AFFINE PSEUDO-PLANES AND CANCELLATION PROBLEM

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Abstract. We define affine pseudo-planes as one class of \( \mathbb{Q} \)-homology planes. It is shown that there exists an infinite-dimensional family of non-isomorphic affine pseudo-planes which become isomorphic to each other by taking products with the affine line \( \mathbb{A}^1 \). Moreover, we show that there exists an infinite-dimensional family of the universal coverings of affine pseudo-planes with a cyclic group acting as the Galois group, which have the equivariant non-cancellation property. Our family contains the surfaces without the cancellation property, due to Danielewski-Fieseler and tom Dieck.

1. Introduction

Let \( G \) be an algebraic group defined over the complex number field \( \mathbb{C} \). We shall consider the following:

Equivariant Cancellation Problem. Let \( X \) and \( Y \) be smooth affine varieties with algebraic \( G \)-actions. If \( X \times W \) is \( G \)-isomorphic to \( Y \times W \) for a \( G \)-module \( W \), is \( X \) then \( G \)-isomorphic to \( Y \)?

If we forget the actions, the problem is simply called the Cancellation Problem. When \( Y \cong \mathbb{A}^2 \), the cancellation holds by the results of Miyanishi-Sugie \([18]\) and Fujita \([7]\). However, the Cancellation Problem for \( Y \cong \mathbb{A}^n \) remains open if \( n \geq 3 \).

In the Equivariant Cancellation Problem, the intriguing case is when \( Y \) is isomorphic to a \( G \)-module, i.e., an affine space with a linear \( G \)-action. In this case, it is known that the answer is negative. In fact, for a reductive algebraic group \( G \), there exist affine spaces with non-linearizable \( G \)-actions which are realized as the total spaces of non-trivial algebraic \( G \)-vector bundles over \( G \)-modules (Schwarz \([19]\), see also references in \([10]\)). By Bass-Haboush \([2]\), every \( G \)-vector bundle over a \( G \)-module is stably trivial, namely, it becomes isomorphic to a trivial \( G \)-vector bundle by adding a certain trivial \( G \)-vector bundle. Hence non-trivial \( G \)-vector bundles over \( G \)-modules, whose total spaces have non-linearizable \( G \)-actions, give rise to counterexamples to the Equivariant Cancellation Problem with \( G \)-modules \( Y \) (cf. Masuda-Miyaniishi \([12]\)). All counterexamples to the Equivariant Cancellation Problem that we have so far for reductive algebraic groups \( G \) and \( G \)-modules \( Y \) are derived from non-trivial \( G \)-vector bundles over \( G \)-modules.

Next, consider the case where \( Y \) is not isomorphic to a \( G \)-module nor an affine space without \( G \)-action. Then there are well-known counterexamples due to Daniel-
Families are of infinite dimension and are derived from $G$ of non-isomorphic affine -varieties without cancellation property. In the last section, we show that there exist families also obtain an infinite-dimensional family of non-isomorphic affine pseudo-planes consists of the universal coverings of affine pseudo-planes and includes the examples additive group $G$ have in fact equivariant non-cancellation property. There exists an action of the Galois group isomorphic to $Z$ and not vector bundles.

Affine surfaces satisfying the assumptions in Lemma 2.1 are characterized as affine pseudo-planes with non-trivial $C^*$-actions. See section 2 for definitions and relevant results.

In the present article, we observe various properties of affine pseudo-planes and their universal coverings. We shall show that affine pseudo-planes can be constructed from the Hirzebruch surfaces, as tom Dieck’s surfaces $V(d,r)$ are constructed from the Hirzebruch surfaces. We also show that the universal coverings of affine pseudo-planes are isomorphic to the hypersurfaces in $A^3$, and give their defining equations explicitly. Using the results on the properties of affine pseudo-planes and their universal coverings, we show that if $d \geq 2$, there exists an infinite-dimensional family of non-isomorphic smooth affine surfaces with actions of $G_a \times Z/dZ$, whose members are mutually equivariantly non-cancellative. The family consists of the universal coverings of affine pseudo-planes and includes the examples due to Danielewski-Fieseler and tom Dieck. By taking their quotients by $Z/dZ$, we also obtain an infinite-dimensional family of non-isomorphic affine pseudo-planes without cancellation property. In the last section, we show that there exist families of non-isomorphic affine $G$-varieties without equivariant cancellation property. The families are of infinite dimension and are derived from $G$-equivariant $A^1$-fibrations and not $G$-vector bundles.

2. AFFINE PSEUDO-PLANES WITH UNIQUE $A^1$-FIBRATIONS

An algebro-geometric characterization of the affine plane $A^2$ is stated as follows: the affine plane $A^2$ is an affine surface such that its coordinate ring $R$ is factorial, $R^* = C^*$, and there exists an $A^1$-fibration with base curve isomorphic to $A^1$. There are many related results on smooth affine surfaces with $A^1$-fibrations (cf. Miyanishi [14]). Here we recall the following.

**Lemma 2.1.** Let $X$ be a smooth affine surface with an $A^1$-fibration $\rho : X \to C \cong A^1$. Suppose that every fiber of $\rho$ is irreducible. Then $\text{Pic} (X) \cong \prod_{P \in C} \mathbb{Z}/d_P \mathbb{Z}$, where $d_P$ is the multiplicity of the fiber $\rho^{-1}(P)$. In particular, if there is only one multiple fiber $dF$ with the multiplicity $d$ and $F \cong A^1$, then $\text{Pic} (X) \cong \mathbb{Z}/d \mathbb{Z}$.

Affine surfaces satisfying the assumptions in Lemma 2.1 are $\mathbb{Q}$-homology planes, and there are many such surfaces. We define affine pseudo-plane as one class of such affine surfaces.

**Definition 2.1.** A smooth affine surface $X$ is an affine pseudo-plane if $X$ satisfies the following conditions.
(1) $X$ has an $\mathbb{A}^1$-fibration $\rho : X \to C$, where $C \cong \mathbb{A}^1$.
(2) The $\mathbb{A}^1$-fibration $\rho$ has a unique multiple fiber $dF$ with multiplicity $d \geq 2$ and $F \cong \mathbb{A}^1$, and every other fiber is isomorphic to $\mathbb{A}^1$.

We say that $X$ has type $(d, n, r)$ if $X$ further satisfies the next condition:

(3) $X$ has a smooth compactification $(V, D)$ such that the boundary divisor $D = V - X$ has simple normal crossings and the dual graph of $D$ is as given in Figure 1 below, where $n \geq 1$ and $r \geq 1$. Furthermore, $\mathcal{F}$ is the closure of $F$ in $V$, and $S'$ is the unique cross-section contained in $D$.

![Figure 1](image)

If $X$ has a smooth compactification $(V, D)$ with the dual graph as in Figure 1 and $(S'^2) = -n$ for $n > 1$, then we can make $(S'^2) = -1$. In fact, choose a point $P$ on the fiber $\ell'_{\infty}$ and blow up the point $P$ to obtain a $(-1)$ curve $E$. Then the proper transform $L$ of $\ell'_{\infty}$ is a $(-1)$ curve. Contract $L$ to obtain the same figure as before with $\ell'_{\infty}$ replaced by the image of $E$ and with $(S'^2) = -n + 1$ if $P \neq S' \cap \ell'_{\infty}$, and $-n - 1$ if $P = S' \cap \ell'_{\infty}$. This operation is called the elementary transformation with center $P$. After several elementary transformations, we obtain $(S'^2) = -1$.

Meanwhile, we have to consider the case $(S'^2) < -1$ as well, e.g., in the proof of Theorem 2.3. We call an affine pseudo-plane of type $(d, 1, r)$ simply an affine pseudo-plane of type $(d, r)$.

**Lemma 2.2.** Let $X$ be an affine pseudo-plane of type $(d, r)$. Then $X$ is isomorphic to the complement of $M_0 \cup C_d$ if $r < d$, and $M_1 \cup C_d$ if $r \geq d$ in the Hirzebruch surface $\Sigma_n$ with $n = |r - d|$, where $M_0$ is the minimal section and where $C_d$ and $M_1$ are specified as follows. In the case $r < d$, $C_d$ is an irreducible member of the linear system $|M_0 + d\ell_0|$ which meets $M_0$ in the point $M_0 \cap \ell_0$ with multiplicity $r$, where $\ell_0$ is a fiber of the $\mathbb{P}^1$-fibration of $\Sigma_n$. In the case $r \geq d$, $M_1$ is a section of $\Sigma_n$ with $(M_1^2) = n$, and $C_d$ is an irreducible member of the linear system $|M_1 + d\ell_0|$ which meets $M_1$ in the point $M_1 \cap \ell_0$ with multiplicity $r$. In both cases, $\ell_0 \cap X = \mathcal{F} \cap X$.

