Abstract. In this paper we explore finite rank perturbations of unilateral weighted shifts $W_\alpha$. First, we prove that the subnormality of $W_\alpha$ is never stable under nonzero finite rank perturbations unless the perturbation occurs at the zeroth weight. Second, we establish that 2-hyponormality implies positive quadratic hyponormality, in the sense that the Maclaurin coefficients of $D_n(s) := \det P_n [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2] P_n$ are nonnegative, for every $n \geq 0$, where $P_n$ denotes the orthogonal projection onto the basis vectors $\{e_0, \ldots, e_n\}$. Finally, for $\alpha$ strictly increasing and $W_\alpha$ 2-hyponormal, we show that for a small finite-rank perturbation $\alpha'$ of $\alpha$, the shift $W_{\alpha'}$ remains quadratically hyponormal.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_\mathcal{H}$, where $N$ is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $T$ is subnormal, then $T$ is also hyponormal. Recall that given a bounded sequence of positive numbers $\alpha : 0, \alpha_1, \cdots$ (called weights), the (unilateral) weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2$. It is straightforward to check that $W_\alpha$ can never be normal, and that $W_\alpha$ is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bram-Halmos criterion for subnormality states that an operator $T$ is subnormal if and only if

$$\sum_{i,j} (T^ix_j, T^jx_i) \geq 0$$
for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2, 11.1.9]). It is easy to see that this is equivalent to the following positivity test:

\[
\begin{pmatrix}
I & T^* & \cdots & T^{*k} \\
T & T^*T & \cdots & T^{*k}T \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^*T^k & \cdots & T^{*k}T^k
\end{pmatrix} \geq 0 \quad \text{for all } k \geq 1.
\] (1.1)

Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (1.1) for all $k$. Let $[A, B] := AB - BA$ denote the commutator of two operators $A$ and $B$, and define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix

\[
M_k(T) := ([T^*, T])_{i,j=1}^k
\] (1.2)

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1.1); the Bram-Halmos criterion can then be rephrased to say that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([16]).

Recall ([1, 16, 5]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be weakly $k$-hyponormal if

\[
LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}
\]

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is weakly positive, i.e. ([16]),

\[
\begin{pmatrix}
M_k(T) & \lambda_0 x \\
\vdots & \vdots \\
\lambda_k x & \lambda_k x
\end{pmatrix} \geq 0 \quad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}.
\] (1.3)

If $k = 2$, then $T$ is said to be quadratically hyponormal, and if $k = 3$, then $T$ is said to be cubically hyponormal. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be polynomially hyponormal if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that $k$-hyponormal $\Rightarrow$ weakly $k$-hyponormal, but the converse is not true in general.

The classes of (weakly) $k$-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([7, 8, 10, 11, 12, 13, 16, 19, 22]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle; in fact, even subnormality for Toeplitz operators has not been characterized (cf. [20, 9]). For weighted shifts, positive results appear in [17] and [12], although no concrete example of a weighted shift which is polynomially hyponormal and not subnormal has yet been found (the existence of such weighted shifts was established in [17] and [18]).

In the present paper we renew our efforts to help describe the above-mentioned gap between subnormality and hyponormality, with particular emphasis on polynomial hyponormality. We focus on the class of unilateral weighted shifts, and initiate a study of how the above-mentioned notions behave under finite perturbations of the weight sequence. We first obtain the following three concrete results.
(i) the subnormality of $W_\alpha$ is never stable under nonzero finite rank perturbations unless the perturbation is confined to the zeroth weight (Theorem 2.1);
(ii) 2-hyponormality implies positive quadratic hyponormality, in the sense that the Maclaurin coefficients of $D_n(s) := \det P_n \left[ (W_\alpha + sW_2^2)^*, W_\alpha + sW_2^2 \right] P_n$ are nonnegative, for every $n \geq 0$, where $P_n$ denotes the orthogonal projection onto the basis vectors $\{e_0, \cdots, e_n\}$ (Theorem 2.2); and
(iii) if $\alpha$ is strictly increasing and $W_\alpha$ is 2-hyponormal, then for $\alpha'$ a small perturbation of $\alpha$, the shift $W_{\alpha'}$ remains positively quadratically hyponormal (Theorem 2.3).

Along the way we establish two related results, each of independent interest:
(iv) an integrality criterion for a subnormal weighted shift to have an $n$-step subnormal extension (Theorem 6.1); and
(v) a proof that the sets of $k$-hyponormal and weakly $k$-hyponormal operators are closed in the strong operator topology (Proposition 6.7).

2. Statement of main results

C. Berger’s characterization of subnormality for unilateral weighted shifts (cf. [21, III.8.16]) states that $W_\alpha$ is subnormal if and only if there exists a Borel probability measure $\mu$ (the so-called Berger measure of $W_\alpha$) supported in $[0, ||W_\alpha||^2]$, with $||W_\alpha||^2 \in \text{supp } \mu$, such that
$$\gamma_n = \int t^n d\mu(t) \quad \text{for all } n \geq 0.$$ Given an initial segment of weights $\alpha : \alpha_0, \cdots, \alpha_m$, the sequence $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\hat{\alpha}_i = \alpha_i$ ($i = 0, \cdots, m$) is said to be recursively generated by $\alpha$ if there exist $r \geq 1$ and $\varphi_0, \cdots, \varphi_{r-1} \in \mathbb{R}$ such that
$$\gamma_{n+r} = \varphi_0 \gamma_n + \cdots + \varphi_{r-1} \gamma_{n+r-1} \quad \text{(for all } n \geq 0),$$ where $\gamma_0 := 1$, $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ ($n \geq 1$). In this case $W_\hat{\alpha}$ with weights $\hat{\alpha}$ is said to be recursively generated. If we let
$$g(t) := t^r - (\varphi_{r-1} t^{r-1} + \cdots + \varphi_0),$$ then $g$ has $r$ distinct real roots $0 \leq s_0 < \cdots < s_{r-1}$ ([11, Theorem 3.9]). Let
$$V := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ s_0 & s_1 & \cdots & s_{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{r-1} & s_{r-1} & \cdots & 1 \end{pmatrix}$$ and let
$$\begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{r-1} \end{pmatrix} := V^{-1} \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$ If the associated recursively generated weighted shift $W_{\hat{\alpha}}$ is subnormal, then its Berger measure is of the form
$$\mu := \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{s_{r-1}}.$$
For example, given \( \alpha_0 < \alpha_1 < \alpha_2 \), \( W_{(\alpha_0, \alpha_1, \alpha_2)} \) is the recursive weighted shift whose weights are calculated according to the recursive relation

\[
\alpha_{n+1}^2 = \varphi_1 + \varphi_0 \frac{1}{\alpha_n},
\]

where

\[
\varphi_0 = -\frac{\alpha_0^2 \alpha_1^2 (\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \varphi_1 = \frac{\alpha_1^2 (\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.
\]

In this case, \( W_{(\alpha_0, \alpha_1, \alpha_2)} \) is subnormal with 2–atomic Berger measure. Let \( W_{x(\alpha_0, \alpha_1, \alpha_2)} \) denote the weighted shift whose weight sequence consists of the initial weight \( x \) followed by the weight sequence of \( W_{(\alpha_0, \alpha_1, \alpha_2)} \).

