THE COHOMOLOGY OF THE STEENROD ALGEBRA AND REPRESENTATIONS OF THE GENERAL LINEAR GROUPS

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Dedicated to Professor Nguyễn Hữu Anh on the occasion of his sixtieth birthday

Abstract. Let $Tr_k$ be the algebraic transfer that maps from the coinvariants of certain $GL_k$-representations to the cohomology of the Steenrod algebra. This transfer was defined by W. Singer as an algebraic version of the geometrical transfer $\pi^*(B_{k}) \to \pi^*(S^0)$. It has been shown that the algebraic transfer is highly nontrivial, more precisely, that $Tr_k$ is an isomorphism for $k = 1, 2, 3$ and that $Tr = \bigoplus_k Tr_k$ is a homomorphism of algebras.

In this paper, we first recognize the phenomenon that if we start from any degree $d$ and apply $Sq^0$ repeatedly at most $(k - 2)$ times, then we get into the region in which all the iterated squaring operations are isomorphisms on the coinvariants of the $GL_k$-representations. As a consequence, every finite $Sq^0$-family in the coinvariants has at most $(k - 2)$ nonzero elements. Two applications are exploited.

The first main theorem is that $Tr_k$ is not an isomorphism for $k \geq 5$. Furthermore, for every $k > 5$, there are infinitely many degrees in which $Tr_k$ is not an isomorphism. We also show that if $Tr_\ell$ detects a nonzero element in certain degrees of $\text{Ker}(Sq^0)$, then it is not a monomorphism and further, for each $k > \ell$, $Tr_k$ is not a monomorphism in infinitely many degrees.

The second main theorem is that the elements of any $Sq^0$-family in the cohomology of the Steenrod algebra, except at most its first $(k - 2)$ elements, are either all detected or all not detected by $Tr_k$, for every $k$. Applications of this study to the cases $k = 4$ and 5 show that $Tr_4$ does not detect the three families $g$, $D_3$ and $p'$, and that $Tr_5$ does not detect the family $\{h_{n+1}g_{n}| n \geq 1\}$.

1. Introduction and statement of results

There have been several efforts, implicit or explicit, to analyze the Steenrod algebra by using modular representations of the general linear groups. (See Mui [22, 23, 24], Madsen-Milgram [19], Adams-Guawardena-Miller [3], Priddy-Wilkerson [27], Peterson [25], Wood [32], Singer [28], Priddy [26], Kuhn [15] and others.) In particular, one of the most direct attempts in studying the cohomology of the Steenrod algebra by means of modular representations of the general linear groups was the surprising work [28] by W. Singer, which introduced a homomorphism, the so-called...
algebraic transfer, mapping from the coinvariants of certain representations of the general linear group to the cohomology of the Steenrod algebra.

Let $\mathbb{V}_k$ denote a $k$-dimensional $\mathbb{F}_2$-vector space, and let $PH_*(BV_k)$ denote the primitive subspace consisting of all elements in $H_*(BV_k)$ that are annihilated by every positive-degree operation in the mod 2 Steenrod algebra, $\mathcal{A}$. Throughout the paper, the homology is taken with coefficients in $\mathbb{F}_2$. The general linear group $GL_k := GL(\mathbb{V}_k)$ acts regularly on $\mathbb{V}_k$ and therefore on the homology and cohomology of $BV_k$. Since the two actions of $\mathcal{A}$ and $GL_k$ upon $H^*(BV_k)$ commute with each other, there are inherited actions of $GL_k$ on $\mathbb{F}_2 \otimes H^*(BV_k)$ and $PH_*(BV_k)$.

In [28], W. Singer defined the algebraic transfer

$$Tr_k : \mathbb{F}_2 \otimes PH_d(BV_k) \rightarrow Ext^k_A((\mathbb{F}_2, \mathbb{F}_2))$$

as an algebraic version of the geometrical transfer $tr_k : \pi_*^\mathcal{A}(BV_k_+ \rightarrow \pi_*^\mathcal{A}(S^0)$ to the stable homotopy groups of spheres.

It has been proved that $Tr_k$ is an isomorphism for $k = 1, 2$ by Singer [28] and for $k = 3$ by Boardman [4]. Among other things, these data together with the fact that $Tr = \bigoplus_k Tr_k$ is an algebra homomorphism (see [28]) show that $Tr_k$ is highly nontrivial. Therefore, the algebraic transfer is expected to be a useful tool in the study of the mysterious cohomology of the Steenrod algebra, $Ext^*_A(\mathbb{F}_2, \mathbb{F}_2)$.

Directly calculating the value of $Tr_k$ on any nonzero element is difficult (see [28], [4], [11]). In this paper, our main idea is to exploit the relationship between the algebraic transfer and the squaring operation $Sq^0$. It is well known (see [13]) that there are squaring operations $Sq^i$ ($i \geq 0$) acting on the cohomology of the Steenrod algebra that share most of the properties with $Sq^i$ on the cohomology of spaces. However, $Sq^0$ is not the identity. On the other hand, there is an analogous squaring operation $\tilde{Sq}^0$, the Kameko one, acting on the domain of the algebraic transfer and commuting with the classical $Sq^0$ on $Ext^*_A(\mathbb{F}_2, \mathbb{F}_2)$ through the algebraic transfer. We refer to Section 2 for its precise meaning.

The key point is that the behaviors of the two squaring operations do not agree in infinitely many certain degrees, called $k$-spikes. A $k$-spike degree is a number that can be written as $(2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$, but cannot be written as a sum of less than $k$ terms of the form $(2^n - 1)$. (See a discussion of this notion after Definition 3.1.) The following result is originally due to Kameko [13]: If $m$ is a $k$-spike, then

$$\tilde{Sq}^0 : PH_*(BV_k)_{m-k} \rightarrow PH_*(BV_k)_m$$

is an isomorphism of $GL_k$-modules, where $\tilde{Sq}^0$ is certain $GL_k$-homomorphism such that $Sq^0 = 1 \otimes \tilde{Sq}^0$. (See Section 2 for an explanation of $\tilde{Sq}^0$.)

We recognize two phenomena on the universality and the stability of $k$-spikes: First, if we start from any degree $d$ that can be written as $(2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$, and apply the function $\delta_k$ with $\delta_k(d) = 2d + k$ repeatedly at most $(k - 1)$ times, then we get a $k$-spike; second, $k$-spikes are mapped by $\delta_k$ to $k$-spikes. Therefore, we have

**Theorem 1.1.** Let $d$ be an arbitrary nonnegative integer. Then

$$(\tilde{Sq}^0)^{i-k+1} : PH_*(BV_k)_{2^{k-2}d + (2^{k-2} - 1)k} \rightarrow PH_*(BV_k)_{2^d + (2^{k-1} - 1)k}$$

is an isomorphism of $GL_k$-modules for every $i \geq k - 2$. 
From the result of Carlisle and Wood \cite{CarlisleWood} on the boundedness conjecture, one can see that, for any degree \(d\), there exists \(t\) such that
\[
(\tilde{Sq}^0)^{t-1} : PH_*(B\mathbb{V}_k)_{2^d + (2^r - 1)k} \to PH_*(B\mathbb{V}_k)_{2^d + (2^r - 1)k}
\]
is an isomorphism of \(GL_k\)-modules for every \(i \geq t\). However, this result does not confirm how large \(t\) should be. Theorem 1.1 shows that a rather small number \(t = k - 2\) commonly serves for every degree \(d\). It will be pointed out in Remark 6.5 that \(t = k - 2\) is the minimum number for this purpose.

An inductive property of \(k\)-spikes, which will also play a key role in the paper, is that if \(m\) is a \(k\)-spike, then \((2^m - 1 + m)\) is a \((k + 1)\)-spike for \(n\) big enough.

Two applications of the study will be exploited in this paper. The first application is the following theorem, which is one of the paper’s main results.

**Theorem 1.2.** \(Tr_k\) is not an isomorphism for \(k \geq 5\). Furthermore, for every \(k > 5\), there are infinitely many degrees in which \(Tr_k\) is not an isomorphism.

That \(Tr_5\) is not an isomorphism in degree 9 is due to Singer \cite{Singer65}.

In order to prove this theorem, using the notion of \(k\)-spike, we introduce the concept of a critical element in Ext\(_A^k(\mathbb{F}_2, \mathbb{F}_2)\) in such a way that if \(d\) is the stem of a critical element, then \(Tr_k\) is not an isomorphism either in degree \(d\) or in degree \(2d + k\). Further, we show that if \(x\) is critical, then so is \(b_n x\) for \(n\) big enough. Our inductive procedure starts with the initial critical element \(Ph_2\) for \(k = 5\).

Combining Theorem 1.2 and the results by Singer \cite{Singer65}, Boardman \cite{Boardman} and Bruner–Hà–Hrung \cite{BrunerHung}, we get

**Corollary 1.3.** (i) \(Tr_k\) is an isomorphism for \(k = 1, 2\) and 3.

(ii) \(Tr_k\) is not an isomorphism for \(k \geq 4\).

(iii) For \(k = 4\) and for each \(k > 5\), there are infinitely many degrees in which \(Tr_k\) is not an isomorphism.

Remarkably, we do not know whether the algebraic transfer fails to be a monomorphism or fails to be an epimorphism for \(k > 5\). Therefore, Singer’s conjecture is still open.

**Conjecture 1.4** \cite{Singer65}. \(Tr_k\) is a monomorphism for every \(k\).

The following theorem is related to this conjecture.

**Theorem 1.5.** If \(Tr_k\) detects a critical element, then it is not a monomorphism and further, for each \(k > \ell\), there are infinitely many degrees in which \(Tr_k\) is not a monomorphism.

A family \(\{a_i\}_{i \geq 0}\) of elements in Ext\(_A^k(\mathbb{F}_2, \mathbb{F}_2)\) (or in \(\mathbb{F}_2 \otimes PH_*(B\mathbb{V}_k)\)) is called a \(Sq^0\)-family if \(a_i = (Sq^0)^i(a_0)\) for every \(i \geq 0\). Recall that, if \(a \in\) Ext\(_A^k(\mathbb{F}_2, \mathbb{F}_2)\), then \(t - k\) is called the stem of \(a\), and denoted by \(\text{Stem}(a)\). The root degree of \(a\) is the maximum nonnegative integer \(r\) such that \(\text{Stem}(a) = 2^r + (2^r - 1)k\), for some nonnegative integer \(d\).

The second application of our study is the following theorem, which is also one of the paper’s main results.