**Proof.** Contract $S', \ell'_{\ell_0}, E_2, \ldots, E_d, E_{d+1}, \ldots, E_{d+r-1}$ in this order. Then the resulting surface is the Hirzebruch surface $\Sigma_n$ with $n = |r - d|$ and the image of $\ell'_{\infty}$ provides $C_d$. The image of $E_1$ provides $M_0$ or $M_1$ according to whether $r - d < 0$ or $r - d \geq 0$, while the image of $\mathcal{F}$ is the fiber $\ell_0$. 

An affine pseudo-plane $X$ of type $(d, n, r)$ with $r \geq 2$ has the distinguished property as stated in the following theorem. An $\mathbb{A}^1$-fibration $\rho : X \to C \cong \mathbb{A}^1$ is called unique if there is another $\mathbb{A}^1$-fibration $\sigma : X \to B \cong \mathbb{A}^1$, then $\sigma = \tau \circ \rho$ for an automorphism $\tau$ of $\mathbb{A}^1$.

The next theorem follows from a theorem of Bertin [3], but we prefer to give a direct proof.

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Theorem 2.3. Let $X$ be an affine pseudo-plane of type $(d, n, r)$ with $r \geq 2$. Then $\rho$ is a unique $\mathbb{A}^1$-fibration on $X$.

Proof. Suppose that there exists another $\mathbb{A}^1$-fibration $\sigma : X \to B$ which is different from the fixed $\mathbb{A}^1$-fibration $\rho : X \to C$. Then $B \cong \mathbb{A}^1$ and every fiber of $\sigma$ is isomorphic to $\mathbb{A}^1$ if taken with the reduced structure. Let $M$ be a linear pencil on $V$ spanned by the closures of general fibers of $\sigma$, where the notations $V, D, \ell, t, \sigma, G$, etc. are the same as in Definition 2.1. Then a general member of $M$ meets the curve $\ell'\infty$, for otherwise the $\mathbb{A}^1$-fibrations $\rho$ and $\sigma$ coincide with each other. Suppose that $M$ has no base points. Then the curve $\ell'\infty$ is a cross-section of $M$ and $S' + t'\ell + E_1 + \cdots + E_{d+r-1}$ supports a reducible fiber of $\rho$. Then $r = d = 1$. Since $d \geq 2$ by the hypothesis, this case does not take place. Hence $M$ has a base point, say $P$, on $\ell'\infty$. Let $Q := \ell'\infty \cap S'$. We consider two cases separately.

Case $P \neq Q$. Then $\ell'\infty + S' + t'\ell + E_1 + E_2 + \cdots + E_{d+r-1}$ will support a reducible member $G_0$ of the pencil $M$. Let $s = (P, G)$, where $G$ is a general member of $M$. By comparing the intersection numbers of $G$ with two fibers of $\ell'\infty$ and the one containing $dP$, it follows that $(\ell'\infty \cdot G) = ds$. Let $\mu$ be the multiplicity of $G$ at $P$, where $P$ is a one-place point of $G$. We have $ds \geq \mu$. Consider first the case $n = 1$. The contraction of $S', t'\ell, E_2, \ldots, E_{d-1}$ makes $E_0$ a $(-1)$ curve meeting three components $\ell_0\infty, E_1, E_{d+1}$, and this is impossible. So, suppose $n \geq 2$. The elimination of the base points of $M$ will be achieved by blowing up the point $P$ and its infinitely near points. After the elimination of the base points of $M$, the proper transform $\tilde{M}$ gives rise to a $\mathbb{P}^1$-fibration, and the proper transform of $\ell'\infty$ is a unique $(-1)$ component. As above, let $G_0$ be a reducible member of $M$ containing $S' + t'\ell + E_1 + E_2 + \cdots + E_{d+r-1}$. If $\ell'\infty$ is not contained in $G_0$, the elimination of the base points of $M$, which is achieved by blowing up the point $P = Q$ and its infinitely near points, yields a $\mathbb{P}^1$-fibration in which the fiber corresponding to $G_0$ is a reducible fiber not containing any $(-1)$ curve. This is a contradiction. Hence $\ell'\infty$ is contained in $G_0$. So, $G_0$ is supported by $S' + t'\ell + E_1 + E_2 + \cdots + E_{d+r-1} + \ell'\infty$. Now apply the elementary transformation with center $P$. Then we obtain the same dual graph as Figure 1 with $(S'^2) = -(n + 1)$ and $\ell'\infty$ is replaced by the image of $E$. After repeating the elementary transformations several times, we are reduced to the case where $P \neq Q$. So, we reach a contradiction in the present case as well.

Since the existence of an $\mathbb{A}^1$-fibration with affine base is equivalent to the existence of an action of the additive group $G_a$, it follows from Theorem 2.3 that there is an essentially unique $G_a$-action on an affine pseudo-plane of type $(d, n, r)$ for $r \geq 2$. On the other hand, an affine pseudo-plane of type $(d, n, 1)$ has two algebraically independent $G_a$-actions, namely, it has trivial Makar-Limanov invariant (cf. [11]). This is a consequence of a more general result in Gurjar-Miyaishi [9], Theorem 3.1] which is stated below. We only note that the boundary divisor $D$ in the case of
type \((d,n,1)\) is a linear chain for the normal compactification in Definition 2.1 and that \(\pi_{1,\infty}(X)\) is then a finite cyclic group of order \(d^2\).

**Theorem 2.4.** Let \(X\) be a smooth affine surface. Then the Makar-Limanov invariant \(ML(X)\) is trivial if and only if \(X\) has a minimal normal compactification \(V\) such that the dual graph of \(D := V - X\) is a linear chain of rational curves and \(\pi_{1,\infty}(X)\) is a finite group.

Lemma 2.2 gives rise to a construction of affine pseudo-planes from the Hirzebruch surfaces. We denote by \(X(d,r)\) an affine pseudo-plane of type \((d,r)\) constructed from the Hirzebruch surface as in Lemma 2.2. Some partial cases of affine pseudo-planes were observed in tom Dieck \(\cite{1}\) as examples of affine surfaces without cancellation property. We shall recall and generalize a little bit his construction. Write \(\Sigma_n = \text{Proj} (\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})\) as the quotient of \((\mathbb{A}^2 \setminus \{0\}) \times \mathbb{P}^1\) under the relation

\[(z_0,z_1),[w_0,w_1] \sim (\nu z_0,\nu z_1),[\nu^n w_0, w_1]\]

for \(\nu \in \mathbb{C}_m = \mathbb{C}^*\). The projection \((\{z_0,z_1\},[w_0,w_1]\} \mapsto [z_0,z_1]\) induces a \(\mathbb{P}^1\)-fibration \(p_n : \Sigma_n \to \mathbb{P}^1\). In the above definition by quotient and in what follows, the integer \(n\) could be negative. If \(n \geq 0\), the curve \(w_0 = 0\) (resp. \(w_1 = 0\)) is a section \(M_1\) of \(p_n\) with \((M_1^2) = n\) (resp. the minimal section \(M_0\) with \((M_0^2) = -n\)).

Meanwhile, if \(n < 0\), then the curve \(w_0 = 0\) (resp. \(w_1 = 0\)) is the minimal section \(M_0\) (resp. a section \(M_1\) with \((M_1^2) = |n|\)) of \(\Sigma_{|n|}\). Let \(d \geq 2\) and \(r = d + n \geq 1\). With the notations of Lemma 2.2, we assume that the fiber \(\ell_0\) is defined by \(z_0 = 0\).

Let \(w = w_0/w_1\). Then \(\{z_0/z_1, w/z_1^n\}\) is a system of local coordinates at the point \(M_1 \cap \ell_0\) (resp. \(M_0 \cap \ell_0\)) if \(n \geq 0\) (resp. \(n < 0\)). Let \(\Lambda\) be a linear subsystem of \(|M_1 + d\ell_0|\) if \(n \geq 0\) (resp. \(|M_0 + d\ell_0|\) if \(n < 0\)), consisting of members which meet the curve \(M_1\) (resp. \(M_0\)) at the point \(M_1 \cap \ell_0\) (resp. \(M_0 \cap \ell_0\)) with multiplicity \(r\) if \(n \geq 0\) (resp. \(n < 0\)). Then any member of \(\Lambda\) is defined by an equation

\[
\frac{w}{z_1^n} \left\{ a_0 + a_1 \left( \frac{z_0}{z_1} \right) + \cdots + a_{d-1} \left( \frac{z_0}{z_1} \right)^{d-1} + a_d \left( \frac{z_0}{z_1} \right)^d \right\} + a_{d+1} \left( \frac{z_0}{z_1} \right)^r = 0
\]

or equivalently by

\[(1) \quad w_0 (a_0 z_1^d + a_1 z_0 z_1^{d-1} + \cdots + a_{d-1} z_0^{d-1} z_1 + a_d z_0^d) + a_{d+1} z_0^r w_1 = 0
\]

for \((a_0,a_1,\ldots,a_{d+1}) \in \mathbb{C}^{d+1}\). In fact, it is readily computed that \(\dim \Lambda = d + 1\).

