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# PRESCRIBING ANALYTIC SINGULARITIES FOR SOLUTIONS OF A CLASS OF VECTOR FIELDS ON THE TORUS

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Abstract. We consider the operator $L = \partial_t + (a(t) + ib(t))\partial_x$ acting on distributions on the two-torus $\mathbb{T}^2$, where $a$ and $b$ are real-valued, real analytic functions defined on the unit circle $\mathbb{T}^1$. We prove, among other things, that when $b$ changes sign, given any subset $\Sigma$ of the set of the local extrema of the local primitives of $b$, there exists a singular solution of $L$ such that the $t-$projection of its analytic singular support is $\Sigma$; furthermore, for any $\tau \in \Sigma$ and any closed subset $F$ of $\mathbb{T}^1_x$ there exists $u \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lu \in C^\omega(\mathbb{T}^2)$ and $\operatorname{sing\,supp}_{\mathrm{A}}(u) = \{\tau\} \times F$. We also provide a microlocal result concerning the trace of $u$ at $t = \tau$.

## 1. Introduction

There are many works about hypoellipticity, local or global, smooth or analytic. In the global case we cite [B], [BCM], [BNZ1], [BNZ2], [DGY], [G], [GW], [Hi], [Ho], and [M]. In these papers, in order to prove that some of the operators under study are not hypoelliptic, singular solutions are constructed. By a singular solution of an operator $L$, we mean a nonanalytic (nonsmooth) distribution $u$ such that $Lu$ is analytic (smooth). However, not much is known about the location and nature of the singularities. The main goal of the present work is to help fill this gap for a class of vector fields on the two-torus in the global analytic setting as we describe below.

Let $c = a + ib$, where $a$ and $b$ are real-valued, real analytic functions defined on the unit circle $\mathbb{T}^1$. We consider the operator

$$L = \partial_t + c(t)\partial_x$$

acting on distributions on the two-torus $\mathbb{T}^2$.

We recall that $L$ is said to be globally analytic hypoelliptic (GAH) if one has $u \in C^\omega(\mathbb{T}^2)$ whenever $u \in \mathcal{D}'(\mathbb{T}^2)$ and $Lu \in C^\omega(\mathbb{T}^2)$.

In [B] one finds a characterization of the global analytic hypoellipticity of the operator $L$; in particular, it is proved there that if $b \not\equiv 0$, then $L$ is globally analytic hypoelliptic if and only if the function $b$ does not change sign (the Nirenberg-Treves condition $(\mathcal{P})$; see [T2]), or equivalently, no primitive of $b$ defined in $\mathbb{R}$ has local extrema.

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We emphasize that the method applied in [B] does not yield all the possible singularities. We now mention two examples of this difficulty. Assume that $b \not\equiv 0$; let $B$ be one of its primitives in $\mathbb{R}$ and suppose that the origin is a local minimum of $B$. If $t_1, t_2 \in (0, 2\pi)$ are points of local maximum of $B_{|[0,2\pi]}$, with $B(t_1) < B(t_2)$, then the singular solution constructed in [B] is real analytic when $t = t_1$. Similarly, if $t_1, t_2 \in (0, 2\pi)$ are points of global maximum of $B_{|[0,2\pi]}$, then the solution constructed in [B] is singular at *both* points $(t_1, 0)$ and $(t_2, 0)$.

In fact, if $b \not\equiv 0$ and $t_\star$ is a point of local maximum of $B$, then in Lemma 2.1 we construct $u \in \mathcal{D}'(\mathbb{T}^2)$ with $\operatorname{sing\,supp_A}(u) = \{(t_\star, 0)\}$ and $Lu \in C^\omega(\mathbb{T}^2)$ and, furthermore, $u(t_\star, x) = \delta_+(x) \doteq \sum_{n\geq 1} e^{inx}$. More generally (see Theorem 2.2), given, instead of $\delta_+$, a distribution $\widetilde{v}(x) = \sum_{n\geq 1} v_n e^{inx}$, there is $u \in \mathcal{D}'(\mathbb{T}^2)$ such that its analytic singular support is $\operatorname{sing\,supp_A}(u) = \{t_\star\} \times \operatorname{sing\,supp_A}(\widetilde{v})$ and $Lu \in C^\omega(\mathbb{T}^2)$. Moreover (see Theorem 2.3), given any nonempty closed subset $F \subset \mathbb{T}^1$, we construct $u \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lu \in C^\omega(\mathbb{T}^2)$ and $\operatorname{sing\,supp_A}(u) = \{t_\star\} \times F$; in fact (see Remark 2.4), the analytic wave front set of the trace $u(t_\star, \cdot)$ is equal to $F \times \mathbb{R}_+$.

We also have the analogous results for the local minima. For instance, if $s_\star$ is such a point and $F \subset \mathbb{T}^1$ is a nonempty closed set, then there is $u \in \mathcal{D}'(\mathbb{T}^2)$ with $\operatorname{sing\,supp_A}(u) = \{s_\star\} \times F$ and $Lu \in C^\omega(\mathbb{T}^2)$ and, furthermore, the analytic wave front set of the trace $u(s_\star, \cdot)$ is equal to $F \times \mathbb{R}_-$.

Theorems 2.9 and 2.10 put together the above results in a more complete way. As an example, suppose $b \not\equiv 0$ and let $B : \mathbb{R} \to \mathbb{R}$ be a real analytic function such that $B' = b$. Let $\Sigma$ be a nonempty subset of the set of points in $[0, 2\pi)$ which are local extrema of $B$. Then there exists a singular solution $u$ such that $\operatorname{sing\,supp_A}(u) = \Sigma \times \{0\}$.

Our results are sharp. Indeed, it follows from [T1] and also from [BT] that if $t$ is neither a local maximum nor a local minimum, then every $u \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lu \in C^\omega(\mathbb{T}^2)$ is real analytic at $(t, x)$ for arbitrary $x \in \mathbb{T}^1$. Furthermore, concerning the microlocal aspect of the singularities, the results of [BT] imply that, for instance, in the case of a local maximum $t_\star$, the trace at $t = t_\star$ of any singular solution has an analytic wave-front set contained in $\mathbb{T}^1 \times \mathbb{R}_+$.

We stress that besides constructing solutions with prescribed singularities, the present work yields a new, simpler, proof of the the hardest case of the main result about global analytic hypoellipticity in [B], that is, when $b \not\equiv 0$. In both works the proofs rely on the method of stationary phase, but now we make use of a suitable function $\psi$ (see the proof of Lemma 2.1) whose role is to simplify the application of this method, similar to what was done in [BNZ1].

We point out that our interest lies primarily in the nonreal case; for the sake of completeness, we also provide, in section 3, information about singularities of solutions when $b \equiv 0$.

For later use, we define $c_0 = \frac{1}{2\pi}\int_0^{2\pi} c(t)dt$, $a_0 = \Re c_0$, and $b_0 = \Im c_0$.

## 2. Construction of singular solutions in the case $b \not\equiv 0$

Suppose that $L = \partial_t + (a(t) + ib(t))\partial_x$ is not GAH and that $b \not\equiv 0$; equivalently, suppose that $b$ changes sign. Let $B : \mathbb{R} \to \mathbb{R}$ be a primitive of $b$. Then the function has at least one point of local maximum. In fact, there exist $r \in \mathbb{Z}_+$ and points $t_0, \dots, t_r$, with $t_0 < t_1 < \cdots < t_r$, and $t_r - t_0 < 2\pi$, such that any point of local maximum of $B$ is equal to $t_k + 2j\pi$, for some $k = 0, \dots, r$ and some $j \in \mathbb{Z}$.

