PRESCRIBING ANALYTIC SINGULARITIES FOR SOLUTIONS OF A CLASS OF VECTOR FIELDS ON THE TORUS

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Abstract. We consider the operator $L = \partial_t + (a(t) + ib(t))\partial_x$ acting on distributions on the two-torus $T^2$, where $a$ and $b$ are real-valued, real analytic functions defined on the unit circle $T^1$. We prove, among other things, that when $b$ changes sign, given any subset $\Sigma$ of the set of the local extrema of the local primitives of $b$, there exists a singular solution of $L$ such that the $t$–projection of its analytic singular support is $\Sigma$; furthermore, for any $\tau \in \Sigma$ and any closed subset $F$ of $T^1_x$ there exists $u \in D'(T^2)$ such that $Lu \in C^\omega(T^2)$ and $\text{sing supp}_A(u) = \{\tau\} \times F$. We also provide a microlocal result concerning the trace of $u$ at $t = \tau$.

1. Introduction

There are many works about hypoellipticity, local or global, smooth or analytic. In the global case we cite [B], [BCM], [BNZ1], [BNZ2], [DGY], [G], [GW], [Hi], [Ho], and [M]. In these papers, in order to prove that some of the operators under study are not hypoelliptic, singular solutions are constructed. By a singular solution of an operator $L$, we mean a nonanalytic (nonsmooth) distribution $u$ such that $Lu$ is analytic (smooth). However, not much is known about the location and nature of the singularities. The main goal of the present work is to help fill this gap for a class of vector fields on the two-torus in the global analytic setting as we describe below.

Let $c = a + ib$, where $a$ and $b$ are real-valued, real analytic functions defined on the unit circle $T^1$. We consider the operator

$$L = \partial_t + c(t)\partial_x$$

acting on distributions on the two-torus $T^2$.

We recall that $L$ is said to be globally analytic hypoelliptic (GAH) if one has $u \in C^\omega(T^2)$ whenever $u \in D'(T^2)$ and $Lu \in C^\omega(T^2)$.

In [B] one finds a characterization of the global analytic hypoellipticity of the operator $L$; in particular, it is proved there that if $b \not\equiv 0$, then $L$ is globally analytic hypoelliptic if and only if the function $b$ does not change sign (the Nirenberg-Treves condition ($P$)); see [T2]), or equivalently, no primitive of $b$ defined in $\mathbb{R}$ has local extrema.
We emphasize that the method applied in [3] does not yield all the possible singularities. We now mention two examples of this difficulty. Assume that \( b \neq 0 \); let \( B \) be one of its primitives in \( \mathbb{R} \) and suppose that the origin is a local minimum of \( B \). If \( t_1, t_2 \in (0, 2\pi) \) are points of local maximum of \( B|_{[0, 2\pi]} \), with \( B(t_1) < B(t_2) \), then the singular solution constructed in [3] is real analytic when \( t = t_1 \). Similarly, if \( t_1, t_2 \in (0, 2\pi) \) are points of global maximum of \( B|_{[0, 2\pi]} \), then the solution constructed in [3] is singular at both points \( (t_1, 0) \) and \( (t_2, 0) \).

In fact, if \( b \neq 0 \) and \( t_* \) is a point of local maximum of \( B \), then in Lemma 2.4 we construct \( u \in \mathcal{D}'(\mathbb{T}^2) \) with \( \text{sing supp}_A(u) = \{(t_*, 0)\} \) and \( Lu = C^\omega(\mathbb{T}^2) \) and, furthermore, \( u(t_*, x) = \delta_+(x) = \sum_{n \geq 1} e^{inx} \). More generally (see Theorem 2.2), given, instead of \( \delta_+ \), a distribution \( \tilde{v}(x) = \sum_{n \geq 1} v_n e^{inx} \), there is \( u \in \mathcal{D}'(\mathbb{T}^2) \) such that its analytic singular support is \( \text{sing supp}_A(u) = \{t_*\} \times \text{sing supp}_A(v) \) and \( Lu \in C^\omega(\mathbb{T}^2) \). Moreover (see Theorem 2.2), given any nonempty closed subset \( F \subset \mathbb{T}^1 \), we construct \( u \in \mathcal{D}'(\mathbb{T}^2) \) such that \( Lu \in C^\omega(\mathbb{T}^2) \) and \( \text{sing supp}_A(u) = \{t_*\} \times F \); in fact (see Remark 2.4), the analytic wave front set of the trace \( u(t_*, \cdot) \) is equal to \( F \times \mathbb{R}_- \).

We also have the analogous results for the local minima. For instance, if \( s_* \) is such a point and \( F \subset \mathbb{T}^1 \) is a nonempty closed set, then there is \( u \in \mathcal{D}'(\mathbb{T}^2) \) with \( \text{sing supp}_A(u) = \{s_*\} \times F \) and \( Lu \in C^\omega(\mathbb{T}^2) \) and, furthermore, the analytic wave front set of the trace \( u(s_*, \cdot) \) is equal to \( F \times \mathbb{R}_- \).

Theorems 2.9 and 2.10 put together the above results in a more complete way. As an example, suppose \( b \neq 0 \) and let \( B : \mathbb{R} \to \mathbb{R} \) be a real analytic function such that \( B' = b \). Let \( \Sigma \) be a nonempty subset of the set of points in \([0, 2\pi]\) which are local extrema of \( B \). Then there exists a singular solution \( u \) such that \( \text{sing supp}_A(u) = \Sigma \times \{0\} \).

Our results are sharp. Indeed, it follows from [11] and also from [3] that if \( t \) is neither a local maximum nor a local minimum, then every \( u \in \mathcal{D}'(\mathbb{T}^2) \) such that \( Lu \in C^\omega(\mathbb{T}^2) \) is real analytic at \((t, x)\) for arbitrary \( x \in \mathbb{T}^1 \). Furthermore, concerning the microlocal aspect of the singularities, the results of [3] imply that, for instance, in the case of a local maximum \( t_* \), the trace at \( t = t_* \) of any singular solution has an analytic wave-front set contained in \( \mathbb{T}^1 \times \mathbb{R}_+ \).

We stress that besides constructing solutions with prescribed singularities, the present work yields a new, simpler, proof of the the hardest case of the main result about global analytic hypoellipticity in [3], that is, when \( b \neq 0 \). In both works the proofs rely on the method of stationary phase, but now we make use of a suitable function \( \psi \) (see the proof of Lemma 2.4) whose role is to simplify the application of this method, similar to what was done in [3].

We point out that our interest lies primarily in the nonreal case; for the sake of completeness, we also provide, in section 3, information about singularities of solutions when \( b = 0 \).

For later use, we define \( c_0 = \frac{1}{\pi} \int_0^{2\pi} c(t)dt, a_0 = \Re c_0, \) and \( b_0 = \Im c_0 \).