**Theorem 1.6.** Let \(\{a_i\}_{i \geq 0}\) be an \(Sq^0\)-family in Ext\(_A^k(\mathbb{F}_2, \mathbb{F}_2)\) and let \(r\) be the root degree of \(a_0\). If \(Tr_k\) detects \(a_n\) for some \(n \geq \max\{k - r - 2, 0\}\), then it detects \(a_i\) for every \(i \geq n\) and detects \(a_j\) modulo \(\text{Ker}(Sq^0)^{n-j}\) for \(\max\{k - r - 2, 0\} \leq j < n\).
An $Sq^0$-family is called finite if it has only finitely many nonzero elements. The existence of finite $Sq^0$-families in $\text{Ext}_{GL_k}^*(F_2,F_2)$ is well known, and that of finite $Sq^0$-families in $F_2 \otimes PH_*(BV_k)$ will be shown in Section 9.

The following is a consequence of Theorem 1.6 and Theorem 1.1.

**Corollary 1.7.**

(i) Every finite $Sq^0$-family in $F_2 \otimes PH_*(BV_k)$ has at most $(k-2)$ nonzero elements.

(ii) If $Tr_k$ is a monomorphism, then it does not detect any element of a finite $Sq^0$-family in $\text{Ext}_{GL_k}^*(F_2,F_2)$ with at least $(k-1)$ nonzero elements.

The following is an application of Theorem 1.6 into the investigation of $Tr_4$.

**Proposition 1.8.** Let $\{b_i| i \geq 0\}$ and $\{d_i| i \geq 0\}$ be the $Sq^0$-families in $\text{Ext}_{GL_4}^*(F_2,F_2)$ with $b_0$ one of the usual five elements $d_0,e_0,p_0,D_3(0),p'_0$, and $b_0$ one of the usual two elements $f_0,g_1$.

(i) If $Tr_4$ detects $b_n$ for some $n \geq 1$, then it detects $b_i$ for every $i \geq 1$.

(ii) If $Tr_4$ detects $d_n$ for some $n \geq 0$, then it detects $d_i$ for every $i \geq 0$.

Based on this event, we prove the following theorem by showing that $Tr_4$ does not detect $g_1,D_3(1),p'_1$.

**Theorem 1.9.** $Tr_4$ does not detect any element in the three $Sq^0$-families $\{g_i| i \geq 1\}$, $\{D_3(i)| i \geq 0\}$ and $\{p'_i| i \geq 0\}$.

This theorem gives further negative information on Minami’s ([21]) conjecture that the localization of the algebraic transfer given by inverting $Sq^0$ is an isomorphism. The first negative answer to this conjecture was given in Bruner–Hà–Hung ([7] by showing that the element in $(Sq^0)^{-1}\text{Ext}_{GL_4}^*(F_2,F_2)$ represented by the family $\{g_i| i \geq 1\}$ is not detected by $(Sq^0)^{-1}Tr_4$. From Theorem 1.6 the two elements in $(Sq^0)^{-1}\text{Ext}_{GL_4}^*(F_2,F_2)$ represented respectively by the two families $\{D_3(i)| i \geq 0\}$ and $\{p'_i| i \geq 0\}$ are also not detected by $(Sq^0)^{-1}Tr_4$.

Recently, T. N. Nam informed the author about his claim that $Tr_4$ does not detect $D_3(0)$.

**Conjecture 1.10.** $Tr_4$ is a monomorphism that detects all elements in $\text{Ext}_{GL_4}^*(F_2,F_2)$ except the ones in the three $Sq^0$-families $\{g_i| i \geq 1\}$, $\{D_3(i)| i \geq 0\}$ and $\{p'_i| i \geq 0\}$.

The following theorem would complete our knowledge in Corollary 1.3 on whether $Tr_5$ is not an isomorphism in infinitely many degrees.

**Theorem 1.11.** If $h_{n+1}g_n$ is nonzero, then it is not detected by $Tr_5$.

It has been claimed by Lin ([16] that $h_{n+1}g_n$ is nonzero for every $n \geq 1$.

The paper is divided into nine sections and organized as follows. Section 2 is a recollection of the Kameko squaring operation. In Section 3, we explain the notion of $k$-spike and then study the Kameko squaring and its iterated operations in $k$-spike degrees. Section 4 deals with an inductive way of producing $k$-spikes, which plays a key role in the proofs of Theorems 1.1, 1.2, 1.5 and 1.6. In Section 5, based on the concept of critical element, we prove Theorems 1.2 and 1.9. Section 6 is devoted to the proofs of Theorems 1.1 and 1.6. Sections 7 and 8 are applications to the study of the fourth and the fifth algebraic transfers. Final remarks and conjectures are given in Section 9.
2. Preliminaries on the squaring operation

To make the paper self-contained, this section is a recollection of the Kameko squaring operation $Sq^0$ on $\mathbb{F}_2 \otimes PH_*(BV_k)$. The most important property of the Kameko $Sq^0$ is that it commutes with the classical $Sq^0$ on $\text{Ext}^*_A(\mathbb{F}_2, \mathbb{F}_2)$ (defined in [18]) through the algebraic transfer (see [4], [21]).

This squaring operation is constructed as follows.

As is well known, $H^*(BV_k)$ is the polynomial algebra, $P_k := \mathbb{F}_2[x_1, \ldots, x_k]$, on $k$ generators $x_1, \ldots, x_k$, each of degree 1. By dualizing,

$$H_*(BV_k) = \Gamma(a_1, \ldots, a_k)$$

is the divided power algebra generated by $a_1, \ldots, a_k$, each of degree 1, where $a_i$ is dual to $x_i \in H^1(BV_k)$. Here the duality is taken with respect to the basis of $H^*(BV_k)$ consisting of all monomials in $x_1, \ldots, x_k$.

In [13] Kameko defined a homomorphism

$$\tilde{Sq}^0 : H_*(BV_k) \rightarrow H_*(BV_k),$$

$$a_1^{i_1} \cdots a_k^{i_k} \mapsto a_1^{(2i_1+1)} \cdots a_k^{(2i_k+1)},$$

where $a_1^{(i_1)} \cdots a_k^{(i_k)}$ is dual to $x_1^{i_1} \cdots x_k^{i_k}$. The following lemma is well known.

**Lemma 2.1.** $\tilde{Sq}^0$ is a homomorphism of $GL_k$-modules.

See e.g. [4] for a proof. Further, there are two well-known relations,

$$Sq^0_{2^k+1} \tilde{Sq}^0 = 0, \quad Sq^0_{2^k} \tilde{Sq}^0 = \tilde{Sq}^0 Sq^0_{2^k}.$$

See [10] for an explicit proof. Therefore, $\tilde{Sq}^0$ maps $PH_*(BV_k)$ to itself.

The Kameko $Sq^0$ is defined by

$$Sq^0 = 1 \otimes \tilde{Sq}^0 : GL_k \otimes PH_*(BV_k) \rightarrow GL_k \otimes PH_*(BV_k).$$

The dual homomorphism $\tilde{Sq}^0 : P_k \rightarrow P_k$ of $\tilde{Sq}^0$ is obviously given by

$$\tilde{Sq}^0_{\alpha_k}(x_1^{j_1} \cdots x_k^{j_k}) = \begin{cases} x_1^{j_1-1} \cdots x_k^{j_k-1}, & j_1, \ldots, j_k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\text{Ker}(\tilde{Sq}^0) = \text{Even},$$

where $\text{Even}$ denotes the vector subspace of $P_k$ spanned by all monomials $x_1^{i_1} \cdots x_k^{i_k}$ with at least one exponent $i_i$ even.

The following lemma is more or less obvious.

**Lemma 2.2 ([7]).** Let $k$ and $d$ be positive integers. Suppose that each monomial $x_1^{i_1} \cdots x_k^{i_k}$ of $P_k$ in degree $2d + k$ with at least one exponent $i_i$ even is hit. Then

$$\tilde{Sq}^0_A : (\mathbb{F}_2 \otimes P_k)_{2d+k} \rightarrow (\mathbb{F}_2 \otimes P_k)_d$$

is an isomorphism of $GL_k$-modules.
Here, as usual, a polynomial is called hit if it is $A$-decomposable in $P_k$.

A proof of this lemma is sketched as follows.

Let $s : P_k \to P_k$ be a right inverse of $\tilde{Sq}_0^*$ defined by

$$s(x_1^{i_1} \cdots x_k^{i_k}) = x_1^{2i_1+1} \cdots x_k^{2i_k+1}.$$  

It should be noted that $s$ does not commute with the doubling map on $A$, that is, in general,

$$Sq^{2t}s \neq sSq^t.$$  

However, $\text{Im}(Sq^{2t}s - sSq^t) \subset \text{Even}.$

Let $A^+$ denote the ideal of $A$ consisting of all positive degree operations. Under the hypothesis of the lemma, we have

$$(A^+ P_k + \text{Even})_{2d+k} \subset (A^+ P_k)_{2d+k}.$$  

Therefore, the map

$$\pi : (F_2 \otimes P_k)_d \to (F_2 \otimes P_k)_{2d+k}$$  

$$\pi[X] = [sX]$$  

is a well-defined linear map. Further, it is the inverse of

$$\tilde{Sq}_*^0 : (F_2 \otimes P_k)_{2d+k} \to (F_2 \otimes P_k)_d.$$  

Therefore, $\tilde{Sq}_*^0$ is an isomorphism in degree $2d + k$.

3. The iterated squaring operations in $k$-spike degrees

The following notion, which is originally due to Kraines [14], formulates some special degrees that we will mainly be interested in.

**Definition 3.1.** A natural number $m$ is called a $k$-spike if

(a) $m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$ with $n_1, \ldots, n_k > 0$, and

(b) $m$ cannot be written as a sum of less than $k$ terms of the form $(2^n - 1)$.

Note that $k$-spike is our terminology. Other authors write $\mu(m) = k$ to say $m$ is a $k$-spike. (See e.g. Wood [33, Definition 4.4].)

One easily checks e.g. that 20 is a 4-spike, 27 is a 5-spike and 58 is a 6-spike.

Let $\alpha(m)$ denote the number of ones in the dyadic expansion of $m$. The following two lemmas are more or less obvious, but useful later.

**Lemma 3.2.** Condition (a) in Definition [14] is equivalent to

$$\alpha(m + k) \leq k \leq m, \ m \equiv k \pmod{2}.$$  

**Proof.** Suppose $m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1)$ with $n_1, \ldots, n_k > 0$. Then

$$m \geq k = (2^1 - 1) + \cdots + (2^1 - 1) \ (k \text{ terms}).$$  

In addition, from $m + k = 2^{n_1} + \cdots + 2^{n_k}$ with $n_1, \ldots, n_k > 0$, it implies

$$\alpha(m + k) \leq k \text{ and } m \equiv k \pmod{2}.$$  

The equality $\alpha(m + k) = k$ occurs if and only if $n_1, \ldots, n_k$ are different from each other.