Similarly we have points of local minimum $s_0, \dots, s_r$ such that, by setting $t_{r+1} \doteq t_0 + 2\pi$ and $s_{r+1} \doteq s_0 + 2\pi$, we may write

$$s_0 < t_0 < s_1 < t_1 < s_2 < \cdots < s_r < t_r < s_{r+1} < t_{r+1}.$$

We will use the same notation, namely, $t_j$ and $s_j$, for the corresponding zeros of $b$ in $\mathbb{T}^1$.

The following crucial lemma will be proved later in this section.

**Lemma 2.1.** *Suppose that $L$ is not GAH and that $b \not\equiv 0$. Let $t_\star \in \{t_0, \dots, t_r\}$. Then there exists $u \in \mathcal{D}'(\mathbb{T}^2)$ with $\operatorname{sing\,supp}_{\mathrm{A}}(u) = \{(t_\star, 0)\}$ and $Lu \in C^\omega(\mathbb{T}^2)$. More precisely, $u$ can be chosen so that $u(t_\star, x) = \delta_+(x) \doteq \sum_{n \geq 1} e^{inx}$.*

Assuming the previous lemma we can prove

**Theorem 2.2.** *Suppose that $L$ is not GAH and $b \not\equiv 0$. Let $t_\star \in \{t_0, \dots, t_r\}$. Given a sequence $(v_n)_{n \in \mathbb{Z}}$ such that $v_n$ is tempered in $n \in \mathbb{Z}$, but not exponentially decaying as $n \to +\infty$, although exponentially decaying in the opposite direction, then there exists $v \in \mathcal{D}'(\mathbb{T}^2) \setminus C^\omega(\mathbb{T}^2)$ such that $\widehat{v}(t_\star, n) = v_n$, $Lv \in C^\omega(\mathbb{T}^2)$, and $\operatorname{sing\,supp}_{\mathrm{A}}(v) = \{t_\star\} \times \operatorname{sing\,supp}_{\mathrm{A}}(\widetilde{v})$, where $\widetilde{v}(x) \doteq \sum_{n \in \mathbb{Z}} v_n e^{inx}$.*

*Proof.* It suffices to take $u$ as in Lemma 2.1 and define

$$v(t, x) = \sum_{n \leq 0} v_n e^{inx} + \sum_{n \geq 1} \widehat{u}(t, n) v_n e^{inx}.$$

□

In Lemma 4.1 we prove that for any given nonempty closed subset $F$ of $\mathbb{T}^1$, there exists $v_o \in \mathcal{D}'(\mathbb{T}^1)$ such that $\operatorname{sing\,supp}_{\mathrm{A}}(v_o) = F$ and $v_o(x) = \sum_{n \geq 1} \widehat{v_o}(n) e^{inx}$. We obtain

**Theorem 2.3.** *Suppose that $L$ is not GAH and $b \not\equiv 0$. Let $t_\star \in \{t_0, \dots, t_r\}$ and let $F$ be a nonempty closed subset of $\mathbb{T}^1$. Then there exists $v \in \mathcal{D}'(\mathbb{T}^2) \setminus C^\omega(\mathbb{T}^2)$ such that $Lv \in C^\omega(\mathbb{T}^2)$ and $\operatorname{sing\,supp}_{\mathrm{A}}(v) = \{t_\star\} \times F$. Furthermore, $\widehat{v}(t_\star, n)$ is tempered in $n \in \mathbb{Z}$ and decays exponentially as $n \to -\infty$.*

*Proof.* Take $v_o \in \mathcal{D}'(\mathbb{T}^1)$ as in Lemma 4.1 corresponding to the given closed subset $F$. Define $v_n = \widehat{v_o}(n)$ for $n \geq 1$ and $v_n = 0$ otherwise. Applying Theorem 2.2 the result follows immediately. □

*Remark* 2.4. Note that the singularity of $v$ in Theorem 2.2 and Theorem 2.3 is due to the positive frequencies $n$. Moreover, any singular solution $w$ of $L$ such that $t_\star$ is in the $t-$projection of its analytic singular support has $\widehat{w}(t_\star, n)$ decaying exponentially as $n \to -\infty$; indeed, this follows from the analogue of Lemma 2.11 with negative indices. Thus, the analytic wave front set of the trace $w(t_\star, \cdot)$ is equal to $F \times \mathbb{R}_+$.

We now proceed to state, without proof, the analogous results for the case of local minima.

**Lemma 2.5.** *Suppose that $L$ is not GAH and that $b \not\equiv 0$. Let $s_\star \in \{s_0, \dots, s_r\}$. Then there exists $u \in \mathcal{D}'(\mathbb{T}^2)$ satisfying $\operatorname{sing\,supp}_{\mathrm{A}}(u) = \{(s_\star, 0)\}$ and $Lu \in C^\omega(\mathbb{T}^2)$. More precisely, $u$ can be chosen so that $u(s_\star, x) = \delta_-(x) \doteq \sum_{n \leq -1} e^{inx}$.*

**Theorem 2.6.** *Suppose that $L$ is not GAH and $b \not\equiv 0$. Let $s_\star \in \{s_0, \ldots, s_r\}$. Given a sequence $(w_n)_{n\in\mathbb{Z}}$ such that $w_n$ is tempered, but not exponentially decaying as $n \to -\infty$, although exponentially decaying in the opposite direction, then there exists $w \in \mathcal{D}'(\mathbb{T}^2)\backslash C^\omega(\mathbb{T}^2)$ such that $\widehat{w}(s_\star, n) = w_n$, $Lw \in C^\omega(\mathbb{T}^2)$, $\operatorname{sing\,supp_A}(w) = \{s_\star\} \times \operatorname{sing\,supp_A}(\widetilde{w})$, where $\widetilde{w}(x) \doteq \sum_{n\in\mathbb{Z}} w_n e^{inx}$.*

**Theorem 2.7.** *Suppose that $L$ is not GAH and $b \not\equiv 0$. Let $s_\star \in \{s_0, \ldots, s_r\}$ and let $F$ be a nonempty closed subset of $\mathbb{T}^1$. Then there exists $w \in \mathcal{D}'(\mathbb{T}^2) \setminus C^\omega(\mathbb{T}^2)$ such that $Lw \in C^\omega(\mathbb{T}^2)$ and $\operatorname{sing\,supp_A}(w) = \{s_\star\} \times F$. Furthermore, $\widehat{w}(s_\star, n)$ is tempered in $n \in \mathbb{Z}$ and decays exponentially as $n \to +\infty$.*

Before proceeding to our main results (Theorems 2.9 and 2.10) we need the following

**Lemma 2.8.** *For each $n \in \mathbb{Z}$ let $A_n = (a_{ij}(n))$ be a $k \times k$ matrix such that $a_{ii}(n) = 1$ and, for $i \neq j$, $a_{ij}(n)$ is exponentially decaying as $n \to \pm\infty$.*

*If $B_n$ is a $k \times 1$ matrix whose entries are tempered in $|n|$, then the equation $A_n X_n = B_n$ has, for all sufficiently large $|n|$, a unique solution. Furthermore, the entries of $X_n$ are tempered in $|n|$.*

*Proof.* All we need is to observe that $\det A_n = 1 + \delta_n$ with $\delta_n$ decaying exponentially as $n \to \pm\infty$. □

**Theorem 2.9.** *Suppose that $L$ is not GAH and $b \not\equiv 0$. Let $u_j(n)$, $j = 0, \ldots, 2r+1$, be tempered in $n \in \mathbb{Z}$ and such that for $j = 0, \ldots, r$, $u_j(n)$ decays exponentially as $n \to -\infty$, and for $j = r+1, \ldots, 2r+1$, $u_j(n)$ decays exponentially as $n \to +\infty$.*