So, the curve \(C_d\) is defined by such an equation with \(a_0 \neq 0\) and \(a_{d+1} \neq 0\). Hence it follows that

\[X(d,r) = \Sigma_{|n|} \setminus (\{w_0 = 0\} \cup C_d)\]

where \(n = r - d\) and \(C_d\) is the curve defined by (1) with \(a_0 \neq 0\) and \(a_{d+1} \neq 0\). We shall verify the following result.

**Lemma 2.5.** Let \(r \geq 2\) and let \(X = X(d,r)\) be an affine pseudo-plane defined as above. Let \(\sigma : G_m \times X \to X\) be a non-trivial action of the algebraic torus \(G_m = \mathbb{C}^*\). Then the following assertions hold true:

1. The action \(\sigma\) induces an action \(\sigma : G_m \times \Sigma_{|n|} \to \Sigma_{|n|}\) such that \(\sigma^{(\mu)} M_i \subseteq M_i\) for \(i = 0,1\), \(\sigma^{(\mu)} C_d \subseteq C_d\) and \(\sigma^{(\mu)} \ell_0 \sim \ell_0\), where \(\sigma^{(\mu)} M_i\) denotes the image of \(M_i\) under the action of \(\mu \in \mathbb{C}^*\), etc.

2. The curve \(C_d\) is defined by an equation

\[z_1^d w_0 + a z_0^r w_1 = 0 \quad \text{for} \quad a \in \mathbb{C}^*\]
Proof: (1) We prove only the case \( n = r - d \geq 0 \). The proof of the case \( n < 0 \) is done in the same manner. Let \( \rho : X \to C \cong A^1 \) be the unique \( A^1 \)-fibration (cf. Theorem 23). Then the fibers of \( \rho \) are permuted by the action \( \sigma \). Hence \( \sigma \) extends to the cross-section \( S' \) and sends \( S' \) into itself. Let \( W \) be a \( G_m \)-equivariant smooth normal compactification of \( X \) whose existence is guaranteed by [24]. We may assume that \( W \setminus X \) contains the cross-section \( S' \). Let \( F_0 \) and \( F_\infty \) be two fibers of the \( P^1 \)-fibration \( \rho : W \to P^1 \) whose supports partly or totally lie outside of \( X \), where \( F_0 \) contains the multiple fiber of \( \rho \). We may assume that all \((-1)\) components of \( F_0 \) and \( F_\infty \) are fixed componentwise under the action \( \sigma \). Then we may assume that \( F_\infty \) is irreducible and \( F_0 \) minus the component \( F \) contains no \((-1)\) components, where \( F \cap X \) gives rise to the multiple fiber of \( \rho \). Then we may assume that \( W \setminus X \) has the dual graph as in Definition 2.1. So, the action \( \sigma \) induces a \( G_m \)-action on \( \Sigma_n \) such that \( \sigma^{(n)}(M_1) \subset M_1, \sigma^{(n)}(C_d) \subset C_d \) and \( \sigma^{(n)}(\ell_0) \subset \ell_0 \) because \( M_1, C_d, \ell_0 \) are the images on \( \Sigma_n \) of the components \( E_1, \ell_\infty, F \), respectively. The minimal section \( M_0 \) is stable under the \( \sigma \)-action because the minimal section is unique on \( \Sigma_n \).

(2) The \( G_m \)-action \( \sigma \) on \( \Sigma_{|n|} \) is given as follows in terms of the coordinates:

\[
\mu \cdot ((z_0, z_1), [w_0, w_1]) = ((\mu^\alpha z_0, \mu^\beta z_1), [\mu^\gamma w_0, \mu^\delta w_1]) \]

for \( \mu \in \mathbb{C}^* \). Since \( C_d \) is stable under the \( \sigma \)-action, the defining equation (1) must be semi-invariant. Note that \( a_0 a_{d+1} \neq 0 \) because \( C_d \) is irreducible. Hence we obtain \( \alpha r + \delta = \beta d + \gamma \). Suppose that \( (a_1, \ldots, a_d) \neq (0, \ldots, 0) \). Then we have an additional relation \( \alpha i + \beta (d - i) + \gamma = \beta d + \gamma \) for some \( 1 \leq i \leq d \). The last relation implies \( \alpha = \beta \). So, the first relation gives \( \gamma = \alpha n + \delta \). Then we have

\[
\mu \cdot ((z_0, z_1), [w_0, w_1]) = ((\mu^\alpha z_0, \mu^\beta z_1), [\mu^\gamma w_0, \mu^\delta w_1]) = ((\mu^\alpha z_0, \mu^\beta z_1), [\mu^{\alpha n + \delta} w_0, \mu^\delta w_1]) \\
\sim ((z_0, z_1), [\mu^{\delta} w_0, \mu^\delta w_1]) = ((z_0, z_1), [w_0, w_1]).
\]

Hence the \( \sigma \)-action is trivial. This proves the second assertion. \( \square \)

Let \( V(d, r) \) be the affine pseudo-plane defined by

\[
V(d, r) = \Sigma_{|n|} \setminus \{ (w_0 = 0) \cup \{ z_1 w_0 + z_0 r w_1 = 0 \} \}
\]

where \( n = r - d \). Then there exists a \( G_m \)-action on \( V(d, r) \) defined by

\[
\mu \cdot ((z_0, z_1), [w_0, w_1]) = ((\mu z_0, z_1), [w_0, \mu^{-r} w_1])
\]

for \( \mu \in \mathbb{C}^* \). For \( r \geq 2 \), one can show that any \( G_m \)-action on \( V(d, r) \) is reduced to the \( G_m \)-action specified as above. In fact, with the notation in the proof of Lemma 2.5(2)

\[
\mu \cdot ((z_0, z_1), [w_0, w_1]) = ((\mu^\alpha z_0, \mu^\beta z_1), [\mu^\gamma w_0, \mu^\delta w_1]) = ((\mu^\alpha z_0, \mu^\beta z_1), [\mu^{\alpha r - \beta d + \delta} w_0, \mu^\delta w_1]) = ((\mu^{\alpha - \beta} z_0, z_1), [\mu^{r (\alpha - \beta)} w_0, w_1]) = ((\mu^{\alpha - \beta} z_0, z_1), [w_0, \mu^{-r (\alpha - \beta)} w_1]).
\]

We shall consider the universal covering \( \tilde{X}(d, r) \) of an affine pseudo-plane \( X(d, r) \).
Lemma 2.6. The following assertions hold true:

1. The universal covering $\tilde{X}(d,r)$ is isomorphic to an affine hypersurface in $\mathbb{A}^3 = \text{Spec} \mathbb{C}[x, y, z]$ defined by an equation

$$x^r z + (y^d + a_1 x y^{d-1} + \cdots + a_{d-1} x^{d-1} y + a_d x^d) = 1.$$ 

2. The projection $\pi : \tilde{X}(d,r) \to X(d,r)$ is a cyclic group $H(d) := \mathbb{Z}/d\mathbb{Z}$ of order $d$ and acts as

$$\lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^{-r} z)$$

for $\lambda \in H(d)$.

3. There is a $G_a$-action on $\tilde{X}(d,r)$ defined by

$$c \cdot (x, y, z) = (x, y + cx^r, z - x^r((y + cx^r)^d + a_1 x(y + cx^r)^{d-1} + \cdots + a_d x^d) - (y^d + a_1 xy^{d-1} + \cdots + a_{d-1} x^{d-1} y + a_d x^d)),$$

where $c \in G_a = \mathbb{C}$.