By the Density Theorem ([11, Theorem 4.2 and Corollary 4.3]), we know that if \( W_\alpha \) is a subnormal weighted shift with weights \( \alpha = \{\alpha_n\} \) and \( \epsilon > 0 \), then there exists a nonzero compact operator \( K \) with \( \|K\| < \epsilon \) such that \( W_\alpha + K \) is a recursively generated subnormal weighted shift; in fact \( W_\alpha + K = W_{\alpha(m)}^\wedge \) for some \( m \geq 1 \), where \( \alpha(m) : \alpha_0, \ldots, \alpha_m \). The following result shows that \( K \) cannot generally be taken to be of finite rank.

**Theorem 2.1** (Finite Rank Perturbations of Subnormal Shifts). If \( W_\alpha \) is a subnormal weighted shift, then there exists no nonzero finite rank operator \( F \) (\( \neq cP(e_0) \)) such that \( W_\alpha + F \) is a subnormal weighted shift. Concretely, suppose \( W_\alpha \) is a subnormal weighted shift with weight sequence \( \alpha = \{\alpha_n\}_{n=0}^\infty \) and assume \( \alpha' = \{\alpha'_n\} \) is a nonzero perturbation of \( \alpha \) in a finite number of weights except the initial weight. Then \( W_{\alpha'} \) is not subnormal.

We next consider the self-commutator \([W_\alpha + s W_\alpha^2]^*, W_\alpha + s W_\alpha^2]\). Let \( W_\alpha \) be a hyponormal weighted shift. For \( s \in \mathbb{C} \), we write

\[
D(s) := [(W_\alpha + s W_\alpha^2)^*, W_\alpha + s W_\alpha^2]
\]

and we let

\[
D_n(s) := P_n[(W_\alpha + s W_\alpha^2)^*, W_\alpha + s W_\alpha^2]P_n
\]

\[
= \begin{pmatrix}
q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\
0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\
0 & 0 & q_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\
0 & 0 & 0 & \cdots & r_{n-1} & q_n
\end{pmatrix},
\]

where \( P_n \) is the orthogonal projection onto the subspace generated by \( \{e_0, \ldots, e_n\} \),

\[
\begin{aligned}
q_n &= u_n + |s|^2 v_n, \\
r_n &= \sqrt{s^2 w_n}, \\
u_n &= \alpha_n^2 - \alpha_{n-1}^2, \\
v_n &= \alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_{n-2}^2, \\
w_n &= \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2)^2,
\end{aligned}
\]

and, for notational convenience, \( \alpha_{-2} = \alpha_{-1} = 0 \). Clearly, \( W_\alpha \) is quadratically hyponormal if and only if \( D_n(s) \geq 0 \) for all \( s \in \mathbb{C} \) and all \( n \geq 0 \). Let \( d_n() := \)
\[ \text{det} (D_n(\cdot)) \]. Then \( d_n \) satisfies the following 2-step recursive formula:

\begin{equation}
\begin{aligned}
&d_0 = q_0, \quad d_1 = q_0 q_1 - |r_0|^2, \quad d_{n+2} = q_n + 2 d_{n+1} - |r_{n+1}|^2 d_n.
\end{aligned}
\end{equation}

If we let \( t := |s|^2 \), we observe that \( d_n \) is a polynomial in \( t \) of degree \( n + 1 \), and if we write \( d_n \equiv c \sum_{i=0}^{n+1} c(n, i) t^i \), then the coefficients \( c(n, i) \) satisfy a double-indexed recursive formula, namely

\begin{equation}
\begin{aligned}
c(n + 2, i) &= u_{n+2} c(n + 1, i) + v_{n+2} c(n + 1, i - 1) - w_{n+1} c(n, i - 1), \\
c(n, 0) &= u_0 \cdots u_n, \\
c(n, n + 1) &= v_0 \cdots v_n, \\
c(1, 1) &= u_1 v_0 + v_1 u_0 - w_0
\end{aligned}
\end{equation}

\((n \geq 0, i \geq 1)\). We say that \( W_\alpha \) is positively quadratically hyponormal if \( c(n, i) \geq 0 \) for every \( n \geq 0 \), \( 0 \leq i \leq n + 1 \) (cf. \cite{9}). Evidently, positively quadratically hyponormal \( \implies \) quadratically hyponormal. The converse, however, is not true in general (cf. \cite{3}).

The following theorem establishes a useful relation between 2-hyponormality and positive quadratic hyponormality.

**Theorem 2.2.** Let \( \alpha \equiv \{\alpha_n\}_{n=0}^\infty \) be a weight sequence and assume that \( W_\alpha \) is 2-hyponormal. Then \( W_\alpha \) is positively quadratically hyponormal. More precisely, if \( W_\alpha \) is 2-hyponormal, then

\begin{equation}
\begin{aligned}
c(n, i) &\geq v_0 \cdots v_{i-1} u_i \cdots u_n \quad (n \geq 0, \ 0 \leq i \leq n + 1).
\end{aligned}
\end{equation}

In particular, if \( \alpha \) is strictly increasing and \( W_\alpha \) is 2-hyponormal, then the Maclaurin coefficients of \( d_n(t) \) are positive for all \( n \geq 0 \).

If \( W_\alpha \) is a weighted shift with weight sequence \( \alpha = \{\alpha_n\}_{n=0}^\infty \), then the *moments* of \( W_\alpha \) are usually defined by \( \beta_0 := 1, \beta_{n+1} := \alpha_n \beta_n \) \((n \geq 0)\) \cite{23}; however, we prefer to reserve this term for the sequence \( \gamma_n := \beta_n^2 \) \((n \geq 0)\). A criterion for \( k \)-hyponormality can be given in terms of these moments (\cite[Theorem 4]{17}): if we build a \((k + 1) \times (k + 1)\) Hankel matrix \( A(n; k) \) by

\begin{equation}
\begin{aligned}
A(n; k) := \begin{pmatrix}
\gamma_n & \cdots & \cdots & \cdots & \cdots & \gamma_{n+k} \\
\gamma_{n+1} & \cdots & \cdots & \cdots & \cdots & \gamma_{n+k+1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\gamma_{n+k} & \gamma_{n+k+1} & \cdots & \cdots & \cdots & \gamma_{n+2k}
\end{pmatrix}
\end{aligned}
\end{equation}

\((n \geq 0)\), then

\begin{equation}
W_\alpha \text{ is } k \text{-hyponormal } \iff \text{det} (A(n; k)) \geq 0 \quad (n \geq 0).
\end{equation}

In particular, for \( \alpha \) strictly increasing, \( W_\alpha \) is 2-hyponormal if and only if

\begin{equation}
\begin{aligned}
\text{det} \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \gamma_{n+2} \\
\gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\
\gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4}
\end{pmatrix} &\geq 0 \quad (n \geq 0).
\end{aligned}
\end{equation}

One might conjecture that if \( W_\alpha \) is a \( k \)-hyponormal weighted shift whose weight sequence is strictly increasing, then \( W_\alpha \) remains weakly \( k \)-hyponormal under a small perturbation of the weight sequence. We will show below that this is true for \( k = 2 \) (Theorem 2.3).

