Proposition 1.1 yield the first part of the assertion. Similarly, one obtains for any $x \in \operatorname{supp} \mu$

$$\begin{aligned}
&\left| \int \varrho(x,y)^{-s}\, f(y)\, \mu(dy) - \int \varrho(z,y)^{-s}\, f(y)\, \mu(dy) \right| \\
&\le \left( \int \left| \varrho(x,y)^{-s} - \varrho(z,y)^{-s} \right|^t \mu(dy) \right)^{1/t} \|f \mid L_r(\mu)\| \ .
\end{aligned}$$

Thus, it remains to apply the arguments of the proof of Proposition 1.3 with $st$ instead of $s$ when using Proposition 1.2. □

One of the basic related properties of the $s$–potential $\mathcal{P}^s_\mu$ is its compactness:

**Theorem 1.6.** *For upper $d$–regular $\mu$ and $0 < s < d$ the operator*

$$\mathcal{P}^s_\mu : L_2(\mu) \to L_2(\mu)$$

*is compact.*

*Proof.* We have to show that any sequence $\mathcal{P}^s_\mu f_n$ with $f_n$ from the unit ball $B_\mu$ contains a subsequence converging in $L_2(\mu)$. Since $L_2(\mu)$ is a Hilbert space it is locally sequentially weakly compact. Choosing a subsequence we may assume that

$$\lim_{n\to\infty} \langle f_n, g\rangle_\mu = \langle f, g\rangle_\mu \ , \ g \in L_2(\mu)\,,$$

for some limit function $f \in B_\mu$. In order to prove that $\mathcal{P}^s_\mu f_n$ converges to $P^s_\mu f$ in $L_2(\mu)$ we estimate for any $\delta > 0$,

$$\begin{aligned}
&\left(\int \left(\int (f_n(y) - f(y))\, \varrho(x,y)^{-s}\, \mu(dy)\right)^2 \mu(dx)\right)^{1/2} \\
&\le \int \left( \int_{\varrho(x,y)>\delta} \big((f_n(y) - f(y))\ \varrho(x,y)^{-s}\, \mu(dy)\big)^2 \mu(dx)\right)^{1/2} \\
&+ \left(\int \int_{\varrho(x,y)\le\delta} \big((f_n(y) - f(y))\ \varrho(x,y)^{-s}\, \mu(dy)\big)^2 \mu(dx)\right)^{1/2} .
\end{aligned}$$

For fixed $\delta$ the first summand on the right–hand side tends to zero as $n \to \infty$, since the inner integral does so by assumption and is bounded in $x$ and $n$ by $2\delta^{-s}$ in view of the Cauchy–Schwarz inequality. Similar to the proof of Theorem 1.4 the second summand may be estimated by

$$\begin{aligned}
&\left(\int U^s_\mu(x) \int 1\{\varrho(x,y) \le \delta\}\, (f_n(y) - f(y))^2\, \varrho(x,y)^{-s}\, \mu(dy)\, \mu(dx)\right)^{1/2} \\
&\le \text{const} \left(\int (f_n(y) - f(y))^2 \int 1\{\varrho(x,y) \le \delta\} \varrho(x,y)^{-s}\, \mu(dx)\, \mu(dy)\right) \\
&\le \text{const} \int (\mu\{x : \varrho(x,y) \le \delta\})^{1/q}\ \mu(dy) \le \text{const}\, \delta^{d/q}
\end{aligned}$$

according to Proposition 1.2 and (1.1). Letting $\delta \to 0$ we obtain the $L_2(\mu)$–convergence stated above. □

By construction, the operator $\mathcal{P}^s_\mu$ under the above conditions is *self–adjoint*. Thus, we can apply the general theory of self–adjoint compact operators in Hilbert spaces (see, e.g., Yosida [23]) in order to conclude a point spectrum of real eigenvalues with finite multiplicities and at most zero as an accumulation point. Moreover, there exists an orthonormal sequence of corresponding eigenvectors which is complete in the image space.

## 2. Riesz potentials of $d$-regular measures in metric spaces

Recall that the upper $d$–regularity (1.1) of the measure $\mu$ implies that for $0 < s < d$ the function $U^s_\mu(x) = \int \varrho(x,y)^{-s}\ \mu(dy)$ is bounded by some constant $c(s)$. Hence,

the $s$–energy $E^s_\mu$ does not exceed $c(s)$. In this section we additionally assume that $d$ is *optimal*, i.e.,

$$d = \sup\left\{ s : \int \varrho(x,y)^{-s}\ \mu(dy) < \infty \right\} \tag{2.1}$$

at $\mu$–almost all $x \in \operatorname{supp}\mu$. Then $d$ will be called the *Riesz dimension* of $\mu$. Obviously, the correlation dimension of $\mu$ given by