4. The $\mathbb{A}^1$-fibration $\tilde{p} : \tilde{X}(d,r) \to \mathbb{A}^1$ is unique for $r \geq 2$.

5. Let $\omega$ be a $d$-th root of unity. Then there exist uniquely determined polynomials $p_\omega(x), q_\omega(x) \in \mathbb{C}[x]$ satisfying the following conditions:

i. $\deg p_\omega(x) \leq r - 1$.
ii. $p_\omega(0) = \omega$.
iii. $x^r p_\omega(x) + p_\omega(x)^d + a_1 x p_\omega(x)^{d-1} + \cdots + a_{d-1} x^{d-1} p_\omega(x) + a_d x^d = 1$.
iv. $p_\lambda(\lambda r x) = \lambda p_\omega(\lambda x), q_\lambda(\lambda x) = \lambda^{-r} q_\omega(x)$ for any $d$-th root $\lambda$ of unity.

By making use of these polynomials, we define the morphism

$$\varphi_\omega : \mathbb{A}^2 \cong \mathbb{A}^1 \times G_a \to \tilde{X}(d,r), \quad (x, c) \mapsto c \cdot (x, p_\omega(x), q_\omega(x))$$

which is an open immersion onto an open set $U_\omega$ which is the complement of $\prod_{\lambda \neq \omega} G_a \cdot (0, \lambda, 0)$. The inverse morphism $\varphi_\omega^{-1}$ on $U_\omega$ is defined by

$$(x, y, z) \mapsto \begin{cases} (x, \frac{y - p_\omega(x)}{x^r}) & \text{if } x \neq 0, \\ (0, \frac{-z + q_\omega(0)}{d \omega^{-1}}) & \text{if } x = 0. \end{cases}$$

6. $\tilde{X}(d,r)$ is obtained by glueing together the $d$-copies of the affine plane $\mathbb{A}^2$ by the transition functions

$$g_{\lambda} := \varphi_\lambda^{-1} \circ \varphi_\omega : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$$

$$(x, c) \mapsto (x, c + \frac{p_\omega(x) - p_\lambda(x)}{x^r}),$$

where $\omega, \lambda \in H(d)$ and $\mathbb{A}_1^1 = \mathbb{A}^1 - \{0\}$.

7. The Galois group $H(d)$ acts as

$$\lambda \cdot \varphi_\omega(x, c) = \varphi_{\lambda \omega}(\lambda x, \lambda^1 - r c).$$
Proof. (1) Recall that $X(d, r)$ is the complement in $\Sigma_{n1}$ of the curves $C_d$ defined by the equation (1) with $a_0 \neq 0$ and $a_{d+1} \neq 0$, and the curve $w_0 = 0$ which is $M_1$ if $r - d \geq 0$ (resp. $M_2$ if $r - d < 0$), where $n = r - d$. Since $w_0 \neq 0$, we can normalize to $w_0 = 1$. We can then normalize
\[
w_0 \left(a_0 z_1^d + a_1 z_0 z_1^{d-1} + \cdots + a_d z_0^{d-1} z_1 + a_d z_0^d\right) + a_{d+1} z_0^d w_1 \neq 0
\]
to the relation
\[
z_0^d w_1 + (a_0 z_1^d + a_1 z_0 z_1^{d-1} + \cdots + a_d z_0^{d-1} z_1 + a_d z_0^d) = 1,
\]
where $a_0 \neq 0$. This normalization comes from the defining equivalence relation
\[
(z_0, z_1), [w_0, w_1] \sim (\nu z_0, \nu z_1), [\nu^a w_0, w_1].
\]
We may assume that $a_0 = 1$. The equivalence relation requires that the points $(\lambda z_0, \lambda z_1, \lambda^{-n} w_1)$ for $\lambda \in H(d)$ should be identified together, where we note that $n = r - d$. Hence the assertion follows. Note that $X(d, r)$ is simply connected.

(3) Let $\delta$ be a derivation on the coordinate ring $\Gamma(X(d, r))$ defined by
\[
\delta(x) = 0, \quad \delta(y) = x^r, \quad \delta(z) = -(dy^{d-1} + (d - 1)a_1 xy^{d-2} + \cdots + a_{d-1} x^{d-1}).
\]
Then $\delta$ is locally nilpotent. Hence it defines a $G_d$-action on $X(d, r)$ by
\[
c \cdot a = \sum_{n=0}^{\infty} \frac{c^n}{n!} \delta^n(a), \quad \text{for } c \in G_d, a \in \Gamma(X(d, r)),
\]
which is as specified in the assertion. One easily verifies that $\ker \delta = \mathbb{C}[x]$ and the inclusion $\ker \delta \hookrightarrow \Gamma(X(d, r))$ induces an $\mathbb{A}^1$-fibration $\hat{\rho} : \hat{X}(d, r) \to \mathbb{A}^1$.

(4) We consider the smooth compactification $(V, D)$ as given in Definition 2.1.

Then we have a linear equivalence
\[
\ell'_\infty \sim \ell'_0 + E_1 + 2E_2 + \cdots + (d - 1)E_{d-1} + dE_d + d(E_{d+1} + \cdots + E_{d+r-1} + \bar{F}),
\]
which is written as follows:
\[
\ell'_\infty \sim (d - 1)(\ell'_0 + E_1) + (d - 2)E_2 + \cdots + E_{d-1}
\]
\[
\sim d \left\{ \ell'_0 + E_1 + E_2 + \cdots + E_{d-1} + E_d + E_{d+1} + \cdots + E_{d+r-1} + \bar{F} \right\}.
\]

Let $q : \hat{V} \to V$ be a $d$-ple cyclic covering which ramifies totally over the branch locus $\ell'_\infty + \ell'_0 + E_1 + E_2 + \cdots + E_{d-1}$. Then $\hat{V}$ has cyclic quotient singularities over the intersection points $\ell'_0 \cap E_2, E_2 \cap E_3, \ldots, E_{d-2} \cap E_{d-1}$. The minimal resolution of these singularities will only insert linear chains of exceptional curves in between the proper transforms of the intersecting curves. Meanwhile, the curve $\bar{E}_d := q^{-1}(E_d)$ of the curve $E_d$ remains irreducible on $\hat{V}$ and the linear branch $E_{d+1} + \cdots + E_{d+r-1} + \bar{F}$ splits into a disjoint union of $d$ linear branches $\bar{E}_d^{(j)} + \cdots + \bar{E}_d^{(j)} + \bar{F}^{(j)} (1 \leq j \leq d)$. Each of the curves $\bar{E}_d^{(j)} (1 \leq j \leq d)$ meets the curve $\bar{E}_d$ transversally in one point. Furthermore, $q^{-1}(X(d, r))$ is the universal covering space $\hat{X}(d, r)$. By the above observations, one knows that the boundary divisor of $\hat{X}(d, r)$ embedded minimally in a smooth projective surface, which is obtained from $\hat{V}$ by resolving minimally the above cyclic quotient singularities and contracting $(-1)$ curves and consecutively contractible curves resulting from the inverse image of $q^{-1}(\ell'_\infty + S' + \ell'_0 + E_1 + \cdots + E_{d-1})$, is not a linear chain, for
\[\tilde{E}_d + \sum_{j=1}^{r} (\tilde{E}_d^{(j)} + \cdots + \tilde{E}_d^{(r)}) \text{ cannot become a part of a linear chain if } d \geq 2 \text{ and } r \geq 2. \] So, we are done by a theorem of Bertin [3].

(5) Write \( p_\omega(x) = \omega + c_1(\omega)x + \cdots + c_{r-1}(\omega)x^{r-1} \), where the coefficients are to be determined by the relation

\[ x^r q_\omega(x) + p_\omega(x)^d + a_1 x p_\omega(x)^{d-1} + \cdots + a_{d-1} x^{d-1} p_\omega(x) + a_d x^d = 1 \]

which is obtained from equation (2) above by substituting \( p_\omega(x), q_\omega(x) \) for \( y, z \).