In \cite[Theorem 4.3]{12}, it was shown that the gap between 2-hyponormality and quadratic hyponormality can be detected by unilateral shifts with a weight sequence \( \alpha : \sqrt{a}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\infty \). In particular, there exists a maximum value \( H_2 \equiv H_2(a, b, c) \)
of $x$ that makes $W_{\sqrt{x}}(\sqrt{\sigma}, \sqrt{\sigma})^\wedge$ 2-hyponormal; $H_2$ is called the modulus of 2-hyponormality (cf. [12]). Any value of $x > H_2$ yields a non-2-hyponormal weighted shift. However, if $x - H_2$ is small enough, $W_{\sqrt{x}}(\sqrt{\sigma}, \sqrt{\sigma})^\wedge$ is still quadratically hyponormal. The following theorem shows that, more generally, for finite rank perturbations of weighted shifts with strictly increasing weight sequences, there always exists a gap between 2-hyponormality and quadratic hyponormality.

**Theorem 2.3** (Finite Rank Perturbations of 2-Hyponormal Shifts). Let $\alpha = \{\alpha_n\}_{n=0}^\infty$ be a strictly increasing weight sequence. If $W_\alpha$ is 2-hyponormal, then $W_\alpha$ remains positively quadratically hyponormal under a small nonzero finite rank perturbation of $\alpha$.

### 3. Proof of Theorem 2.1

**Proof of Theorem 2.1.** It suffices to show that if $T$ is a weighted shift whose restriction to $\{e_n, e_{n+1}, \ldots\}$ $(n \geq 2)$ is subnormal, then there is at most one $\alpha_{n-1}$ for which $T$ is subnormal.

Let $W := T|_{\{e_n, e_{n+1}, \ldots\}}$ and $S := T|_{\{e_{n-1}, e_{n+1}, \ldots\}}$, where $n \geq 2$. Then $W$ and $S$ have weights $\alpha_k(W) := \alpha_{k+n-1}$ and $\alpha_k(S) := \alpha_{k+n}$ ($k \geq 0$). Thus the corresponding moments are related by the equation

$$
\gamma_k(S) = \alpha_2 \cdots \alpha_{n+k-1} = \frac{\gamma_{k+1}(W)}{\alpha_n^{2}}.
$$

We now adapt the proof of [7, Proposition 8]. Suppose $S$ is subnormal with associated Berger measure $\mu$. Then $\gamma_k(S) = \int_0^{||T||^2} t^k d\mu$. Thus $W$ is subnormal if and only if there exists a probability measure $\nu$ on $[0, ||T||^2]$ such that

$$
\frac{1}{\alpha_n^{2}} \int_0^{||T||^2} t^{k+1} d\nu(t) = \int_0^{||T||^2} t^k d\mu(t) \quad \text{for all} \ k \geq 0,
$$

which readily implies that $t d\nu = \alpha_n^{2} d\mu$. Thus $W$ is subnormal if and only if the formula

$$
d\nu := \lambda \cdot \delta_0 + \frac{\alpha_n^{2}}{t} d\mu
$$

defines a probability measure for some $\lambda \geq 0$, where $\delta_0$ is the point mass at the origin. In particular $\frac{1}{t} \in L^1(\mu)$ and $\mu(\{0\}) = 0$ whenever $W$ is subnormal. If we repeat the above argument for $W$ and $V := T|_{\{e_{n-2}, e_{n-1}, \ldots\}}$, then we should have that $\nu(\{0\}) = 0$ whenever $V$ is subnormal. Therefore we can conclude that if $V$ is subnormal, then $\lambda = 0$, and hence

$$
d\nu = \frac{\alpha_n^{2}}{t} d\mu.
$$

Thus we have

$$
1 = \int_0^{||T||^2} d\nu(t) = \alpha_n^{2} \int_0^{||T||^2} \frac{1}{t} d\mu(t),
$$

so that

$$
\alpha_n^{2} = \left(\int_0^{||T||^2} \frac{1}{t} d\mu(t)\right)^{-1},
$$

(3.3)
which implies that \( \alpha_{n-1} \) is determined uniquely by \( \{ \alpha_n, \alpha_{n+1}, \cdots \} \) whenever \( T \) is subnormal. This completes the proof. \( \square \)

Theorem 2.1 says that a nonzero finite rank perturbation of a subnormal shift is never subnormal unless the perturbation occurs at the initial weight. However, this is not the case for \( k \)-hyponormality. To see this we use a close relative of the Bergman shift \( B_+ \) (whose weights are given by \( \alpha = \{ \frac{n+1}{n+2} \}_{n=0}^\infty \)); it is well known that \( B_+ \) is subnormal.

**Example 3.1.** For \( x > 0 \), let \( T_x \) be the weighted shift whose weights are given by

\[
\alpha_0 := \sqrt{\frac{T}{2}}, \quad \alpha_1 := \sqrt{x}, \quad \text{and} \quad \alpha_n := \sqrt{\frac{n+1}{n+2}} \quad (n \geq 2).
\]

Then we have:

(i) \( T_x \) is subnormal \( \iff \) \( x = \frac{2}{3} \);  
(ii) \( T_x \) is 2-hyponormal \( \iff \) \( \frac{35 - \sqrt{129}}{80} \leq x \leq \frac{24}{35} \).

**Proof.** Assertion (i) follows from Theorem 2.1. For assertion (ii) we use (2.12): \( T_x \) is 2-hyponormal if and only if

\[
\det \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2}x & \frac{1}{2}x^2 \\
\frac{1}{2}x & \frac{3}{8}x & \frac{3}{8}x & \frac{3}{8}x^2 \\
\frac{1}{2}x^2 & \frac{3}{8}x^2 & \frac{3}{10}x & \frac{3}{10}x^2 \\
\frac{1}{2}x^3 & \frac{3}{8}x^3 & \frac{3}{10}x^2 & \frac{1}{4}x^4
\end{pmatrix} \geq 0 
\]

or equivalently, \( \frac{35 - \sqrt{129}}{80} \leq x \leq \frac{24}{35} \). \( \square \)

For perturbations of recursive subnormal shifts of the form \( W_{(\sqrt{a}, \sqrt{b}, \sqrt{c})} \), subnormality and 2-hyponormality coincide.

**Theorem 3.2.** Let \( \alpha = \{ \alpha_n \}_{n=0}^\infty \) be recursively generated by \( \sqrt{a}, \sqrt{b}, \sqrt{c} \). If \( T_x \) is the weighted shift whose weights are given by \( \alpha_x : \alpha_0, \cdots, \alpha_{j-1}, \sqrt{a}, \alpha_j, \cdots \), then we have

\[
T_x \text{ is subnormal} \iff T_x \text{ is 2-hyponormal} \iff \begin{cases} 
x = \alpha_j^2 & \text{if } j \geq 1; \\
x \leq a & \text{if } j = 0.
\end{cases}
\]

**Proof.** Since \( \alpha \) is recursively generated by \( \sqrt{a}, \sqrt{b}, \sqrt{c} \), we have that \( \alpha_0^2 = a, \alpha_1^2 = b, \alpha_2^2 = c \),

\[
\alpha_3^2 = \frac{b(c^2 - 2ac + ab)}{c(b - a)}, \quad \text{and} \quad \\
\alpha_4^2 = \frac{bc^3 - 4abc^2 + 2ab^2c + a^2bc - a^2b^2 + a^2c^2}{(b - a)(c^2 - 2ac + ab)}.
\]