*Then there exists $u \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lu \in C^\omega(\mathbb{T}^2)$ and $\widehat{u}(t_j, n) = u_j(n)$ for $j = 0, \ldots, r$, and $\widehat{u}(s_{j-r}, n) = u_j(n)$, for $j = r+1, \ldots, 2r+1$, for all $n$. Moreover,*

$$\operatorname{sing\,supp_A}(u) \subset \left(\bigcup_{j=0}^{r} \{t_j\} \times \operatorname{sing\,supp_A}(\widetilde{u}_j)\right) \cup \left(\bigcup_{j=1}^{r+1} \{s_j\} \times \operatorname{sing\,supp_A}(\widetilde{u}_j)\right),$$

*where $\widetilde{u}_j(x) = \sum_{n\in\mathbb{Z}} u_j(n) e^{inx}$, $j = 0, \ldots, 2r+1$.*

*In particular, if $I$ is the set of indices $j \in \{0, \ldots, r\}$ such that $u_j(n)$ does not decay exponentially as $n \to +\infty$ and $J$ is the set of indices $j \in \{r+1, \ldots, 2r\}$ such that $u_j(n)$ does not decay exponentially as $n \to -\infty$, then*

$$\operatorname{sing\,supp_A}(u) = \left(\bigcup_{j\in I} \{t_j\} \times \operatorname{sing\,supp_A}(\widetilde{u}_j)\right) \cup \left(\bigcup_{j\in J} \{s_j\} \times \operatorname{sing\,supp_A}(\widetilde{u}_j)\right).$$

*Proof.* For $j = 0, \ldots, r$, we use Lemma 2.1 and obtain $v_j \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lv_j \in C^\omega(\mathbb{T}^2)$, $\widehat{v_j}(t_j, n) = 1$, for all $n \geq 1$, $\widehat{v_j}(t_j, n)$ is exponentially decaying when $n \to -\infty$, and, for $t \neq t_j$, $\widehat{v_j}(t, n)$ is exponentially decaying as $|n| \to +\infty$.

For $j = r+1, \ldots, 2r+1$, we use Lemma 2.5 and obtain $w_j \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lw_j \in C^\omega(\mathbb{T}^2)$, $\widehat{w_j}(s_{j-r}, n) = 1$, for all $n \leq -1$, $\widehat{w_j}(s_{j-r}, n)$ is exponentially decaying when $n \to +\infty$, and, for $t \neq s_{j-r}$, $\widehat{w_j}(t, n)$ is exponentially decaying as $|n| \to +\infty$.

Let $A_n = (a_{ij}(n))$ be given by

$$a_{ij}(n) = \begin{cases} \widehat{v_i}(t_j, n), i = 0, \dots, r, \quad j = 0, \dots, r, \\ \widehat{v_i}(s_{j-r}, n), i = 0, \dots, r, \quad j = r+1, \dots, 2r+1, \\ \widehat{w_i}(t_j, n), i = r+1, \dots, 2r+1, \quad j = 0, \dots, r, \\ \widehat{w_i}(s_{j-r}, n), i = r+1, \dots, 2r+1, \quad j = r+1, \dots, 2r+1. \end{cases}$$

Define $B_n = (b_{i1}(n))$, where $b_{i1}(n) = u_i(n)$. Let $X_n = (\alpha_{i1}(n))$ be the solution of $A_n X_n = B_n$, for all $|n| > n_0$, for some $n_0$ as in Lemma 2.8. Then $v \in \mathcal{D}'(\mathbb{T}^2)$ defined by

$$\widehat{v}(t, n) = \sum_{i=0}^{r} \alpha_{i1}(n) \widehat{v_i}(t, n) + \sum_{i=r+1}^{2r+1} \alpha_{i1}(n) \widehat{w_i}(t, n)$$

satisfies $Lv \in C^\omega(\mathbb{T}^2)$, $\widehat{v}(t_j, n) = u_j(n)$ for $j = 0, \dots, r$, and $\widehat{v}(s_{j-r}, n) = u_j(n)$, for $j = r+1, \dots, 2r+1$, for $|n| > n_0$.

Let

$$z_j = \begin{cases} e^{it_j}, \qquad j = 0, \dots, r, \\ e^{is_{j-r}}, \qquad j = r+1, \dots, 2r+1, \end{cases}$$

and, for $|n| \le n_0$, let $p_n(z)$ be a polynomial such that $p_n(z_j) = u_j(n)$, $j = 0, \dots, 2r+1$.

Let $u(t, x) = v(t, x) + \sum_{|n| \le n_0} p_n(e^{it}) e^{inx}$. Then $u \in \mathcal{D}'(\mathbb{T}^2)$, $Lu \in C^\omega(\mathbb{T}^2)$ and $\widehat{u}(t_j, n) = u_j(n)$ for $j = 0, \dots, r$, and $\widehat{u}(s_{j-r}, n) = u_j(n)$, for $j = r+1, \dots, 2r+1$, for all $n$. □

We also have

**Theorem 2.10.** *Suppose that $L$ is not GAH and $b \not\equiv 0$. Then, given $\{p_1, \dots, p_k\} \subset \{t_0, \dots, t_r, s_1, \dots, s_r\}$ and nonempty closed subsets $F_j \subset \mathbb{T}^1$, $j = 1, \dots, k$, there exists $u \in \mathcal{D}'(\mathbb{T}^2)$ such that $Lu \in C^\omega(\mathbb{T}^2)$ and $\operatorname{sing\,supp}_{\mathrm{A}}(u) = \bigcup_{j=1}^{k} \{p_j\} \times F_j$.*

Before we present the proof of Lemma 2.1, we need one more result.

**Lemma 2.11.** *Let $c = a + ib$ be a continuous function defined on a bounded interval $J$, where $a$ and $b$ are real valued and $b$ has only isolated zeros. Let $C = A + iB$, where $A$ and $B$ are primitives of $a$ and $b$, respectively. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of continuous functions defined on $J$. Let $t_0 \in J$ and set*

$$t_1 = \inf\{t \in J; B \text{ is nondecreasing on } [t, t_0]\}$$

*and*

$$t_2 = \sup\{t \in J; B \text{ is nonincreasing on } [t_0, t]\}.$$

*Then all the solutions of the ODE*

$$\frac{d}{dt} v_n(t) + inc(t) v_n(t) = f_n(t) \tag{2.1}$$

*satisfy the following:*

1. *If $v_n(t_0)$ is tempered in $n$ and if $f_n(t)$ is tempered in $n$ uniformly in $t$, then $v_n(t)$ is tempered in $n$ uniformly in $[t_1, t_2] \cap J$.*
2. *If $v_n(t_0)$ is exponentially decaying in $n$ and if $f_n(t)$ is exponentially decaying in $n$ uniformly in $t$, then $v_n(t)$ is exponentially decaying in $n$ uniformly in $[t_1, t_2] \cap J$.*

(3) *Suppose $t_0$ is not a minimum. If $v_n(t_0)$ is tempered in $n$ and if $f_n(t)$ is exponentially decaying in $n$ uniformly in $t$, then, for any sufficiently small $\varepsilon > 0$, $v_n(t)$ is exponentially decaying in $n$ uniformly in*

$$[t_1, t_2] \cap J \setminus (t_0 - \varepsilon, t_0 + \varepsilon).$$

*Proof.* If $v_n$ is a solution of (2.1), then

$$v_n(t) = e^{-in(C(t)-C(t_0))} v_n(t_0) + \int_{t_0}^{t} e^{-in(C(t)-C(s))} f_n(s)\, ds, \qquad t \in J.$$

Note that for $t \in [t_1, t_2] \cap J$ we have $B(t) \leq B(s)$ whenever $s \in [t, t_0] \cap J$ or $s \in [t_0, t] \cap J$.