$$\sup\left\{ s : \iint \varrho(x,y)^{-s}\ \mu(dy)\mu(dx) < \infty \right\}$$

does not exceed such a $d$. Moreover, by (1.1) the lower pointwise dimensions

$$\liminf_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r}$$

at all $x \in \operatorname{supp}\mu$ are not less than $d$. Therefore, given (1.1) a sufficient condition for (2.1) is

$$\limsup_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r} \le d$$

at almost all points. In this case $\mu$ is dimension regular and $d$ agrees with its Hausdorff, packing and correlation dimensions. Note that the $d$–regularity (1.2) may be considered as a special case.

In analogy to the Euclidean case (see, e.g., [1], [18], [19]) we now introduce the *Riesz potentials* $I^s_\mu$ *of order* $s$ for measures $\mu$ as before with Riesz dimension $d$:

$$I^s_\mu f(x) := \int \varrho(x,y)^{-(d-s)}\, f(y)\, \mu(dy)\,,$$

$f \in L_2(\mu),\ 0 < s < d$. These are nothing but the operators $\mathcal{P}^{d-s}_\mu$ from above.

In order to determine an inverse operator we have to eliminate the kernel $N(I^s_\mu)$ of the Riesz potential. Recall that $\widetilde{L}^s_2(\mu)$ denotes the orthogonal complement of $N(I^s_\mu)$ in $L_2(\mu)$. Then $N(I^s_\mu)$ and $\widetilde{L}^s_2(\mu)$ are closed subspaces of $L_2(\mu)$ and we have

$$L_2(\mu) = N(I^s_\mu) \oplus \widetilde{L}^s_2(\mu)\,.$$

Moreover, $I^s_\mu$ is a bijection from $\widetilde{L}^s_2(\mu)$ onto the *space of Riesz potentials*

$$L^s_2(\mu) := I^s_\mu\ (L_2(\mu))\ .$$

Denote the *inverse operator* by $D^s_\mu$, i.e.,

$$D^s_\mu : L^s_2(\mu) \to \widetilde{L}^s_2(\mu)\,.$$

In Theorem 3.2 below we will show that in the Euclidean case $\widetilde{L}^s_2(\mu)$ agrees with the whole $L_2(\mu)$, i.e., $I^s_\mu$ is invertible.

According to the arguments at the end of section 1 the operator $I^s_\mu$ on $\widetilde{L}^s_2(\mu)$ has a countable complete orthonormal system of eigenvectors corresponding to nonzero eigenvalues. Therefore $L^s_2(\mu)$ is dense in $\widetilde{L}^s_2(\mu)$.

For $s := 1 < d$ we denote

$$\nabla_\mu := D^1_\mu.$$

$\nabla_\mu$ may be interpreted as a nonlinear variant of a gradient operator if it is locally defined. The mapping

$$\Delta_\mu := -\left(\nabla_\mu\right)^2$$

with domain $\mathrm{dom}(\Delta_\mu) := \{f \in L_2^1(\mu) : \nabla_\mu f \in L_2^1(\mu)\}$ may be called a (boundary free) *Laplace operator* with respect to the measure $\mu$ if it is a local operator. By construction, $\mathrm{dom}(\Delta_\mu)$ is dense in $\widetilde{L}_2^1(\mu)$ and $\Delta_\mu$ is a nonpositive self–adjoint operator in this Hilbert space. We have

$$\langle \Delta_\mu f, g\rangle_\mu = -\langle \nabla_\mu f, \nabla_\mu g\rangle_\mu \,, \tag{2.2}$$

$f \in \mathrm{dom}(\Delta_\mu)\,,\ g \in L_2^1(\mu)$.

The eigenvalues of $\Delta_\mu$ are nonpositive, countable with finite multiplicities and at most $-\infty$ as an accumulation point.

## 3. Riesz potentials, Bessel potentials and spectral properties on $d$-sets in Euclidean space

Throughout this section we suppose that $X = \mathbb{R}^n$ with the Euclidean metric and $\Gamma$ is a compact $d$–set, i.e., there is a $d$–regular finite Borel measure $\mu$ with support $\Gamma$. Recall that all such measures are equivalent to $\mathcal{H}^d \,|\, \Gamma$. Therefore the theory of *Besov spaces* on compact $d$–sets of $\mathbb{R}^n$ is available. These spaces were introduced by Jonsson and Wallin [11] as natural traces of associated Besov spaces on $\mathbb{R}^n$. In subsequent papers they studied properties of these fractal Besov spaces. Recent developments may be found in Triebel [22]. For definitions, notations and more details we refer to this monograph. In particular, we will use the Besov spaces $B^\alpha_{p,q}(\Omega)$ for $\Omega = \mathbb{R}^n$ or $\Omega = \Gamma$. The latter are introduced for $s > 0$, $0 < p \le \infty$, $0 < q \le \infty$ by