**Case 1** \( (j = 0) \): It is evident that \( T_x \) is subnormal if and only if \( x \leq a \). For 2-hyponormality observe by (2.12) that \( T_x \) is 2-hyponormal if and only if

\[
\det \begin{pmatrix}
1 & x & bx \\
x & bx & bxc \\
bx & bxc & \alpha_3^2 bxc
\end{pmatrix} \geq 0,
\]

or equivalently, \( x \leq a \).
Case 2 ($j \geq 1$): Without loss of generality we may assume that $j = 1$ and $a = 1$. Thus $\alpha_1 = \sqrt{x}$. Then by Theorem 2.1, $T_x$ is subnormal if and only if $x = b$. On the other hand, by (2.12), $T_x$ is 2-hyponormal if and only if
\[
\det \begin{pmatrix} 1 & x & cx \\ 1 & x & cx \\ x & cx & \alpha^2 cx \end{pmatrix} \geq 0 \quad \text{and} \quad \det \begin{pmatrix} 1 & x & cx \\ x & cx & \alpha^2 cx \\ cx & \alpha^2 cx & \alpha^2 \alpha^2 cx \end{pmatrix} \geq 0.
\]
Thus a direct calculation with the specific forms of $\alpha_3, \alpha_4$ given in (3.4) shows that $T_x$ is 2-hyponormal if and only if $(x - b) \left( x - \frac{b(c^2 - 2c + b)}{b - 1} \right) \leq 0$ and $x \leq b$. Since $b = \frac{b(c^2 - 2c + b)}{b - 1}$, it follows that $T_x$ is 2-hyponormal if and only if $x = b$. This completes the proof. \hfill \Box

4. Proof of Theorem 2.2

With the notation in (2.6), we let
\[
p_n := u_n v_{n+1} - w_n \quad (n \geq 0).
\]
We then have:

**Lemma 4.1.** If $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ is a strictly increasing weight sequence, then the following statements are equivalent:

(i) $W_\alpha$ is 2-hyponormal;
(ii) $\alpha_{n+2}^2 (u_{n+1} + u_{n+2})^2 \leq u_{n+1} v_{n+2}$ \quad $(n \geq 0)$;
(iii) $\alpha_{n+2} \frac{u_{n+2}}{v_{n+3}} \leq \frac{u_{n+1}}{u_{n+2}}$ \quad $(n \geq 0)$;
(iv) $p_n \geq 0$ \quad $(n \geq 0)$.

**Proof.** This follows from a straightforward calculation. \hfill \Box

**Proof of Theorem 2.2.** If $\alpha$ is not strictly increasing, then $\alpha$ is flat, by the argument of [7, Corollary 6], i.e., $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$. Then
\[
D_n(s) = \left( \frac{\alpha_{n+2}^2 + |s|^2 \alpha_{n+1}^2}{\alpha_{n+2}} \right) \oplus 0_n
\]
(cf. (2.5)), so that (2.9) is evident. Thus we may assume that $\alpha$ is strictly increasing, so that $u_n > 0$, $v_n > 0$ and $w_n > 0$ for all $n \geq 0$. Recall that if we write $d_n(t) := \sum_{i=0}^{n+1} c(n, i) t^i$, then the $c(n, i)$’s satisfy the following recursive formulas (cf. (2.8)):
\[
c(n + 2, i) = u_{n+2} c(n + 1, i) + v_{n+2} c(n + 1, i - 1) \\
- w_{n+1} c(n, i - 1) \quad (n \geq 0, 1 \leq i \leq n).
\]
Also, $c(n, n+1) = v_0 \cdots v_n$ (again by (2.8)) and $p_n := u_n v_{n+1} - w_n \geq 0$ $(n \geq 0)$, by Lemma 4.1. A straightforward calculation shows that
\[
d_0(t) = u_0 + v_0 t;
\]
\[
d_1(t) = u_0 u_1 + (v_0 u_1 + p_0) t + v_0 v_1 t^2;
\]
\[
d_2(t) = u_0 u_1 u_2 + (v_0 u_1 u_2 + u_0 p_1 + u_2 p_0) t + (v_0 v_1 u_2 + v_0 p_1 + v_2 p_0) t^2 + v_0 v_1 v_2 t^3.
\]
Evidently,
\[
c(n, i) \geq 0 \quad (0 \leq n \leq 2, 0 \leq i \leq n + 1).
\]
Define
\[ \beta(n, i) := c(n, i) - v_0 \cdots v_{i-1} u_i \cdots u_n \quad (n \geq 1, 1 \leq i \leq n). \]

For every \( n \geq 1 \), we now have
\[ c(n, i) = \begin{cases} u_0 \cdots u_n & (i = 0), \\ v_0 \cdots v_{i-1} u_i \cdots u_n + \beta(n, i) & (1 \leq i \leq n), \\ v_0 \cdots v_n & (i = n+1). \end{cases} \tag{4.5} \]

For notational convenience we let \( \beta(n, 0) := 0 \) for every \( n \geq 0 \).

Claim 1. For \( n \geq 1 \),
\[ c(n, n) \geq u_n c(n-1, n) \geq 0. \tag{4.6} \]

Proof of Claim 1. We use mathematical induction. For \( n = 1 \),
\[ c(1, 1) = v_0 u_1 + p_0 \geq u_1 c(0, 1) \geq 0, \]
and
\[ c(n + 1, n + 1) = u_{n+1} c(n, n + 1) + v_{n+1} c(n, n) - w_n c(n-1, n) \]
\[ \geq u_{n+1} c(n, n + 1) + v_{n+1} u_n c(n - 1, n) - w_n c(n - 1, n) \]
(by the inductive hypothesis)
\[ = u_{n+1} c(n, n + 1) + p_n c(n - 1, n) \]
\[ \geq u_{n+1} c(n, n + 1), \]
which proves Claim 1.

Claim 2. For \( n \geq 2 \),
\[ \beta(n, i) \geq u_n \beta(n - 1, i) \geq 0 \quad (0 \leq i \leq n - 1). \tag{4.7} \]

Proof of Claim 2. We use mathematical induction. If \( n = 2 \) and \( i = 0 \), this is trivial. Also,
\[ \beta(2, 1) = u_0 p_1 + u_2 p_0 = u_0 p_1 + u_2 \beta(1, 1) \geq u_2 \beta(1, 1) \geq 0. \]
Assume that (4.7) holds. We shall prove that
\[ \beta(n + 1, i) \geq u_{n+1} \beta(n, i) \geq 0 \quad (0 \leq i \leq n). \]

For,
\[ \beta(n + 1, i) + v_0 \cdots v_{i-1} u_i \cdots u_{n+1} = c(n + 1, i) \quad \text{(by (4.2))} \]
\[ = u_{n+1} c(n, i) + v_{n+1} c(n, i - 1) - w_n c(n - 1, i - 1) \]
\[ = u_{n+1} \left( \beta(n, i) + v_0 \cdots v_{i-1} u_i \cdots u_n \right) \]
\[ + v_{n+1} \left( \beta(n, i - 1) + v_0 \cdots v_{i-2} u_{i-2} \cdots u_n \right) \]
\[ - w_n \left( \beta(n - 1, i - 1) + v_0 \cdots v_{i-2} u_{i-2} \cdots u_{n-1} \right), \]
so that
\[
\beta(n + 1, i) = u_{n+1} \beta(n, i) + v_{n+1} \beta(n, i - 1) - w_n \beta(n - 1, i - 1) \\
+ v_0 \cdots v_{i-2} u_{i-1} \cdots u_{n-1} (u_n v_{n+1} - w_n) \\
= u_{n+1} \beta(n, i) + v_{n+1} \beta(n, i - 1) - w_n \beta(n - 1, i - 1) \\
+ (v_0 \cdots v_{i-2} u_{i-1} \cdots u_{n-1}) p_n \\
\geq u_{n+1} \beta(n, i) + v_{n+1} u_n \beta(n - 1, i - 1) - w_n \beta(n - 1, i - 1) \\
\quad \text{(by the inductive hypothesis and Lemma 4.1;)} \\
\quad \text{observe that } i - 1 \leq n - 1, \text{ so (4.7) applies)} \\
\geq u_{n+1} \beta(n, i),
\]
which proves Claim 2.