We prove statements (1) and (2). We have

$$|v_n(t)| \leq |v_n(t_0)| + \sup_{s \in J} |f_n(s)|\, \ell(J),$$

for all $t \in [t_1, t_2] \cap J$, $n \in \mathbb{N}$, where $\ell(J)$ is the length of $J$. It follows that $v_n(t)$ is tempered in $n$ for $t \in [t_1, t_2] \cap J$ in case (1) and exponentially decaying in $n$ for $t \in [t_1, t_2] \cap J$ in case (2).

We now prove (3). Since $b$ has only isolated zeros, $B$ is strictly increasing in $[t_1, t_0] \cap J$ and strictly decreasing in $[t_0, t_2] \cap J$ (provided the intervals under consideration are nonempty).

Let $\varepsilon_0 > 0$ such that $J \setminus (t_0 - \varepsilon_0, t_0 + \varepsilon_0) \neq \emptyset$. Let $0 < \varepsilon < \varepsilon_0$; if $t_0 - \varepsilon \in J$ we set $\eta_1 = B(t_0) - B(t_0 - \varepsilon) > 0$. If $t \in J \cap [t_1, t_0 - \varepsilon]$, and $n \in \mathbb{N}$, we have

$$\begin{aligned} |v_n(t)| &\leq e^{n(B(t)-B(t_0))} |v_n(t_0)| + \sup_{s \in J} |f_n(s)|\, \ell(J) \\ &\leq e^{-n\eta_1} |v_n(t_0)| + \sup_{s \in J} |f_n(s)|\, \ell(J). \end{aligned}$$

Similarly, if $t \in J \cap [t_0 + \varepsilon, t_2]$, we have $\eta_2 \doteq B(t_0) - B(t_0 + \varepsilon) > 0$, and

$$\begin{aligned} |v_n(t)| &\leq e^{-in(C(t)-C(t_0))} |v_n(t_0)| + \sup_{s \in J} |f_n(s)|\, \ell(J) \\ &\leq e^{-n\eta_2} |v_n(t_0)| + \sup_{s \in J} |f_n(s)|\, \ell(J). \end{aligned}$$

The result follows since $e^{-n\eta_j} |v_n(t_0)|$ is exponentially decaying in $n$, $j = 1, 2$. □

*Remark* 2.12. If $t_1$ is an interior point of $J$ and $f_n$ and $v_n$ satisfy the assumptions of (2) (respectively, (3)) in the previous lemma, then $v_n(t)$ is exponentially decaying in $n$ for a slightly larger interval $[s_1, t_2] \cap J$ (respectively, the larger set $[s_1, t_2] \cap J \setminus (t_0 - \varepsilon, t_0 + \varepsilon)$) having $t_1$ in its interior. Indeed, there exists $\varepsilon_1 > 0$ such that $e^{n\varepsilon_1} f_n(t)$ is still exponentially decaying in $n$ and uniformly in $t$. Thus, if $s_1 < t_1$ is such that $B(t) - B(s) \leq \varepsilon_1$ for all $s, t \in [s_1, t_1] \cap J$ we get

$$\begin{aligned} \left| \int_{t_0}^{t} e^{-in(C(t)-C(s))} f_n(s)\, ds \right| &\leq \left| \int_{t_0}^{t} e^{n(B(t)-B(s)-\varepsilon_1)} e^{n\varepsilon_1} |f_n(s)|\, ds \right| \\ &\leq e^{n\varepsilon_1} \sup_{s \in J} |f_n(s)| \ell(J). \end{aligned}$$

A similar result holds for $t_2$.

*Remark* 2.13. Results similar to the ones in Lemma 2.11 and in Remark 2.12 hold when $n \leq 0$ and $t_1$ and $t_2$ are defined instead by

$$t_1 = \inf\{t \in J; B \text{ is nonincreasing on } [t, t_0]\}$$

and

$$t_2 = \sup\{t \in J; B \text{ is nondecreasing on } [t_0, t]\}.$$

We now proceed to prove Lemma 2.1.

By reordering we may assume that the singularity is to be put at $t_0$ and by translating we may assume that $t_0 = 0$. We may also assume that $A(0) = B(0) = 0$.

The proof is divided into the cases $b_0 \neq 0$ and $b_0 = 0$.

2.1. **Case** $b_0 \neq 0$**.** We will prove only the case when $b_0 < 0$, the case $b_0 > 0$ being similar (one uses different, albeit similar, formulas for the solutions).

We will first give the proof in the case $r \geq 1$. The case $r = 0$ is simpler, and at the end we will explain how its proof can be read from the proof of the case $r \geq 1$.

If we are to have $u \in \mathcal{D}'(\mathbb{T}^2)$ and $Lu = f \in C^\omega(\mathbb{T}^2)$, then we must have

$$\partial_t \widehat{u}(t,n) + inc(t)\widehat{u}(t,n) = \widehat{f}(t,n), \quad n \geq 1, \tag{2.2}$$

and $\widehat{u}(t,n)$ must be given in terms of $\widehat{f}(t,n)$ by

$$\widehat{u}(t,n) = e^{-inC(t)} \left[ \int_0^t e^{inC(s)} \widehat{f}(s,n)\, ds + (e^{2\pi inc_0} - 1)^{-1} \int_0^{2\pi} e^{inC(s)} \widehat{f}(s,n)\, ds \right] \tag{2.3}$$

or

$$\widehat{u}(t,n) = e^{-inC(t)} (1 - e^{-2\pi inc_0})^{-1} \int_0^t e^{inC(s)} \widehat{f}(s,n)\, ds + e^{-inC(t)} (e^{i2\pi nc_0} - 1)^{-1} \int_t^{2\pi} e^{inC(s)} \widehat{f}(s,n)\, ds. \tag{2.4}$$

We also have a formula for expressing $\widehat{u}(t,n)$ in terms of $\widehat{u}(t',n)$, where $t, t' \in \mathbb{R}$, namely,

$$\widehat{u}(t,n) = e^{-in\{C(t)-C(t')\}} \widehat{u}(t',n) + \int_{t'}^t e^{-in\{C(t)-C(s)\}} \widehat{f}(s,n)\, ds. \tag{2.5}$$

Ideally we would like to construct $\widehat{u}(t,n)$, tempered in $n$ uniformly in $t$ so that we have

$$\widehat{u}(t_0,n) = 1, \quad \widehat{u}(t_1,n) = \cdots = \widehat{u}(t_r,n) = 0. \tag{2.6}$$

Now by using (2.6), (2.3) and (2.5), we are led to find $u$ satisfying the following conditions:

$$\int_0^{t_1} e^{inC(s)} \widehat{f}(s,n) ds = -1; \tag{2.7}$$

$$\int_{t_{k-1}}^{t_k} e^{inC(s)} \widehat{f}(s,n) ds = 0, \quad 2 \leq k \leq r; \tag{2.8}$$

$$\int_{t_r}^{2\pi} e^{in[C(s)-C(2\pi)]} \widehat{f}(s,n) ds = 1. \tag{2.9}$$

We begin by defining $v_n(t)$ as the right-hand side of (2.3) (or (2.4)) with $\widehat{f}(\cdot,n)$ replaced by a piecewise continuous function $g_n$ given by

$$g_n(t) = e^{in(\psi(t)+i\varphi(t))}\left[C_n\chi_{[0,t_1)} + D_n\chi_{[t_r,2\pi)}\right], \tag{2.10}$$

where

$$\varphi(t) = K(1-\cos(t-s_1))(1-\cos(t-s_{r+1})),$$
$$\psi(t) = a_0t - A(t) + \frac{a_0}{2}\sin(2t-s_1-s_{r+1}), \quad \text{if } s_{r+1}-s_1=\pi$$

and

$$\psi(t) = a_0t - A(t) - \frac{a_0}{\cos((s_{r+1}-s_1)/2)}\sin(t-\frac{s_1+s_{r+1}}{2}), \quad \text{if}$$

$s_{r+1}-s_1 \neq \pi$, and, moreover, $K, C_n$, and $D_n$ will be chosen later.