2. Construction of singular solutions in the case \( b \neq 0 \)

Suppose that \( L = \partial_t + (a(t) + ib(t))\partial_x \) is not GAH and that \( b \neq 0 \); equivalently, suppose that \( b \) changes sign. Let \( B : \mathbb{R} \to \mathbb{R} \) be a primitive of \( b \). Then the function has at least one point of local maximum. In fact, there exist \( r \in \mathbb{Z}_+ \) and points \( t_0, \ldots, t_r \), with \( t_0 < t_1 < \cdots < t_r \), and \( t_r - t_0 < 2\pi \), such that any point of local maximum of \( B \) is equal to \( t_k + 2j\pi \), for some \( k = 0, \ldots, r \) and some \( j \in \mathbb{Z} \).
Similarly we have points of local minimum $s_0, \ldots, s_r$ such that, by setting $t_{r+1} \equiv t_0 + 2\pi$ and $s_{r+1} \equiv s_0 + 2\pi$, we may write

$$s_0 < t_0 < s_1 < t_1 < s_2 < \cdots < s_r < t_r < s_{r+1} < t_{r+1}.$$  

We will use the same notation, namely, $t_j$ and $s_j$, for the corresponding zeros of $b$ in $\mathbb{T}^1$.

The following crucial lemma will be proved later in this section.

**Lemma 2.5.** Suppose that $L$ is not GAH and that $b \not\equiv 0$. Let $t_* \in \{t_0, \ldots, t_r\}$. Then there exists $u \in \mathcal{D}'(\mathbb{T}^1)$ with $\text{sing supp}_A(u) = \{(t_*, 0)\}$ and $Lu \in C^\omega(\mathbb{T}^2)$. More precisely, $u$ can be chosen so that $u(t_*, x) = \delta_+(x) \doteq \sum_{n \geq 1} e^{inx}$.

Assuming the previous lemma we can prove

**Theorem 2.2.** Suppose that $L$ is not GAH and $b \not\equiv 0$. Let $t_* \in \{t_0, \ldots, t_r\}$. Given a sequence $(v_n)_{n \in \mathbb{Z}}$ such that $v_n$ is tempered in $n \in \mathbb{Z}$, but not exponentially decaying as $n \to +\infty$, although exponentially decaying in the opposite direction, then there exists $v \in \mathcal{D}'(\mathbb{T}^1) \setminus C^\omega(\mathbb{T}^2)$ such that $\tilde{v}(t_*, n) = v_n$, $Lv \in C^\omega(\mathbb{T}^2)$, and $\text{sing supp}_A(v) = \{t_*\} \times \text{sing supp}_A(\tilde{v})$, where $\tilde{v}(x) \doteq \sum_{n \in \mathbb{Z}} v_n e^{inx}$.

**Proof.** It suffices to take $u$ as in Lemma 2.1 and define

$$v(t, x) = \sum_{n \leq 0} v_n e^{inx} + \sum_{n \geq 1} \hat{u}(t, n) v_n e^{inx}.$$  

In Lemma 4.1 we prove that for any given nonempty closed subset $F$ of $\mathbb{T}^1$, there exists $v_o \in \mathcal{D}'(\mathbb{T}^1)$ such that $\text{sing supp}_A(v_o) = F$ and $v_o(x) = \sum_{n \geq 1} \hat{v}_o(n) e^{inx}$. We obtain

**Theorem 2.3.** Suppose that $L$ is not GAH and $b \not\equiv 0$. Let $t_* \in \{t_0, \ldots, t_r\}$ and let $F$ be a nonempty closed subset of $\mathbb{T}^1$. Then there exists $v \in \mathcal{D}'(\mathbb{T}^2) \setminus C^\omega(\mathbb{T}^2)$ such that $Lv \in C^\omega(\mathbb{T}^2)$ and $\text{sing supp}_A(v) = \{t_*\} \times F$. Furthermore, $\tilde{v}(t_*, n)$ is tempered in $n \in \mathbb{Z}$ and decays exponentially as $n \to -\infty$.

**Proof.** Take $v_o \in \mathcal{D}'(\mathbb{T}^1)$ as in Lemma 4.1 corresponding to the given closed subset $F$. Define $v_n = \tilde{v}_o(n)$ for $n \geq 1$ and $v_n = 0$ otherwise. Applying Theorem 2.2 the result follows immediately.

**Remark 2.4.** Note that the singularity of $v$ in Theorem 2.2 and Theorem 2.3 is due to the positive frequencies $n$. Moreover, any singular solution $w$ of $L$ such that $t_*$ is in the $t-$projection of its analytic singular support has $\tilde{w}(t_*, n)$ decaying exponentially as $n \to -\infty$; indeed, this follows from the analogue of Lemma 2.1 with negative indices. Thus, the analytic wave front set of the trace $w(t_*, \cdot)$ is equal to $F \times \mathbb{R}_+$.

We now proceed to state, without proof, the analogous results for the case of local minima.

**Lemma 2.5.** Suppose that $L$ is not GAH and that $b \not\equiv 0$. Let $s_* \in \{s_0, \ldots, s_r\}$. Then there exists $u \in \mathcal{D}'(\mathbb{T}^2)$ satisfying $\text{sing supp}_A(u) = \{(s_*, 0)\}$ and $Lu \in C^\omega(\mathbb{T}^2)$. More precisely, $u$ can be chosen so that $u(s_*, x) = \delta_-(x) \doteq \sum_{n \leq -1} e^{inx}$.
Theorem 2.6. Suppose that $L$ is not GAH and $b \neq 0$. Let $s_n \in \{s_0, \ldots, s_r\}$. Given a sequence $(w_n)_{n \in \mathbb{Z}}$ such that $w_n$ is tempered, but not exponentially decaying as $n \to -\infty$, then there exists $w \in \mathcal{D}'(\mathbb{T}^2) \setminus C^\infty(\mathbb{T}^2)$ such that $\hat{w}(s_n, n) = w_n$, $Lw \in C^\omega(\mathbb{T}^2)$, and $\text{sing supp}_A(w) = \{s_n\} \times \text{sing supp}_A(\hat{w})$, where $\hat{w}(x) = \sum_{n \in \mathbb{Z}} w_n e^{inx}$.

Theorem 2.7. Suppose that $L$ is not GAH and $b \neq 0$. Let $s_n \in \{s_0, \ldots, s_r\}$ and let $F$ be a nonempty closed subset of $\mathbb{T}^1$. Then there exists $w \in \mathcal{D}'(\mathbb{T}^2) \setminus C^\infty(\mathbb{T}^2)$ such that $Lw \in C^\omega(\mathbb{T}^2)$ and $\text{sing supp}_A(w) = \{s_n\} \times F$. Furthermore, $\hat{w}(s_n, n)$ is exponentially decaying as $n \to +\infty$.

Before proceeding to our main results (Theorems 2.9 and 2.10), we need the following

Lemma 2.8. For each $n \in \mathbb{Z}$ let $A_n = (a_{ij}(n))$ be a $k \times k$ matrix such that $a_{ii}(n) = 1$ and, for $i \neq j$, $a_{ij}(n)$ is exponentially decaying as $n \to \pm \infty$.