$$B^s_{p,q}(\Gamma) = \mathrm{tr}_\Gamma \left( B^{s+\frac{n-d}{p}}_{p,q}(\mathbb{R}^n) \right)$$

for the *trace operator* $\mathrm{tr}_\Gamma$ mapping a distribution from $B^{\frac{n-d}{p}}_{p,1}(\mathbb{R}^n)$ to an element of $L_2(\Gamma) = L_2(\mu)$, where $1 < p < \infty$. (In [22] the reference measure is $\mu := \mathcal{H}^d|\Gamma$.)

The corresponding norm is given by

$$\|f \,|\, B^s_{p,q}(\Gamma)\| = \inf \|g \,|\, B^{s+\frac{n-d}{p}}_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in B^{s+\frac{n-d}{p}}_{p,q}(\mathbb{R}^n)$ with $\mathrm{tr}_\Gamma f = g$.

It is well known that for $\sigma > 0$ the classical Besov space $B^\sigma_{2,2}(\mathbb{R}^n)$ via norm equivalence agrees with the Hilbert space $H^\sigma(\mathbb{R}^n) = \mathcal{I}^\sigma\,(L_2(\mathbb{R}^n))$, where the operator $\mathcal{I}^\sigma (\sigma \in \mathbb{R})$ is determined by the distributional Fourier transform $\mathcal{F}$:

$$\mathcal{I}^\sigma = \mathcal{F}^{-1}\left((1+|\xi|^2)^{-\sigma/2}\mathcal{F}\right),$$

i.e.,

$$\mathcal{I}^\sigma = (\mathrm{id} - \Delta)^{-\sigma/2}$$

for the Euclidean Laplacian $\Delta$ and the identity map id. The scalar product in $H^\sigma(\mathbb{R}^n)$ is introduced by

$$\langle f, g\rangle_{H^\sigma(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \mathcal{I}^{-\sigma} f(x)\, \mathcal{I}^{-\sigma} g(x)\, dx\,.$$

Hence, $\mathcal{I}^\sigma$ acts as an isometry:

$$\langle \mathcal{I}^\sigma f,\ \mathcal{I}^\sigma g\rangle_{H^\sigma(\mathbb{R}^n)} = \langle f, g\rangle_{L_2(\mathbb{R}^n)}\,.$$

For $\sigma > 0$ the operators $\mathcal{I}^\sigma$ and $\mathcal{I}^{-\sigma}$ may be realized by means of Bessel potentials and hypersingular integrals of order $\sigma$, respectively (up to certain exceptional orders for $\mathcal{I}^{-\sigma}$; cf. [19]). In general, one obtains the *mapping property*

$$\mathcal{I}^\sigma \left(B^s_{p,q}(\mathbb{R}^n)\right) = B^{s+\sigma}_{p,q}(\mathbb{R}^n), \tag{3.1}$$

$s, \sigma \in \mathbb{R},\ 0 < p \le \infty,\ 0 < q \le \infty$, in the sense of equivalent quasinorms (cf. the references in [22]).

Similarly, for the fractional powers of minus the Euclidean Laplacian we have

$$(-\Delta)^{-\sigma/2} = \mathcal{F}^{-1}\left(|\xi|^{-\sigma}\mathcal{F}\right),\ \sigma \in \mathbb{R},$$

and in the case $\sigma > 0$,

$$(-\Delta)^{-\sigma/2}\varphi(x) = \gamma_n(\sigma)^{-1}\int_{R^n} |x-y|^{-(n-\sigma)}\ \varphi(y)\ dy$$

with $\gamma_n(\sigma) = 2^\sigma \pi^{n/2}\ \Gamma(\sigma/2)/\Gamma\left(\frac{n-\sigma}{2}\right)$ for all Schwartz functions $\varphi$. These are the classical *Riesz potentials* $I^\sigma$ of order $\sigma$ in $\mathbb{R}^n$. The Riesz (or Bessel) potentials of $\mu$ in the sense of section 2 may be obtained by traces of associated potentials in $\mathbb{R}^n$. The mapping properties (3.1) of $\mathcal{I}^\sigma$ are an essential tool for these purposes. Unfortunately, the Riesz potentials $I^\sigma$ do not share this behavior because of the worse asymptotics of the Riesz kernel at infinity. Therefore we introduce the *modified Riesz potentials* $\widetilde{I}^\sigma,\ \sigma > 0$, as follows: Let $g^\sigma_R : (0, +\infty) \to \mathbb{R}$ be any smooth function such that the Fourier transform of the function $G^\sigma_R(x) := g^\sigma_R(|x|)$ on $\mathbb{R}^n$, which depends only on $|\xi|$, satisfies