Conversely, suppose that $\alpha(m + k) \leq k \leq m$ and $m \equiv k \pmod{2}$. Let $i = \alpha(m + k)$. Then we have

$$m + k = 2^{m_1} + \cdots + 2^{m_i},$$  

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where \( m_1, \ldots, m_i > 0 \), as \( m + k \) is even.

If at least one exponent \( m_j > 1 \), then we write \( (m + k) \) as a sum of \((i + 1)\) terms of 2-powers as follows:

\[
m + k = 2^{m_1} + \cdots + 2^{m_{i-1}} + 2^{m_i-1} + \cdots + 2^m.
\]

This procedure can be continued if at least one of the exponents \( m_1, \ldots, m_j - 1, m_j - 1, \ldots, m_i \) is bigger than 1. After each step, the number of terms in the sum increases by 1. The procedure stops only in the case when the sum becomes \( m + k = 2 + \cdots + 2 \) with the number of terms \( (m + k)/2 \geq 2k/2 = k \). In particular, we reached at some step a sum of exactly \( k \) terms

\[
m + k = 2^{n_1} + \cdots + 2^{n_k}
\]

with \( n_1, \ldots, n_k > 0 \), or equivalently

\[
m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1).
\]

The lemma is proved. \( \square \)

The following lemma helps to recognize \( k \)-spikes.

**Lemma 3.3.** A natural number \( m \) is a \( k \)-spike if and only if

1. \( \alpha(m + k) \leq k \leq m, \ m \equiv k \pmod{2} \), and
2. \( \alpha(m + i) > i \) for \( 1 \leq i < k \).

**Proof.** From Lemma 3.2 if \( m \) satisfies (i), then \( m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1) \) with \( n_1, \ldots, n_k > 0 \). Also by Lemma 3.2 if \( m \) satisfies (ii), then it cannot be written as a sum of less than \( k \) terms of the form \((2^n - 1)\).

So, if \( m \) satisfies (i) and (ii), then it is a \( k \)-spike.

Conversely, suppose \( m \) is a \( k \)-spike. Then (i) holds by Lemma 3.2.

It suffices to show (ii). Suppose to the contrary that \( \alpha(m + i) \leq i \) for some \( i \) with \( 1 \leq i < k \). We then have \( \alpha(m + i) \leq i < k \leq m \). Let us consider the two cases.

**Case 1:** \( m \equiv i \pmod{2} \). Then, by Lemma 3.2 we get \( m = (2^{n_1} - 1) + \cdots + (2^{n_i} - 1) \) with \( n_1, \ldots, n_i > 0 \). This contradicts to the definition of a \( k \)-spike.

**Case 2:** \( m \equiv i - 1 \pmod{2} \). It implies \( i > 1 \). Indeed, if \( i = 1 \), combining the hypothesis \( \alpha(m + 1) \leq 1 \) with the fact \( m + 1 \) is odd, we get \( m + 1 = 1 \). This contradicts the hypothesis that \( m \) is a natural number.

By Lemma 4.3 below, we have

\[
\alpha(m + (i - 1)) = \alpha(m + i) - 1 \leq i - 1.
\]

As \( m \equiv i - 1 \pmod{2} \), we apply Lemma 3.2 again to see that \( m \) can be written as a sum of \((i - 1)\) terms of the form \((2^n - 1)\). This is also a contradiction.

Combining the two cases, we see that if \( m \) is a \( k \)-spike, then (i) and (ii) hold.

The lemma follows. \( \square \)

The following proposition is originally due to Kameko [13]. We give a proof of it to make the paper self-contained.

**Proposition 3.4.** If \( m \) is a \( k \)-spike, then

\[
\tilde{Sq}^0: (\mathbb{F}_2 \otimes P_k)_m \to (\mathbb{F}_2 \otimes P_k)_{m-k}
\]

is an isomorphism of \( GL_k \)-modules.
Proof. By using Lemma 2.2, it suffices to show that any monomial \( R \) of \( P_k \) in degree \( m \) with at least one even exponent is hit. Such a monomial \( R \) can be written, up to a permutation of variables, in the form

\[
R = x_1 \cdots x_i Q^2,
\]

with \( 0 \leq i < k \), where \( Q \) is a monomial in degree \((m - i)/2\).

If \( i = 0 \), then \( R = Q^2 \) is simply in the image of \( Sq^1 \). (It is also in the image of \( Sq^2 \), as \( R = Q^2 = Sq^2 Q \).) So, it suffices to consider the case \( 0 < i < k \).

Let \( \chi \) be the anti-homomorphism in the Steenrod algebra. The so-called \( \chi \)-trick, which was known to Brown and Peterson in the mid-sixties, states that

\[
u Sq^n(v) \equiv \chi(Sq^n)(u)v \mod A^+ M,
\]

for \( u, v \) in any \( A \)-algebra \( M \). (See also Wood [32].) In our case, it claims that \( R = x_1 \cdots x_i Q^2 = x_1 \cdots x_i Sq^m - i Q \) is hit if and only if \( \chi(Sq^m - i)(x_1 \cdots x_i) = 0 \).

As \( A \) is a commutative coalgebra, \( \chi \) is a homomorphism of coalgebras (see [20, Proposition 8.6]). Then we have the Cartan formula

\[
\chi(Sq^n)(uv) = \sum_{i+j=n} \chi(Sq^i)(u)\chi(Sq^j)(v).
\]

Furthermore, it is shown by Brown and Peterson in [3] that

\[
\chi(Sq^n)(x_j) = \begin{cases} 
x_j^{2^n}, & \text{if } n = 2^q - 1 \text{ for some } q, \\
0, & \text{otherwise},
\end{cases}
\]

for \( x_j \) in degree 1.

So, in order to prove \( \chi(Sq^m - i)(x_1 \cdots x_i) = 0 \) we need only to show that \( \frac{m-i}{2} \) cannot be written in the form

\[
\frac{m-i}{2} = (2^{\ell_1} - 1) + \cdots + (2^{\ell_i} - 1)
\]

with \( \ell_1, \ldots, \ell_i \geq 0 \). This equation is equivalent to

\[
m = (2^{\ell_1+1} - 1) + \cdots + (2^{\ell_i+1} - 1).
\]

Since \( 0 < i < k \), this equality contradicts the hypothesis that \( m \) is a \( k \)-spike. The proposition is completely proved. \( \square \)

The following lemma is the base for an iterated application of Proposition 3.4.

**Lemma 3.5.** If \( m \) is a \( k \)-spike, then so is \( (2m + k) \).

**Proof.** (a) From the definition of \( k \)-spike,

\[
m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1),
\]

for \( n_1, \ldots, n_k > 0 \). It implies that

\[
2m + k = (2^{n_1+1} - 1) + \cdots + (2^{n_k+1} - 1).
\]

So, \( 2m + k \) satisfies the first condition in the definition of \( k \)-spike.

(b) Also by this definition, we have

\[
\alpha(m + k - j) > k - j,
\]
for $1 \leq j < k$. Hence
\[
\alpha(2m + k + (k - 2j)) = \alpha(2(m + k - j)) = \alpha(m + k - j) > k - j > k - 2j,
\]
\[
\alpha(2m + k + (k - 2j + 1)) = \alpha(2(m + k - j) + 1) = \alpha(2(m + k - j)) + 1 \quad \text{(by Lemma 1.3)}
\]
\[
= \alpha(m + k - j) + 1 > (k - j) + 1 > k - 2j + 1.
\]
Note that each $i$ satisfying $1 \leq i < k$ can be written either in the form $i = k - 2j$ (for $1 \leq j \leq \frac{k-1}{2}$) or in the form $i = k - 2j + 1$ (for $1 \leq j \leq \frac{k}{2}$). So, the above two inequalities show that
\[
\alpha(2m + k + i) > i,
\]
for $1 \leq i < k$. Thus, $2m + k$ satisfies the second condition in Definition 3.1.

Combining parts (a) and (b), we see that $2m + k$ is a $k$-spike. \hfill \Box

**Remark 3.6.** The converse of Lemma 3.5 is false. For instance, $27$ is a $5$-spike, whereas $11 = (27 - 5)/2$ is not.

**Proposition 3.7.** If $m$ is a $k$-spike, then
\[
(\widetilde{Sq}^0)^{i+1} : PH_*(BV_k)_{m-k} \to PH_*(BV_k)_{2^i m + (2^i - 1)k}
\]
is an isomorphism of $GL_k$-modules for every $i \geq 0$.

**Proof.** If $m$ is a $k$-spike, then by the dual of Proposition 3.4, we have an isomorphism of $GL_k$-modules
\[
\widetilde{Sq}^0 : PH_*(BV_k)_{m-k} \to PH_*(BV_k)_m.
\]

On the other hand, from Lemma 3.5 if $m$ is a $k$-spike, then so is $2^i m + (2^i - 1)k$ for every $i \geq 0$. Hence, repeatedly applying the dual of Proposition 3.4 we get an isomorphism of $GL_k$-modules
\[
(\widetilde{Sq}^0)^{i+1} : PH_*(BV_k)_{m-k} \to PH_*(BV_k)_{2^i m + (2^i - 1)k}.
\]
The proposition is proved. \hfill \Box

**Corollary 3.8.** If $m$ is a $k$-spike, then
\[
(Sq^0)^{i+1} : (\mathbb{F}_2 \otimes_{GL_k} PH_*(BV_k))_{m-k} \to (\mathbb{F}_2 \otimes_{GL_k} PH_*(BV_k))_{2^i m + (2^i - 1)k}
\]
is an isomorphism for every $i \geq 0$.

4. **Recognition of $k$-spikes**

In this section, we introduce an inductive way of producing $k$-spikes, which will play a key role in the proofs of Theorems 1.1, 1.2, 1.5 and 1.6 in the next two sections.

**Lemma 4.1.** If $m$ is a $k$-spike, then $(2^n - 1 + m)$ is a $(k + 1)$-spike for every $n$ with $2^n \geq m + k - 1$. 

To prove this lemma, we need the following two lemmas.

**Lemma 4.2.** If \(2^n \geq a\), then
\[
\alpha(2^n - 1 + a) \geq \alpha(a).
\]

**Proof.** The proof proceeds by induction on \(\alpha(a)\). If \(\alpha(a) = 1\), then \(a\) is a power of 2, say \(a = 2^p \leq 2^n\). We have
\[
2^n - 1 + 2^p = 2^n + (2^p - 1) = 2^n + (2^{p-1} + \cdots + 2^0).
\]
Thus \(\alpha(2^n - 1 + 2^p) = 1 + p \geq 1 = \alpha(a)\).