If $B_n$ is a $k \times 1$ matrix whose entries are tempered in $|n|$, then the equation $A_n x_n = B_n$ has, for all sufficiently large $|n|$, a unique solution. Furthermore, the entries of $x_n$ are tempered in $|n|$.

Proof. All we need is to observe that det $A_n = 1 + \delta_n$ with $\delta_n$ decaying exponentially as $n \to \pm \infty$. \qed

Theorem 2.9. Suppose that $L$ is not GAH and $b \neq 0$. Let $u_j(n), j = 0, \ldots, 2r+1$, be tempered in $\mathbb{Z}$ and such that for $j = 0, \ldots, r$, $u_j(n)$ decays exponentially as $n \to -\infty$, and for $j = r+1, \ldots, 2r+1$, $u_j(n)$ decays exponentially as $n \to +\infty$.

Then there exists $u \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lu \in C^\omega(\mathbb{T}^2)$ and $\hat{u}(t_j, n) = u_j(n)$ for $j = 0, \ldots, r$, and $\hat{u}(s_{j-r}, n) = u_j(n)$, for $j = r+1, \ldots, 2r+1$, for all $n$. Moreover,

$$\text{sing supp}_A(u) \subseteq \bigcup_{j=0}^r \{t_j\} \times \text{sing supp}_A(\tilde{u}_j) \cup \bigcup_{j=1}^{r+1} \{s_j\} \times \text{sing supp}_A(\tilde{u}_j),$$

where $\tilde{u}_j(x) = \sum_{n \in \mathbb{Z}} u_j(n) e^{inx}$, $j = 0, \ldots, 2r+1$.

In particular, if $I$ is the set of indices $j \in \{0, \ldots, r\}$ such that $u_j(n)$ does not decay exponentially as $n \to +\infty$ and $J$ is the set of indices $j \in \{r+1, \ldots, 2r\}$ such that $u_j(n)$ does not decay exponentially as $n \to -\infty$, then

$$\text{sing supp}_A(u) = \bigcup_{j \in I} \{t_j\} \times \text{sing supp}_A(\tilde{u}_j) \cup \bigcup_{j \in J} \{s_j\} \times \text{sing supp}_A(\tilde{u}_j).$$

Proof. For $j = 0, \ldots, r$, we use Lemma 2.11 and obtain $v_j \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lv_j \in C^\omega(\mathbb{T}^2)$, $\hat{v}_j(t_j, n) = 1$, for all $n \geq 1$, $\hat{v}_j(t_j, n)$ is exponentially decaying when $n \to -\infty$, and, for $t \neq t_j$, $\hat{v}_j(t, n)$ is exponentially decaying as $|n| \to +\infty$.

For $j = r+1, \ldots, 2r+1$, we use Lemma 2.20 and obtain $w_j \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lw_j \in C^\omega(\mathbb{T}^2)$, $\hat{w}_j(s_{j-r}, n) = 1$, for all $n \leq -1$, $\hat{w}_j(s_{j-r}, n)$ is exponentially decaying when $n \to +\infty$, and, for $t \neq s_{j-r}$, $\hat{w}_j(t, n)$ is exponentially decaying as $|n| \to +\infty$.\qed
Let $A_n = (a_{ij}(n))$ be given by
\[
a_{ij}(n) = \begin{cases} 
\hat{v}_i(t_j, n), & i = 0, \ldots, r, \quad j = 0, \ldots, r, \\
\hat{v}_i(s_{j-r}, n), & i = 0, \ldots, r, \quad j = r + 1, \ldots, 2r + 1, \\
\bar{w}_i(t_j, n), & i = r + 1, \ldots, 2r + 1, \quad j = 0, \ldots, r, \\
\bar{w}_i(s_{j-r}, n), & i = r + 1, \ldots, 2r + 1, \quad j = r + 1, \ldots, 2r + 1.
\end{cases}
\]

Define $B_n = (b_{11}(n))$, where $b_{11}(n) = u_z(n)$. Let $X_n = (\alpha_{11}(n))$ be the solution of $A_n X_n = B_n$, for all $|n| > n_0$, for some $n_0$ as in Lemma 2.8. Then $v \in D'(\mathbb{T}^2)$ defined by
\[
\hat{v}(t, n) = \sum_{i=0}^{r} \alpha_{i1}(n) \hat{v}_i(t, n) + \sum_{i=r+1}^{2r+1} \alpha_{i1}(n) \bar{w}_i(t, n)
\]
satisfies $Lv \in C^\infty(\mathbb{T}^2)$, $\hat{v}(t, n) = u_j(n)$ for $j = 0, \ldots, r$, and $\hat{v}(s_{j-r}, n) = u_j(n)$, for $j = r + 1, \ldots, 2r + 1$, for $|n| > n_0$.

Let
\[
z_j = \begin{cases} 
e^{it_j}, & j = 0, \ldots, r, \\
e^{is_{j-r}}, & j = r + 1, \ldots, 2r + 1,
\end{cases}
\]
and, for $|n| \leq n_0$, let $p_n(z)$ be a polynomial such that $p_n(z_j) = u_j(n)$, $j = 0, \ldots, 2r + 1$.

Let $u(t, x) = v(t, x) + \sum_{|n| \leq n_0} p_n(e^{it}) e^{inx}$. Then $u \in D'(\mathbb{T}^2)$, $Lu \in C^\infty(\mathbb{T}^2)$ and $\hat{u}(t_1, n) = u_j(n)$ for $j = 0, \ldots, r$, and $\hat{u}(s_{j-r}, n) = u_j(n)$, for $j = r + 1, \ldots, 2r + 1$, for all $n$.

We also have

**Theorem 2.10.** Suppose that $L$ is not GAH and $b \neq 0$. Then, given $\{p_1, \ldots, p_k\} \subset \{t_0, \ldots, t_r, s_1, \ldots, s_r\}$ and nonempty closed subsets $F_j \subset \mathbb{T}^1$, $j = 1, \ldots, k$, there exists $u \in D'(\mathbb{T}^2)$ such that $Lu \in C^\infty(\mathbb{T}^2)$ and $\text{sing supp}_A(u) = \bigcup_{j=1}^{k} (p_j) \times F_j$.

Before we present the proof of Lemma 2.11 we need one more result.

**Lemma 2.11.** Let $c = a + ib$ be a continuous function defined on a bounded interval $J$, where $a$ and $b$ are real valued and $b$ has only isolated zeros. Let $C = A + iB$, where $A$ and $B$ are primitives of $a$ and $b$, respectively. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of continuous functions defined on $J$. Let $t_0 \in J$ and set
\[
t_1 = \inf \{t \in J; B \text{ is nondecreasing on } [t, t_0] \}
\]
and
\[
t_2 = \sup \{t \in J; B \text{ is nonincreasing on } [t_0, t] \}.
\]
Then all the solutions of the ODE
\[
(2.1) \quad \frac{d}{dt} v_n(t) + i c(t)v_n(t) = f_n(t)
\]
satisfy the following:

1. If $v_n(t_0)$ is tempered in $n$ and if $f_n(t)$ is tempered in $n$ uniformly in $t$, then $v_n(t)$ is tempered in $n$ uniformly in $[t_1, t_2] \cap J$.
2. If $v_n(t_0)$ is exponentially decaying in $n$ and if $f_n(t)$ is exponentially decaying in $n$ uniformly in $t$, then $v_n(t)$ is exponentially decaying in $n$ uniformly in $[t_1, t_2] \cap J$. 