Note that $A(t)-a_0t$ defines an element of $C^\omega(\mathbb{T}^1,\mathbb{R})$.

By our choice of $g_n$ we have

$$\int_{t_{k-1}}^{t_k} e^{inC(s)}g_n(s)ds = 0, \quad 2\le k\le r.$$

We now define $C_n$ and $D_n$ by the following equations:

$$C_n\int_0^{t_1} e^{in(C(s)+\psi(s)+i\varphi(s))}ds = -1 \tag{2.11}$$

and

$$D_n\int_{t_r}^{2\pi} e^{in(C(s)+\psi(s)+i\varphi(s)-C(2\pi))}ds = 1.$$

We claim that both $C_n$ and $D_n$ are exponentially decaying as $n\to\infty$, uniformly in $t\in\mathbb{T}^1$. We prove this for $C_n$ employing the stationary phase method, as in [Sj].

Let

$$\begin{aligned}\beta(s) &= -i(A(s)+iB(s)+\psi(s)) + \varphi(s) - B(s_1) + i(A(s_1)+\psi(s_1))\\ &= B(s)+\varphi(s) - i(A(s)+\psi(s)) - B(s_1) + i(A(s_1)+\psi(s_1)).\end{aligned}$$

Hence, $\beta(s_1)=0$.

The function $\beta$ has a holomorphic extension to a complex neighborhood $V_\delta = \{z=x+iy; |x-s_1|<\delta, |y|<\delta\}$ of $s_1$, given by

$$\tilde\beta(z) = B(z)+\varphi(z) - i(A(z)+\psi(z)) - B(s_1) + i(A(s_1)+\psi(s_1)).$$

We have

$$\tilde\beta'(s_1) = b(s_1)+\varphi'(s_1) - i(a(s_1)+\psi'(s_1)) = 0$$

and

$$\begin{aligned}\tilde\beta''(s_1) &= b'(s_1)+\varphi''(s_1) - i(a'(s_1)+\psi''(s_1))\\ &= b'(s_1) + K(1-\cos(s_1-s_{r+1})) - i(a'(s_1)+\psi''(s_1)).\end{aligned}$$

By taking $\delta>0$ small enough, we may suppose that $s_1$ is the only critical point of $\tilde\beta$ in $V_\delta$, and, for sufficiently large $K>0$, we have $\tilde\beta''(s_1)\neq 0$.

If $t\in I_\delta = [s_1-\delta, s_1+\delta] = \overline{V_\delta}\cap\mathbb{R}$, then

$$\Re\tilde\beta(t) = B(t)-B(s_1)+\varphi(t) \ge B(t)-B(s_1)\ge 0$$

if we take a smaller $\delta$, if necessary, since then $s_1$ is the unique point of global minimum of $B(t)-B(s_1)$, $t\in I_\delta$. Finally, taking $0<\delta<(s_{r+1}-s_1)/2$ we have $\varphi(t)>0$ if $t\in I_\delta$, $t\neq s_1$, $\varphi(s_1)=0$, and the same is true for $\Re\tilde\beta$.

We may now apply the stationary phase theorem (see [Sj]) to get that

$$\int_{I_\delta} e^{-n\beta(s)}\,ds = \sqrt{2\pi}(\tilde{\beta}''(0))^{-1/2}n^{-1/2}(1+\mathcal{O}(n^{-1})).$$

Let

$$\eta = K(1-\cos\delta)\min\{1-\cos(s_1-s_{r+1}-\delta), 1-\cos(s_1-s_{r+1}+\delta)\}.$$

We have $\eta > 0$ and if $t \in [0,t_1]\setminus I_\delta$, then

$$\Re\tilde{\beta}(t) \geq \varphi(t) \geq \eta.$$

Thus,

$$\left|\int_{[0,t_1]\setminus I_\delta} e^{-n\beta(s)}\,ds\right| \leq 2\pi e^{-n\eta}.$$

Hence,

$$\int_0^{t_1} e^{in(C(s)+\psi(s)+i\varphi(s))}ds$$
$$= \sqrt{2\pi}e^{-n(B(s_1)-i(A(s_1)+\psi(s_1)))}(\tilde{\beta}''(0))^{-1/2}n^{-1/2}(1+\mathcal{O}(n^{-1}))$$

which grows exponentially with $n \geq 1$ since $B(s_1) < 0$.

Let

$$h_{1,n}(t) = C_n\chi_1(t)e^{in(\psi(t)+i\varphi(t))},$$

where $\chi_1$ is a $C^\infty$ $2\pi-$periodic function satisfying

(1) $0 \leq \chi_1 \leq 1$,
(2) $\chi_1(t) = 0$ for $t_1 \leq t \leq 2\pi$,
(3) $\chi_1(t) = 1$ for $t \in J_1 \doteq [s_1-\delta_1, s_1+\delta_1] \subset I_\delta$.

Clearly,

$$\int_{t_{k-1}}^{t_k} e^{inC(s)}h_{1,n}(s)ds = \int_{t_r}^{2\pi} e^{in(C(s)-C(2\pi))}h_{1,n}(s)ds = 0, \quad 2 \leq k \leq r.$$

By (2.11) we have

$$\int_0^{t_1} e^{inC(s)}h_{1,n}(s)ds = \alpha_n - 1 \tag{2.12}$$

with

$$\alpha_n \doteq \int_0^{t_1} e^{inC(s)}(h_{1,n}(s)-g_n(s))ds$$
$$= C_n\int_0^{t_1} e^{in(C(s)+\psi(s)+i\varphi(s))}\,(\chi_1(s)-1)\,ds.$$

Thus

$$\left|\int_0^{t_1} e^{in(C(s)+\psi(s)+i\varphi(s))}\,(\chi_1(s)-1)\,ds\right|$$
$$= \left|\int_{[0,t_1]\setminus J_1} e^{in(C(s)+\psi(s)+i\varphi(s))}\,(\chi_1(s)-1)\,ds\right|$$
$$\leq \int_{[0,t_1]\setminus I_\delta} e^{-n\Re\beta(s)}\,ds \leq 2\pi e^{-n\eta},$$

and, therefore, $\alpha_n$ decays exponentially.

Setting $\lambda_n \doteq 1-\alpha_n$, we see that $|\lambda_n|$ is bounded above and below by positive constants and (2.12) can be rewritten as

$$\int_0^{t_1} e^{inC(s)} h_{1,n}(s)ds = -\lambda_n.$$

Defining

$$h_{r,n}(t) = D_n \chi_r(t) e^{in(\psi(t)+i\varphi(t))},$$

where $\chi_r$ is a $C^\infty$ $2\pi-$periodic function satisfying

(1) $0 \leq \chi_r \leq 1$,
(2) $\chi_r(t) = 0$ for $0 \leq t \leq t_r$,
(3) $\chi_r(t) = 1$ for $t$ near $s_r$,

we have

$$\int_{t_{k-1}}^{t_k} e^{inC(s)} h_{r,n}(s)ds = 0, \quad 1 \leq k \leq r.$$

In the same fashion as above one can prove that

$$\mu_n \doteq \int_{t_r}^{2\pi} e^{in(C(s)-C(2\pi))} h_{r,n}(s)ds$$

has its absolute value bounded above and below by positive constants.