Suppose inductively that the lemma is valid for \(\alpha(a) = t\). We now consider the case \(\alpha(a) = t + 1 > 1\). That is, \(a = 2^{n+1} + 2^n + \cdots + 2^n\) with \(n_{t+1} > n_t > \cdots > n_1\).

Set \(b = 2^n + \cdots + 2^n < 2^{n+1}\); then \(a = 2^{n+1} + b\), and \(\alpha(b) = t\). From \(2^n \geq a\), it implies \(2^n > 2^{n+1}\). Therefore, we obtain
\[
\alpha(2^n - 1 + a) = \alpha(2^n + 2^{n+1} - 1 + b)
\]
\[
= 1 + \alpha(2^{n+1} - 1 + b)
\]
\[
\geq 1 + \alpha(b) \quad \text{(by the inductive hypothesis)}
\]
\[
= 1 + t = \alpha(a).
\]
The lemma is proved. \(\square\)

The following lemma is an obvious observation.

**Lemma 4.3.** If \(e\) is an even number, then
\[
\alpha(e + 1) = \alpha(e) + 1.
\]

**Proof of Lemma 4.1.** (a) Since \(m = (2^{n_1} - 1) + \cdots + (2^{n_k} - 1)\), we get
\[
(2^n - 1) + m = (2^n - 1) + (2^n - 1) + \cdots + (2^n - 1).
\]
So the first condition in Definition 3.1 holds for \((2^n - 1 + m)\).

(b) If \(1 \leq i < k\), then \(2^n \geq m + k - 1 \geq m + i\). By Lemma 4.2, we have
\[
\alpha(2^n - 1 + m + i) \geq \alpha(m + i) > i.
\]
The last inequality comes from the hypothesis that \(m\) is a \(k\)-spike.

Finally, we need to show \(\alpha(2^n - 1 + m + k) > k\). Recall that, as \(m\) is a \(k\)-spike, then \(m \equiv k \pmod{2}\). Hence, \(e = (2^n - 1) + m + (k - 1)\) is even. By Lemma 4.3, we have
\[
\alpha(2^n - 1 + m + k) = \alpha(2^n - 1 + m + (k - 1) + 1)
\]
\[
= \alpha(2^n - 1 + m + (k - 1)) + 1.
\]
Now, applying Lemma 4.2 to the case \(2^n \geq m + k - 1\), we get
\[
\alpha(2^n - 1 + m + (k - 1)) + 1 \geq \alpha(m + (k - 1)) + 1
\]
\[
> (k - 1) + 1 = k.
\]
The last inequality comes from the fact that \(m\) is a \(k\)-spike.

In summary, the second condition in Definition 3.1 holds for \((2^n - 1 + m)\).

Combining parts (a) and (b), we see that \((2^n - 1 + m)\) is a \((k + 1)\)-spike. The lemma is proved. \(\square\)
Remark 4.4. Lemma 4.1 cannot be improved in the meaning that the hypothesis $2^{n+1} \geq m + k - 1$ does not imply $(2^n - 1 + m)$ to be a $(k + 1)$-spike. This is the case of $k = 5, m = 27$ and $2^n = 16$, because $15 + 27 = 42$ is not a $6$-spike.

The following corollary is a key point in the proof of Lemma 6.3 and therefore in the proofs of Theorems 1.1 and 1.6.

Corollary 4.5. $2^k - k$ is a $k$-spike for every $k > 0$.

Proof. We prove this by induction on $k$. The corollary holds trivially for $k = 1$.

Suppose inductively that $2^k - k$ is a $k$-spike. Then, as $2^{k+1} = (2^k - k) + k - 1$, applying Lemma 4.2 to the case $n = k$ and $m = 2^{k+1} - k$, we have

$$2^{k+1} - (k + 1) = (2^k - 1) + (2^k - k)$$

to be a $(k + 1)$-spike. The corollary follows.

5. THE ALGEBRAIC TRANSFER IS NOT AN ISOMORPHISM FOR $k \geq 4$

We first briefly recall the definition of the algebraic transfer. Let $\widehat{P}_1$ be the submodule of $F_2[x_1, x_1^{-1}]$ spanned by all powers $x_1^i$ with $i \geq -1$. The usual $A$-action on $P_1 = F_2[x_1]$ is canonically extended to an $A$-action on $F_2[x_1, x_1^{-1}]$ (see Adams [2], Wilkerson [31]). $\widehat{P}_1$ is an $A$-submodule of $F_2[x_1, x_1^{-1}]$. The inclusion $P_1 \subset \widehat{P}_1$ gives rise to a short exact sequence of $A$-modules:

$$0 \rightarrow P_1 \rightarrow \widehat{P}_1 \rightarrow \Sigma^{-1}F_2 \rightarrow 0.$$ 

Let $e_1$ be the corresponding element in $\text{Ext}^1_A(\Sigma^{-1}F_2, P_1)$. Singer set $e_k = e_1 \otimes \cdots \otimes e_1 \in \text{Ext}^k_A(\Sigma^{-k}F_2, P_k)$ ($k$ times). Then, he defined $Tr^*_k : \text{Tor}_k^A(F_2, \Sigma^{-k}F_2) \rightarrow \text{Tor}_k^A(F_2, P_k) = \Sigma^{-k}F_2 \otimes P_k$ by $Tr^*_k(z) = e_k \cap z$. Its image is a submodule of $(\Sigma^{-k}F_2 \otimes P_k)^{G_{k+1}}$.

The $k$-th algebraic transfer is defined to be the dual of $Tr^*_k$.

We will need to apply the following theorem by D. Davis [9].

Let $h_n$ be the nonzero element in $\text{Ext}^1_A(\Sigma^{-2n}F_2, F_2)$.

Theorem 5.1 (D). If $x$ is a nonzero element in $\text{Ext}^k_A(F_2, F_2)$ with $4 \leq d \leq 2j$, then $h_n x \neq 0$ for every $n \geq 2j + 1$.

The following concept plays a key role in this section.

Definition 5.2. A nonzero element $x \in \text{Ext}^*_A(F_2, F_2)$ is called critical if

(a) $Sq^0(x) = 0$, and

(b) $2\text{Stem}(x) + k$ is a $k$-spike.

Note that, by Lemma 3.5 if $\text{Stem}(x)$ is a $k$-spike, then so is $2\text{Stem}(x) + k$.

Lemma 5.3. If $x \in \text{Ext}^k_A(F_2, F_2)$ is critical, then so is $h_n x$ for every $n$ with $2^n \geq \max\{4d^2, d + k\}$, where $d = \text{Stem}(x)$.

Proof. First, we show that if $x$ is critical, then $\text{Stem}(x) > 0$. Indeed, suppose to the contrary that $\text{Stem}(x) = 0$; then $x = h_0^1$. As $x$ is critical, $Sq^0(x) = Sq^0(h_0^1) = h_1^k = 0$. This implies that $k \geq 4$, as $h_1^1, h_1^3, h_1^5$ all are nonzero, whereas $h_1^2 = 0$. However, $2\text{Stem}(x) + k = k$ is not a $k$-spike for $k \geq 4$, because it can be written as a sum $k = 3 + 1 + \cdots + 1$ of $(k - 2)$ terms of the form $(2^n - 1)$. This contradicts the definition of a critical element.
Now we have $\text{Stem}(x) > 0$. Combining the fact that $Sq^0$ is a monomorphism in positive stems of $\text{Ext}_A^k(\mathbb{F}_2, \mathbb{F}_2)$ for $k \leq 4$, and that $x$ is critical, we get $k > 4$. As $x$ is a nonzero element of positive stem in $\text{Ext}_A^k(\mathbb{F}_2, \mathbb{F}_2)$ with $k > 4$, by the vanishing line theorem (see [1]) we have $\text{Stem}(x) > 7$. Therefore, $x$ satisfies the hypothesis of Theorem 5.1 that $d = \text{Stem}(x) \geq 4$.

Let $j$ be the smallest positive integer such that $2^j \geq d$. Then, the smallest positive integer $i$ with $2^i \geq d^2$ should be either $2j$ or $2j - 1$. From the hypothesis $2^n \geq 4d^2$, it implies that $2^{n-2} \geq d^2$. Hence, we get $n - 2 \geq i \geq 2j - 1$, or equivalently, $n \geq 2j + 1$.

Therefore, by Theorem 5.1, $h_n x \neq 0$ if $2^n \geq 4d^2$.

As $Sq^0$ is a homomorphism of algebras, we have

$$Sq^0(h_n x) = Sq^0(h_n) Sq^0(x) = Sq^0(h_n) \cdot 0 = 0.$$  

Since $x$ is critical, $m := 2d + k$ is a $k$-spike. We need to show that $2\text{Stem}(h_n x) + (k+1)$ is a $(k+1)$-spike. We have

$$2\text{Stem}(h_n x) + (k+1) = 2^{n-1} + \text{Stem}(x) = 2^n - 1 + d.$$  

A routine calculation shows

$$2\text{Stem}(h_n x) + (k+1) = 2(2^n - 1 + d) + (k+1) = 2^{n+1} - 2 + (2d + k) + 1 = 2^{n+1} - 1 + m.$$  

By Lemma 4.1, this number is a $(k+1)$-spike for every $n$ with $2^n \geq m + k - 1 = 2(d + k) - 1$, or equivalently $2^n \geq d + k$.

In summary, $h_n x$ is critical for every $n$ with

$$2^n \geq \max\{4d^2, d + k\}.$$  

The lemma is proved.

Remark 5.4.  

(a) Suppose $h_n x \neq 0$ although $2^n < 4(\text{Stem}(x))^2$. If $x$ is critical and $2^n \geq \text{Stem}(x) + k$, then $h_n x$ is also critical.

(b) There is no critical element for $k \leq 4$, as $Sq^0$ is a monomorphism in positive stems of $\text{Ext}_A^k(\mathbb{F}_2, \mathbb{F}_2)$ for $k \leq 4$.

Proposition 5.5.  

(i) For $k = 5$, there is at least one number, which is the stem of a critical element.

(ii) For each $k > 5$, there are infinitely many numbers, which are stems of critical elements.

Proof. For $k = 5$, $P h_2 \in \text{Ext}_A^5(\mathbb{F}_2, \mathbb{F}_2)$ is critical. Indeed, it is well known (see e.g. Tangora [30]) that $\text{Ext}_A^5(\mathbb{F}_2, \mathbb{F}_2) = 0$, so we get

$$Sq^0(P h_2) = 0.$$  

Further, by Lemma 5.3, $2\text{Stem}(P h_2) + 5 = 27$ is a $5$-spike.

We can start the inductive argument of Lemma 5.5 with the initial critical element $P h_2$. The proposition follows.

The following theorem is also numbered as Theorem 1.2 in the Introduction.