(3) Suppose $t_0$ is not a minimum. If $v_n(t_0)$ is tempered in $n$ and if $f_n(t)$ is exponentially decaying in $n$ uniformly in $t$, then, for any sufficiently small $\varepsilon > 0$, $v_n(t)$ is exponentially decaying in $n$ uniformly in $\mathbb{R}^n$.

Proof. If $v_n$ is a solution of (2.1), then

$$v_n(t) = e^{-in(C(t)-C(t_0))}v_n(t_0) + \int_{t_0}^{t} e^{-in(C(t)-C(s))} f_n(s) \, ds, \quad t \in J.$$  

Note that for $t \in [t_1, t_2] \cap J$ we have $B(t) \leq B(s)$ whenever $s \in [t, t_0] \cap J$ or $s \in [t_0, t] \cap J$.

We prove statements (1) and (2). We have

$$|v_n(t)| \leq |v_n(t_0)| + \sup_{s \in J} |f_n(s)| \ell(J),$$

for all $t \in [t_1, t_2] \cap J$, $n \in \mathbb{N}$, where $\ell(J)$ is the length of $J$. It follows that $v_n(t)$ is tempered in $n$ for $t \in [t_1, t_2] \cap J$ in case (1) and exponentially decaying in $n$ for $t \in [t_1, t_2] \cap J$ in case (2).

We now prove (3). Since $b$ has only isolated zeros, $B$ is strictly increasing in $[t_1, t_0] \cap J$ and strictly decreasing in $[t_0, t_2] \cap J$ (provided the intervals under consideration are nonempty).

Let $\varepsilon_0 > 0$ such that $J \setminus (t_0 - \varepsilon_0, t_0 + \varepsilon_0) \neq \emptyset$. Let $0 < \varepsilon < \varepsilon_0$; if $t_0 - \varepsilon \in J$ we set $\eta_1 = B(t_0) - B(t_0 - \varepsilon) > 0$. If $t \in J \cap [t_1, t_0 - \varepsilon]$, and $n \in \mathbb{N}$, we have

$$|v_n(t)| \leq e^{-n\eta_1} |v_n(t_0)| + \sup_{s \in J} |f_n(s)| \ell(J),$$

Similarly, if $t \in J \cap [t_0 + \varepsilon, t_2]$, we have $\eta_2 \equiv B(t_0) - B(t_0 + \varepsilon) > 0$, and

$$|v_n(t)| \leq e^{-n\eta_2} |v_n(t_0)| + \sup_{s \in J} |f_n(s)| \ell(J).$$

The result follows since $e^{-n\eta_j} |v_n(t_0)|$ is exponentially decaying in $n$, $j = 1, 2$.  \[ \Box \]

Remark 2.12. If $t_1$ is an interior point of $J$ and $f_n$ and $v_n$ satisfy the assumptions of (2) (respectively, (3)) in the previous lemma, then $v_n(t)$ is exponentially decaying in $n$ for a slightly larger interval $[s_1, t_2] \cap J$ (respectively, the larger set $[s_1, t_2] \cap J \setminus (t_0 - \varepsilon, t_0 + \varepsilon)$) having $t_1$ in its interior. Indeed, there exists $\varepsilon_1 > 0$ such that $e^{n\varepsilon_1} f_n(t)$ is still exponentially decaying in $n$ and uniformly in $t$. Thus, if $s_1 < t_1$ is such that $B(t) - B(s) \leq \varepsilon_1$ for all $s, t \in [s_1, t_1] \cap J$ we get

$$\left| \int_{t_0}^{t} e^{-in(C(t)-C(s))} f_n(s) \, ds \right| \leq e^{n\varepsilon_1} \sup_{s \in J} |f_n(s)| \ell(J).$$

A similar result holds for $t_2$.  \[ \Box \]
Remark 2.13. Results similar to the ones in Lemma 2.11 and in Remark 2.12 hold when $n \leq 0$ and $t_1$ and $t_2$ are defined instead by

$$t_1 = \inf \{ t \in J ; B \text{ is nonincreasing on } [t, t_0] \}$$

and

$$t_2 = \sup \{ t \in J ; B \text{ is nondecreasing on } [t_0, t] \}.$$
We begin by defining \( v_n(t) \) as the right-hand side of (2.9) (or (2.10)) with \( \tilde{f}(\cdot, n) \) replaced by a piecewise continuous function \( g_n \) given by

\[
g_n(t) = e^{i\psi(t)} \left[ C_n \chi_{[0,t_1]} + D_n \chi_{[t_r,2\pi]} \right],
\]

where

\[
\psi(t) = \begin{cases} a_0 t - A(t) + \frac{a_0}{2} \sin(2t - s_1 - s_{r+1}), & \text{if } s_{r+1} - s_1 = \pi \\
- \frac{a_0}{\cos((s_{r+1} - s_1)/2)} \sin(t - s_1), & \text{if } s_{r+1} - s_1 \neq \pi
\end{cases}
\]

and

\[
\psi(t) = \begin{cases} a_0 t - A(t) + \frac{a_0}{2} \sin(2t - s_1 - s_{r+1}), & \text{if } s_{r+1} - s_1 = \pi \\
- \frac{a_0}{\cos((s_{r+1} - s_1)/2)} \sin(t - s_1 + s_{r+1}), & \text{if } s_{r+1} - s_1 \neq \pi
\end{cases}
\]

We now define \( C_n \) and \( D_n \) by the following equations:

\[
C_n \int_0^{t_1} e^{i\psi(s)} ds = -1
\]

and

\[
D_n \int_{t_r}^{2\pi} e^{i\psi(s)} ds = 1.
\]

We claim that both \( C_n \) and \( D_n \) are exponentially decaying as \( n \to \infty \), uniformly in \( t \in \mathbb{T}^1 \). We prove this for \( C_n \) employing the stationary phase method, as in [Sj].

Let

\[
\beta(s) = -i(A(s) + B(s) + \psi(s)) + \varphi(s) - B(s_1) + i(A(s_1) + \psi(s_1))
\]

\[
= B(s) + \varphi(s) - i(A(s) + \psi(s)) - B(s_1) + i(A(s_1) + \psi(s_1)).
\]

Hence, \( \beta(s_1) = 0 \).