By Claim 2 and (4.5), we can see that \(c(n, i) \geq 0\) for all \(n \geq 0\) and \(1 \leq i \leq n - 1\).
Therefore (4.4), (4.5), Claim 1 and Claim 2 imply
\[
c(n, i) \geq v_0 \cdots v_{i-1} u_i \cdots u_n \quad (n \geq 0, \ 0 \leq i \leq n + 1).
\]
This completes the proof.

5. PROOF OF THEOREM 2.3

To prove Theorem 2.3 we need:

**Lemma 5.1 ([13] Lemma 2.3).** Let \(\alpha = \{\alpha_n\}_{n=0}^\infty\) be a strictly increasing weight sequence. If \(W_\alpha\) is 2-hyponormal, then the sequence of quotients
\[
\Theta_n := \frac{u_{n+1}}{u_{n+2}} \quad (n \geq 0)
\]
is bounded away from 0 and from \(\infty\). More precisely,
\[
1 \leq \Theta_n \leq \frac{u_1}{u_2} \left(\frac{|W_\alpha|^2}{\alpha_0 \alpha_1}\right)^2 \quad \text{for sufficiently large } n.
\]
In particular, \(\{u_n\}_{n=0}^\infty\) is eventually decreasing.

**Proof of Theorem 2.3.** By Theorem 2.2, \(W_\alpha\) is strictly positively quadratically hyponormal, in the sense that all coefficients of \(d_n(t)\) are positive for all \(n \geq 0\). Note that finite rank perturbations of \(\alpha\) affect a finite number of values of \(u_n\), \(v_n\) and \(w_n\). More concretely, if \(\alpha'\) is a perturbation of \(\alpha\) in the weights \(\{\alpha_0, \cdots, \alpha_N\}\), then \(u_n\), \(v_n\), \(w_n\) and \(p_n\) are invariant under \(\alpha'\) for \(n \geq N + 3\). In particular, \(p_n \geq 0\) for \(n \geq N + 3\).

**Claim 1.** For \(n \geq 3, \ 0 \leq i \leq n + 1,\)
\[
c(n, i) = u_n c(n - 1, i) + p_{n-1} c(n - 2, i - 1) \\
+ \sum_{k=4}^{n} p_{k-2} \left(\prod_{j=k}^{n} v_j\right) c(k - 3, i - n + k - 2) + v_n \cdots v_3 p_{i-n+1},
\]

where
\[
\rho_{i-n+1} = \begin{cases} 
0 & (i < n - 1), \\
u_0 p_1 & (i = n - 1), \\
v_0 p_1 + v_2 p_0 & (i = n), \\
v_0 v_1 v_2 & (i = n + 1)
\end{cases}
\]

(cf. [12] Proof of Theorem 4.3).

Proof of Claim 1. We use induction. For \( n = 3, \ 0 \leq i \leq 4, \)
\[
c(3, i) = u_3 c(2, i) + v_3 c(2, i - 1) - w_2 c(1, i - 1)
\]
\[
= u_3 c(2, i) + v_3 \left( u_2 c(1, i - 1) + v_2 c(1, i - 2) - w_1 c(0, i - 2) \right)
\]
\[
- w_2 c(1, i - 1)
\]
\[
= u_3 c(2, i) + p_2 c(1, i - 1) + v_3 \left( v_2 c(1, i - 2) - w_1 c(0, i - 2) \right)
\]
\[
= u_3 c(2, i) + p_2 c(1, i - 1) + v_3 \rho_{i-2},
\]

where by (4.3),
\[
\rho_{i-2} = \begin{cases} 
0 & (i < 2), \\
u_0 p_1 & (i = 2), \\
v_0 p_1 + v_2 p_0 & (i = 3), \\
v_0 v_1 v_2 & (i = 4).
\end{cases}
\]

Now,
\[
c(n + 1, i) = u_{n+1} c(n, i) + v_{n+1} c(n, i - 1) - w_n c(n - 1, i - 1)
\]
\[
= u_{n+1} c(n, i) + v_{n+1} \left( u_n c(n - 1, i - 1) + p_{n-1} c(n - 2, i - 2) \right)
\]
\[
+ \sum_{k=4}^{n} p_{k-2} \left( \prod_{j=k}^{n} v_j \right) c(k - 3, i - n + k - 3) + v_n \cdots v_3 \rho_{i-n}
\]
\[
- w_n c(n - 1, i - 1)
\]
\[
= u_{n+1} c(n, i) + p_n c(n - 1, i - 1) + v_{n+1} p_{n-1} c(n - 2, i - 2)
\]
\[
+ \sum_{k=4}^{n} p_{k-2} \left( \prod_{j=k}^{n} v_j \right) c(k - 3, i - n + k - 3) + v_{n+1} \cdots v_3 \rho_{i-n}
\]

(by the inductive hypothesis)
\[
= u_{n+1} c(n, i) + p_n c(n - 1, i - 1)
\]
\[
+ \sum_{k=4}^{n+1} p_{k-2} \left( \prod_{j=k}^{n+1} v_j \right) c(k - 3, i - n + k - 3) + v_{n+1} \cdots v_3 \rho_{i-n},
\]

which proves Claim 1.
Write $u'_{n}$, $v'_{n}$, $w'_{n}$, $p'_{n}$, $\rho'_{n}$, and $c'(\cdot, \cdot)$ for the entities corresponding to $\alpha'$. If $p_{n} > 0$ for every $n = 0, \cdots, N + 2$, then in view of Claim 1, we can choose a small perturbation such that $p'_{n} > 0$ ($0 \leq n \leq N + 2$) and therefore $c'(n, i) > 0$ for all $n \geq 0$ and $0 \leq i \leq n + 1$, which implies that $W_{\alpha'}$ is also positively quadratically hyponormal. If instead $p_{n} = 0$ for some $n = 0, \cdots, N + 2$, careful inspection of (5.3) reveals that without loss of generality we may assume $p_{0} = \cdots = p_{N+2} = 0$. By Theorem 2.2, we have that for a sufficiently small perturbation $\alpha'$ of $\alpha$,

\[(5.4) \quad c'(n, i) > 0 \quad (0 \leq n \leq N + 2, \ 0 \leq i \leq n + 1) \quad \text{and} \quad c'(n, n + 1) > 0 \quad (n \geq 0).\]

Write

\[k_{n} := \frac{v_{n}}{u_{n}} \quad (n = 2, 3, \cdots).\]

**Claim 2.** $\{k_{n}\}_{n=2}^{\infty}$ is bounded.

**Proof of Claim 2.** Observe that

\begin{equation}
\begin{aligned}
k_{n} &= \frac{v_{n}}{u_{n}} = \frac{\alpha_{n}^{2} - \alpha_{n-1}^{2}}{\alpha_{n}^{2} - \alpha_{n-1}^{2}} \quad (n = 2, 3, \cdots) \\
&= \alpha_{n}^{2} + \alpha_{n-1}^{2} + \alpha_{n}^{2} - \alpha_{n-1}^{2} + \alpha_{n-1}^{2} - \alpha_{n-2}^{2}.
\end{aligned}
\end{equation}

Therefore if $W_{\alpha}$ is 2-hyponormal, then by Lemma 5.1, the sequences

\[\left\{\frac{\alpha_{n+1}^{2} - \alpha_{n}^{2}}{\alpha_{n}^{2} - \alpha_{n-1}^{2}}\right\}_{n=2}^{\infty} \quad \text{and} \quad \left\{\frac{\alpha_{n-1}^{2} - \alpha_{n-2}^{2}}{\alpha_{n}^{2} - \alpha_{n-1}^{2}}\right\}_{n=2}^{\infty}
\]

are both bounded, so that $\{k_{n}\}_{n=2}^{\infty}$ is bounded. This proves Claim 2.