By condition (i), it is easy to see that \( p_\omega(x) \) is uniquely determined. Namely the coefficients \( c_1(\omega), \ldots, c_{r-1}(\omega) \) are uniquely determined by putting the coefficients of the terms \( x^i \) \((1 \leq i \leq r-1)\) to be zero in the left-hand side of equation (3) above. Then \( q_\omega(x) \) is uniquely determined as well. By multiplying \( \lambda^d = 1 \) to the relation (3), we obtain

\[ (\lambda x)^r \lambda^{-r} q_\omega(\lambda^{-1}(\lambda x)) + (\lambda p_\omega(\lambda^{-1}(\lambda x)))^d + a_1(\lambda x)(\lambda p_\omega(\lambda^{-1}(\lambda x)))^{d-1} + \cdots + a_d(\lambda x)^d = 1. \]

Replace \( \lambda x \) by \( x \) in the above relation. Then the uniqueness of the polynomials \( p_{\lambda \omega}(x), q_{\lambda \omega}(x) \) imply that \( p_{\lambda \omega}(x) = \lambda p_\omega(\lambda^{-1} x) \) and \( q_{\lambda \omega}(x) = \lambda^{-r} q_\omega(\lambda^{-1} x) \). Now replace \( x \) by \( \lambda x \). Then we obtain the relation (iv). Note that \( \varphi_\omega : \mathbb{A}^2 \to U_\omega \) is injective and \( U_\omega \cong \mathbb{A}^2 \). Hence \( \varphi_\omega \) is an isomorphism by [1]. The \( \varphi_\omega(\omega \in H(d)) \) are bundle charts of \( \tilde{X}(d, r) \) and \( \tilde{X}(d, r) \) is obtained by gluing \( d \)-copies \( \{ \omega \} \times \mathbb{A}^3 \) \((\omega \in H(d)) \) under the identification

\[(\omega, x, c) \sim (\lambda, x, c + \frac{p_\omega(x) - p_\lambda(x)}{x^r}) \]

for \( \omega, \lambda \in H(d) \) and \((x, c) \in \mathbb{A}^1 \times \mathbb{A}^1 \). The other assertions are verified in a straightforward manner.

We say that a homogeneous polynomial \( y^d + a_1 xy^{d-1} + \cdots + a_d x^d \) with the coefficient of the \( y^d \)-term equal to 1 is monic. Let \( \tilde{X}(d, r) \) and \( \tilde{X}'(d, s) \) be the affine hypersurfaces in \( \mathbb{A}^3 \) defined by \( x^r z + f(x, y) = 1 \) and \( x^s z + h(x, y) = 1 \), respectively, where \( f(x, y) \) and \( h(x, y) \) are monic homogeneous polynomials of degree \( d \). If \( r \neq s \), then \( \tilde{X}(d, r) \not\cong \tilde{X}'(d, s) \) since \( \pi_{1, \infty}(\tilde{X}(d, r)) \neq \pi_{1, \infty}(\tilde{X}'(d, s)) \). Concerning the isomorphism classes of the hypersurfaces \( \tilde{X}(d, r) \), we have the following result. Note that we may assume that \( f(x, y) \) and \( h(x, y) \) are of the form \( y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d \) by changing the coordinates.

**Lemma 2.7.** Let \( d \geq 2 \) and \( r > d \). Let \( \tilde{X}_1(d, r) \) and \( \tilde{X}_2(d, r) \) be the hypersurfaces defined by the equations \( x^r z + y^d + a_1 x y^{d-1} + \cdots + a_d x^d = 1 \) and \( x^r z + y^d + b_1 x y^{d-1} + \cdots + b_d x^d = 1 \), respectively. Then there is an isomorphism \( \tilde{X}_1(d, r) \cong \tilde{X}_2(d, r) \) if and only if

\[ a_i = \mu^i b_i, \quad \text{for } \mu \in \mathbb{C}^*, \quad 2 \leq i \leq d. \]

**Proof.** Let \( f : \tilde{X}_1(d, r) \to \tilde{X}_2(d, r) \) be an isomorphism and let \( \varphi \) be the induced isomorphism of the coordinate rings. Note that \( f \) preserves the unique \( \mathbb{A}^3 \)-fibrations of \( \tilde{X}_1(d, r) \) and \( \tilde{X}_2(d, r) \) as well as the reduced reducible fibers. Hence \( f \) induces an automorphism of \( \mathbb{C}[x] \) and \( \varphi(x) = \mu x \) with \( \mu \in \mathbb{C}^* \). Then \( \varphi \) induces an automorphism of the polynomial ring \( \mathbb{C}[x, x^{-1}][y] \). So, one can write \( \varphi(y) = uy^e + F(x) \) with \( u \in \mathbb{C}^*, e \in \mathbb{Z} \) and \( F(x) \in \mathbb{C}[x, x^{-1}] \). Since \( \varphi \) is an isomorphism of the coordinate rings, it follows that \( e \geq 0 \) and \( F \in \mathbb{C}[x] \). Furthermore, since \( f \) maps
isomorphically the unique reducible fiber of \( \tilde{X}_1(d, r) \) to the unique reducible fiber of \( \tilde{X}_2(d, r) \), it follows that \( e = 0, F(0) = 0 \) and \( u^d = 1 \). Since
\[
z = x^{-r} \{ 1 - (y^d + b_2 x^2 y^{d-2} + \cdots + b_d x^d) \}
\in \Gamma(\tilde{X}_2(d, r)) \otimes \mathbb{C}[x, x^{-1}] = \mathbb{C}[x, x^{-1}][y],
\]
it follows that \( \varphi(z) = \mu x^{-r} \{ 1 - (\varphi(y)^d + b_2 (\mu x)^2 \varphi(y)^{d-2} + \cdots + b_d (\mu x)^d) \} \) in \( \Gamma(\tilde{X}_1(d, r)) \otimes \mathbb{C}[x, x^{-1}] = \mathbb{C}[x, x^{-1}][y] \). While, \( \varphi \) is an isomorphism from \( \Gamma(\tilde{X}_2(d, r)) \) to \( \Gamma(\tilde{X}_1(d, r)) \), \( \varphi(z) \) is written in a form \( \sum_{k \geq 0} \varphi_k(x, y) z^k \), where \( \varphi_k(x, y) = \sum_{0 \leq j < d} \phi_k(j)(x) y^j \) for \( \phi_k(j)(x) \in \mathbb{C}[x] \). Hence we have \( \varphi(z) = \mu x^{-r} z + \varphi_0(x, y) \), and
\[
(\varphi(y))^d + b_2 \mu^2 x^2 (\varphi(y))^{d-2} + \cdots + b_d \mu^d x^d - 1
\]
costs with
\[
y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d - 1
\]
modulo \( x^r \). The comparison of the coefficients of the terms \( x^i y^{d-i} \) for \( 1 \leq i \leq d \) implies that \( F \) is a multiple of \( x^r \) and \( a_i = \mu^i u_i b_i \) for \( 2 \leq i \leq d \). Replacing \( \mu u^{-1} \) by a new \( \mu \), we obtain \( a_i = \mu^i b_i \) for \( \mu \in \mathbb{C}^* \), \( 2 \leq i \leq d \).

Conversely, if \( a_i = \mu^i b_i \) \((2 \leq i \leq d)\) for \( \mu \in \mathbb{C}^* \), then we can determine an isomorphism \( \varphi \) by
\[
\begin{align*}
\varphi(x) &= \mu x, \\
\varphi(y) &= y + x^r G(x), \\
\varphi(z) &= \mu x^{-r} \left[ z - x^{-r} \left( (\varphi(y)^d + b_2 (\mu x)^2 \varphi(y)^{d-2} + \cdots + b_d (\mu x)^d) \right. \right. \\
&\quad \quad \quad \left. \left. - (y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d) \right) \right],
\end{align*}
\]
where \( G(x) \in \mathbb{C}[x] \). \( \square \)

Let \( \tilde{X}(d, r) \) be the affine hypersurface in \( \mathbb{A}^3 \) defined by equation (2) in Lemma 2.6 which has the transition functions given in assertion (6) of the same lemma. Let \( \tilde{X}'(d, s) \) be a similar affine hypersurface in \( \mathbb{A}^3 \) with the equation
\[
x^s + (y^d + a_1 x y^{d-1} + \cdots + a_{d-1} x^{d-1} y + a_d x^d) = 1
\]
and the transition functions
\[
g_{\lambda, \omega} := \varphi_{-1, \lambda} \circ \varphi_{\omega, \omega} : \mathbb{A}^1_x \times \mathbb{A}^1_x \to \mathbb{A}^1_x \times \mathbb{A}^1_x, \quad (x, c) \mapsto (x, c + \frac{p_s(x) - p_s(x)}{x^s}).
\]
As in [5], we define a 3-dimensional affine variety \( \tilde{X}(d, r, s) \) by glueing together \( d \)-copies of the affine 3-space \( \{ \omega \} \times \mathbb{A}^3 \) (\( \omega \in \mathbb{H}(d) \)) by the following identification:
\[
(\omega, x, c_1, c_2) \sim (\lambda, x, c_1 + \frac{p_s(x) - p_s(x)}{x^s}, c_2 + \frac{p_s(x) - p_s(x)}{x^s}), \quad x \neq 0.
\]
The projection \( (\omega, x, c_1, c_2) \mapsto \omega, x, c_1 \) yields a morphism \( \pi_1 : \tilde{X}(d, r, s) \to \tilde{X}(d, r) \) which is a principal \( G_a \)-bundle over \( \tilde{X}(d, r) \) with \( G_a \) acting naturally on the coordinate \( c_2 \). Similarly, the projection \( (\omega, x, c_1, c_2) \mapsto (\omega, x, c_2) \) gives rise to a principal \( G_a \)-bundle \( \pi_2 : \tilde{X}(d, r, s) \to \tilde{X}'(d, s) \) with \( G_a \) acting naturally on the coordinate \( c_1 \). Since every principal \( G_a \)-bundle over an affine variety is trivial [20], it follows that
\[
\tilde{X}(d, r) \times \mathbb{A}^1_x \cong \tilde{X}(d, r, s) \cong \tilde{X}'(d, s) \times \mathbb{A}^1_x.
\]
Hence the surfaces \( \tilde{X}(d, r) \) have the non-cancellation property.
Theorem 2.8. Let $d \geq 2$ and let $r, s > d$. Let $\tilde{X}(d, r)$ and $\tilde{X}^\prime(d, s)$ be the affine hypersurfaces defined by the equations $x^r z + (y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d) = 1$ and $x^s z + (y^d + a'_2 x^2 y^{d-2} + \cdots + a'_d x^d) = 1$, respectively. Then the following assertions hold:

1. For any $r$ and $s$,
   \[ \tilde{X}(d, r) \times \mathbb{A}^1 \cong \tilde{X}^\prime(d, s) \times \mathbb{A}^1. \]

2. The isomorphism $\tilde{X}(d, r) \cong \tilde{X}^\prime(d, s)$ holds if and only if $r = s$ and $a'_i = \mu^i a_i$ for $\mu \in \mathbb{C}^*$ and $2 \leq i \leq d$.

At this point, we do not know whether the isomorphism $\tilde{X}(d, r) \times \mathbb{A}^1 \cong \tilde{X}^\prime(d, s) \times \mathbb{A}^1$ in Theorem 2.8 is $H(d)$-equivariant or not. We shall show that the isomorphism in Theorem 2.8(1) is in fact $H(d)$-equivariant in some cases. The $H(d)$-action specified in assertion (1) of Lemma 2.6 is lifted to $\tilde{X}(d, r, s)$ on $\omega$-charts as follows so that $\pi_1$ and $\pi_2$ are $H(d)$-equivariant:

\[ \lambda \cdot (\omega, x, c_1, c_2) = (\lambda \omega, \lambda x, \lambda^{1-r} c_1, \lambda^{1-s} c_2) \quad \text{for} \quad \lambda \in H(d). \]

We look for $H(d)$-equivariant sections of $\pi_1$ and $\pi_2$. A section of $\pi_1 : \tilde{X}(d, r, s) \rightarrow \tilde{X}(d, r)$ is expressed on $\omega$-chart as

\[ \{\omega\} \times \mathbb{A}^2 \rightarrow \{\omega\} \times \mathbb{A}^3, \quad (x, c) \mapsto (x, c, \sigma_\omega(x, c)). \]

Hence an $H(d)$-equivariant section of $\pi_1$ is a family of $\sigma_\omega \in \mathbb{C}[x, c]$ ($\omega \in H(d)$), which is compatible with glueing maps and $H(d)$-actions.

Lemma 2.9. For the principal $G_a$-bundle $\pi_1 : \tilde{X}(d, r, s) \rightarrow \tilde{X}(d, r)$, an $H(d)$-equivariant section is given by a family of polynomials $\sigma_\omega \in \mathbb{C}[x, c]$ ($\omega \in H(d)$), satisfying the following conditions:

1. For all $\omega, \lambda \in H(d)$ and $(x, c) \in \mathbb{A}^1 \times \mathbb{A}^1$,
   \[ \sigma_\omega(x, c) + \frac{p_\omega(x) - p_\lambda(x)}{x^s} = \sigma_\lambda(x, c) + \frac{p_\omega(x) - p_\lambda(x)}{x^r}. \]

2. For any $\omega, \lambda \in H(d)$,
   \[ \lambda^{1-s} \sigma_\omega(x, c) = \sigma_\lambda(x, \lambda^{1-r} c). \]

We can use relation (2) in the above lemma to compute $\sigma_\lambda$ from $\sigma_1$:

\[ \sigma_\lambda(x, c) = \lambda^{1-s} \sigma_1(\lambda^{-1} x, \lambda^{r-1} c). \]

In terms of the function $\sigma_1$, conditions (1) and (2) in Lemma 2.9 are reformulated as in the following result. The proof is essentially the same as in [5] if one takes into account relation (5)(iv) of Lemma 2.6.

Lemma 2.10. Given a polynomial $\sigma = \sigma_1 \in \mathbb{C}[x, c]$, define polynomials \{ $\sigma_\lambda$ | $\lambda \in H(d)$ \} by equation (4) above. Then conditions (1) and (2) in Lemma 2.9 are satisfied if and only if $\sigma$ satisfies

\[ \lambda^{1-s} x^s \sigma(\lambda^{-1} x, \lambda^{r-1} c + \frac{p_1(x) - p_\lambda(x)}{x^r}) = x^s \sigma(x, c) + p'_1(x) - p'_\lambda(x) \]

for all $\lambda \in H(d), (x, c) \in \mathbb{A}^1 \times \mathbb{A}^1$. 

If there exists a polynomial \( \sigma \) satisfying condition (5) in Lemma 2.10, then there is an \( H(d) \)-equivariant isomorphism
\[
\widetilde{X}(d, r, s) \cong \widetilde{X}(d, r) \times \mathbb{A}^1(1 - s),
\]
where \( \mathbb{A}^1(a) \) denotes the one-dimensional \( H(d) \)-module of weight \( a \). In fact, an \( H(d) \)-equivariant isomorphism \( \widetilde{X}(d, r) \times \mathbb{A}^1(1 - s) \cong \widetilde{X}(d, r, s) \) is defined as follows on the \( \omega \)-chart for \( \omega \in H(d) \):
\[
(\omega \times \mathbb{A}^2) \times \mathbb{A}^1 \rightarrow \omega \times \mathbb{A}^3, \\
((\omega, x, c_1), c_2) \mapsto (\omega, x, c_1, \sigma_\omega(x, c_1) + c_2).
\]

Let \( \widetilde{V}(d, r) \) be the affine surface defined by \( x^r z + y^d = 1 \). Then \( \widetilde{V}(d, r) \) is the universal covering of the affine pseudo-plane \( V(d, r) \). The polynomial \( p_\omega(x) \) corresponding to \( \widetilde{V}(d, r) \) is \( p_\omega(x) = \omega \). Let \( \tilde{V}(d, r, s) \) be the affine variety glueing \( \{\omega\} \times \mathbb{A}^3 \) for \( \omega \in H(d) \) with transition functions of \( \tilde{V}(d, r) \) and \( \tilde{V}(d, s) \) just as we constructed \( \widetilde{X}(d, r, s) \). In \([5]\), it is shown that there exist \( H(d) \)-equivariant sections of \( \tilde{V}(d, r, s) \rightarrow \tilde{V}(d, r) \) for any \( r \) and \( s \). The next result is a key fact in finding \( H(d) \)-equivariant sections, which is due to tom Dieck in the case \( u = 1 \).

**Proposition 2.11.** Let \( u \) and \( t \) be positive integers and let \( 1 \leq u \leq d - 1 \).

(1) There exists a unique polynomial \( Q_{u,t}(x) \in \mathbb{C}[x] \) satisfying the following properties:
\[
(i) \quad Q_{u,t}(\lambda x) = \lambda^u Q_{u,t}(x) \text{ for any } \lambda \in H(d).
\]
\[
(ii) \quad \deg Q_{u,t}(x) = u + (t - 1)d.
\]
\[
(iii) \quad Q_{u,t}(1 + x) - 1 \text{ is divisible by } x^t.
\]
(2) Let \( P_{u,t}(x) \) be the polynomial defined by the equation \( Q_{u,t}(1 + x) - 1 = x^t P_{u,t}(x) \). Then for any \( \lambda \in H(d) \),
\[
\lambda^{-t}(x + 1 - \lambda)^t P_{u,t}(\lambda^{-1}(x + 1 - \lambda)) = \lambda^{-u}(x^t P_{u,t}(x) + 1 - \lambda^u).
\]