$$c'\left(1+|\xi|^2\right)^{-\sigma/2} \le \mathcal{F}G^\sigma_R(\xi) \le c''\left(1+|\xi|^2\right)^{-\sigma/2} \tag{3.2}$$

for some constants $c', c'' > 0$ and

$$G^\sigma_R(x) = \gamma_n(\sigma)^{-1}\,|x|^{-(n-\sigma)} \text{ if } |x| \le \operatorname{diam}\Gamma\ . \tag{3.3}$$

This means that the kernel $G^\sigma_R(x)$ behaves like the Bessel kernel $G^\sigma_B(x)$ and coincides with the Riesz kernel on a ball containing $\Gamma$. For the existence of such functions cf. the Appendix.

We now define for $\sigma \in \mathbb{R}$

$$\widetilde{I}^\sigma := \mathcal{F}^{-1}\left(\mathcal{F}(G^\sigma_R)^{\operatorname{sgn}\sigma}\cdot\mathcal{F}\right) \tag{3.4}$$

in the sense of distributions, in particular,

$$\widetilde{I}^\sigma f(x) = \int_{\mathbb{R}^n} G^\sigma_R(x-y)\ f(y)\ dy\,,\ f \in L_2(\mathbb{R}^n)\,,$$

if $\sigma > 0$. An advantage of $\widetilde{I}^\sigma$ is that it has the same mapping properties between the Besov spaces as $\mathcal{I}^\sigma$ in (3.1). This can be proved by the same Fourier transformation methods for the quasinorms in $B^\alpha_{p,q}(\mathbb{R}^n)$. Moreover, in the Hilbert space structure in $H^\sigma(\mathbb{R}^n)$ mentioned above the operator $\mathcal{I}^\sigma$ may be replaced by $\widetilde{I}^\sigma$.

Recall that $\operatorname{tr}_\Gamma$ is a bounded operator from $B^\sigma_{p,q}(\mathbb{R}^n)$ onto $B^{\sigma-\frac{n-d}{2}}_{p,q}(\Gamma)$ if $\sigma > \frac{n-d}{2}$. On the other hand, up to norm equivalence, $L_2(\mu)$ agrees with the space $B^{-\frac{n-d}{2},\Gamma}_{2,\infty}(\mathbb{R}^n)$ of distributions from $B^{-\frac{n-d}{2}}_{2,\infty}(\mathbb{R}^n)$ vanishing on Schwartz functions with zero trace on $\Gamma$. This is realized by the identification $\mathrm{id}_\mu$ of $f \in L_2(\mu)$ with

the distribution $f\mu$ given by $f\mu(\varphi) := \int \varphi\, f\, d\mu$ for any Schwartz function $\varphi$. The latter extends to a mapping

$$\mathrm{tr}^{\mu} : B_{2,1}^{\frac{n-d}{2}}(\mathbb{R}^n) \to B_{2,\infty}^{-\frac{n-d}{2}}(\mathbb{R}^n)$$

via the same integrals

$$\mathrm{tr}^{\mu}\ f(\varphi) := \int \varphi\, f\, d\mu\,.$$

(For details see [22], Theorems 18.2 and 18.6. Note that H. Triebel always uses the superscript $\Gamma$ choosing $\mu = \mathcal{H}^d \,|\, \Gamma$. For purposes of application to quasidiffusion processes on fractals in the present paper we distinguish between different equivalent measures $\mu$.)

We are now ready to state the *trace property* of the Riesz potentials $I_\mu^s$ normalized by the constant factor $\gamma_n(s+n-d)^{-1}$ and the Bessel potentials defined by

$$\mathcal{I}_\mu^s\, f(x) := \int G_B^{s+n-d}(x-y)\, f(y)\, \mu(dy)$$

at $\mu$–almost all $x$.

**Theorem 3.1.** *For any $d$–regular Borel measure $\mu$ in $\mathbb{R}^n$ with compact support $\Gamma$ and $0 < s < d < n$ we have*

(i) $\mathrm{tr}_\Gamma \circ \widetilde{I}^{s+n-d} \circ \mathrm{id}_\mu \;=\; I_\mu^s$,

(ii) $\mathrm{tr}_\Gamma \circ \mathcal{I}^{s+n-d} \circ \mathrm{id}_\mu \;=\; \mathcal{I}_\mu^s$

*acting as bounded operators from $L_2(\Gamma)$ into $B_{2,\infty}^s(\Gamma)$.*

*Proof.* The arguments are the same for all potentials whose kernel is equivalent to the Bessel kernel in the sense of (3.2). Therefore we consider only the case (i).