Write $k := \sup_{n} k_{n}$. Without loss of generality we assume $k < 1$ (this is possible from the observation that $\alpha$ induces $\{c^{2}k_{n}\}$). Choose a sufficiently small perturbation $\alpha'$ of $\alpha$ such that if we let

\begin{equation}
\begin{aligned}
h := \sup_{0 \leq \ell \leq N+2} \left| \sum_{0 \leq m \leq 1}^{N+4} p'_{k-2} \left( \prod_{j=k}^{N+3} v'_{j} \right) c'(k - 3, \ell) + v'_{N+3} \cdots v'_{N+3} \rho'_{m} \right|,
\end{aligned}
\end{equation}

then

\begin{equation}
\begin{aligned}
c'(N + 3, i) - \frac{1}{1 - k} h > 0 \quad (0 \leq i \leq N + 3)
\end{aligned}
\end{equation}

(this is always possible because, by Theorem 2.2, we can choose a sufficiently small $|p'_{i}|$ such that

\[c'(N+3, i) > v_{0} \cdots v_{i-1} u_{i} \cdots u_{N+3} - \epsilon \quad \text{and} \quad |h| < (1-k)(v_{0} \cdots v_{i-1} u_{i} \cdots u_{N+3} - \epsilon)
\]

for any small $\epsilon > 0$).
Claim 3. For $j \geq 4$ and $0 \leq i \leq N + j$,

\begin{equation}
(5.8) \quad c'(N + j, i) \geq u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-3} k^n h \right).
\end{equation}

Proof of Claim 3. We use induction. If $j = 4$, then by Claim 1 and (5.6),

\[
c'(N + 4, i) = u'_{N+4} c'(N + 3, i) + p'_{N+3} c'(N + 2, i - 1)
+ v'_{N+4} \sum_{k=4}^{N+4} p'_{k-2} \left( \prod_{j=k}^{N+3} v'_j \right) c'(k - 3, i - N + k - 6)
+ v'_{N+4} \cdots v'_{3} p'_{i-(N+3)}
\]

\[
\geq u'_{N+4} c'(N + 3, i) + p'_{N+3} c'(N + 2, i - 1) - v'_{N+4} h
\geq u_{N+4} (c'(N + 3, i) - k_{N+4} h)
\geq u_{N+4} (c'(N + 3, i) - k h),
\]

because $u'_{N+4} = u_{N+4}$, $v'_{N+4} = v_{N+4}$ and $p'_{N+3} = p_{N+3} \geq 0$. Now suppose (5.8) holds for some $j \geq 4$. By Claim 1, we have that for $j \geq 4$,

\[
c'(N + j + 1, i) = u'_{N+j+1} c'(N + j, i) + p'_{N+j} c'(N + j - 1, i - 1)
+ \sum_{k=4}^{N+j+1} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_j \right) c'(k - 3, i - N + k - j - 3)
+ v'_{N+j+1} \cdots v'_{3} p'_{i-(N+j)}
= u'_{N+j+1} c'(N + j, i) + p'_{N+j} c'(N + j - 1, i - 1)
+ \sum_{k=N+5}^{N+j+1} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_j \right) c'(k - 3, i - N + k - j - 3)
+ \sum_{k=4}^{N+4} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_j \right) c'(k - 3, i - N + k - j - 3)
+ v'_{N+j+1} \cdots v'_{3} p'_{i-(N+j)}.
\]

Since $p'_n = p_n > 0$ for $n \geq N + 3$ and $c'(n, \ell) > 0$ for $0 \leq n \leq N + j$ by the inductive hypothesis, it follows that

\begin{equation}
(5.9) \quad p'_{N+j} c(N+j-1, i-1) + \sum_{k=N+5}^{N+j+1} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_j \right) c'(k-3, i-N+k-j-3) \geq 0.
\end{equation}
By the inductive hypothesis and (5.9),

\[ c'(N + j + 1, i) \]

\[ \geq u'_{N+j+1} c'(N + j, i) + \sum_{k=4}^{N+4} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_i \right) c'(k - 3, i - N + k - j - 3) \]

\[ + v'_N \cdots v'_{N+j} \rho'_{N+j} \]

\[ \geq u_{N+j+1} u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-3} k^n h \right) \]

\[ + v_N \cdots v_{N+j} v_{N+4} \left( \sum_{k=4}^{N+4} p'_{k-2} \left( \prod_{j=k}^{N+3} v'_i \right) c'(k - 3, i - N + k - j - 3) \right) \]

\[ \geq u_{N+j+1} u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-3} k^n h \right) \]

\[ = u_{N+j+1} u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-3} k^n h - k_{N+j+1} k_{N+j} \cdots k_{N+4} h \right) \]

\[ \geq u_{N+j+1} u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-2} k^n h \right), \]

which proves Claim 3.

Since \( \sum_{n=1}^{j-3} k^n < \frac{1}{1 - k} \) for every \( j > 1 \), it follows from Claim 3 and (5.7) that

(5.10) \[ c'(N + j, i) > 0 \] for \( j \geq 4 \) and \( 0 \leq i \leq N + j \).

It thus follows from (5.4) and (5.10) that \( c'(n, i) > 0 \) for every \( n \geq 0 \) and \( 0 \leq i \leq n + 1 \). Therefore \( W_{\alpha'} \) is also positively quadratically hyponormal. This completes the proof. \( \square \)

**Corollary 5.2.** Let \( W_{\alpha} \) be a weighted shift such that \( \alpha_{j-1} < \alpha_j \) for some \( j \geq 1 \), and let \( T_x \) be the weighted shift with weight sequence

\[ \alpha_x : \alpha_0, \cdots, \alpha_{j-1}, x, \alpha_{j+1}, \cdots. \]

Then \( \{ x : T_x \text{ is 2-hyponormal} \} \) is a proper closed subset of \( \{ x : T_x \text{ is quadratically hyponormal} \} \) whenever the latter set is nonempty.