Theorem 5.6. $Tr_k$ is not an isomorphism for $k \geq 5$. Furthermore, for every $k > 5$, there are infinitely many degrees in which $Tr_k$ is not an isomorphism.
Proof. In order to prove the theorem, by means of Proposition 5.5 it suffices to show that $Tr_k$ is not an isomorphism either in degree $d$ or in degree $2d+k$, where $d$ denotes the stem of a critical element $x \in \text{Ext}^k_A(\mathbb{F}_2, \mathbb{F}_2)$.

We consider the following two cases:

Case 1: $x$ is not in the image of $Tr_k$.
Then, $Tr_k$ is not an epimorphism in degree $d$.

Case 2: $x = Tr_k(y)$ for some $y \in \mathbb{F}_2 \otimes \text{PH}_k(BV_k)$.

From $x \neq 0$, it implies that $y \neq 0$. According to Boardman [4, Thm 6.9 and Cor 6.12] and Minami [21, Thm 4.4], we have a commutative diagram

\[
\begin{array}{ccc}
(F_2 \otimes \text{PH}_k(BV_k))_d & \xrightarrow{Tr_k} & \text{Ext}^{k,k+d}_A(F_2, F_2) \\
\downarrow Sq^0 & & \downarrow Sq^0 \\
(F_2 \otimes \text{PH}_k(BV_k))_{2d+k} & \xrightarrow{Tr_k} & \text{Ext}^{k,2(k+d)}_A(F_2, F_2)
\end{array}
\]

where the left vertical arrow is the Kameko $Sq^0$ and the right vertical one is the classical squaring operation.

As $m = 2d+k$ is a $k$-spike, by Corollary 3.8 the Kameko $Sq^0$ is an isomorphism. So, from $y \neq 0$, we have

$z = Sq^0(y) \neq 0$.

Now, by the commutativity of the diagram, we get

$Tr_k(z) = Tr_k(Sq^0(y)) = Sq^0(Tr_k(y)) = Sq^0(x) = 0$.

This means that $Tr_k$ is not a monomorphism in degree $2d+k$. The theorem is completely proved. \hfill \Box

Remark 5.7. (a) We can show that $\mathbb{F}_2 \otimes \text{PH}_5(BV_5)_{11} = 0$. It implies that $Ph_2$ is not detected by $Tr_5$.

(b) By Lemma 5.3, $h_n Ph_2$ is critical for every $n \geq 9$, as $\text{Stem}(Ph_2) + 5 < 4(\text{Stem}(Ph_2))^2 = 4 \cdot 11^2 = 484 < 2^9 = 512$. Also, by Remark 5.4, $h_n Ph_2$ is critical for $n = 4, 5, 6$, as it is nonzero (see [6]) and $2^4 \geq \text{Stem}(Ph_2) + 5 = 16$. R. Bruner privately claimed $h_7 Ph_2 \neq 0$. It seems likely that $h_8 Ph_2 \neq 0$. If so, by the same argument, these two elements are also critical.

The following corollary is also numbered as Corollary 1.3 in the Introduction.

Corollary 5.8. (i) $Tr_k$ is an isomorphism for $k = 1, 2$ and 3.

(ii) $Tr_k$ is not an isomorphism for $k \geq 4$.

(iii) For $k = 4$ and for each $k > 5$, there are infinitely many degrees in which $Tr_k$ is not an isomorphism.

This result is due to Singer [28] for $k = 1, 2$, to Boardman [4] for $k = 3$, and to Bruner–Há–Hu [7] for $k = 4$. That $Tr_5$ is not an isomorphism in degree 9 is also due to Singer [28]. The remaining part is shown by Theorem 5.6.

Our knowledge’s gap on whether $Tr_5$ is not an isomorphism in infinitely many degrees will be studied in Section 8.
The following theorem is also numbered as Theorem 1.5 in the Introduction.

**Theorem 5.9.** If \( Tr_\ell \) detects a critical element, then it is not a monomorphism, and further, for each \( k > \ell \), there are infinitely many degrees in which \( Tr_k \) is not a monomorphism.

**Proof.** The proof proceeds by induction on \( k \geq \ell \).

For \( k = \ell \), suppose \( Tr_\ell \) detects a critical element \( x_\ell \in \text{Ext}_A^\ell (\mathbb{F}_2, \mathbb{F}_2) \). Then, by Case 2 in the proof of Theorem 5.6, \( Tr_\ell \) is not a monomorphism in degree \( 2\text{Stem}(x_\ell) + \ell \).

By means of this argument, it suffices to show that if \( Tr_k \) detects a critical element \( x_k \), then \( Tr_{k+1} \) detects infinitely many critical elements, whose stems are different from each other.

From the hypothesis, \( x_k = Tr_k(y_k) \) for some \( y_k \in \mathbb{F}_2 \otimes PH_*(BV_k) \). With ambiguity of notation, let \( h_n \) also denote the element in \( \mathbb{F}_2 \otimes PH_*(BV_1) \), whose image under \( Tr_1 \) is the usual \( h_n \in \text{Ext}_A^1(\mathbb{F}_2, \mathbb{F}_2) \). As \( Tr = \bigoplus_k Tr_k \) is a homomorphism of algebras (see [28]), we have

\[
Tr_{k+1}(h_n y_k) = Tr_1(h_n)Tr_k(y_k) = h_n x_k.
\]

By Lemma 5.3 the element \( h_n x_k \) is critical for every \( n \) with \( 2^n \geq \max\{4d^2, d + k\} \).

By the first part of the theorem, since \( Tr_{k+1} \) detects the critical element \( h_n x_k \), it is not a monomorphism in degree \( 2\text{Stem}(h_n x_k) + (k + 1) \) for every \( n \) with \( 2^n \geq \max\{4d^2, d + k\} \). Thus, \( Tr_{k+1} \) is not a monomorphism in infinitely many degrees. The theorem follows. \( \square \)

6. The stability of the iterated squaring operations

The following theorem, which is also numbered as Theorem 1.1 in the Introduction, shows that \( Sq^0 \) is eventually isomorphic on \( \mathbb{F}_2 \otimes PH_*(BV_k) \). More precisely, it claims that if we start from any degree \( d \) of this module, and apply \( Sq^0 \) repeatedly at most \( (k - 2) \) times, then we get into the region, in which all the iterated squaring operations are isomorphisms.

**Theorem 6.1.** Let \( d \) be an arbitrary nonnegative integer. Then

\[
(Sq^0)^{i-k+2} : PH_*(BV_k)_{2^k-2d+(2^k-2-1)k} \rightarrow PH_*(BV_k)_{2^i d+(2^i-1)k}
\]

is an isomorphism of \( GL_k \)-modules for every \( i \geq k - 2 \).

In the theorem, for \( k = 1 \) we take the convention that \( 2^{1-2}d + (2^{1-2} - 1)k = d \).

Let us denote

\[
(Sq^0)^{-1}(\mathbb{F}_2 \otimes PH_*(BV_k))_d = \lim_{i \rightarrow \infty} \cdots (\mathbb{F}_2 \otimes PH_*(BV_k))_{2^i d+(2^i-1)k} \rightarrow \cdots \rightarrow \mathbb{F}_2 \otimes PH_*(BV_k)_{2^k-2d+(2^k-2-1)k}.
\]

The following corollary is an immediate consequence of Theorem 6.1.

**Corollary 6.2.** Let \( d \) be an arbitrary nonnegative integer. Then,

(i) The following iterated operation is an isomorphism for every \( i \geq k - 2 \):

\[
(Sq^0)^{i-k+2} : \mathbb{F}_2 \otimes PH_*(BV_k)_{2^k-2d+(2^k-2-1)k} \rightarrow \mathbb{F}_2 \otimes PH_*(BV_k)_{2^i d+(2^i-1)k}.
\]
(ii) 
\((Sq^0)^{-1}(\mathbb{F}_2 \otimes \text{PH}_* (BV_k))_d \cong (\mathbb{F}_2 \otimes \text{PH}_* (BV_k))_{2^{k-2}d + (2^{k-2} - 1)k}.\)

(iii) If \(d = 2^{k-2}d' + (2^{k-2} - 1)k\) for some nonnegative integer \(d'\), then 
\((Sq^0)^{-1}(\mathbb{F}_2 \otimes \text{PH}_* (BV_k))_d \cong (\mathbb{F}_2 \otimes \text{PH}_* (BV_k))_d.\)

In order to prove Theorem 6.1, we need the following lemma. Let \(\delta_k\) denote the function given by \(\delta_k(d) = 2d + k\).

**Lemma 6.3.** If \(d\) is a nonnegative integer with \(\alpha(d + k) \leq k\), then \(\delta_k^{d+k-1}(d) = 2^{k-1}d + (2^{k-1} - 1)k\) is a \(k\)-spike.

**Proof.** The lemma holds trivially for \(k = 1\). Indeed, from the hypothesis \(\alpha(d+1) \leq 1\) it implies that \(d = 2^n - 1\) for some \(n\). Then \(\delta_1^d(d) = d = 2^n - 1\) is an \(1\)-spike.

We now consider the case of \(k \geq 2\). First, we observe that \(k \leq 2^{k-1}d + (2^{k-1} - 1)k \equiv k \pmod{2}\) and 
\[\alpha(2^{k-1}d + (2^{k-1} - 1)k + k) = \alpha(2^{k-1}(d + k)) = \alpha(d + k) \leq k.\]

By Lemma 3.2, \(\delta_k^{d+k-1}(d) = 2^{k-1}d + (2^{k-1} - 1)k\) satisfies condition (a) of Definition 3.1

So, in order to prove the lemma, it suffices to show that 
\[\alpha(2^{k-1}d + (2^{k-1} - 1)k + i) > i\] for \(1 \leq i < k\).

We now work modulo \(2^{k-1}\). First, we have 
\[2^{k-1}d + (2^{k-1} - 1)k \equiv (2^{k-1} - 1)k \pmod{2^{k-1}}.\]

Let \(k = 2^{n_1} + \cdots + 2^{n_t}\) be the dyadic expansion of \(k\) with \(n_1 > \cdots > n_t\). We get 
\[(2^{k-1} - 1)k = 2^{k-1}(2^{n_1} + \cdots + 2^{n_t}) + (2^{k-1} - (2^{n_1} + \cdots + 2^{n_t})).\]

Thus 
\[(2^{k-1} - 1)k \equiv 2^{k-1} + (2^{k-1} - 1)k \equiv 2^{k-1} + (2^{k-1} - 1)k \equiv 2^{k-1} - k \pmod{2^{k-1}},\]

where \(2^{k-1} - k \geq 0\) because \(k \geq 2\).