In terms of integrals the assertion means that the trace on $\Gamma$ of the function $h$ defined at Lebesgue almost all $x$ by

$$h(x) := \int G_R^{s+n-d}(x-y)\, f(y)\, \mu(dy)$$

agrees with $I_\mu^s f$, where $f \in L_2(\mu)$ by assumption and $h \in B_{2,\infty}^{\alpha}(\mathbb{R}^n)$ with $\alpha = -\frac{n-d}{2} + s + n - d = s + \frac{n-d}{2}$ by the above mapping properties. (Observe that for any Schwartz function $\varphi$ we have by Fubini

$$\begin{aligned}
\widetilde{I}^{s+n-d}\, f\mu(\varphi) &= f\mu\left(\widetilde{I}^{s+n-d}\varphi\right)\\
&= \iint G_R^{s+n-d}(y-x)\, \varphi(x)\, dx\, f(y)\, \mu(dy)\\
&= \int \varphi(x) \int G_R^{s+n-d}(y-x)\, f(y)\, \mu(dy)\, dx\ .)
\end{aligned}$$

Further, for Schwartz functions the trace on $\Gamma$ is defined pointwise and norm estimates w.r.t. $L_2(\mu)$ and $B_{2,1}^{\frac{n-d}{2}}(\mathbb{R}^n)$ are shown. Then for arbitrary functions from $B_{2,1}^{\frac{n-d}{2}}(\mathbb{R}^n)$ the trace operator is determined by continuous extension (cf. [22], Theorem 18.6). Let $\varkappa_\varepsilon(x) = \varepsilon\varkappa(\varepsilon x)$ be standard smoothing kernels with respect to Lebesgue measure converging to the $\delta$–function as $\varepsilon \to 0$. We approximate $h$ by the smooth functions $h * \varkappa_\varepsilon$, i.e.,

$$\lim_{\varepsilon\to 0} \left\| h * \varkappa_\varepsilon - h \,|\, B_{2,\infty}^{s+\frac{n-d}{2}}(\mathbb{R}^n) \right\| = 0\ .$$

Since $s > 0$, convergence holds also in $B_{2,1}^{\frac{n-d}{2}}(\mathbb{R}^n)$. Then it suffices to show that

$$\lim_{\varepsilon\to 0} \|h * \varkappa_\varepsilon - h_\Gamma \mid L_2(\mu)\| = 0,$$

where

$$\begin{aligned} h_\Gamma(x) &= \int G_R^{s+n-d}(x-y)\, f(y)\, \mu(dy) \\ &= \int \gamma_n(s+n-d)^{-1}|x-y|^{-(d-s)}\, f(y)\, \mu(dy) \\ &= I_\mu^s f(x) \end{aligned}$$

at $\mu$–almost all $x$, since by construction the kernel $G_R^\sigma(x-y)$ for $x, y \in \Gamma$ agrees with the Riesz kernel $\gamma_n(\sigma)^{-1}\,|x-y|^{-(n-\sigma)}$ and for $\sigma = s+n-d$ we have $n-\sigma = d-s$. For, we estimate

$$\begin{aligned} &\int (h * \varkappa_\varepsilon - h_\Gamma)^2\, d\mu \\ &= \int \left( \int (G_R^\sigma(x-u) - G_R^\sigma(x))\, \varkappa_\epsilon(u)\, du \right)^2 \mu(dx) \\ &\le \iint (G_R^\sigma(x-u) - G_R^\sigma(x))^2\, \varkappa_\epsilon(u)\, du\, \mu(dx) \\ &= \iint (G_R^\sigma(x-u) - G_R^\sigma(x))^2\, \mu(dx)\, \varkappa_\epsilon(u)\, du\ . \end{aligned}$$

Theorem 1.4(iii) implies

$$\lim_{u\to 0} \int (G_R^\sigma(x-u) - G_R^\sigma(x))^2\, \mu(dx) = 0\ .$$

(In that assertion the original Riesz kernel may be replaced by the modified version. It also follows directly, because the latter may be replaced by the former as $u \to 0$.) Therefore the above expression tends to zero as $\varepsilon \to 0$. □

For the rest of the paper we will restrict ourselves to Riesz potentials. The arguments for Bessel potentials are completely analogous. Recall the notation $L_2^s(\mu) = I_\mu^s\,(L_2(\mu))$ .