**Proof.** Write

\( H_2 := \{ x : T_x \text{ is 2-hyponormal} \}. \)

Without loss of generality, we can assume that \( H_2 \) is nonempty, and that \( j = 1 \). Recall that a 2-hyponormal weighted shift with two equal weights is of the form \( \alpha_0 = \alpha_1 = \alpha_2 = \cdots \) or \( \alpha_0 < \alpha_1 = \alpha_2 = \cdots \). Let \( x_m := \inf H_2 \). By Proposition 6.7 below, \( T_{x_m} \) is hyponormal. Then \( x_m > \alpha_0 \). By assumption, \( x_m < \alpha_2 \). Thus \( \alpha_0, x_m, \alpha_2, \alpha_3, \cdots \) is strictly increasing. Now we apply Theorem 2.3 to obtain \( x' \) such that \( \alpha_0 < x' < x_m \) and \( T_{x'} \) is quadratically hyponormal. However \( T_{x'} \) is not 2-hyponormal by the definition of \( x_m \). The proof is complete. \( \square \)
The following question arises naturally:

**Question 5.3.** Let $\alpha$ be a strictly increasing weight sequence and let $k \geq 3$. If $W_\alpha$ is a $k$-hyponormal weighted shift, does it follow that $W_\alpha$ is weakly $k$-hyponormal under a small perturbation of the weight sequence?

### 6. Other related results

#### 6.1. Subnormal extensions

Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence, let $x_i > 0$ for $1 \leq i \leq n$, and let $(x_n, \cdots, x_1)\alpha : x_n, \cdots, x_1, \alpha_0, \alpha_1, \cdots$ be the augmented weight sequence. We say that $W_{(x_n, \cdots, x_1)\alpha}$ is an extension (or $n$-step extension) of $W_\alpha$. Observe that

$$W_{(x_n, \cdots, x_1)\alpha} | \{e_n, e_{n+1}, \cdots \} \cong W_\alpha.$$  

The hypothesis $F \neq c P_{(e_0)}$ in Theorem 2.1 is essential. Indeed, there exist infinitely many one-step subnormal extensions of a subnormal weighted shift whenever one such extension exists. Recall ([7, Proposition 8]) that if $W_\alpha$ is a weighted shift whose restriction to $\{e_1, e_2, \cdots \}$ is subnormal with associated measure $\mu$, then $W_\alpha$ is subnormal if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;

(ii) $\alpha_0^2 \leq \left(\frac{1}{|t|L^1(\mu)}\right)^{-1}$.  

Also note that there may not exist any one-step subnormal extension of the subnormal weighted shift: for example, if $W_\alpha$ is the Bergman shift, then the corresponding Berger measure is $\mu(t) = t$, and hence $\frac{1}{t}$ is not integrable with respect to $\mu$; therefore $W_\alpha$ does not admit any subnormal extension. A similar situation arises when $\mu$ has an atom at $\{0\}$. More generally we have:

**Theorem 6.1 (Subnormal Extensions).** Let $W_\alpha$ be a subnormal weighted shift with weights $\alpha : \alpha_0, \alpha_1, \cdots$ and let $\mu$ be the corresponding Berger measure. Then $W_{(x_n, \cdots, x_1)\alpha}$ is subnormal if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;

(ii) $x_j = \left(\frac{\alpha^2}{\|x_j\|^2L^1(\mu)}\right)^{\frac{1}{2}}$ for $1 \leq j \leq n - 1$;

(iii) $x_n = \left(\frac{\alpha^2}{\|x_n\|^2L^1(\mu)}\right)^{\frac{1}{2}}$.  

In particular, if we put

$$S := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : W_{(x_n, \cdots, x_1)\alpha} \text{ is subnormal}\},$$

then either $S = \emptyset$ or $S$ is a line segment in $\mathbb{R}^n$.

**Proof.** Write $W_j := W_{(x_n, \cdots, x_j)\alpha} | \{e_n, e_{n-1}, \cdots \}$ $(1 \leq j \leq n)$ and hence $W_{(x_n, \cdots, x_1)\alpha}$. By the argument used to establish (3.2) we have that $W_1$ is subnormal with associated measure $\nu_1$ if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;

(ii) $d\nu_1 = \frac{\alpha^2}{\|x_1\|^2}d\mu$, or equivalently, $x_1^2 = \left(\int_0^\infty \frac{\alpha^2}{1} d\mu(t)\right)^{-1}$.  

Inductively $W_{n-1}$ is subnormal with associated measure $\nu_{n-1}$ if and only if

(i) $W_{n-2}$ is subnormal;

(ii) $\frac{1}{\nu_{n-1}} \in L^1(\mu)$;
(iii) \( d\nu_{n-1} = \frac{x^2_{n-1}}{t_{n-1}} d\nu_{n-2} = \cdots = \frac{x^2_{n-1} \cdots x^2_1}{t_{n-1} \cdots t_1} d\mu, \) or equivalently,
\[
x^2_{n-1} = \frac{\int |W_\alpha|^2}{\int |W_\alpha|^2} \frac{1}{t_{n-1}} d\mu(t) = \frac{\int |W_\alpha|^2}{\int |W_\alpha|^2} \frac{1}{t_n} d\mu(t).
\]

Therefore \( W_n \) is subnormal if and only if

(i) \( W_{n-1} \) is subnormal;
(ii) \( \frac{1}{t_n} \in L^1(\mu); \)
(iii) \( x^2_n \leq \left( \int |W_\alpha|^2 \frac{1}{t_n} d\nu_{n-1} \right)^{-1} = \left( \int |W_\alpha|^2 \frac{x^2_{n-1} \cdots x^2_1}{t_n} d\mu(t) \right)^{-1} = \frac{\int |W_\alpha|^2}{\int |W_\alpha|^2} \frac{1}{t_n} d\mu(t) \).

\[ \Box \]

Corollary 6.2. If \( W_\alpha \) is a subnormal weighted shift with associated measure \( \mu \), there exists an \( n \)-step subnormal extension of \( W_n \) if and only if \( \frac{1}{t_n} \in L^1(\mu) \).

For the next result we refer to the notation in (2.1) and (2.2).

Corollary 6.3. A recursively generated subnormal shift with \( \varphi_0 \neq 0 \) admits an \( n \)-step subnormal extension for every \( n \geq 1 \).

Proof. The assumption about \( \varphi_0 \) implies that the zeros of \( g(t) \) are positive, so that \( s_0 > 0 \). Thus for every \( n \geq 1 \), \( \frac{1}{t_n} \) is integrable with respect to the corresponding Berger measure \( \mu = \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{s_{r-1}} \). By Corollary 6.2, there exists an \( n \)-step subnormal extension.

We need not expect that for arbitrary recursively generated shifts, \( 2 \)-hyponormality and subnormality coincide as in Theorem 3.2. For example, if \( \alpha : \sqrt{\frac{3}{2}}, \sqrt{x}, \)
\[
(\sqrt{3}, \sqrt{2}, \sqrt{x}), \text{ then by (2.12) and Theorem 6.1,}
\]

(i) \( T_x \) is \( 2 \)-hyponormal \( \iff \) \( 4 - \sqrt{6} \leq x \leq 2 \);
(ii) \( T_x \) is subnormal \( \iff \) \( x = 2 \).

A straightforward calculation shows, however, that \( T_x \) is \( 3 \)-hyponormal if and only if \( x = 2 \); for,
\[
A(0; 3) := \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\
\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 5x & 17x \\
5x & 17x & 58x
\end{pmatrix} \geq 0 \iff x = 2.
\]

This behavior is typical of general recursively generated weighted shifts: we show in [L3] that subnormality is equivalent to \( k \)-hyponormality for some \( k \geq 2 \).

6.2. Convexity and closedness. Next, we will show that canonical rank-one perturbations of \( k \)-hyponormal weighted shifts which preserve \( k \)-hyponormality form a convex set. To see this we need an auxiliary result.