Setting $h_n = h_{1,n}/\lambda_n + h_{r,n}/\mu_n$, we obtain that equations (2.7), (2.8), and (2.9) are satisfied with $\widehat{f}(\cdot,n)$ replaced by $h_n$. Furthermore,

$$\begin{aligned} |h_n| &\leq C(|h_{1,n}|+|h_{2,n}|) \leq C\left(\left|e^{in(\psi+i\varphi)}C_n\chi_1\right| + \left|e^{in(\psi+i\varphi)}D_n\chi_r\right|\right) \\ &\leq Ce^{-n\varphi}(|C_n|+|D_n|) \leq C(|C_n|+|D_n|), \end{aligned} \tag{2.13}$$

hence, $h_n$ is uniformly exponentially decaying.

Let $m = \min_{[0,2\pi]} B$ and $M = \max_{[0,2\pi]} B$. Note that $M \geq 0$. Fix $\sigma > M - m$ and for each $n \in \mathbb{N}$ let $\widetilde{f}_n$ be a partial sum of the Fourier series of $h_n$ satisfying

$$||\widetilde{f}_n - h_n||_{C^0} \leq e^{-\sigma n}. \tag{2.14}$$

By (2.13) and our choice of $\widetilde{f}_n$, we see that $\widetilde{f}_n$ is uniformly exponentially decaying as $n \to +\infty$.

Let $\widetilde{u}_n(t)$ represent the right-hand side of (2.3) with $\widehat{f}(s,n)$ replaced by $\widetilde{f}_n(s)$. We have

$$\begin{aligned} \widetilde{u}_n(t_0) &= (e^{2\pi inc_0}-1)^{-1}\int_0^{2\pi} e^{inC(s)}\widetilde{f}_n(s)\,ds \\ &= 1 + (e^{2\pi inc_0}-1)^{-1}\int_0^{2\pi} e^{inC(s)}\left[\widetilde{f}_n(s)-h_n(s)\right]\,ds = 1+\gamma_n, \quad \text{say.} \end{aligned}$$

Since $\sigma > -m$, $b_0 < 0$, and

$$\begin{aligned} |\gamma_n| &\leq \left|e^{2\pi inc_0}-1\right|^{-1}\int_0^{2\pi} e^{-nB(s)}\left|\widetilde{f}_n(s)-h_n(s)\right|ds \\ &\leq \frac{1}{e^{-2\pi nb_0}-1}\int_0^{2\pi} e^{-nm}e^{-n\sigma}\,ds \leq \frac{2\pi}{e^{-2\pi b_0}-1}e^{-(m+\sigma)n}, \end{aligned}$$

we see that $\gamma_n$ is exponentially decaying as $n \to +\infty$.

For $k = 1, \dots, r$,

$$\widetilde{u}_n(t_k) = e^{-inC(t_k)} \left[ \int_0^{t_k} e^{inC(s)} \left[ \widetilde{f}_n(s) - h_n(s) \right] ds + (e^{i2\pi n c_0} - 1)^{-1} \int_0^{2\pi} e^{inC(s)} \left[ \widetilde{f}_n(s) - h_n(s) \right] ds \right]$$

satisfies, as one can easily check,

$$|\widetilde{u}_n(t_k)| \leq 2\pi \frac{1}{1 - e^{2\pi b_0}} e^{(M-m-\sigma)n},$$

which is also exponentially decaying as $n \to +\infty$.

Let $f_n \doteq \widetilde{f_n}/(1+\gamma_n)$. Since $f_n$ is uniformly exponentially decaying as $n \to +\infty$,

$$f(t,x) \doteq \sum_{n \geq 1} f_n(t) e^{inx}$$

defines a real analytic function on $\mathbb{T}^2$ whose partial Fourier coefficients are given by $\widehat{f}(t,n) = f_n(t)$. Thus, if $u_n(t)$ stands for the right-hand side of (2.3), we get that $u_n(t_0) = 1$ and $u_n(t_k)$ is exponentially decaying as $n \to +\infty$, for $k = 1, \dots, r$. Moreover, by Lemma 2.11,

$$u(t,x) \doteq \sum_{n \geq 1} u_n(t) e^{inx}$$

defines an element of $\mathcal{D}'(\mathbb{T}^2)$. A simple computation shows that the partial Fourier coefficients of $u$, which are given by $\widehat{u}(t,n) = u_n(t)$, satisfy (2.2), whence, $Lu = f$.

Applying Lemma 2.11 one more time we see that the analytic singular support of $u$ is contained in $\{t_0\} \times \mathbb{T}^1$. In fact, $\operatorname{sing\,supp}_{\mathrm{A}}(u) = \{(t_0, 0)\}$ since $\widehat{u}(t_0, n) = 1$, that is, $u(t_0, x) = \delta_+(x)$. This concludes the proof of the case $r \geq 1$.

We now now say a few words about the case $r = 0$. All we need is $\widehat{u}(0,n) = 1$ and we are led to

$$\int_0^{2\pi} e^{inC(s)} \widehat{f}(s,n) ds = -1.$$

Instead of (2.10), take

$$g_n(t) = C_n e^{in(\psi(t) + i\varphi(t))},$$

where now

$$\varphi(t) = K(1 - \cos(t - s_1)),$$
$$\psi(t) = a_0 t - A(t) - a_0 \sin(t - s_1),$$

and $K$ and $C_n$ are similarly chosen as in the case $r \geq 1$. We emphasize that in this case there is no need for approximations, that is, we can take $\widehat{f}(t,n) = g_n(t) = h_n(t)$.

2.2. **Case** $b_0 = 0$**.** We will only treat the case $r \geq 1$. We divide the proof into the following three cases:

*Case* 1: $a_0 \in \mathbb{R} \setminus \mathbb{Q}$. This case is quite similar to the case $b_0 \neq 0$, since

$$d_n \doteq e^{2\pi i n c_0} - 1 = e^{2\pi i n a_0} - 1 \neq 0.$$

Note that $|d_n| \leq 2$.

The only difference here is that we choose $\widetilde{f}_n$ satisfying

$$||\widetilde{f}_n - h_n||_{C^0} \leq e^{-\sigma n}|d_n|$$

instead of (2.14), to get

$$\begin{aligned} |\gamma_n| &\leq \left|e^{2\pi i n c_0} - 1\right|^{-1} \int_0^{2\pi} e^{-nB(s)} \left|\widetilde{f}_n(s) - h_n(s)\right| ds \\ &\leq \int_0^{2\pi} e^{-nm} e^{-n\sigma}\, ds \leq 2\pi e^{-(m+\sigma)n}. \end{aligned}$$

The rest of the proof follows as in the case $b_0 \neq 0$.

*Case* 2: $a_0 \in \mathbb{Z}$. In this case we deal with

$$\widehat{u}(t,n) = \int_0^t e^{in[C(s)-C(t)]}\widehat{f}(s,n)\, ds + e^{-inC(t)} \tag{2.15}$$

instead of (2.3). Note that $e^{\pm inC(t)}$ is an element of $C^\omega(\mathbb{T})$.

We proceed as in the case $b_0 \neq 0$ to get $\widetilde{f}_n$ as in (2.14).

Let $u_n(t)$ be the right-hand side of (2.15) with $f_n \doteq \widetilde{f}_n$ instead of $\widehat{f}(\cdot, n)$.

We have $u_n(0) = 1$ and for $k = 1, \ldots, r$,

$$\begin{aligned} u_n(t_k) &= e^{-inC(t_k)} \left[\int_0^{t_k} e^{inC(s)} f_n(s) ds + 1\right] \\ &= e^{-inC(t_k)} \int_0^{t_k} e^{inC(s)} \left[f_n(s) - h_n(s)\right] ds, \end{aligned}$$

which leads to $|u_n(t_k)| \leq 2\pi e^{n(M-m-\sigma)}$.

The rest of the proof follows as in previous cases.