As a consequence, we get 
\[2^{k-1}d + (2^{k-1} - 1)k + i \equiv 2^{k-1} - k + i \pmod{2^{k-1}}\]

for \(1 \leq i < k\). Since \(k \geq 2\) and \(d \geq 0\) we have 
\[2^{k-1}d + (2^{k-1} - 1)k + i \geq 2^{k-1} + 1 > 2^{k-1}.\]

From this inequality it implies that, in the dyadic expansion of \(2^{k-1}d + (2^{k-1} - 1)k + i\), there is at least one nonzero term \(2^n\) with \(n \geq k-1\). On the other hand, as \(2^{k-1} - k + i < 2^{k-1}\) for \(1 \leq i < k\), the dyadic expansion of \(2^{k-1} - k + i\) is just a combination of the 2-powers \(2^0, 2^1, \dotsc, 2^{k-2}\). Therefore, in order to prove 
\[\alpha(2^{k-1}d + (2^{k-1} - 1)k + i) > i\]

for \(1 \leq i < k\), we need only to show that 
\[\alpha(2^{k-1} - k + i) \geq i.\]
From Corollary 6.3, $2^{k-1} - (k - 1)$ is a $(k - 1)$-spike. Then we have
\[
\alpha(2^{k-1} - (k - 1) + j) > j
\]
for $1 \leq j < k - 1$. Set $i = j + 1$; we get
\[
\alpha(2^{k-1} - k + i) \geq i
\]
for $2 \leq i < k$. In addition, it is obvious that
\[
\alpha(2^{k-1} - k + 1) \geq 1.
\]
In summary, we have shown that
\[
\alpha(2^{k-1} - k + i) \geq i
\]
for $1 \leq i < k$. The lemma is proved. \(\Box\)

Remark 6.4. (a) Lemma 6.3 cannot be improved in the meaning that the number
\[
\delta_k^{k-1}(d) = 2^{k-2}d + (2^{k-2} - 1)k
\]
is not a $k$-spike in general.
Indeed, taking $d = 2^t + 1 - k$ with $t$ big enough so that $d \geq 0$, we have
\[
\alpha(2^{k-2}d + (2^{k-2} - 1)k + (k - 1)) = \alpha(2^{t+k-2} + (2^{k-2} - 1)) = k - 1.
\]
By Lemma 3.3, $2^{k-2}d + (2^{k-2} - 1)k$ is not a $k$-spike.

(b) However, a number could be a $k$-spike although it is not of the form $\delta_k^{k-1}(d)$
for any nonnegative integer $d$. For instance, this is the case of the following numbers
with $k = 4$:
\[
\text{Stem}(e_2) = 80, \quad \text{Stem}(f_1) = 40, \quad \text{Stem}(p_2) = 144, \quad \text{Stem}(D_3(2)) = 256, \quad \text{Stem}(p_2') = 288,
\]
where $e_2, f_1, p_2, D_3(2), p_2'$ are the usual elements in $\text{Ext}_A^4(\mathbb{F}_2, \mathbb{F}_2)$. This observation
will be helpful in the proof of Proposition 7.2 below.

Proof of Theorem 6.1. According to Wood’s theorem [32] (it was originally Peterson’s conjecture),
the primitive part $PH_*(B\mathbb{V}_k)$ is concentrated in the degrees $d$
with $\alpha(d + k) \leq k$. This fact together with the equality
\[
\alpha(\delta_k^i(d + k)) = \alpha(2^i(d + k)) = \alpha(d + k)
\]
show that, if $\alpha(d + k) > k$, then the domain and the target of the homomorphism
in the theorem are both zero.

If $\alpha(d + k) \leq k$, then the theorem is an immediate consequence of Lemma 6.3
and Proposition 5.7. The theorem is proved. \(\Box\)

Remark 6.5. Let $k = 5$ and $d = 0$. As $\delta_5^{5-2}(0) = 35$, Theorem 6.1 claims that
\[
(Sq^0)^{i-3} : PH_*(B\mathbb{V}_5)_{35} \longrightarrow PH_*(B\mathbb{V}_5)_{5(2^i-1)}
\]
is an isomorphism of $GL_5$-modules for $i \geq 3$. In the final section we will see that
\[
Sq^0 : \mathbb{F}_2 \otimes_{GL_5} PH_*(B\mathbb{V}_5)_{15} \longrightarrow \mathbb{F}_2 \otimes_{GL_5} PH_*(B\mathbb{V}_5)_{35}
\]
is not a monomorphism. This shows that Theorem 6.1 cannot be improved in the
meaning that $(k - 2)$ is, in general, the minimum times that we must repeatedly
apply $Sq^0$ to get into “the isomorphism region” of the iterated squaring operations.

A family $\{a_i \mid i \geq 0\}$ of elements in $\text{Ext}_A^k(\mathbb{F}_2, \mathbb{F}_2)$ is called an $Sq^0$-family
if $a_i = (Sq^0)^i(a_0)$ for every $i \geq 0$. An $Sq^0$-family in $\mathbb{F}_2 \otimes_{GL_5} PH_*(B\mathbb{V}_k)$ is similarly
defined.
Definition 6.6. Let $a_0 \in \text{Ext}^k_A(\mathbb{F}_2, \mathbb{F}_2)$. The root degree of $a_0$ is the maximum nonnegative integer $r$ such that $\text{Stem}(a_0)$ can be written in the form

$$\text{Stem}(a_0) = \delta_k^r(d) = 2^r d + (2^r - 1)k,$$

for some nonnegative integer $d$.

The following theorem is also numbered as Theorem 6.6 in the Introduction.

Theorem 6.7. Let $\{a_i \mid i \geq 0\}$ be an $Sq^0$-family in $\text{Ext}^k_A(\mathbb{F}_2, \mathbb{F}_2)$ and let $r$ be the root degree of $a_0$. If $\text{Tr}_k$ detects $a_n$ for some $n \geq \max\{k-r-2, 0\}$, then it detects $a_i$ for every $i \geq n$ and detects $a_j$ modulo $\text{Ker}(Sq^0)^{n-j}$ for $\max\{k-r-2, 0\} \leq j < n$.

Proof. It is easy to see that

$$\alpha(\text{Stem}(a_i) + k) = \alpha(2^i(\text{Stem}(a_0) + k)) = \alpha(\text{Stem}(a_0) + k).$$

Suppose $\alpha(\text{Stem}(a_i) + k) > k$; then we have $\alpha(\text{Stem}(a_i) + k) > k$ for every $i \geq 0$. By Wood’s theorem [32] (it was originally Peterson’s conjecture), $PH_*(BV_k)_t = 0$ in any degree $t$ with $\alpha(t + k) > k$. So, all elements of the family $\{a_i \mid i \geq 0\}$ are not detected by $\text{Tr}_k$.

Now we consider the case where $\alpha(\text{Stem}(a_0) + k) \leq k$. We observe that

$$\alpha(\text{Stem}(a_0) + k) = \alpha(2^r(d + k)) = \alpha(d + k) \leq k.$$

Set $q = \max\{k - r - 2, 0\}$, and we have

$$\text{Stem}(a_{q+1}) = \delta_k^{q+1}(\text{Stem}(a_0)) = \delta_k^{q+r+1}(d).$$

Note that

$$q + r + 1 = \max\{k - r - 2, 0\} + r + 1 \geq (k - r - 2) + r + 1 = k - 1.$$

So, by Lemmas 6.3 and 3.5, $\text{Stem}(a_{q+1})$ is a $k$-spike.

According to Theorem 6.4 if $c = \text{Stem}(a_q)$, then

$$\hat{(S^0)}^{i-n} : PH_*((BV_k)_c) \rightarrow PH_*((BV_k)_{2^{-i}c+(2^{-i}-1)k})$$

is an isomorphism of $GL_k$-modules for every $i \geq q$.

Suppose $\text{Tr}_k$ detects $a_n$ with $n \geq q$, that is, $a_n = \text{Tr}_k(\tilde{a}_n)$ for some $\tilde{a}_n$ in $\mathbb{F}_2 \otimes PH_*(BV_k)$. If $i \geq n$, then we set $\tilde{a}_i = (S^0)^{i-n}(\tilde{a}_n)$. As the squaring operations commute with each other through the algebraic transfer, we have

$$a_i = (S^0)^{i-n}(a_n) = (S^0)^{i-n}\text{Tr}_k(\tilde{a}_n) = \text{Tr}_k(S^0)^{i-n}(\tilde{a}_n) = \text{Tr}_k(\tilde{a}_i).$$

Thus, $a_i$ is detected by $\text{Tr}_k$ for every $i \geq n$.

Next we consider $j$ with $\max\{k - r - 2, 0\} \leq j < n$. Then we set

$$\tilde{a}_j = [(S^0)^{n-j}]^{-1}(\tilde{a}_n).$$

This makes sense, as it is shown above that $(S^0)^{n-j}$ is isomorphic in the degree of $\tilde{a}_j$. Again, as the squaring operations commute with each other through the algebraic transfer, we have

$$(S^0)^{n-j}\text{Tr}_k(\tilde{a}_j) = \text{Tr}_k(S^0)^{n-j}(\tilde{a}_j) = \text{Tr}_k(\tilde{a}_n) = a_n = (S^0)^{n-j}(a_j).$$
As a consequence, we get

\[ Tr_k(\bar{a}_j) = a_j \pmod{\ker(Sq^0)^{n-j}}. \]

This means that \( Tr_k \) detects \( a_j \) modulo \( \ker(Sq^0)^{n-j} \). The theorem is proved. \( \square \)

**Remark 6.8.** (a) Under the hypothesis of Theorem 6.7, let

\[ a'_i = Tr_k(Sq^0)^{-n}(\bar{a}_n) \]

for every \( i \geq \max\{k-r-2,0\} \) no matter whether \( i \geq n \) or \( i < n \). Then we get a new \( Sq^0 \)-family \( \{a'_i, i \geq \max\{k-r-2,0\}\} \), whose every element is detected by \( Tr_k \) and

\[ a'_i = \begin{cases} a_i, & \text{if } i \geq n, \\ a_i \pmod{\ker(Sq^0)^{n-i}}, & \text{if } i < n. \end{cases} \]

The new \( Sq^0 \)-family is called the adjustment of the original one.

(b) Theorem 6.7 is still valid and can be shown by the same proof if we replace \( \max\{k-r-2,0\} \) by any number \( q \) such that \( \text{Stem}(a_{q+1}) \) is a \( k \)-spike. This remark will be useful in the proof of Proposition 7.2 for the case \( k = 4 \).