**Theorem 3.2.** *Under the conditions of Theorem* 3.1 *the mapping* $I_\mu^s : L_2(\mu) \to L_2^s(\mu)$ *is invertible and* $\langle I_\mu^s f, f\rangle_\mu \ge 0$, *i.e.,* $I_\mu^s$ *is positive. Moreover,*

$$\langle I_\mu^s f,\, g\rangle_\mu = \langle I^{\frac{s+n-d}{2}}(f\mu),\, I^{\frac{s+n-d}{2}}(g\mu)\rangle_{L_2(\mathbb{R}^n)}$$

*for any* $f, g \in L_2(\mu)$.

*Proof.* We use the convolution property of the Riesz kernel

$$\gamma_n(\sigma)^{-1}|x-y|^{-(n-\sigma)} = (\gamma_n(\sigma/2))^{-2} \int_{\mathbb{R}^n} |x-z|^{-(n-\sigma/2)}\, |y-z|^{-(n-\sigma/2)}\, dz$$

for $0 < \sigma < n$ (see, e.g., [19], (25.38)). Again setting $\sigma = s + n - d$ we infer by Fubini that

$$\begin{aligned}
\langle I^s_\mu f,\, g\rangle_\mu &= \int \gamma_n(\sigma)^{-1} \int |x-y|^{-(n-\sigma)}\, f(x)\, \mu(dx)\, g(y)\, \mu(dy) \\
&= \iint f(x)\, g(y)\, \gamma_n(\sigma/2)^{-2} \int |x-z|^{-(n-\sigma/2)}\, |y-z|^{-(n-\sigma/2)} dz\, \mu(dx)\, \mu(dy) \\
&= \int \gamma_n(\sigma/2)^{-1} \int |x-z|^{-(n-\sigma/2)}\, f(x)\, \mu(dx) \\
&\quad \gamma_n(\sigma/2)^{-1} \int |y-z|^{-(n-\sigma/2)}\, g(y)\, \mu(dy)\, dz \\
&= \int I^{\sigma/2}(f\mu)(z)\, I^{\sigma/2}(g\mu)(z)\, dz = \langle I^{\sigma/2}(f\mu),\, I^{\sigma/2}(g\mu)\rangle_{L_2(\mathbb{R}^n)}\ .
\end{aligned}$$

For $f = g$ this means

$$\langle I^s_\mu f, f\rangle_\mu = \int \gamma_n(\sigma/2)^{-2} \left( \int |x-z|^{-(n-\sigma/2)}\, f(x)\, \mu(dx) \right)^2 dz \geq 0\ .$$

Therefore $\langle I^s_\mu f,\, f\rangle_\mu = 0$ implies

$$\widetilde{h}(z) := \int \gamma_n(\sigma/2)^{-1}\, |x-z|^{-(n-\sigma/2)}\, f(x)\, \mu(dx) = 0$$

at Lebesgue almost all $z$. Since

$$\mathcal{F}(\widetilde{h})(\xi) = |\xi|^{-\sigma/2}\, \mathcal{F}(f\mu)(\xi)$$

we get $\mathcal{F}(f\mu) = 0$, i.e. $f(x) = 0$ at $\mu$–almost all $x$. In particular, $I^s_\mu f = 0$ yields $f = 0$ in $L_2(\mu)$. □

Denote the inverse operator of $I^s_\mu$ by $D^s_\mu$ . Considering $I^s_\mu$ as an *isometry* between $L_2(\mu)$ and $L^s_2(\mu)$ the latter is provided with a *Hilbert space structure* according to

$$\langle f, g\rangle_{L^s_2(\mu)} := \langle D^s_\mu f, D^s_\mu g\rangle_\mu\ . \tag{3.5}$$

It will be called the *space of Riesz potentials of order* $s$ w.r.t. the measure $\mu$.

In order to derive spectral properties of $I^s_\mu$ (and hence, of $D^s_\mu$) we first consider its restriction to the Hilbert space $H^{s/2}(\Gamma)$. The latter agrees with $B^{s/2}_{2,2}(\Gamma)$ provided with the scalar product induced by the norm in $H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n)$ (see [22], 25.1). The proof of the following spectral result uses a method of Triebel developed for another operator. In our case we need no dimension bounds.

**Theorem 3.3.** *For $0 < s < d < n$ the Riesz potential $I^s_\mu$ is a positive self–adjoint compact operator in the Hilbert space $H^{s/2}(\Gamma)$. There are positive constants $C_1$ and $C_2$ such that the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$ repeated according to multiplicity satisfy*

$$C_1\, k^{-s/d} \leq \lambda_k \leq C_2\, k^{-s/d}\ .$$

*Proof.* Self–adjointness is obvious. By the arguments at the end of section 1 and Theorem 3.2 the eigenvalues of the operator $I^s_\mu$ on $L_2(\mu)$ are countable and positive. In case of compactness of $I^s_\mu$ in $H^{s/2}(\Gamma)$ this implies its positivity.