The function \( \beta \) has a holomorphic extension to a complex neighborhood \( V_\delta = \{ z = x + iy : |x - s_1| < \delta, |y| < \delta \} \) of \( s_1 \), given by

\[
\tilde{\beta}(z) = B(z) + \varphi(z) - i(A(z) + \psi(z)) - B(s_1) + i(A(s_1) + \psi(s_1)).
\]

We have

\[
\tilde{\beta}'(s_1) = b(s_1) + \varphi'(s_1) - i(a(s_1) + \psi'(s_1)) = 0
\]

and

\[
\tilde{\beta}''(s_1) = b'(s_1) + \varphi''(s_1) - i(a'(s_1) + \psi''(s_1))
\]

\[
= b'(s_1) + K(1 - \cos(s_1 - s_{r+1})) - i(a'(s_1) + \psi''(s_1)).
\]

By taking \( \delta > 0 \) small enough, we may suppose that \( s_1 \) is the only critical point of \( \tilde{\beta} \) in \( V_\delta \), and, for sufficiently large \( K > 0 \), we have \( \tilde{\beta}''(s_1) \neq 0 \).

If \( t \in I_\delta = [s_1 - \delta, s_1 + \delta] \), then

\[
\Re \tilde{\beta}(t) = B(t) - B(s_1) + \varphi(t) \geq B(t) - B(s_1) \geq 0
\]

if we take a smaller \( \delta \), if necessary, since then \( s_1 \) is the unique point of global minimum of \( B(t) - B(s_1) \), \( t \in I_\delta \). Finally, taking \( 0 < \delta < (s_{r+1} - s_1)/2 \) we have \( \varphi(t) > 0 \) if \( t \in I_\delta, t \neq s_1, \varphi(s_1) = 0 \), and the same is true for \( \Re \tilde{\beta} \).
We may now apply the stationary phase theorem (see [S]) to get that
\[
\int_{I_k} e^{-n\beta(s)} \, ds = \sqrt{2\pi} (\beta''(0))^{-1/2} n^{-1/2} (1 + O(n^{-1})).
\]

Let
\[
\eta = K (1 - \cos \delta) \min \{1 - \cos(s_1 - s_{r+1} - \delta), 1 - \cos(s_1 - s_{r+1} + \delta)\}.
\]
We have \(\eta > 0\) and if \(t \in [0, t_1] \setminus I_k\), then
\[
\Re \beta(t) \geq \varphi(t) \geq \eta.
\]
Thus,
\[
\left| \int_{[0, t_1] \setminus I_k} e^{-n\beta(s)} \, ds \right| \leq 2\pi e^{-n\eta}.
\]

Hence,
\[
\int_0^{t_1} e^{in(C(s) + \psi(s) + i\varphi(s))} \, ds
\]
\[
= \sqrt{2\pi} e^{-n(B(s_1) - i(A(s_1) + \psi(s_1)))(\beta''(0))^{-1/2} n^{-1/2} (1 + O(n^{-1}))}
\]

which grows exponentially with \(n \geq 1\) since \(B(s_1) < 0\).

Let
\[
h_{1,n}(t) = C_n \chi_1(t) e^{i\varphi(t) + i\psi(t)},
\]
where \(\chi_1\) is a \(C^\infty\) \(2\pi\)-periodic function satisfying
\[
(1) \quad 0 \leq \chi_1 \leq 1,
\]
\[
(2) \quad \chi_1(t) = 0 \text{ for } t_1 \leq t \leq 2\pi,
\]
\[
(3) \quad \chi_1(t) = 1 \text{ for } t \in J_1 = [s_1 - \delta_1, s_1 + \delta_1] \subset I_k.
\]

Clearly,
\[
\int_{I_{k-1}}^{t_k} e^{i\varphi(C(s))} h_{1,n}(s) \, ds = \int_{t_r}^{2\pi} e^{i\varphi(C(s))} h_{1,n}(s) \, ds = 0, \quad 2 \leq k \leq r.
\]

By (2.11) we have
\[
\int_0^{t_1} e^{i\varphi(C(s))} h_{1,n}(s) \, ds = \alpha_n - 1
\]
with
\[
\alpha_n \doteq \int_0^{t_1} e^{i\varphi(C(s))} (h_{1,n}(s) - g_n(s)) \, ds
\]
\[
= C_n \int_0^{t_1} e^{i\varphi(C(s))} (\chi_1(s) - 1) \, ds.
\]

Thus
\[
\left| \int_0^{t_1} e^{i\varphi(C(s))} (\chi_1(s) - 1) \, ds \right|
\]
\[
= \left| \int_{[0, t_1] \setminus I_k} e^{i\varphi(C(s))} (\chi_1(s) - 1) \, ds \right|
\]
\[
\leq \int_{[0, t_1] \setminus I_k} e^{-n\Re \beta(s)} \, ds \leq 2\pi e^{-n\eta},
\]

and, therefore, \(\alpha_n\) decays exponentially.
Setting \(\lambda_n \equiv 1 - \alpha_n\), we see that \(|\lambda_n|\) is bounded above and below by positive constants and (2.12) can be rewritten as

\[
\int_0^{t_1} e^{inC(s)} h_{1,n}(s) ds = -\lambda_n.
\]

Defining

\[
h_{r,n}(t) = D_n \chi_r(t)e^{in(\psi(t)+i\varphi(t))},
\]

where \(\chi_r\) is a \(C^\infty 2\pi\)-periodic function satisfying

1. \(0 \leq \chi_r \leq 1\),
2. \(\chi_r(t) = 0\) for \(0 \leq t \leq t_r\),
3. \(\chi_r(t) = 1\) for \(t\) near \(s_r\),

we have

\[
\int_{t_{k-1}}^{t_k} e^{inC(s)} h_{r,n}(s) ds = 0, \quad 1 \leq k \leq r.
\]

In the same fashion as above one can prove that

\[
\mu_n = \int_{t_r}^{2\pi} e^{in(C(s) - C(2\pi))} h_{r,n}(s) ds
\]

has its absolute value bounded above and below by positive constants.

Setting \(h_n = h_{1,n}/\lambda_n + h_{r,n}/\mu_n\), we obtain that equations (2.7), (2.8), and (2.9) are satisfied with \(\tilde{f}(\cdot, n)\) replaced by \(h_n\). Furthermore,

\[
|h_n| \leq C(|h_{1,n}| + |h_{2,n}|) \leq C \left( |e^{in(\psi+i\varphi)} C_n \chi_1| + |e^{in(\psi+i\varphi)} D_n \chi_r| \right) \leq C e^{-\sigma n} (|C_n| + |D_n|),
\]

(2.13)

hence, \(h_n\) is uniformly exponentially decaying.

Let \(m = \min_{[0,2\pi]} B\) and \(M = \max_{[0,2\pi]} B\). Note that \(M \geq 0\). Fix \(\sigma > M - m\) and for each \(n \in \mathbb{N}\) let \(\tilde{f}_n\) be a partial sum of the Fourier series of \(h_n\) satisfying

\[
||\tilde{f}_n - h_n||_{C^0} \leq e^{-\sigma n}.
\]

(2.14)

By (2.13) and our choice of \(\tilde{f}_n\), we see that \(\tilde{f}_n\) is uniformly exponentially decaying as \(n \to +\infty\).