**Proof.** (1) By the property (i) and (ii), \( Q_{u,t}(x) \) is written as
\[
Q_{u,t}(x) = \sum_{j=0}^{t-1} a_j x^{u+jd}.
\]
By property (iii), the coefficients \( a_j \) must satisfy linear equations. The determinant of the coefficient matrix of the system of the linear equations in \( a_0, \ldots, a_{t-1} \) is
\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
(1 \ldots u) & (1 \ldots u+d) & \cdots & (1 \ldots u+(t-1)d) \\
\vdots & \vdots & \ddots & \vdots \\
(1 \ldots u_{t-1}) & (1 \ldots u+d_{t-1}) & \cdots & (1 \ldots u+(t-1)d_{t-1})
\end{vmatrix},
\]
where the binomial coefficient \( \binom{u+jd}{i} \) for \( a < b \) is defined to be zero. Note that \( \binom{u+jd}{i} \) is a polynomial in \( jd \) of degree \( i \). By adding a linear combination of the first \( i \) rows to the \((i+1)\)-th row, the \((i+1)\)-th row becomes \( 1/i! \) times of \((0, d^i, (2d)^i, \ldots, (t-1)^id^i) \). Hence the determinant reduces to a non-zero multiple of the Vandermonde determinant and its value is non-zero. Thus we can determine the coefficients \( a_j \), and the polynomial \( Q_{u,t}(x) \) is uniquely determined.
(2) By the definition of $P_{u,t}(x)$, $Q_{u,t}(x)$ is written as

$$Q_{u,t}(x) = 1 + (x - 1)^t P_{u,t}(x - 1).$$

Then the required relation follows from property (i) of $Q_{u,t}(x).$ \hfill $\square$

**Theorem 2.12** (cf. tom Dieck [5]). Let $d \geq 2$ and $r, s \geq 1$. Then for any $r$ and $s$,

there exists an $H(d)$-equivariant isomorphism

$$\tilde{V}(d, r) \times \mathbb{A}^1(1 - s) \cong \tilde{V}(d, s) \times \mathbb{A}^1(1 - r).$$

**Proof.** It suffices to find $\sigma(x, c)$ satisfying

$$\lambda^{1-s} x^s \sigma(\lambda^{-1}x, \lambda^{-1}(c + \frac{1 - \lambda}{x^r})) = x^s \sigma(x, c) + 1 - \lambda$$

for all $\lambda \in H(d)$, $(x, c) \in \mathbb{A}^1 \times \mathbb{A}^1$. Let $a$ and $t$ be integers such that $a = -s + rt$, $t > 0$ and $a \geq 0$. Set $\sigma(x, c) = x^a c^t P_{1,t}(x^r c)$, where $P_{1,t}(x)$ is the polynomial defined in Proposition 2.11(2). Then one easily verifies that $\sigma(x, c)$ satisfies the above condition, and the assertion follows. \hfill $\square$

**Remark.** There is a $\mathbb{C}^*$-action on $\tilde{V}(d, r)$ defined by

$$\mu \cdot (x, y, z) = (\mu x, y, \mu^{-r} z) \quad \text{for} \quad \mu \in \mathbb{C}^*,$$

which is the lift-up of the $\mathbb{C}^*$-action on the affine pseudo-plane $V(d, r)$ (cf. Lemma 2.5 and the statement below it). One verifies that the isomorphism in Theorem 2.12 is in fact an $H(d) \times \mathbb{C}^*$-equivariant isomorphism

$$\tilde{V}(d, r) \times \mathbb{A}^1(1 - s, -s) \cong \tilde{V}(d, s) \times \mathbb{A}^1(1 - r, -r),$$

where $\mathbb{A}^1(a, b)$ denotes the one-dimensional $H(d) \times \mathbb{C}^*$-module with weight $a$ for $H(d)$ and with weight $b$ for $\mathbb{C}^*$.

In some cases, we can find a polynomial $\sigma$ satisfying the condition in Lemma 2.10 and write down $\sigma$ explicitly. First, we consider the case $r = s = 2$.

**Lemma 2.13.** Let $\tilde{X}(d, 2)$ and $\tilde{X}'(d, 2)$ be the affine surfaces defined by $x^2z + f(x, y) = 1$ and $x^2z + h(x, y) = 1$, respectively, where $f(x, y)$ and $h(x, y)$ are monic homogeneous polynomials of degree $d$. Then for any $d \geq 2$ there is an $H(d)$-equivariant isomorphism

$$\tilde{X}(d, 2) \times \mathbb{A}^1(-1) \cong \tilde{X}'(d, 2) \times \mathbb{A}^1(-1).$$

**Proof.** Let $\tilde{X}(d, 2, 2)$ be the affine variety obtained by $\tilde{X}(d, 2)$ and $\tilde{X}'(d, 2)$. For the principal $G_a$-bundle $\tilde{X}(d, 2) \rightarrow \tilde{X}(d, 2), p_1(x)$ and $p_1'(x)$ in Lemma 2.9 are both of the form $1 + ax$ for $a \in \mathbb{C}$. Since $p_1(x) = \lambda p_1(\lambda^{-1}x)$ and $p_1'(x) = \lambda p_1'(\lambda^{-1}x)$, condition (5) in Lemma 2.10 is reduced to

$$\lambda^{-1} x^2 \sigma(\lambda^{-1}x, \lambda(c + \frac{1 - \lambda}{x^2})) = x^2 \sigma(x, c) + 1 - \lambda$$

for all $\lambda \in H(d)$, $(x, c) \in \mathbb{A}^1 \times \mathbb{A}^1$. Set $\sigma(x, c) = x^{d-1} + c$. Then $\sigma$ satisfies the above condition and it follows that $\tilde{X}(d, 2, 2) \cong \tilde{X}(d, 2) \times \mathbb{A}^1(-1)$. Since $\sigma(x, c)$ gives rise to an $H(d)$-equivariant section of the principal $G_a$-bundle $\tilde{X}(d, 2, 2) \rightarrow \tilde{X}'(d, 2)$, we have the assertion. \hfill $\square$

Next, we consider the case where $\tilde{X}(d, r)$ is defined by the equation $x^r z + y^d + ax^d = 1$ with $a \in \mathbb{C}$. 

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Lemma 2.14. Let $d < r \leq 2d$. Suppose that $\tilde{X}(d, r)$ is defined by $x^r z + y^d + ax^d = 1$ with $a \in \mathbb{C}$. Then there exists an $H(d)$-equivariant isomorphism
\[
\tilde{X}(d, r) \times \mathbb{A}^1(1 - r) \cong \tilde{V}(d, r) \times \mathbb{A}^1(1 - r).
\]

Proof. Let $\tilde{X}(d, r, r)$ be the affine variety obtained by $\tilde{X}(d, r)$ and $\tilde{V}(d, r)$. The polynomial $\sigma$ which gives rise to an $H(d)$-equivariant section of the principal $G_\alpha$-bundle $\tilde{X}(d, r, r) \to \tilde{X}(d, r)$ must satisfy condition (5) in Lemma 2.10 with $r = s$, $p_\lambda(x) = \lambda(1 - (a/d)x^d)$ and $p'_\lambda(x) = \lambda$. Define $\sigma(x, c) \in \mathbb{C}[x, c]$ by
\[
\sigma(x, c) = -\frac{a^2}{d^2}x^{2d-r} + \left(1 + \frac{a}{d}x^d\right)c.
\]
Then $\sigma$ satisfies the condition and we obtain an $H(d)$-equivariant isomorphism $\tilde{X}(d, r, r) \cong \tilde{X}(d, r) \times \mathbb{A}^1(1 - r)$. Similarly, one easily verifies that
\[
\tilde{X}(r)(1 - s, x, c) \equiv \tilde{X}(d, s) \times \mathbb{A}^1(1 - r) \times \mathbb{A}^1(1 - s).
\]

Combining Lemma 2.14 and Theorem 2.12, we obtain the following.

Lemma 2.15. Let $d < r \leq 2d$ and $d < s \leq 2d$. Suppose that $\tilde{X}(d, r)$ and $\tilde{X}'(d, s)$ are defined by $x^r z + y^d + ax^d = 1$ and $x^s z + y^d + a'x^d = 1$ with $a, a' \in \mathbb{C}$, respectively. Then there exists an $H(d)$-equivariant isomorphism
\[
\tilde{X}(d, r) \times \mathbb{A}^1(1 - r) \times \mathbb{A}^1(1 - s) \cong \tilde{X}'(d, s) \times \mathbb{A}^1(1 - r) \times \mathbb{A}^1(1 - s).
\]

Now, resume the set-up in Lemmas 2.9 and 2.10, and suppose that $r \equiv s \equiv 1 \pmod{d}$. Then we can find a polynomial $\sigma$ as satisfying the condition in Lemma 2.10 so that there exists an $H(d)$-equivariant isomorphism $\tilde{X}(d, r) \times \mathbb{A}^1 \cong \tilde{X}'(d, s) \times \mathbb{A}^1$, where $\mathbb{A}^1$ is the trivial $H(d)$-module.