The remaining arguments are the same as in the proof of Theorem 28.6 in [22] when replacing the exponent $\varkappa$ by $(s+n-d)/2$, the bounded domain $\Omega$ by the

whole $\mathbb{R}^n$, and the operator $A^{-\varkappa/m}$ for $m = 1$ by our modified Euclidean Riesz potential $\widetilde{I}^{2\varkappa}$. This is justified by Theorem 3.1 above and the construction of $\widetilde{I}^{2\varkappa}$. The arguments at some places are simpler, since we can work with the whole $\mathbb{R}^n$ as an embedding space. In particular, we need no dimension restrictions in order to apply duality results for Besov spaces. The desired inequality $\varkappa > (n-d)/2$ is automatically fulfilled.

We will briefly summarize the main steps:

One first proves that the operator $B := \widetilde{I}^{s+n-d} \circ tr^{\mu}$ has the null space

$$N(B) = \left\{ f \in H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n) : tr_{\Gamma} f = 0 \right\}$$

and satisfies

$$\langle tr_{\Gamma} f, tr_{\Gamma} g \rangle_{\mu} = \langle Bf, g \rangle_{H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n)}$$

for $f, g \in H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n)$, where the Hilbert space structure of the last space is generated by $\widetilde{I}^{s+n-d}$ instead of $\mathcal{I}^{s+n-d}$ (cf. steps 1 and 2 in [22], 28.6).

Then by (25.6) in [22] one obtains the decomposition

$$H^{\frac{s}{2}+\frac{n-d}{2}}(\mathbb{R}^n) = N(B) \oplus H^{\frac{s}{2}}(\Gamma).$$

Therefore the eigenvalues of the operator $tr_{\Gamma} \circ B$ restricted to $H^{s/2}(\Gamma)$ coincide with those of $B$. Further, $tr_{\Gamma} \circ B$ on $H^{s/2}(\Gamma)$ agrees with $I^s_{\mu}$ by Theorem 3.1 above.

Using the relationship between eigenvalues and approximation numbers of self–adjoint compact operators in Hilbert spaces and the above isometry property of $\sqrt{B}$ one infers the lower estimate $\lambda_k \geq C_1\, k^{-s/d}$. Here the $d$–regularity of $\mu$ and localization properties in Besov spaces on $\mathbb{R}^n$ play a central role. (See step 4 in [22], 28.6. Compactness of $B$ will be indicated below.)

Further, $I^s_{\mu} : H^{s/2}(\Gamma) \to H^{s/2}(\Gamma)$ may be factorized by

$$I^s_{\mu} = \mathrm{id}_2 \circ tr_{\Gamma} \circ \widetilde{I}^{s+n-d}_{\mu} \circ \mathrm{id}_{\mu} \circ \mathrm{id}_1,$$

where

$$\begin{aligned}
\mathrm{id}_1 &: H^{s/2}(\Gamma) \to L_2(\mu), \\
\mathrm{id}_{\mu} &: L_2(\mu) \to B^{-\frac{n-d}{2}}_{2,\infty}(\mathbb{R}^n), \\
\widetilde{I}^{s+n-d} &: B^{-\frac{n-d}{2}}_{2,\infty}(\mathbb{R}^n) \to B^{s+\frac{n-d}{2}}_{2,\infty}(\mathbb{R}^n), \\
tr_{\Gamma} &: B^{s+\frac{n-d}{2}}_{2,\infty}(\mathbb{R}^n) \to B^{s}_{2,\infty}(\Gamma), \\
\mathrm{id}_2 &: B^{s}_{2,\infty}(\Gamma) \to H^{s/2}(\Gamma).
\end{aligned}$$

Both the embeddings $\mathrm{id}_1$ and $\mathrm{id}_2$ are compact with $k$-th entropy numbers equivalent to $k^{-s/2d}$ (see [22], Theorem 20.6). The remaining operators are bounded. Therefore $I^s_{\mu}$ is compact on $H^{s/2}(\Gamma)$ and its $k$-th entropy numbers do not exceed $\mathrm{const}\, k^{-s/d}$. The known upper estimate of the eigenvalues by $\sqrt{2}$ times the entropy numbers yields $\lambda_k \leq C_2\, k^{-s/d}$ (cf. [22], 28.6, step 3).

Finally, compactness of the above operator $B$ follows from that of $\mathrm{id}_1$. $\square$

If $\lambda_k$ is an eigenvalue of $I_\mu^s$ on the whole space $L_2(\mu)$ with eigenvector $e_k \in L_2(\mu)$, then Theorem 3.1 implies $e_k \in B_{2,\infty}^s(\Gamma)$, hence $e_k \in H^{s/2}(\Gamma)$. Consequently, the eigenvalues of $I_\mu^s$ on $L_2(\mu)$ agree with those given in Theorem 3.3 and we obtain the following.