**Corollary 6.9.** Let \( \{a_i, i \geq 0\} \) be an \( Sq^0 \)-family in \( \text{Ext}^k_A(\mathbb{F}_2, \mathbb{F}_2) \) and let \( r \) be the root degree of \( a_0 \). Suppose the classical \( Sq^0 \) is a monomorphism in the stems of the elements \( \{a_i, i \geq \max\{k-r-2,0\}\} \). If \( Tr_k \) detects \( a_n \) for some \( n \geq \max\{k-r-2,0\} \), then it detects \( a_i \) for every \( i \geq \max\{k-r-2,0\} \).

An \( Sq^0 \)-family is called finite if it has only finitely many nonzero elements, infinite if all of its elements are nonzero. The following is also numbered as Corollary 1.7 in the Introduction.

**Corollary 6.10.**

(i) Every finite \( Sq^0 \)-family in \( \mathbb{F}_2 \otimes_{GL_k} PH_* (B\mathbb{V}_k) \) has at most \((k-2)\) nonzero elements.

(ii) If \( Tr_k \) is a monomorphism, then it does not detect any element of a finite \( Sq^0 \)-family in \( \text{Ext}^k_A(\mathbb{F}_2, \mathbb{F}_2) \) with at least \((k-1)\) nonzero elements.

**Proof.** (i) Suppose that \( \{\bar{a}_i, i \geq 0\} \) is an \( Sq^0 \)-family in \( \mathbb{F}_2 \otimes_{GL_k} PH_* (B\mathbb{V}_k) \) with at least \((k-1)\) nonzero elements. Then \( \bar{a}_0, \bar{a}_1, ..., \bar{a}_{k-2} \) are its first \((k-1)\) nonzero elements. Set \( d = \deg(\bar{a}_0) \); then \( \deg(\bar{a}_{k-2}) = 2^{k-2}d + (2^{k-2} - 1)k \). Therefore, by Proposition 6.2

\[ (Sq^0)^{i-k+2} : \mathbb{F}_2 \otimes_{GL_k} PH_* (B\mathbb{V}_k)_{2^{k-2}d+(2^{k-2} - 1)k} \to \mathbb{F}_2 \otimes_{GL_k} PH_* (B\mathbb{V}_k)_{2d+(2^{k-2} - 1)k} \]

is an isomorphism for every \( i \geq k-2 \). Therefore, from \( \bar{a}_{k-2} \neq 0 \) it implies that \( \bar{a}_i = (Sq^0)^{i-k+2}(\bar{a}_{k-2}) \) is nonzero for every \( i \geq k-2 \). Thus, the \( Sq^0 \)-family is infinite.

(ii) Let \( a_0, a_1, ..., a_{k-2} \) be the last \((k-1)\) nonzero elements of the given finite \( Sq^0 \)-family in \( \text{Ext}^k_A(\mathbb{F}_2, \mathbb{F}_2) \). As \( a_{k-2} \) is the last nonzero element in the \( Sq^0 \)-family, we have \( Sq^0(a_{k-2}) = 0 \). Set \( d = \text{Stem}(a_0) \); then by Lemma 6.3, \( 2\text{Stem}(a_{k-2}) + k = 2^{k-1}d + (2^{k-1} - 1)k \) is a \( k \)-spike. So, \( a_{k-2} \) is critical.

Suppose to the contrary that \( Tr_k \) detects some (nonzero) element in the \( Sq^0 \)-family. Then, as the squaring operations commute with each other through the algebraic transfer, \( Tr_k \) also detects the critical element \( a_{k-2} \). According to Theorem 5.9, this contradicts the hypothesis that \( Tr_k \) is a monomorphism. The corollary is proved. \( \square \)
7. On Behavior of the Fourth Algebraic Transfer

This section is an application of the previous section into the study of \( Tr_4 \). We refer to \([30], [6], [17]\) for an explanation of the generators of \( \text{Ext}_4^A(\mathbb{F}_2, \mathbb{F}_2) \).

It has been known (see \([17]\)) that the graded module \( \text{Ext}_4^A(\mathbb{F}_2, \mathbb{F}_2) \) is generated by \( h_1h_jh_km, h_icj, d_i, e_i, f_i, g_i+1, p_i, D_3(i), p'_i \) and subject to the relations

\[
\begin{align*}
&h_1h_{i+1} = 0, \quad h_1h_{i+2} = 0, \quad h_{i+1}^3 = h_{i+1}^2h_i, \\
&h_{i+1}^2h_{i+3} = 0, \quad h_icj = 0 \quad \text{for } i = j-1, j, j+2, j+3.
\end{align*}
\]

The following is also numbered as Conjecture 1.10 in the Introduction.

**Conjecture 7.1.** \( Tr_4 \) is a monomorphism that detects all elements in \( \text{Ext}_4^A(\mathbb{F}_2, \mathbb{F}_2) \) except the ones in the three \( Sq^0 \)-families \( \{g_i \mid i \geq 1\} \), \( \{D_3(i) \mid i \geq 0\} \) and \( \{p'_i \mid i \geq 0\} \).

That \( Tr_4 \) does not detect the family \( \{g_i \mid i \geq 1\} \) is due to Bruner–Hâ–Hung \([7]\). Recently, T. N. Nam informed the author about his claim that \( Tr_4 \) does not detect the element \( D_3(0) \).

The following proposition, which is also numbered as Proposition 1.8 in the Introduction, is an attempt to prepare for a proof of Conjecture 7.1.

**Proposition 7.2.** Let \( \{b_i \mid i \geq 0\} \) and \( \{\overline{b}_i \mid i \geq 0\} \) be the \( Sq^0 \)-families in \( \text{Ext}_4^A(\mathbb{F}_2, \mathbb{F}_2) \) with \( b_0 \) one of the usual five elements \( d_0, e_0, p_0, D_3(0), p'_0 \), and \( \overline{b}_0 \) one of the usual two elements \( f_0, g_1 \).

(i) If \( Tr_4 \) detects \( b_n \) for some \( n \geq 1 \), then it detects \( b_i \) for every \( i \geq 1 \).

(ii) If \( Tr_4 \) detects \( \overline{b}_n \) for some \( n \geq 0 \), then it detects \( \overline{b}_i \) for every \( i \geq 0 \).

**Proof.** Although the stems of \( b_2 \) and \( \overline{b}_1 \) cannot be written as \( \delta_2^4(d) \) for some non-negative integer \( d \) (except for \( b_2 = d_2 \) and \( \overline{b}_1 = g_2 \)), it is easy to check by using Lemma 5.6 that they all are 4-spikes.

Following part (b) of Remark 6.8, we can show this proposition by the same argument as given in the proof of Theorem 6.4. Furthermore, as \( Sq^0 \) is a monomorphism in positive stems of \( \text{Ext}_4^A(\mathbb{F}_2, \mathbb{F}_2) \) (see e.g. \([17]\)), the proposition has the strong formulation as in Corollary 6.9.

The proposition is proved. \( \square \)

By means of Proposition 7.2, to prove Conjecture 7.1, it suffices to show that:

1. \( Tr_4 \) detects \( d_0, e_0, f_0, p_0 \);
2. \( Tr_4 \) does not detect \( g_1, D_3(1), p'_1 \); and
3. \( Tr_4 \) is a monomorphism.

The following theorem is also numbered as Theorem 1.9 in the Introduction.

**Theorem 7.3.** \( Tr_4 \) does not detect any element in the three \( Sq^0 \)-families \( \{g_i \mid i \geq 1\} \), \( \{D_3(i) \mid i \geq 0\} \) and \( \{p'_i \mid i \geq 0\} \).

**Outline of the proof.** First, we show that \( \mathbb{F}_2 \otimes_{GL_4} PH_*(BV_4) \) is zero in degree 20. Therefore, \( Tr_4 \) does not detect \( g_1 \) of stem 20 and, by Proposition 7.2, does not detect any element in the \( Sq^0 \)-family \( \{g_i \mid i \geq 1\} \). (Note again that this part of the theorem is due to Bruner–Hâ–Hung \([7]\).)

Second, as the stems of \( D_3(1) \) and \( p'_1 \) are respectively 126 and 142, we focus to the \( GL_4 \)-module \( PH_*(BV_4) \) in degrees 126 and 142. By routine computations, we show that \( PH_*(BV_4) \) has dimension 80 and 285 in degrees 126 and 142, respectively, and further that \( \mathbb{F}_2 \otimes_{GL_4} PH_*(BV_4) \) is of dimension 1 in these two degrees.


Note that, as $Tr_1$ detects the family $\{h_n \mid n \geq 0\}$ (see [28]), the homomorphism of algebras $Tr = \bigoplus_k Tr_k$ detects the subalgebra generated by the family $\{h_n \mid n \geq 0\}$. So, $Tr_4$ definitely sends the two generators of its domain in degrees 126 and 142 to the nonzero elements $h^2_0 h^2_6$ and $h^2_0 h_4 h_7$, respectively. Therefore, the two elements $D_3(1)$ and $p'_1$ of, respectively, stems 126, 142 are not detected by $Tr_4$.

The theorem is proved by combining this fact and Proposition 7.2. □

8. An observation on the fifth algebraic transfer

From Corollary 5.8, the following conjecture naturally comes up.

**Conjecture 8.1.** There are infinitely many degrees in which $Tr_5$ is not an isomorphism.

The fact that $g_n$ is not detected by $Tr_4$ and that $Tr = \bigoplus_k Tr_k$ is a homomorphism of algebras do not imply that $h^i g_n$ is not detected by $Tr_5$. For instance, $h^0_0 g_1 = h^2_0 e_0$ and $h^1_0 g_1 = h^2_0 f_0$ are presumably detected by $Tr_5$, as $e_0$ and $f_0$ are expectedly detected by $Tr_4$.

The purpose of this section is to prove the following, which is also numbered as Theorem 1.11 in the Introduction.

**Theorem 8.2.** If $h_{n+1} g_n$ is nonzero, then it is not detected by $Tr_5$.

**Outline of the proof.** We first observe that, as $Sq^0$ is a homomorphism of algebras, $\{h_{n+1} g_n \mid n \geq 1\}$ is an $Sq^0$-family, that is,

$$(Sq^0)^{n-1}(h_2 g_1) = h_{n+1} g_n,$$

for every $n \geq 1$.

Next, using Lemma 3.3 we easily show that $\text{Stem}(h_2 g_1) = 23$ is not a 5-spike, but $\delta_5(23) = 2 \cdot 23 + 5 = 51$ is. So, by Proposition 3.7,

$$(\widetilde{Sq}^0)^i : PH_*(BV_5)_{23} \to PH_*(BV_5)_{2i \cdot 23 + (2^i - 1)5}$$

is an isomorphism of $GL_5$-modules for every $i \geq 0$.