Theorem 2.16. Let $d \geq 2$ and $r \equiv s \equiv 1 \pmod{d}$. Then there is an $H(d)$-isomorphism
\[
\tilde{X}(d, r) \times \mathbb{A}^1 \cong \tilde{X}'(d, s) \times \mathbb{A}^1.
\]

Proof. We first show that $\tilde{X}(d, r) \times \mathbb{A}^1 \cong \tilde{V}(d, 1) \times \mathbb{A}^1$ for any $r \equiv 1 \pmod{d}$. Consider the affine variety $\tilde{X}(d, r, 1)$ obtained by $\tilde{X}(d, r)$ and $\tilde{V}(d, 1)$. Then, the polynomial $\sigma$ which gives rise to an $H(d)$-equivariant section of the principal $G_\alpha$-bundle $\tilde{X}(d, r, 1) \to \tilde{X}(d, r)$ must satisfy
\[
x \sigma(\lambda^{-1} x, c + \frac{p_1(x) - p_\lambda(x)}{x^r}) = x \sigma(x, c) + 1 - \lambda
\]
for all $\lambda \in H(d), (x, c) \in \mathbb{A}^1 \times \mathbb{A}^1$, where $p_1(x) = 1 + a_1 x + \cdots + a_{r-1} x^{r-1}$ with $a_i \in \mathbb{C}$. It follows from $p_\lambda(x) = \lambda p_1(\lambda^{-1} x)$ that
\[
p_1(x) - p_\lambda(x) = p_1(x) - \lambda p_1(\lambda^{-1} x)
\]
\[
= (1 - \lambda) + \sum_{i=2}^{kd} a_i (1 - \lambda^{1-i}) x^i,
\]
where \( k \) is a non-negative integer such that \( r = 1 + kd \). Set

\[
\sigma(x, c) = f_0(x) + f_1(x)c,
\]

where

\[
f_0(x) = a_2x + \cdots + a_{kd}x^{kd-1}, \quad f_1(x) = x^{kd}.
\]

Then \( \sigma \) satisfies the above condition, and it follows that \( \tilde{X}(d, r, 1) \cong \tilde{X}(d, r) \times \mathbb{A}^1 \) as \( H(d) \)-varieties. Next, we find a polynomial \( \tau \) which gives rise to an \( H(d) \)-equivariant section of the principal \( G_\alpha \)-bundle \( \tilde{X}(d, r, 1) \to \tilde{V}(d, 1) \). The polynomial \( \tau \) must satisfy

\[
x^r \tau(\lambda^{-1}x, c + \frac{1-\lambda}{x}) = x^r \tau(x, c) + p_1(x) - p_\lambda(x)
\]

for all \( \lambda \in H(d) \), \((x, c) \in \mathbb{A}^1_x \times \mathbb{A}^1 \). Note that

\[
p_1(x) - p_\lambda(x) = (1 - \lambda) + \sum_{2 \leq i \leq d, 0 \leq j \leq k-1} a_{i+jd}(1 - \lambda^{1-i})x^{i+jd}.
\]

Set

\[
\tau(x, c) = c^rP_{1,r}(xc) + \sum_{2 \leq i \leq d, 0 \leq j \leq k-1} a_{i+jd}c^{r-i-jd}P_{d-i+1,r-i-jd}(xc),
\]

where \( P_{1,j}(x) \) is the polynomial defined in Proposition 2.11(2). Then \( \tau \) satisfies the above equation, and an \( H(d) \)-equivariant isomorphism \( \tilde{X}(d, r, 1) \cong \tilde{V}(d, 1) \times \mathbb{A}^1 \) holds. Hence we obtain an \( H(d) \)-equivariant isomorphism \( \tilde{X}(d, r) \times \mathbb{A}^1 \cong \tilde{V}(d, 1) \times \mathbb{A}^1 \) for any \( r \equiv 1 \pmod{d} \). Since we have an \( H(d) \)-isomorphism \( \tilde{X}'(d, s) \times \mathbb{A}^1 \cong \tilde{V}(d, 1) \times \mathbb{A}^1 \) for \( s \equiv 1 \pmod{d} \), there exists an \( H(d) \)-isomorphism \( \tilde{X}(d, r) \times \mathbb{A}^1 \cong \tilde{X}'(d, s) \times \mathbb{A}^1 \).

By Lemma 2.7 and Theorem 2.16, we obtain families of non-isomorphic affine surfaces \( X(d, r) \) with equivariant non-cancellation property. By taking the quotients by \( H(d) \), we obtain families of affine pseudo-planes with non-cancellation property.

**Theorem 2.17.** Let \( d \geq 2 \) and let \( r, s > 1 \) and \( r \equiv s \equiv 1 \pmod{d} \). Let \( \tilde{X}(d, r; f) \) be the affine hypersurface defined by \( x^2z + f(x, y) = 1 \), where \( f(x, y) \) is of the form \( y^d + a_2x^2y^{d-2} + \cdots + a_dx^d \) with \( a_j \in \mathbb{C} \). Then the quotient of \( X(d, r; f) \) by the Galois group \( H(d) \) is an affine pseudo-plane \( X(d, r; f) \) of type \((d, r)\), and the following assertions hold:

1. For any \( r \) and \( s \),

\[
X(d, r; f_1) \times \mathbb{A}^1 \cong X(d, s; f_2) \times \mathbb{A}^1,
\]

where \( f_1 \) and \( f_2 \) are monic homogeneous polynomials of the form stated above.

2. The isomorphism \( X(d, r; f_1) \cong X(d, s; f_2) \) holds if and only if \( r = s \) and \( f_1(x, y) = f_2(\mu x, y) \) for \( \mu \in \mathbb{C}^* \).
3. An application

Let $G$ be a reductive algebraic group. As an application of the results in the previous section, we construct the examples of families of affine $G$-varieties without equivariant cancellation property.

Let $Y = \text{Spec } R$ be an affine $G$-variety such that the invariant subring $R^G$ is a polynomial of degree $d$. Let $Y(d, r; f)$ be a hypersurface in $Y \times \mathbb{A}^2 = \text{Spec } R[y, z]$ defined by $x^r z + f(x, y) = 1$, where $f(x, y)$ is a monic homogeneous polynomial of degree $d$. Then $Y(d, r; f)$ has a $G_{a}$-action induced by a locally nilpotent $R$-derivation $D = x^r \partial_y - f_y \partial_z$. Since the $G_{a}$-action commutes with the $G$-action, the inclusion $\text{Ker } D = R \hookrightarrow R[y, z]$ induces a $G$-equivariant $\mathbb{A}^1$-fibration $\tilde{Y}(d, r; f) \to Y$. Let $\tilde{\pi} : \tilde{Y}(d, r; f) \to \tilde{X}(d, r; f)$ and $\pi : Y \to \mathbb{A}^1 = \text{Spec } \mathbb{C}[x]$ be the algebraic quotients by $G$, where $\tilde{X}(d, r; f)$ is the affine hypersurface in $\mathbb{A}^3 = \text{Spec } \mathbb{C}[x, y, z]$ defined by $x^r z + f(x, y) = 1$. Then it follows that $\tilde{Y}(d, r; f) = \tilde{X}(d, r; f) \times_{\text{Spec } \mathbb{C}[x]} Y$.

\[ \begin{array}{ccc} Y & \downarrow \pi & \tilde{Y}(d, r; f) \\ \downarrow & & \tilde{\pi} \downarrow \rho & \to & \tilde{X}(d, r; f) & \to \mathbb{A}^1 \end{array} \]

**Theorem 3.1.** Let $d \geq 2$ and let $r, s > d$. Let $\tilde{Y}(d, r; f_1)$ and $\tilde{Y}(d, s; f_2)$ be affine $G$-varieties defined by $f_1(x, y)$ and $f_2(x, y)$ as above, respectively, where $f_1$ and $f_2$ are homogeneous polynomials of the form $y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d$ for $a_i \in \mathbb{C}$ ($2 \leq i \leq d$). Then the following assertions hold:

1. For any $r$ and $s$, there is a $G$-equivariant isomorphism \[ \tilde{Y}(d, r; f_1) \times \mathbb{A}^1 \cong \tilde{Y}(d, s; f_2) \times \mathbb{A}^1 \].

2. The isomorphism of $G$-varieties $\tilde{Y}(d, r; f_1) \cong \tilde{Y}(d, s; f_2)$ holds if and only if $r = s$ and $f_1(x, y) = f_2(\mu x, y)$ for $\mu \in \mathbb{C}^*$.

**Proof.** The assertions follows from Theorem 2.8. \qed

**References**


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