**Corollary 3.4.** *For any $d$–regular $\mu$ with compact support and $0 < s < d < n$, the operator $D_\mu^s$ inverse to the Riesz potential $I_\mu^s$ is self–adjoint and positive in $L_2(\mu)$ with domain $L_2^s(\mu)$. Its eigenvalues are given by $\mu_k = \lambda_k^{-1}$ for the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$ of $I_\mu^s$ on $L_2(\mu)$ repeated according to multiplicities and satisfy*

$$C_2^{-1}\, k^{s/d} \leq \mu_k \leq C_1^{-1}\, k^{s/d}\ .$$

## Appendix—Construction of Riesz-Bessel potentials

In order to construct a Riesz-Bessel kernel $G_R^\sigma$ on $\mathbb{R}^n$ with the properties (3.2) and (3.3) first note the following: For any continuous function $G$ which coincides on the ball $B(0,R)$ with the Riesz kernel of order $\sigma$, i.e. satisfies (3.3), and rapidly decreases at infinity, one obtains by standard arguments the Fourier transform estimates (3.2), i.e.,

$$c'(1+|\xi|^2)^{-\sigma/2} \leq \mathcal{F}G(\xi) \leq c''(1+|\xi|^2)^{-\sigma/2}$$

for sufficiently small and sufficiently large $\xi$. In order to get these inequalities for all $\xi$ (with modified constants) it is enough to find such a $G = G_R^\sigma$, where $\mathcal{F}G(\xi) > 0$, $\xi \in \mathbb{R}^n$.

We use the polar coordinates representation of the Fourier transform of a rapidly decreasing continuous radial function $G(x) = g(|x|)$ in $\mathbb{R}^n$. Observe that $\mathcal{F}G$ is also a radial function and may be evaluated at $\xi = (0,\dots,0,|\xi|)$:

$$\begin{aligned}
(2\pi)^{n/2}\mathcal{F}G(\xi) &= \int_{-\infty}^{\infty} e^{-i|\xi|x_n} \int_{\mathbb{R}^{n-1}} g(|x|)\ dx_1 \dots dx_n \\
&= |S^{n-2}| \int_{-\infty}^{\infty} e^{i|\xi|r} \int_0^\infty g(\sqrt{t^2+r^2}) t^{n-2}\ dt\ dr \\
&= 2|S^{n-2}| \int_0^\infty \cos(|\xi|r) \int_r^\infty u g(u)(u^2-r^2)^{(n-3)/2}\ du\ dr \\
&= 2|S^{n-2}| \int_0^\infty \cos(|\xi|r) r^{n-1} \int_1^\infty v g(vr)(v^2-1)^{(n-3)/2}\ dv\ dr \\
&= 2|S^{n-2}| \int_1^\infty v(v^2-1)^{(n-3)/2} \int_0^\infty r^{n-1} g(vr)\cos(|\xi|r)\ dr\ dv \\
&= (2\pi)^{n/2}|S^{n-2}| \int_1^\infty v^{-(n-1)}(v^2-1)^{(n-3)/2}(\hat{f}(\frac{|\xi|}{v}) + \hat{f}(-\frac{|\xi|}{v}))\ dv,
\end{aligned}$$

where $|S^{n-2}|$ is the surface area of the unit sphere $S^{n-2}$ and $f(s) = f_R^\sigma(s) := s_+^{n-1} g_R^\sigma(s)$ with

$$f_R^\sigma(s) = \begin{cases} 0\ , & s \leq 0, \\ \gamma_n(\sigma)^{-1} s^{\sigma-1}\ , & 0 < s < R, \\ \text{rapidly decreasing at infinity.} & \end{cases}$$

Thus, it suffices to choose $f_R^\sigma$ in such a way that its one-dimensional Fourier transform $\hat{f_R^\sigma}$ is nonnegative. Taking

$$h_R^\sigma(s) := \begin{cases} 0\ , & s \le 0, \\ \gamma_n(\sigma/2)^{-1} s^{\sigma/2-1}\ , & 0 < s < R, \\ \text{rapidly decreasing at infinity}, & \end{cases}$$

we obtain this property for $f_R^\sigma := h_R^\sigma * h_R^\sigma$, since in this case $\hat{f_R^\sigma} = |\hat{h_R^\sigma}|^2$.

## Note added in proof

Relationships of the above operators to Dirichlet forms and Markovian jump processes on $d$-sets are considered in [25], and the corresponding Laplace operators and diffusions are considered in a subsequent paper.

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