Let \(\tilde{u}_n(t)\) represent the right-hand side of (2.3) with \(\tilde{f}(s, n)\) replaced by \(\tilde{f}_n(s)\). We have

\[
\tilde{u}_n(t_0) = (e^{2\pi inc_0} - 1)^{-1} \int_0^{2\pi} e^{inC(s)} \tilde{f}_n(s) ds
\]

\[
= 1 + (e^{2\pi inc_0} - 1)^{-1} \int_0^{2\pi} e^{inC(s)} \left[ \tilde{f}_n(s) - h_n(s) \right] ds = 1 + \gamma_n, \quad \text{say}.
\]

Since \(\sigma > -m\), \(b_0 < 0\), and

\[
|\gamma_n| \leq |e^{2\pi inc_0} - 1|^{-1} \int_0^{2\pi} e^{-nB(s)} \left| \tilde{f}_n(s) - h_n(s) \right| ds
\]

\[
\leq \frac{1}{e^{-2\pi b_0} - 1} \int_0^{2\pi} e^{-nm} e^{-\sigma s} ds \leq \frac{2\pi}{e^{-2\pi b_0} - 1} e^{-(m+\sigma)n},
\]

we see that \(\gamma_n\) is exponentially decaying as \(n \to +\infty\).
For \( k = 1, \ldots, r \),
\[
\tilde{u}_n(t_k) = e^{-in\Theta(t_k)} \left[ \int_0^{t_k} e^{i\pi(n \Theta(s))} \left( \tilde{f}_n(s) - h_n(s) \right) ds + (e^{i2\pi n\sigma} - 1)^{-1} \int_0^{2\pi} e^{i\pi(n \Theta(s))} \left( \tilde{f}_n(s) - h_n(s) \right) ds \right]
\]
satisfies, as one can easily check,
\[
|\tilde{u}_n(t_k)| \leq 2\pi \frac{1}{1 - e^{2\pi b_0}} e^{(M - m - \sigma) \pi},
\]
which is also exponentially decaying as \( n \to +\infty \).

Let \( f_n = \tilde{f}_n/(1 + \gamma_n) \). Since \( f_n \) is uniformly exponentially decaying as \( n \to +\infty \),
\[
f(t, x) \doteq \sum_{n \geq 1} f_n(t) e^{i\pi(n \Theta(x))}
\]
defines a real analytic function on \( \mathbb{T}^2 \) whose partial Fourier coefficients are given by \( \hat{f}(t, n) = f_n(t) \). Thus, if \( u_n(t) \) stands for the right-hand side of (2.2), we get that \( u_n(t_0) = 1 \) and \( u_n(t_k) \) is exponentially decaying as \( n \to +\infty \), for \( k = 1, \ldots, r \). Moreover, by Lemma 2.11
\[
u(t, x) \doteq \sum_{n \geq 1} u_n(t) e^{i\pi(n \Theta(x))}
\]
defines an element of \( D'(\mathbb{T}^2) \). A simple computation shows that the partial Fourier coefficients of \( u \), which are given by \( \hat{u}(t, n) = u_n(t) \), satisfy (2.2), whence, \( Lu = f \).

Applying Lemma 2.11 one more time we see that the analytic singular support of \( u \) is contained in \( \{t_0\} \times \mathbb{T}^1 \). In fact, \( \text{sing supp}_A(u) = \{t_0, 0\} \) since \( \hat{u}(t_0, n) = 1 \), that is, \( u(t_0, x) = \delta_0(x) \). This concludes the proof of the case \( r \geq 1 \).

We now now say a few words about the case \( r = 0 \). All we need is \( \hat{u}(0, n) = 1 \) and we are led to
\[
\int_0^{2\pi} e^{i\pi(n \Theta(s))} \hat{f}(s, n) ds = -1.
\]
Instead of (2.10), take
\[
g_n(t) = C_n e^{i\pi(n \gamma(t) + i\phi(t))},
\]
where now
\[
\phi(t) = K(1 - \cos(t - s_1)),
\]
\[
\psi(t) = a_0 t - A(t) - a_0 \sin(t - s_1),
\]
and \( K \) and \( C_n \) are similarly chosen as in the case \( r \geq 1 \). We emphasize that in this case there is no need for approximations, that is, we can take \( \hat{f}(t, n) = g_n(t) = h_n(t) \).

2.2. Case \( b_0 = 0 \). We will only treat the case \( r \geq 1 \). We divide the proof into the following three cases:

Case 1: \( a_0 \in \mathbb{R} \setminus \mathbb{Q} \). This case is quite similar to the case \( b_0 \neq 0 \), since
\[
d_n = e^{i2\pi \sigma} - 1 = e^{2\pi i a_0} - 1 \neq 0.
\]
Note that \( |d_n| \leq 2 \).
The only difference here is that we choose \( \tilde{f}_n \) satisfying
\[
||\tilde{f}_n - h_n||_{C^0} \leq e^{-\sigma n} |d_n|
\]
instead of (2.14), to get
\[
|\gamma_n| \leq |e^{2\pi in c_0} - 1|^{-1} \int_0^{2\pi} e^{-nB(s)} |\tilde{f}_n(s) - h_n(s)| \, ds
\leq \int_0^{2\pi} e^{-nm} e^{-\sigma n} \, ds \leq 2\pi e^{-(m+\sigma)n}.
\]

The rest of the proof follows as in the case \( b_0 \neq 0 \).

Case 2: \( a_0 \in \mathbb{Z} \). In this case we deal with
\[
(2.15) \quad \tilde{u}(t, n) = \int_0^t e^{in[C(s)-C(t)]} \tilde{f}(s, n) \, ds + e^{-inC(t)}
\]
instead of (2.14). Note that \( e^{\pm inC(t)} \) is an element of \( C^\omega(\mathbb{T}) \).

We proceed as in the case \( b_0 \neq 0 \) to get \( \tilde{f}_n \) as in (2.14).

Let \( u_n(t) \) be the right-hand side of (2.15) with \( f_n = \tilde{f}_n \) instead of \( \hat{f} (t, n) \).

We have \( u_n(0) = 1 \) and for \( k = 1, \ldots, r \),
\[
u_n(t_k) = e^{-inC(t_k)} \left[ \int_0^{t_k} e^{inC(s)} f_n(s) \, ds + 1 \right]
= e^{-inC(t_k)} \int_0^{t_k} e^{inC(s)} \left[ f_n(s) - h_n(s) \right] \, ds,
\]
which leads to \( |u_n(t_k)| \leq 2\pi e^{n(M-m-\sigma)} \).

The rest of the proof follows as in previous cases.

Case 3: \( a_0 \in \mathbb{Q} \setminus \mathbb{Z} \). Write \( a_0 = p/q \) where \( p \in \mathbb{Z} \), \( q \in \mathbb{N} \), and g.c.d.\( \{p,q\} = 1 \).

Let \( A = \{ nq; n \in \mathbb{N} \} \) and \( B = \mathbb{N} \setminus A \).