*Case* 3: $a_0 \in \mathbb{Q} \setminus \mathbb{Z}$. Write $a_0 = p/q$ where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and g.c.d.$\{p,q\} = 1$. Let $\mathcal{A} \doteq \{nq; n \in \mathbb{N}\}$ and $\mathcal{B} \doteq \mathbb{N} \setminus \mathcal{A}$.

Let $f_n, u_n$ be as in the previous case with $a_0 = q$. Define

$$\widetilde{f}_n = \begin{cases} f_n & \text{if } n \in \mathcal{A}, \\ 0 & \text{if } n \in \mathcal{B}, \end{cases} \qquad \widetilde{u}_n = \begin{cases} u_n & \text{if } n \in \mathcal{A}, \\ 0 & \text{if } n \in \mathcal{B}. \end{cases}$$

Then $(\partial_t + inC')\widetilde{u}_n = \widetilde{f}_n$ and

$$U_0(t,x) \doteq \sum_{n\in\mathbb{N}} \widetilde{u}_n(t) e^{inx} = \sum_{n\in\mathcal{A}} u_n(t) e^{inx} = \sum_{n\in\mathbb{N}} u_{nq}(t) e^{inqx}$$

defines an element of $\mathcal{D}'(\mathbb{T}^2)$ with $LU_0 \in C^\omega(\mathbb{T}^2)$ and $\operatorname{sing\,supp}_{\mathrm{A}}(U_0) = \{(t_0, 0)\}$.

We note that if $n \in \mathcal{B}$, then $e^{2\pi i n c_0} = e^{2\pi i n p/q} \neq 1$. Thus, we are allowed to proceed as in the case $b_0 \neq 0$ but working only with frequencies in $\mathcal{B}$. In this manner we obtain $U_1 \in \mathcal{D}'(\mathbb{T}^2)$ satisfying $\widehat{U}_1(t,n) = 1$ if $n \in \mathcal{B}$ and $\widehat{U}_1(t,n) = 0$ if $n \in \mathcal{A}$. Also, $LU_1 \in C^\omega(\mathbb{T}^2)$, $\operatorname{sing\,supp}_{\mathrm{A}}(U_1) = \{(t_0, 0)\}$.

Finally, taking $U = U_0 + U_1$ we get a singular solution of $L$ such that $U(0,x) = \delta_+(x)$.

This completes the proof of Lemma 2.1. □

## 3. Construction of singular solutions in the case $b \equiv 0$

In this section $L$ is a nonsingular, real vector field. Hence, if $Lu = 0$ and $\operatorname{sing\,supp}_{\mathrm{A}}(u)$ contains a point $p$, then it contains the complete integral curve of $L$ through $p$.

By using, if necessary, the change of variables $(t, x) \mapsto (t, x - A(t) + a_0 t)$, we may assume $a(t) = a_0$. Thus, the study of $Lu = f$ amounts to study the algebraic equation

$$i(m + a_0 n)\widehat{u} = \widehat{f}, \qquad (m, n) \in \mathbb{Z}^2.$$

We need a definition. We say that an irrational number $\alpha$ is exponential Liouville if there exist $\varepsilon > 0$ and a sequence $\{(m_k, n_k)\}_{k\geq 1}$ of points in $\mathbb{Z} \times \mathbb{N}$ satisfying $|\alpha + m_k/n_k| \leq e^{-\varepsilon n_k}$; the set of all such numbers is denoted by $\mathcal{EL}$.

We divide the proof into the following three cases:

*Case* 1: $a_0 \in \mathbb{Q}$. Write $a_0 = p/q$, $q \in \mathbb{N}$, $p \in \mathbb{Z}$ with g.c.d. $\{p, q\} = 1$.

Let $\mathcal{A} \doteq \{(m, n) \in \mathbb{Z}^2; m + a_0 n = 0\} = \{(m, n) \in \mathbb{Z}^2; qm + pn = 0\}$.

Let $v(m, n)$ be a tempered sequence of complex numbers satisfying, for some $\varepsilon > 0$ and $C > 0$, $|v(m, n)| \leq Ce^{-\varepsilon(|m|+|n|)}$, for all $(m, n) \notin \mathcal{A}$, i.e., $|qm + pn| \geq 1$.

Let $u(x, t) = \sum_{m,n\in\mathbb{Z}} v(m, n)e^{i(mt+nx)}$. It is clear that $u$ defines a distribution in $\mathbb{T}^2$.

Set $g(m, n) = i(m + a_0 n)\widehat{u}(m, n) = i(m + a_0 n)v(m, n)$. We see that $g(m, n) = 0$ for $(m, n) \in \mathcal{A}$ and for $(m, n) \notin \mathcal{A}$,

$$|g(m, n)| = |m + a_0 n||v(m, n)| \leq C|m + a_0 n|e^{-\varepsilon(|m|+|n|)}.$$

Since there exists $C_1 > 0$ such that $|m + a_0 n| \leq C_1 e^{\frac{\varepsilon}{2}(|m|+|n|)}$, we obtain

$$|g(m, n)| \leq C_0 e^{-\frac{\varepsilon}{2}(|m|+|n|)}$$

for all $(m, n) \in \mathbb{Z}^2$, where $C_0 = CC_1$.

Thus, $f(x, t) = \sum_{m,n\in\mathbb{Z}} g(m, n)e^{i(mt+nx)}$ defines an element of $C^\omega(\mathbb{T}^2)$ and $Lu = f$. In particular, if $v(m, n)$ is not exponentially decaying, then $u$ is a singular solution of $L$.

We now explain the location of the singularities. Given a distribution $v = v(x)$ in $\mathbb{T}^1$ with $v(x - \frac{2\pi}{q}) = v(x)$ there exists a distribution $u = u(x, t)$ in $\mathbb{T}^2$ such that $Lu = 0$ and $u(x, 0) = v(x)$. In fact, we may take $u(x, t) = v(qx - pt)$.

Alternatively, if $v(x) = \sum_{\ell\in\mathbb{Z}} \widehat{v}(\ell)e^{-iqx\ell}$, then

$$u(x, t) = v(x - \frac{p}{q}t) = \sum_{\ell\in\mathbb{Z}} \widehat{v}(\ell)e^{-i(qx-pt)\ell}$$

is such that $Lu = 0$ and $u(x, 0) = v(x)$. Note that if we set $Z = Z(x, t) = e^{i(-qx+pt)}$, then $u = h \circ Z$ where

$$h(\xi) = \sum_{\ell\in\mathbb{Z}} \widehat{v}(\ell)\xi^\ell.$$

In any case, $\operatorname{sing\,supp}_{\mathrm{A}}(u)$ is equal to the union of all integral curves of $L$ which meet $\operatorname{sing\,supp}_{\mathrm{A}}(v)$.

*Case* 2: $a_0 \in \mathbb{R} \setminus (\mathbb{Q} \cup \mathcal{EL})$. In this case we prove that $L$ is GAH. Suppose $Lu = f$ with $f \in C^\omega(\mathbb{T}^2)$ and $u \in \mathcal{D}'(\mathbb{T}^2)$.

There exists $k_0 \geq 0$ such that $|\widehat{f}(m, n)| \leq C_0 e^{-\varepsilon(|m|+|n|)}$ for all integers $m, n$ with $|m| + |n| \geq k_0$.