In addition, a routine computation shows that $PH_*(BV_5)$ is of dimension 1245 in degree 23, and further that

$$\mathbb{F}_2 \otimes_{GL_5} PH_*(BV_5)_{23} = 0.$$

As a consequence, we get

$$\mathbb{F}_2 \otimes_{GL_5} PH_*(BV_5)_{2i \cdot 23 + (2^i - 1)5} = 0,$$

for every $i \geq 0$. So, the domain of $Tr_5$ is zero in the degree that equals

$$\text{Stem}(h_{n+1} g_n) = 2^{n-1} \cdot 23 + (2^{n-1} - 1)5$$

for every $n \geq 1$.

Therefore, if $h_{n+1} g_n$ is nonzero, then it is not detected by $Tr_5$. The theorem is proved. □

**Corollary 8.3.** If $h_{n+1} g_n$ is nonzero for every $n \geq 1$, then there are infinitely many degrees, namely the degrees of $h_{n+1} g_n$ for $n \geq 1$, in which $Tr_5$ is not an epimorphism.

The corollary’s hypothesis is claimed to be true by Lin [16]. So, Conjecture 8.1 is established.
Remark 8.4. As $h_3g_2 = h_5g_1$ (see [30]) and $Sq^0$ is a homomorphism of algebras, Theorem 8.2 also shows that if $h_{n+4}g_n$ is nonzero, then it is not detected by $Tr_5$.

Which elements in $\text{Ext}^4_1(F_2, F_2)$ are detected by $Tr_5$?

This question can be partially answered by using the fact that $Tr = \bigoplus_k Tr_k$ is an algebra homomorphism and the information on elements detected by $Tr_k$ for $k \leq 4$. For instance, $h_3D_3(0) = h_0d_2$ (see [30]) is presumably detected by $Tr_5$, as $h_0$ is detected by $Tr_1$ and $d_2$ is expectedly detected by $Tr_4$ (see Conjecture 6.4).

Based on Theorem 6.7 and concrete calculations, the following conjecture presents some “new” families, which are expectedly detected by $Tr_5$.

Conjecture 8.5. $Tr_5$ detects every element in the $Sq^0$-families initiated by the classes $n, x, h_0g_2, D_1, H_1, h_1D_3(0), h_2D_3(0), Q_3, h_4D_3(0), h_6g_1, h_6g_3$ of stems 31, 37, 44, 52, 62, 64, 67, 76, 83, 92, respectively.

Conjectures 8.4 and 7.4 together with the fact that $Tr = \bigoplus_k Tr_k$ is an algebra homomorphism, predict that $Tr_5$ detects all $Sq^0$-families initiated by the classes of stems $< 125$, except possibly the three families, which are respectively initiated by $Ph_1$, $Ph_2$ and $h_0p'$.

Since $Sq^0(Ph_1) = h_2g_1$, every element of the $Sq^0$-family initiated by $Ph_1$ is not detected by $Tr_5$ (see [28] for $Ph_1$ and Theorem 8.2 for $h_{n+1}g_n$). It has been known that $Tr_5$ does not detect the $Sq^0$-family of exactly one nonzero element $\{Ph_2\}$ (see Remark 5.4). We have no prediction on whether the $Sq^0$-family initiated by $h_0p'$ of stem 69 is detected or not.

9. Final remarks

Remark 9.1. We still do not know whether $Tr_k$ fails to be a monomorphism for $k > 5$.

As in the proof of Theorem 5.9, let $h_n$ also denote the element in $F_2 \otimes PH_*(BV_k)$, whose image under the isomorphism $Tr_1$ is the usual $h_n \in \text{Ext}^4_1(F_2, F_2)$. In the following remark, we will use the product of $\bigoplus_k (F_2 \otimes PH_*(BV_k))$ defined by Singer in [28] and his result that $Tr = \bigoplus_k Tr_k : \bigoplus_k (F_2 \otimes PH_*(BV_k)) \to \text{Ext}_A^4(F_2, F_2)$ is a homomorphism of algebras.
 Remark 9.2. (a) Let $t_5 = h_0^4 h_4 \in (\mathbb{F}_2 \otimes_{GL_5} PH^*(BV_5))_{15}$. Then $Sq^0(t_5) = 0$ and $Tr_5(t_5) = h_0^4 h_4 \neq 0$.

(b) If $t_k \in \mathbb{F}_2 \otimes_{GL_k} PH^*(BV_k)$ is a positive degree element with $Sq^0(t_k) = 0$ and $Tr_k(t_k) \neq 0$, then $Sq^0(h_n t_k) = 0$ and $Tr_{k+1}(h_n t_k) \neq 0$ for every $n$ with $2^n \geq 4(\deg(t_k))^2$.

"Proof of Remark 9.2". Part (a) of this proof proceeds under two hypotheses:

(i) $\mathbb{F}_2 \otimes_{GL_5} PH^*(BV_5)$ has respectively dimension 2 and 1 in degrees 15 and 35.

(This is known by a computer calculation as written above.)

(ii) $d_0 \in \text{Im}(Tr_4)$. (This is a part of Conjecture 1.10. When this paper was being revised, Lê M. Hà privately informed the author that he proved this claim.)

It would be better to write a direct proof in the framework of invariant theory for the fact $Sq^0(h_0^4 h_4) = 0$ in $\mathbb{F}_2 \otimes_{GL_5} PH^*(BV_5)$.

(a) As is well known, $\text{Ext}^{5\lambda + 15}_A(F_2, F_2) = \text{Span}\{h_0^4 h_4, h_1 d_0\}$. Combining the fact that $Tr = \bigoplus_k Tr_k$ is an algebra homomorphism with the one that $h_n$ is in the image of $Tr_1$ for every $n$, and $d_0$ is in the image of $Tr_4$, we conclude that $h_0^4 h_4$ and $h_1 d_0$ are both in the image of $Tr_5$. On the other hand, the domain of $Tr_5$ has dimension 2 in degree 15. So, $Tr_5$ is an isomorphism in degree 15. As $\mathbb{F}_2 \otimes_{GL_5} PH^*(BV_5)$ has respectively dimension 2 and 1 in degrees 15 and 35, there exists a nonzero element $t_5 \in \mathbb{F}_2 \otimes_{GL_5} PH^*(BV_5)$ in degree 15 such that $Sq^0(t_5) = 0$.

Since $Tr_5$ is an isomorphism in degree 15, $Tr_5(t_5) \neq 0$.

Next, we show that $Tr_5(t_5) = h_0^4 h_4$. Indeed, we suppose to the contrary that $Tr_5(t_5) = \lambda h_0^4 h_4 + h_1 d_0$ for some $\lambda \in F_2$. Then, as $Sq^0(h_0^4) = h_1^4$ in $\text{Ext}_A^*(F_2, F_2)$ and $Sq^0$ is an algebra homomorphism, we have

$$Sq^0 Tr_5(t_5) = \lambda Sq^0(h_0^4 h_4) + Sq^0(h_1 d_0) = Sq^0(h_1 d_0) = h_2 d_1.$$ 

Since $Tr_5$ commutes with the squaring operations, we get

$$Tr_5 Sq^0(t_5) = Sq^0 Tr_5(t_5) = h_2 d_1 \neq 0.$$ 

This contradicts the above conclusion that $Sq^0(t_5) = 0$. Therefore,

$$Tr_5(t_5) \neq \lambda h_0^4 h_4 + h_1 d_0$$

for any $\lambda \in F_2$. Combining this with the fact that $Tr_5(t_5) \neq 0$ in $\text{Span}\{h_0^4 h_4, h_1 d_0\}$, we get $Tr_5(t_5) = h_0^4 h_4$.

With ambiguity of notation, we also have $Tr_5(h_0^4 h_4) = h_0^4 h_4 = Tr_5(t_5)$. As $Tr_5$ is an isomorphism in degree 15, we obtain $t_5 = h_0^4 h_4$.

(b) As $Sq^0$ is an algebra homomorphism, we have

$$Sq^0(h_n t_k) = Sq^0(h_n) Sq^0(t_k) = 0.$$ 

On the other hand, as $Tr = \bigoplus_k Tr_k$ is also an algebra homomorphism, we get

$$Tr_{k+1}(h_n t_k) = Tr_1(h_n) Tr_k(t_k) = h_n Tr_k(t_k).$$
As shown in the proof of Lemma 5.3, a consequence of Davis’ Theorem 5.1 claims that, if $Tr_k(t_k) \neq 0$, then $h_n Tr_k(t_k) \neq 0$ for every $n$ with
\[2^n \geq 4(\text{Stem}(Tr_k(t_k)))^2 = 4(\text{deg}(t_k))^2.\]
The remark is proved.

As an immediate consequence, we have

**Corollary 9.3.** (i) $Ker(Sq^0) \cap (F_2 \otimes PH_*(BV_k))$ is nonzero for $k = 5$ and has an infinite dimension for $k > 5$.
(ii) For $k = 5$, $Tr_k$ detects a nonzero element in the kernel of $Sq^0$, and for each $k > 5$, it detects infinitely many nonzero elements in this kernel.

It has been known (see [28], [4]) that $Sq^0$ is injective on $F_2 \otimes PH_*(BV_k)$ for $k \leq 3$.

**Conjecture 9.4.** $Sq^0$ is a monomorphism in positive degrees of $F_2 \otimes PH_*(BV_k)$.
In other words, $Sq^0$ is a monomorphism in positive degrees of $F_2 \otimes PH_*(BV_k)$ if and only if $k \leq 4$.

The following is an analogue of Corollary 6.2 and is related to Corollary 6.10:

**Conjecture 9.5** ($Sq^0$ is eventually isomorphic on the Ext groups). Let $\text{Im}(Sq^0)^i$ denote the image of $(Sq^0)^i$ on $\text{Ext}_A^k(F_2, F_2)$. There is a number $t$ depending on $k$ such that
\[(Sq^0)^{i-t} : \text{Im}(Sq^0)^i \to \text{Im}(Sq^0)^i\]
is an isomorphism for every $i > t$.

In other words, $\text{Ker}(Sq^0)^i = \text{Ker}(Sq^0)^t$ on $\text{Ext}_A^k(F_2, F_2)$ for every $i > t$. As a consequence, any finite $Sq^0$-family in $\text{Ext}_A^k(F_2, F_2)$ has at most $t$ nonzero elements.

The conjecture true for $t = k - 2$?

An observation on the known generators of the Ext groups supports the above conjecture with $t$ much smaller than $k - 2$.

It also leads us to the question on whether $Sq^0$ is an isomorphism on
\[\text{Im}(Sq^0)^t \subset F_2 \otimes PH_*(BV_k)\]
for some $t < k - 2$. (This question has an affirmative answer given by Corollary 6.2 for $t = k - 2$.)

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