Let \( f_n, u_n \) be as in the previous case with \( a_0 = q \). Define
\[
\tilde{f}_n = \begin{cases} f_n & \text{if } n \in A, \\ 0 & \text{if } n \in B, \end{cases} \quad \tilde{u}_n = \begin{cases} u_n & \text{if } n \in A, \\ 0 & \text{if } n \in B. \end{cases}
\]

Then \( (\partial_t + inC') \tilde{u}_n = \tilde{f}_n \) and
\[
U_0(t, x) = \sum_{n\in\mathbb{N}} \tilde{u}_n(t) e^{inx} = \sum_{n\in A} u_n(t) e^{inx} = \sum_{n\in B} u_{nq}(t) e^{inx}
\]
defines an element of \( D'(\mathbb{T}^2) \) with \( LU_0 \in C^\omega(\mathbb{T}^2) \) and \( \text{sing supp}_A(U_0) = \{(t_0, 0)\} \).

We note that if \( n \in B \), then \( e^{2\pi in c_0} = e^{2\pi i np/q} \neq 1 \). Thus, we are allowed to proceed as in the case \( b_0 \neq 0 \) but working only with frequencies in \( B \). In this manner we obtain \( U_1 \in D'(\mathbb{T}^2) \) satisfying \( \hat{U}_1(t, n) = 1 \) if \( n \in B \) and \( \hat{U}_1(t, n) = 0 \) if \( n \in A \). Also, \( LU_1 \in C^\omega(\mathbb{T}^2) \), \( \text{sing supp}_A(U_1) = \{(t_0, 0)\} \).

Finally, taking \( U = U_0 + U_1 \) we get a singular solution of \( L \) such that \( U(0, x) = \delta_+(x) \).

This completes the proof of Lemma 2.3. \( \square \)
3. Construction of singular solutions in the case $b \equiv 0$

In this section $L$ is a nonsingular, real vector field. Hence, if $Lu = 0$ and $\text{sing supp}_A(u)$ contains a point $p$, then it contains the complete integral curve of $L$ through $p$.

By using, if necessary, the change of variables $(t, x) \mapsto (t, x - A(t) + a_0t)$, we may assume $a(t) = a_0$. Thus, the study of $Lu = f$ amounts to study the algebraic equation

$$i(m + a_0n)\hat{w} = \hat{f}, \quad (m, n) \in \mathbb{Z}^2.$$  

We need a definition. We say that an irrational number $\alpha$ is exponential Liouville if there exist $\varepsilon > 0$ and a sequence $\{(m_k, n_k)\}_{k \geq 1}$ of points in $\mathbb{Z} \times \mathbb{N}$ satisfying $|\alpha + m_k/n_k| \leq e^{-\varepsilon n_k}$; the set of all such numbers is denoted by $\mathcal{EL}$.

We divide the proof into the following three cases:

Case 1: $a_0 \in \mathbb{Q}$. Write $a_0 = p/q$, $q \in \mathbb{N}$, $p \in \mathbb{Z}$ with $\text{g.c.d.} \{p, q\} = 1$.

Let $A = \{(m, n) \in \mathbb{Z}^2; m + a_0n = 0\} \subset \{(m, n) \in \mathbb{Z}^2; qm + pn = 0\}$.

Let $v(m, n)$ be a tempered sequence of complex numbers satisfying, for some $\varepsilon > 0$ and $C > 0$, $|v(m, n)| \leq Ce^{-\varepsilon(|m| + |n|)}$, for all $(m, n) \notin A$, i.e., $|qm + pn| \geq 1$.

Let $u(x, t) = \sum_{m,n \in \mathbb{Z}} v(m, n)e^{i(mt + nx)}$. It is clear that $u$ defines a distribution in $\mathbb{T}^2$.

Set $g(m, n) = i(m + a_0n)\hat{u}(m, n) = i(m + a_0n)v(m, n)$. We see that $g(m, n) = 0$ for $(m, n) \in A$ and for $(m, n) \notin A$,

$$|g(m, n)| = |m + a_0n||v(m, n)| \leq C|m + a_0n|e^{-\varepsilon(|m| + |n|)}.$$  

Since there exists $C_1 > 0$ such that $|m + a_0n| \leq C_1 e^{2(|m| + |n|)}$, we obtain

$$|g(m, n)| \leq C_0e^{-\varepsilon_0(|m| + |n|)}$$

for all $(m, n) \in \mathbb{Z}^2$, where $C_0 = CC_1$.

Thus, $f(x, t) = \sum_{m,n \in \mathbb{Z}} g(m, n)e^{i(mt + nx)}$ defines an element of $C^\infty(\mathbb{T}^2)$ and $Lu = f$ in particular, if $v(m, n)$ is not exponentially decaying, then $u$ is a singular solution of $L$.

We now explain the location of the singularities. Given a distribution $v = v(x)$ in $\mathbb{T}^1$ with $v(x + 2\pi/q) = v(x)$ there exists a distribution $u = u(x, t)$ in $\mathbb{T}^2$ such that $Lu = 0$ and $u(x, 0) = v(x)$. In fact, we may take $u(x, t) = v(qx - pt)$.

Alternatively, if $v(x) = \sum_{\ell \in \mathbb{Z}} \bar{v}(\ell)e^{-i\varphi x + i\ell t}$, then

$$u(x, t) = v(x - \frac{p}{q}t) = \sum_{\ell \in \mathbb{Z}} \bar{v}(\ell)e^{-i(qx - pt)\ell}$$

is such that $Lu = 0$ and $u(x, 0) = v(x)$. Note that if we set $Z = Z(x, t) = e^{i(-qx + pt)}$, then $u = h \circ Z$ where

$$h(\xi) = \sum_{\ell \in \mathbb{Z}} \bar{v}(\ell)\xi^\ell.$$  

In any case, $\text{sing supp}_A(u)$ is equal to the union of all integral curves of $L$ which meet $\text{sing supp}_A(v)$.

Case 2: $a_0 \notin \mathbb{R} \setminus (\mathbb{Q} \cup \mathcal{EL})$. In this case we prove that $L$ is GAH. Suppose $Lu = f$ with $f \in C^\infty(\mathbb{T}^2)$ and $u \in D'(\mathbb{T}^2)$.

There exists $k_0 \geq 0$ such that $|\hat{f}(m, n)| \leq C_0e^{-\varepsilon(|m| + |n|)}$ for all integers $m, n$ with $|m| + |n| \geq k_0$. 


Since \( a_0 \not\in \mathcal{E}L \), there exists \( k \geq k_0 \) such that
\[
\left| a_0 + \frac{m}{n} \right| > e^{-\tilde{\xi}|n|}, \quad |m| + |n| \geq k.
\]
Hence, for all \((m, n)\) with \(|m| + |n| \geq k\), we get
\[
|\tilde{u}(m, n)| = \left| \frac{f(m, n)}{|m + a_0 n|} \right| = \frac{1}{|n|} \left| \frac{f(m, n)}{a_0 + \frac{m}{n}} \right| \leq e^{\tilde{\xi}|n|} C_0 e^{-\varepsilon (|m| + |n|)} = C_0 e^{-\tilde{\xi}(|m| + |n|)}.
\]

Case 3: \( a_0 \in \mathbb{R} \setminus \mathbb{Q} \cap \mathcal{E}L \). In this case there exist \( \varepsilon > 0 \) and a sequence \( \{(m_k, n_k)\}_{k \geq 1} \) of points in \( \mathbb{Z} \times \mathbb{N} \) satisfying \( |a_0 + m_k/n_k| \leq e^{-\varepsilon n_k} \). Since \( m_k/n_k \to -a_0 \), there exist \( c > 0 \) and \( k_o \) such that
\[
\frac{1}{c} n_k \leq |m_k| \leq c n_k, \quad \text{for all } k \geq k_o.
\]

Therefore, there exist positive constants \( \varepsilon \) and \( C \) such that \( |a_0 n_k + m_k| \leq C e^{-\varepsilon (n_k + |m_k|)} \).