Since $a_0 \notin \mathcal{EL}$, there exists $k \geq k_0$ such that

$$\left|a_0 + \frac{m}{n}\right| > e^{-\frac{\varepsilon}{2}|n|}, \qquad |m| + |n| \geq k.$$

Hence, for all $(m,n)$ with $|m|+|n| \geq k$, we get

$$\begin{aligned} |\widehat{u}(m,n|) &= \frac{\left|\widehat{f}(m,n)\right|}{|m+a_0 n|} = \frac{1}{|n|}\frac{\left|\widehat{f}(m,n)\right|}{\left|a_0 + \frac{m}{n}\right|} \\ &\leq e^{\frac{\varepsilon}{2}|n|} C_0 e^{-\varepsilon(|m|+|n|)} = C_0 e^{-\frac{\varepsilon}{2}(|m|+|n|)}. \end{aligned}$$

*Case* 3: $a_0 \in (\mathbb{R} \setminus \mathbb{Q}) \cap \mathcal{EL}$. In this case there exist $\varepsilon > 0$ and a sequence $\{(m_k, n_k)\}_{k \geq 1}$ of points in $\mathbb{Z} \times \mathbb{N}$ satisfying $|a_0 + m_k/n_k| \leq e^{-\varepsilon n_k}$. Since $m_k/n_k \to -a_0$, there exist $c > 0$ and $k_o$ such that

$$\frac{1}{c} n_k \leq |m_k| \leq c n_k, \qquad \text{for all } k \geq k_o.$$

Therefore, there exist positive constants $\varepsilon$ and $C$ such that $|a_0 n_k + m_k| \leq Ce^{-\varepsilon(n_k + |m_k|)}$.

Let $\mathcal{A} = \{(m_k, n_k); k \geq 1\}$.

Let $v(m,n)$ be a tempered sequence of complex numbers satisfying, for some $\eta > 0$ and $C_0 > 0$, $|v(m,n)| \leq C_0 e^{-\eta(|m|+|n|)}$, for all $(m,n) \notin \mathcal{A}$.

Let $u(x,t) = \sum_{m,n \in \mathbb{Z}} v(m,n) e^{i(mt+nx)}$. It is clear that $u$ defines a distribution in $\mathbb{T}^2$.

Set $g(m,n) = i(m + a_0 n)\widehat{u}(m,n) = i(m + a_0 n) v(m,n)$.

Since $v(m,n)$ is tempered, there exists $C_1 > 0$ with $|v(m,n)| \leq C_1 e^{\frac{\varepsilon}{2}(|m|+|n|)}$.

Thus, for $(m,n) \in \mathcal{A}$

$$|g(m,n)| \leq CC_1 e^{-\frac{\varepsilon}{2}(|m|+|n|)}.$$

Now for $(m,n) \notin \mathcal{A}$,

$$|g(m,n)| = |m + a_0 n||v(m,n)| \leq C_0 |m + a_0 n| e^{-\eta(|m|+|n|)},$$

and since there exists $C_2 > 0$ such that $|m + a_0 n| \leq C_2 e^{\frac{\eta}{2}(|m|+|n|)}$, we obtain

$$|g(m,n)| \leq C_0 C_2 e^{-\frac{\eta}{2}(|m|+|n|)}.$$

Thus, $f(x,t) = \sum_{m,n \in \mathbb{Z}} g(m,n) e^{i(mt+nx)}$ defines an element of $C^\omega(\mathbb{T}^2)$ and $Lu = f$. In particular, if $v(m,n)$ is not exponentially decaying, then $u$ is a singular solution of $L$; in fact, $\operatorname{sing\,supp}_{\mathrm{A}}(u) = \mathbb{T}^2$ since the orbit is dense.

## 4. Appendix

In this Appendix we present a one-dimensional periodic version of Theorem 8.4.14 in [H].

**Lemma 4.1.** *Let $F$ be a nonempty closed subset of $\mathbb{T}^1$. Then there exists $v \in \mathcal{D}'(\mathbb{T}^1)$ such that $\mathrm{WF}_{\mathrm{A}}(v) = F \times \mathbb{R}_+$. Furthermore, $v$ can be chosen continuous and $v(x) = \sum_{n \geq 1} c_n e^{inx}$.*

*Proof.* Let $f(z) = z + (1-z)\log(1-z)$, where log is a branch of the logarithm defined in $\Omega = \mathbb{C} \setminus \{z = x + iy \in \mathbb{C}; y = 0, x \geq 1\}$. Note that $\lim_{z \in \Omega, z \to 1} f(z) = 1$.

It is easy to see that

$$f(z) = \sum_{n \geq 2} \frac{1}{n(n-1)} z^n, \qquad \text{for } |z| \leq 1.$$

Let $M = \sum_{n\geq 2} 1/n(n-1)$. Clearly, $|f(z)| \leq M$ in the closed unit disk $\bar{D}$.

Let $\{z_k\} \subset F$ be at most countable and dense in $F$ such that $z_k \neq z_l$ if $k \neq l$. Write $z_k = e^{ix_k}$ with $x_k \in [0, 2\pi)$.

Consider

$$u(z) = \sum_{k\geq 1} \frac{1}{3^k} f(ze^{-ix_k}), \qquad z \in \bar{D}.$$

We see that $u$ is holomorphic in $D$ and continuous in $\bar{D}$. In particular, $v = \flat(u)$, the boundary value of $u$, defines an element of $\mathcal{D}'(\mathbb{T}^1)$.

We have

$$u''(z) = \sum_{k\geq 1} \frac{e^{-2ix_k}}{3^k(1 - ze^{-ix_k})}, \qquad \text{for } z \in D.$$

For each $\ell \geq 1$ write

$$u''(z) = \sum_{k=1}^{\ell-1} \frac{e^{-2ix_k}}{3^k(1 - ze^{-ix_k})} + \frac{e^{-2ix_\ell}}{3^\ell(1 - ze^{-ix_\ell})} + R(z),$$

where

$$R(z) = \sum_{k\geq \ell+1} \frac{e^{-2ix_k}}{3^k(1 - ze^{-ix_k})}.$$

For $|z| < 1$, we have

$$|R(z)| \leq \frac{1}{1 - |z|} \sum_{k\geq \ell+1} \frac{1}{3^k} = \frac{1}{2 \cdot 3^\ell} \cdot \frac{1}{1 - |z|}.$$

For $z = te^{ix_\ell}$, $0 \leq t < 1$, we have

$$\left| \frac{e^{-2ix_\ell}}{3^\ell(1 - ze^{-ix_\ell})} \right| = \frac{1}{3^\ell} \frac{1}{1 - t}.$$

For such values of $z$ we obtain

$$\left| \sum_{k\geq \ell} \frac{e^{-2ix_k}}{3^k(1 - ze^{-ix_k})} \right| \geq \frac{1}{2 \cdot 3^\ell} \cdot \frac{1}{1 - |z|},$$

hence such a sum is unbounded when $z$ tends to $z_\ell$ along the ray $tz_\ell$.

Since

$$\sum_{k=1}^{\ell-1} \frac{e^{-2ix_k}}{3^k(1 - ze^{-ix_k})}$$

is holomorphic, we conclude that $u$ cannot be extended holomorphically in a neighborhood of $z_\ell$. We have shown that $F \times \mathbb{R}_+ \subset \mathrm{WF_A}(v)$.

Let $\zeta = e^{i\xi} \in \mathbb{T}^1 \setminus F$. Let $S = \{re^{ix}; |r - 1| < \varepsilon, |x - \xi| < \varepsilon\}$ be such that $\bar{S} \cap F = \emptyset$, for some $\varepsilon > 0$. Consider $D_1 = \{z; |z| \leq 1 + \varepsilon\} \setminus e^{-i\xi}S$. Since $D_1 \subset \Omega$ is compact we have $|f(z)| \leq C$ for some $C > 0$ and all $z \in D_1$.

Since, for $z \in S$,

$$\sum_{k\geq 1} \frac{1}{3^k} \left| f(ze^{-ix_k}) \right| \leq C/2,$$

it follows that $u$ can be continued holomorphically in a full neighborhood of $\zeta$. This means that $F \times \mathbb{R}_+ \supset \mathrm{WF_A}(v)$. $\square$

*Remark* 4.2. An analogous result is true for negative frequencies.

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