Lemma 6.4. Let \( I = \{1, \cdots, n\} \times \{1, \cdots, n\} \) and let \( J \) be a symmetric subset of \( I \). Let \( A = (a_{ij}) \in M_n(\mathbb{C}) \) and let \( C = (c_{ij}) \in M_n(\mathbb{C}) \) be given by
\[
c_{ij} = \begin{cases} 
ca_{ij} & \text{if } (i,j) \in J \\
a_{ij} & \text{if } (i,j) \in I \setminus J 
\end{cases} \quad (c > 0).
\]
If $A$ and $C$ are positive semidefinite, then $B = (b_{ij}) \in M_n(\mathbb{C})$ defined by
\[
b_{ij} = \begin{cases} 
ba_{ij} & \text{if } (i, j) \in J \\
a_{ij} & \text{if } (i, j) \in I \setminus J
\end{cases} \quad (b \in [1, c] \text{ or } [c, 1])
\]
is also positive semidefinite.

Proof. Without loss of generality we may assume $c > 1$. If $b = 1$ or $b = c$, the assertion is trivial. Thus we assume $1 < b < c$. The result is now a consequence of the following observation. If $[D]_{(i,j)}$ denotes the $(i,j)$-entry of the matrix $D$, then
\[
\begin{pmatrix}
\frac{c-b}{c-1} & \frac{b-1}{c} \\
\frac{c-b}{c-1} & \frac{1}{c} \\
r & r
\end{pmatrix}_{(i,j)} = \begin{cases}
\frac{c-b}{c-1} \left(1 + \frac{b-1}{c} \right) a_{ij} & \text{if } (i, j) \in J, \\
\frac{c-b}{c-1} \left(1 + \frac{b-1}{c} \right) a_{ij} & \text{if } (i, j) \in I \setminus J,
\end{cases}
\]
which is positive semidefinite because positive semidefinite matrices in $M_n(\mathbb{C})$ form a cone. 

An immediate consequence of Lemma 6.4 is that positivity of a matrix forms a convex set with respect to a fixed diagonal location; i.e., if
\[
A_x = \begin{pmatrix}
* & * & * \\
x & * & * \\
* & * & *
\end{pmatrix},
\]
then $\{x : A_x \text{ is positive semidefinite}\}$ is convex.

We now have:

**Theorem 6.5.** Let $\alpha = \{\alpha_n\}_{n=0}^\infty$ be a weight sequence, let $k \geq 1$, and let $j \geq 0$. Define $\alpha^{(j)}(x) : \alpha_0, \cdots, \alpha_{j-1}, x, \alpha_{j+1}, \cdots$. Assume $W_\alpha$ is $k$-hyponormal and define
\[
\Omega^{k,j}_\alpha := \{x : W^{(j)}_{\alpha(x)} \text{ is } k\text{-hyponormal}\}.
\]

Then $\Omega^{k,j}_\alpha$ is a closed interval.

Proof. Suppose $x_1, x_2 \in \Omega^{k,j}_\alpha$ with $x_1 < x_2$. Then by (2.11), the $(k+1) \times (k+1)$ Hankel matrix
\[
A_{x_i}(n; k) := \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\
\gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k}
\end{pmatrix} \quad (n \geq 0; \; i = 1, 2)
\]
is positive, where $A_{x_i}$ corresponds to $\alpha^{(j)}(x_i)$. We must show that $tx_1 + (1-t)x_2 \in \Omega^{k,j}_\alpha$ $(0 < t < 1)$, i.e.,
\[
A_{tx_1 + (1-t)x_2}(n; k) \geq 0 \quad (n \geq 0, \; 0 < t < 1).
\]
Observe that it suffices to establish the positivity of the $2k$ Hankel matrices corresponding to $\alpha^{(j)}(tx_1 + (1-t)x_2)$ such that $tx_1 + (1-t)x_2$ appears as a factor in at least one entry but not in every entry. A moment’s thought reveals that without loss of generality we may assume $j = 2k$. Observe that
\[
A_{x_1}(n; k) - A_{x_2}(n; k) = (z_1^2 - z_2^2)H(n; k)
\]
for some Hankel matrix $H(n; k)$. For notational convenience, we abbreviate $A_x(n; k)$ as $A_x$. Then

$$A_{tx_1+(1-t)x_2} = \begin{cases} t^2 A_{x_1} + (1-t)^2 A_{x_2} + 2t(1-t)A_{\sqrt{x_1x_2}} & \text{for } 0 \leq n \leq 2k, \\ \left( t + (1-t)\frac{x_2}{x_1} \right)^2 A_{x_1} & \text{for } n \geq 2k+1. \end{cases}$$

Since $A_{x_1} \geq 0$, $A_{x_2} \geq 0$ and $A_{\sqrt{x_1x_2}}$ have the form described by Lemma 6.4 and since $x_1 < \sqrt{x_1x_2} < x_2$, it follows from Lemma 6.4 that $A_{\sqrt{x_1x_2}} \geq 0$. Thus evidently, $A_{tx_1+(1-t)x_2} \geq 0$, and therefore $tx_1 + (1-t)x_2 \in \Omega_{\alpha}^{k,j}$. This shows that $\Omega_{\alpha}^{k,j}$ is an interval. The closedness of the interval follows from Proposition 6.7 below.

In [17] and [18], it was shown that there exists a nonsubnormal polynomially hyponormal operator. Also in [22], it was shown that there exists a nonsubnormal polynomially hyponormal operator if and only if there exists one which is also a weighted shift. However, no concrete weighted shift has yet been found. As a strategy for finding such a shift, we would like to suggest the following:

**Question 6.6.** Does it follow that the polynomial hyponormality of a weighted shift is stable under small perturbations of the weight sequence?

If the answer to Question 6.6 were affirmative then we would easily find a polynomially hyponormal nonsubnormal (even non-2-hyponormal) weighted shift; for example, if

$$\alpha : 1, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$$

and $T_\alpha$ is the weighted shift associated with $\alpha$, then by Theorem 3.2, $T_\alpha$ is subnormal \iff $x = 2$, whereas $T_\alpha$ is polynomially hyponormal \iff $2 - \delta_1 < x < 2 + \delta_2$ for some $\delta_1, \delta_2 > 0$ provided the answer to Question 6.6 is yes; therefore for sufficiently small $\epsilon > 0$,

$$\alpha_\epsilon : 1, \sqrt{2+\epsilon}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$$

would induce a non-2-hyponormal weighted shift.

The answer to Question 6.6 for weak $k$-hyponormality is negative. In fact we have:

**Proposition 6.7.** (i) The set of $k$-hyponormal operators is sot-closed.

(ii) The set of weakly $k$-hyponormal operators is sot-closed.

**Proof.** Suppose $T_\eta \in \mathcal{L}(\mathcal{H})$ and $T_\eta \rightarrow T$ in sot. Then, by the Uniform Boundedness Principle, $\{||T_\eta||\}_\eta$ is bounded. Thus $T_\eta^* T_j^* \rightarrow T^* T_j$ in sot for every $i, j$, so that $M_k(T_\eta) \rightarrow M_k(T)$ in sot (where $M_k(T)$ is as in (1.2)).

(i) In this case $M_k(T_\eta) \geq 0$ for all $\eta$, so $M_k(T) \geq 0$, i.e., $T$ is $k$-hyponormal.

(ii) Here, $M_k(T_\eta)$ is weakly positive for all $\eta$. By (1.3), $M_k(T)$ is also weakly positive, i.e., $T$ is weakly $k$-hyponormal. \qed

**References**


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