Let \( \mathcal{A} = \{(m_k, n_k); k \geq 1\} \).

Let \( v(m, n) \) be a tempered sequence of complex numbers satisfying, for some \( \eta > 0 \) and \( C_0 > 0 \), \( |v(m, n)| \leq C_0 e^{-\eta(|m| + |n|)} \), for all \((m, n) \not\in \mathcal{A}\).

Let \( u(x, t) = \sum_{m, n \in \mathbb{Z}} v(m, n) e^{i(m+nx)} \). It is clear that \( u \) defines a distribution in \( T^2 \).

Set \( g(m, n) = i(m + a_0 n) \tilde{u}(m, n) = i(m + a_0 n) v(m, n) \).

Since \( v(m, n) \) is tempered, there exists \( C_1 > 0 \) with \( |v(m, n)| \leq C_1 e^{\tilde{\xi}(|m| + |n|)} \).

Thus, for \((m, n) \in \mathcal{A}\)
\[
|g(m, n)| \leq C C_1 e^{\tilde{\xi}(|m| + |n|)}.
\]

Now for \((m, n) \not\in \mathcal{A}\),
\[
|g(m, n)| = |m + a_0 n| |v(m, n)| \leq C_0 |m + a_0 n| e^{-\eta(|m| + |n|)},
\]
and since there exists \( C_2 > 0 \) such that \( |m + a_0 n| \leq C_2 e^{\tilde{\xi}(|m| + |n|)} \), we obtain
\[
|g(m, n)| \leq C_0 C_2 e^{-\tilde{\xi}(|m| + |n|)}.
\]

Thus, \( f(x, t) = \sum_{m, n \in \mathbb{Z}} g(m, n) e^{i(m+nx)} \) defines an element of \( C^\omega(T^2) \) and \( Lu = f \). In particular, if \( v(m, n) \) is not exponentially decaying, then \( u \) is a singular solution of \( L \); in fact, \( \text{sing supp}_\mathcal{A}(u) = T^2 \) since the orbit is dense.

4. Appendix

In this Appendix we present a one-dimensional periodic version of Theorem 8.4.14 in [H].

Lemma 4.1. Let \( F \) be a nonempty closed subset of \( T^1 \). Then there exists \( v \in D'(T^1) \) such that \( \text{WF}_A(v) = F \times \mathbb{R}_+ \). Furthermore, \( v \) can be chosen continuous and \( v(x) = \sum_{n \geq 1} c_n e^{inx} \).

Proof. Let \( f(z) = z + (1 - z) \log(1 - z) \), where \( \log \) is a branch of the logarithm defined in \( \Omega = \mathbb{C} \setminus \{z = x + iy \in \mathbb{C}; y = 0, x \geq 1\} \). Note that \( \lim_{z \to \Omega, z \to -1} f(z) = 1 \).

It is easy to see that
\[
f(z) = \sum_{n \geq 2} \frac{1}{n(n - 1)} z^n, \quad \text{for } |z| \leq 1.
\]
Let $M = \sum_{n \geq 2} 1/n(n-1)$. Clearly, $|f(z)| \leq M$ in the closed unit disk $\bar{D}$.

Let $\{z_k\} \subset F$ be at most countable and dense in $F$ such that $z_k \neq z_l$ if $k \neq l$.

Write $z_k = e^{ix_k}$ with $x_k \in [0, 2\pi)$.

Consider

$$u(z) = \sum_{k \geq 1} \frac{1}{3^k} f(z e^{-ix_k}), \quad z \in \bar{D}.$$

We see that $u$ is holomorphic in $D$ and continuous in $\bar{D}$.

In particular, $v = b(u)$, the boundary value of $u$, defines an element of $\mathcal{D}'(\mathbb{T}^1)$.

We have

$$u''(z) = \sum_{k \geq 1} \frac{e^{-2ix_k}}{3^k (1 - z e^{-ix_k})}, \quad \text{for } z \in D.$$

For each $\ell \geq 1$ write

$$u''(z) = \sum_{k=1}^{\ell-1} \frac{e^{-2ix_k}}{3^k (1 - z e^{-ix_k})} + \frac{e^{-2ix_{\ell}}}{3^\ell (1 - z e^{-ix_{\ell}})} + R(z),$$

where

$$R(z) = \sum_{k \geq \ell+1} \frac{e^{-2ix_k}}{3^k (1 - z e^{-ix_k})}.$$

For $|z| < 1$, we have

$$|R(z)| \leq \frac{1}{1 - |z|} \sum_{k \geq \ell+1} \frac{1}{3^k} = \frac{1}{2 \cdot 3^\ell} \cdot 1 - |z|.$$

For $z = te^{ix}, \ 0 \leq t < 1$, we have

$$\left| \frac{e^{-2ix_{\ell}}}{3^\ell (1 - z e^{-ix_{\ell}})} \right| = \frac{1}{3^\ell 1 - t}.$$

For such values of $z$ we obtain

$$\sum_{k \geq \ell} \frac{e^{-2ix_k}}{3^k (1 - z e^{-ix_k})} \geq \frac{1}{2 \cdot 3^\ell} \cdot 1 - |z|,$$

hence such a sum is unbounded when $z$ tends to $z_\ell$ along the ray $tz_\ell$.

Since

$$\sum_{k=1}^{\ell-1} \frac{e^{-2ix_k}}{3^k (1 - z e^{-ix_k})}$$

is holomorphic, we conclude that $u$ cannot be extended holomorphically in a neighborhood of $z_\ell$.

Let $\zeta = e^{i\xi} \in \mathbb{T}^1 \setminus F$. Let $S = \{re^{ix}; |r-1| < \varepsilon, |x - \xi| < \varepsilon\}$ be such that $S \cap F = \emptyset$, for some $\varepsilon > 0$. Consider $D_1 = \{z; |z| \leq 1 + \varepsilon\} \setminus e^{-i\xi}S$. Since $D_1 \subset \Omega$ is compact we have $|f(z)| \leq C$ for some $C > 0$ and all $z \in D_1$.

Since, for $z \in S$,

$$\sum_{k \geq 1} \frac{1}{3^k} |f(z e^{-ix_k})| \leq C/2,$$

it follows that $u$ can be continued holomorphically in a full neighborhood of $\zeta$. This means that $F \times \mathbb{R}_+ \supset WF_A(v)$. \hfill $\Box$

**Remark 4.2.** An analogous result is true for negative frequencies.
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